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MULTIPLICITY RESULTS FOR A KIRCHHOFF SINGULAR PROBLEM INVOLVING THE FRACTIONAL P-LAPLACIAN

Mounir Hsini[1,†]

Abstract The aim of this paper is to study the multiplicity of solutions for a Kirchhoff singular problem involving the fractional p-Laplacian operator. Using the concentration compactness principle and Ekeland's variational principle, we obtain two positive weak solutions.

Keywords Fractional p-Laplacian operator, multiple positive solutions, Fibering maps, Nehari manifold.

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1. Introduction

In this project, we are motivated to study the existence of multiple weak solutions for the following Kirchhoff problem containing singular term

$$(P_\lambda)\begin{cases} M\left(\int_{\Omega^2}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)(-\Delta_p)^s u(x)=\lambda u^q+u^{-\alpha} \text{ in } \Omega,\\ u=0 \qquad \text{in } \partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 2$ is a bounded smooth domain, $1<p<q+1<p^*$, $s\in(0,1)$ and $0<\alpha<1$ while M is a continuous function and the fractional p-Laplacian operator $(\Delta)^s_p$ is given by

$$(-\Delta_p)^s u(x)=2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}dy, \text{ for } x\in\mathbb{R}^n.$$

Before giving our results, let us briefly recall literature concerning related problem (P_λ).

Equations and variational problems involving the fractional and non local operators have captured a special attention in the recent last years. Indeed, there are some physical phenomena which can be modeled by such kind of equations. In this context, many results have been obtained on this kind of problems, specially in finance, thin obstacle problem, optimization, quasi-geostrophic flow, geomorphological, electrorheological fluids [2,9,15,18,19,28,34].

Problems of Type (P_λ) are called a nonlocal problems due to the term M, which implies that these equations are no longer a pointwise equation. This causes

†the corresponding author.mounir.hsini@ipeit.rnu.tn
1Department of Mathematics,Elmanar University, Faculty of Sciences, Campus Universitaire 2092, Tunis, Tunisia

some mathematical difficulties which gives particularly interesting for the study of these problems. Note that these equations (nonlocal differential equations) are investigated in the following form

$$\rho\frac{\partial^2 u}{\partial t^2} - (\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2)\frac{\partial^2 u}{\partial x^2} = 0,$$

which extends the classical Alembert's wave equation.

Up to these days, a great deal of results have been obtained for solutions to equations related to the Kirchhoff problem. Precisely, in the past decade, many people have extensively studied the nonlinear boundary value problems involving this kind of equations. We will only state in this introduction those results that are related to the ones we address in this paper.

At 1978, Lions [22], proposed an abstract framework for the Kirchhoff type equations. Of course, we do not forget the contributions of Simon [29] and Berestycki and Brezis [6] which appeared in this period and they have studied the non local boundary conditions. Later Corrêa and all. studied the Kirchhoff problem.

Recently, the studies of Kirchhoff Dirichlet problems have been considered by variational methods, In this case we find the Kirchhoff equation involving p-Laplacian operator and operators in divergence form [3, 10, 23, 30].

Problem involving fractional power of the Laplace operator have been studied in a large number of works. Caffarelli and Silvestre [7] investigated the fractional Laplacian through extension theory. The existence, non existence and uniqueness of positive solution for the fractional laplacian was treated by Chen and all in [8]. Moreover they obtained a symmetry property of solutions for equations involving the fractional Laplacian.

Using the Nahari manifold, Ghanmi and Saoudi [17] studied the multiplicity of weak positive solutions for a semi linear problem involving the fractional Laplace operator.

The existence and multiplicity results of weak solutions for Singular elliptic problems have been studied by Crandall [12], [11], Liao [21] in the case of Laplace operator and Qin Li and all [20] in the case of the Kirchhoff p-laplacian operator. Precisely, by means of the concentration compactness principle and Ekeland's variational principle, Qin and all proved the existence of multiple solution for the following problem

$$\begin{cases} M\left(\|u\|^p\right)\Delta_p u(x) = \lambda u^{p^*} + \rho(x)u^{-\gamma} \text{ in } \Omega, \\ u = 0 \qquad \text{in } \partial\Omega, \end{cases}$$

where $M(t) = a + bt^k$ and $0 < \gamma < 1 < p$.

In [16], Ghanmi studied the following problem

$$M\left(\int_\Omega A(x, \nabla u)dx\right) \operatorname{div}(a(x, \nabla u)) = \lambda h(x)\frac{\partial F}{\partial u}(x, u), \text{ in } \Omega, u = 0 \text{ in } \partial\Omega.$$

Using variational arguments and the theory of variable exponent Sobolev spaces, the author proved the existence of non trivial weak solutions for some $\lambda \leq \lambda_0$.

In [5], the authors dealt with the singular Kirchhoff problem involving the $p(x)$-Laplacian operator

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_p u(x) = g(x)u^{-\gamma(x)} - \lambda f(x, u), \text{ in } \Omega, \ \ u = 0, \ \ \text{in } \partial\Omega,$$

where $M(t)$ is a positive continuous function on $(0,\infty)$ which is $L^1(0,T)$ for some $T>0$.

In [4], the authors were interested with the singular Dirichlet problem

$$(-\Delta)^s u = \lambda f(x)u^{-\gamma} + Mu^p,\ u>0 \text{ in } \Omega,\quad u=0 \text{ in } \partial\Omega,$$

where $M\geq 0,\ 0<s<1, \gamma>0, \lambda>0, 1<p<2_s^*-1$ and $f\in L^m(\Omega), m\geq 1$ is a non negative function. The authors studied the existence of the regularized problems with singular term $u^{-\gamma}$ replaced by the sequence $(u+\frac{1}{n})^{-\gamma}$. Note that the critical singular problem (case $p=2_s^*-1$) is studied in [26], where the multiplicity results are obtained using the Nehari manifold method.

There are many papers which are devoted to the study of p-fractional laplacian with polynomial type nonlinearities, where they study the subcritical problems using Nehari manifold and fibering maps, in this way, we cite [7,24,27] and the reference therein. In [25], Brezis-Nirenberg type critical exponent equation was investigated.

Inspired by the above results, we study the singular kirchhoff problem (P_λ) involving the p-fractional laplacian, which generalizes, improves and extends the above mentioned references under suitable other conditions. This gives and makes importance and significance to this project. Note that we need the concentration compactness principle and Ekeland's variational principle to obtain two positive weak solutions.

This paper is divided into four sections. In the next part, we introduce necessary notations, fundamental hypothesis and the spaces on which we work. We introduce also the main result. In the third section, we explore the Nehari manifold and Fibering maps. Note that we give some elementary results which will be useful to the proof of our principal Theorem, that is the object of section 4.

2. Preliminaries and Main results

In this section, we introduce some preliminary results which will be needed in the proof of the main result. For all $1<r\leq\infty$, we denote $\|\cdot\|_r$ the norm of the space $L^r(\Omega)$. In addition, if $0<s<1<p<\infty$ are real numbers, the fractional critical exponent is defined by

$$p_s^* = \frac{np}{n-sp}.$$

Due to the non-localness of the operator fractional p-Laplacian, we introduce the functional space E by

$$E=\{u:\mathbb{R}^n\to\mathbb{R},\ \text{is measurable},\quad u_{|\Omega}\in L^p(\Omega) \text{ and } \frac{u(x)-u(y)}{|x-y|^{\frac{n+sp}{p}}}\in L^p(X)\},$$

where $X=\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega\times\mathbb{R}^n\setminus\Omega)$.

The Gagliardo semi-norm for any measurable function $u:\mathbb{R}^n\to\mathbb{R}$ is given by

$$[u]_{s,p}=\Big(\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy\Big)^{\frac{1}{p}}$$

and the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ is defined by

$$W^{s,p}(\mathbb{R}^n)=\Big\{u\in L^p(\mathbb{R}^n) \text{ measurable and } [u]_{s,p}<\infty\Big\}.$$

The space $W^{s,p}(\mathbb{R}^n)$ is endowed with the norm

$$\|u\|_{s,p} := \Big(\|u\|_p + [u]_{s,p}^p\Big)^{\frac{1}{p}}.$$

Our basic space on which we shall work is

$$E = \Big\{u \in W^{s,p}(\mathbb{R}^n),\ u(x) = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\Big\}.$$

It's well known that E can be equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$. Moreover, $(E, \|\cdot\|)$ is a Banach space which is uniformly convex and the embedding $E \hookrightarrow L^q(\Omega)$ is continuous for all $1 \le q \le p_s^*$ and compact in the case $1 \le q < p_s^*$. The dual space of $(E, \|\cdot\|)$ is denoted by $(E^*, \|\cdot\|_*)$ and $< \cdot, \cdot >$ represents the duality product between E and E^*. Finally, for $1 \le q \le p_s^*$ we denote by S the best Sobolev constant for the operator $E \hookrightarrow L^q(\Omega)$, that is

$$S\|u\|_q^p \le \|u\|^p. \tag{2.1}$$

Definition 2.1. A function $u \in E$ is termed a positive weak solution of (P_λ) if $u > 0$ and

$$\begin{aligned} &M\Big(\|u\|^p\Big)\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}((u(x)-u(y))(v(x)-v(y)))}{|x-y|^{n+sp}}dxdy \\ &= \int_\Omega (u(x)^{-\alpha} + \lambda u(x)^q)v(x)\mathrm{d}x. \end{aligned} \tag{2.2}$$

We will associate to the problem (P_λ) the functional $J_\lambda : E \to \mathbb{R}$ defined as follow

$$J_\lambda(u) = \frac{1}{p}\widehat{M}\Big(\|u\|^p\Big) - \frac{1}{1-\alpha}\int_\Omega |u(x)|^{1-\alpha}dx - \frac{\lambda}{1+q}\int_\Omega |u(x)|^{q+1}\mathrm{d}x, \tag{2.3}$$

where

$$\widehat{M}(t) = \int_0^t M(s)ds.$$

Throughout, this paper we suppose that $M : (0, +\infty) \to (0, +\infty)$ is a continuous function defined by

$$M(s) = as^m, \quad a > 0, \tag{2.4}$$

where $q + 1 > p(m + 1)$. Under the assumption (2.4) needed on the function M, our fundamental result can be described as follow.

Theorem 2.1. *Assume that (2.1), (2.4) and* $0 < \alpha < 1 < p < p(m+1) < q+1 < p^* - 1$ *are fulfilled. Then, problem* (P_λ) *has at least two positive solutions for all* $\lambda \in (0, T_{q,\alpha})$, *where*

$$T_{q,\alpha} = ABa^{\frac{\alpha+q}{p(m+1)+\alpha-1}}S^{(1-\alpha)\frac{q+1-p(m+1)}{p(m+1)+\alpha-1}},$$

$$A = \left(\frac{p(m+1)+\alpha-1}{q+1-p(m+1)}\right) \quad \text{and} \quad B = \left(\frac{q+1-p(m+1)}{(q+\alpha)}\right)^{\frac{\alpha+q}{p(m+1)+\alpha-1}}.$$

3. Nehari manifold and fibering maps

It's well known that the singular term leads to the fact that functional $J_\lambda \notin C^1(\mathcal{W}^{1,p}(\Omega), \mathbb{R})$. Nevertheless, we get the multiplicity of solutions for the problem (P_λ) by investigating suitable minimization problems for the functional J_λ.

Notice that any solution u of the problem (P_λ) must be positive and satisfies the equation

$$M\Big(\|u\|^p\Big)\|u\|^p - \int_\Omega |u(x)|^{1-\alpha}dx - \lambda \int_\Omega |u(x)|^{q+1}\mathrm{d}x = 0. \tag{3.1}$$

Therefore, any weak solution must be in the Nehari manifold defined as follow

$$\mathcal{N}_\lambda = \Big\{u \in E,\ M\Big(\|u\|^p\Big)\|u\|^p - \int_\Omega |u(x)|^{1-\alpha}\,\mathrm{d}x - \lambda \int_\Omega |u(x)|^{1+q}\,\mathrm{d}x = 0\Big\}.$$

To obtain the multiplicity of solutions, we decompose $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection, are measurable sets defined as follows:

$$\mathcal{N}_\lambda = \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-,$$

where

$$\mathcal{N}_\lambda^0 = \Big\{u \in \mathcal{N}_\lambda,\ a(p(m+1)+\alpha-1)\|u\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |u(x)|^{1+q}\,\mathrm{d}x = 0\Big\},$$

$$\mathcal{N}_\lambda^+ = \Big\{u \in \mathcal{N}_\lambda,\ a(p(m+1)+\alpha-1)\|u\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |u(x)|^{1+q}\,\mathrm{d}x > 0\Big\},$$

and

$$\mathcal{N}_\lambda^- = \Big\{u \in \mathcal{N}_\lambda,\ a(p(m+1)+\alpha-1)\|u\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |u(x)|^{1+q}\,\mathrm{d}x < 0\Big\}.$$

Finally, we prove that a minimizer of J_λ on $\mathcal{N}_\lambda^+$ (and in $\mathcal{N}_\lambda^-$ respectively) is a positive solution of (2.2).

Our first preliminary result deals with the coercivity of the functional J_λ.

Lemma 3.1. *J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. The proof is immediately deduced from (2.1) and the definition of the set $\mathcal{N}_\lambda$. □

Lemma 3.2. *Under the assumptions (2.1), (2.4) and* $(0 < \alpha < 1 < p < p(m+1) < q+1 < p^*-1)$. *Then, for all $\lambda \in (0, T_{q,\alpha})$, there exist t_0^+ and t_0^- such that $t_0^+u \in \mathcal{N}_\lambda^+$ and $t_0^-u \in \mathcal{N}_\lambda^-$.*

Proof. Let us introduce the function

$$\Phi(t) = a\|u\|^{p(m+1)}t^{p(m+1)-q-1} - t^{-\alpha-q}\int_\Omega |u(x)|^{1-\alpha}\,\mathrm{d}x.$$

It's an easy task to see that $\Phi'(t) = 0$ if and only if

$$t = t_{max} = \left(\frac{(\alpha+q)\int_\Omega |u(x)|^{1-\alpha}\,\mathrm{d}x}{a(q+1-p(m+1))\|u\|^{p(m+1)}}\right)^{\frac{1}{\alpha+p(m+1)-1}}.$$

A Straightforward calculation gives as :

$$\Phi(t_{max}) = A.B.a^{\frac{\alpha+q}{p(m+1)+\alpha-1}} \|u\|^{\frac{p(m+1)(\alpha+q)}{p(m+1)+\alpha-1}} \left(\int_\Omega |u(x)|^{1-\alpha}\,\mathrm{d}x\right)^{\frac{p(m+1)-q-1}{p(m+1)+\alpha-1}}, \qquad (3.2)$$

where

$$A = \left(\frac{p(m+1)+\alpha-1}{q+1-p(m+1)}\right) \text{ and } B = \left(\frac{(q+1-p(m+1))}{(q+\alpha)}\right)^{\frac{\alpha+q}{p(m+1)+\alpha-1}}. \qquad (3.3)$$

In view of assumption (2.1) we obtain :

$$\Phi(t_{max})-\lambda\int_\Omega |u(x)|^{1+q}\,\mathrm{d}x > ABS^{(1-\alpha)\frac{q+1-p(m+1)}{p(m+1)+\alpha-1}} \frac{\|u\|^{\frac{p(m+1)(\alpha+q)}{p(m+1)+\alpha-1}}}{\|u\|^{(1-\alpha)\frac{q+1-p(m+1)}{p(m+1)+\alpha-1}}} - \lambda\|u\|^{1+q}.$$

That is

$$\Phi(t_{max}) - \lambda\int_\Omega |u(x)|^{1+q}\,\mathrm{d}x > ABS^{(1-\alpha)\frac{q+1-p(m+1)}{p(m+1)+\alpha-1}} \|u\|^{1+q} - \lambda\|u\|^{1+q}$$

and then

$$\Phi(t_{max}) - \lambda\int_\Omega |u(x)|^{1+q}\,dx \ > (T_{q,\alpha}-\lambda)\,\|u\|^{1+q} \ > 0, \text{ for all } \lambda \in (0, T_{q,\alpha}). \qquad (3.4)$$

Hence, there exist $0 < t_0^+ < t_{max} < t_0^-$ satisfying

$$\Phi(t_0^+) = \lambda\int_\Omega |u(x)|^{1+q}\,\mathrm{d}x \ = \Phi(t_0^-)$$

and

$$\Phi'(t_0^+) \ < 0 \ < \ \Phi'(t_0^-).$$

Therefore, $t_0^+u \in \mathcal{N}_\lambda^+$ and $t_0^-u \in \mathcal{N}_\lambda^-$, as required. □

Lemma 3.3. *Suppose that $\lambda \in (0, T_{q,\alpha})$, then the sets $\mathcal{N}_\lambda^\pm$ are non empty and $\mathcal{N}_\lambda^0 = \{0\}$. In addition, $\mathcal{N}_\lambda^-$ is a closed set in E-topology.*

Proof. Due to the Lemma 3.2, we obtain $\mathcal{N}_\lambda^\pm \neq \emptyset$, for any $\lambda \in (0, T_{q,\alpha})$. Let us argue by contradiction that $\mathcal{N}_\lambda^0 = \{0\}$ and suppose that there exits $v \neq 0$ in the set $\mathcal{N}_\lambda^0$. Then,

$$a(p(m+1)+\alpha-1)\|v\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |v(x)|^{1+q}\,\mathrm{d}x = 0$$

and

$$a\|v\|^{p(m+1)} - \int_\Omega |v(x)|^{1-\alpha}\,\mathrm{d}x - \lambda\int_\Omega |v(x)|^{1+q}\,\mathrm{d}x = 0.$$

Consequently, we obtain

$$\int_\Omega |v(x)|^{1-\alpha}\,dx = \frac{a(q+1-p(m+1))}{\alpha+q}\|v\|^{p(m+1)}. \qquad (3.5)$$

Combining equations (3.2) and (3.5), we get

$$\begin{aligned}
0 <&\Phi(t_{max}) - \lambda \int_\Omega |v(x)|^{1+q}\, dx \\
=&A.Ba^{\frac{\alpha+q}{p(m+1)+\alpha-1}} \frac{\|v\|^{\frac{p(m+1)(\alpha+q)}{p(m+1)+\alpha-1}}}{\left(\int_\Omega |v(x)|^{1-\alpha}\,\mathrm{d}x\right)^{\frac{q+1-p(m+1)}{p(m+1)+\alpha-1}}} - \lambda \int_\Omega |v(x)|^{1+q}\, dx \\
=&A.B.a^{\frac{\alpha+q}{p(m+1)+\alpha-1}} \frac{\|v\|^{\frac{p(m+1)(\alpha+q)}{p(m+1)+\alpha-1}}}{\left(\frac{a(q+1-p(m+1))}{\alpha+q}\|v\|^{p(m+1)}\right)^{\frac{q+1-p(m+1)}{p(m+1)+\alpha-1}}} \\
&-\frac{a(p(m+1)+\alpha-1)}{(q+\alpha)}\|v\|^{p(m+1)} = 0, \qquad (3.6)
\end{aligned}$$

where A and B are defined in (3.3), which is a contradiction. Hence, $\mathcal{N}_\lambda^0 = \{0\}$, for all $\lambda \in (0, T_{q,\alpha})$ as required. Now let us consider a sequence $(u_n) \subset \mathcal{N}_\lambda^-$ such that $(u_n) \to u$ in E. From the definition of the set $\mathcal{N}_\lambda^-$, we get

$$a(p(m+1)+\alpha-1)\|u_n\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |u_n(x)|^{1+q}\,\mathrm{d}x < 0 \qquad (3.7)$$

and

$$a\|u_n\|^{p(m+1)} - \int_\Omega |u_n(x)|^{1-\alpha}\,\mathrm{d}x - \lambda\int_\Omega |u_n(x)|^{1+q}\,\mathrm{d}x = 0.$$

Consequently,

$$a(p(m+1)+\alpha-1)\|u\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |u(x)|^{1+q}\,\mathrm{d}x \leq 0$$

and

$$a\|u\|^{p(m+1)} - \int_\Omega |u(x)|^{1-\alpha}\,\mathrm{d}x - \lambda\int_\Omega |u(x)|^{1+q}\,\mathrm{d}x = 0.$$

Thus $u \in \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$. If $u \in \mathcal{N}_\lambda^0$, then $u = 0$. By equation (3.7) we obtain

$$\|u\| \geq \left[\frac{p(m+1)+\alpha-1}{\lambda(q+\alpha)}\right] > 0. \qquad (3.8)$$

It's in contradiction with the fact that $u = 0$. Therefore $u \in \mathcal{N}_\lambda^-$, for all $\lambda \in (0, T_{q,\alpha})$ and the proof is completed. ☐

Lemma 3.4. *Given $u \in \mathcal{N}_\lambda^-$ (respectively $\mathcal{N}_\lambda^+$) with $u \geq 0$, for all $v \in E$ with $v \geq 0$, there exist $\varepsilon > 0$ and a continuous function h such that for all $\beta \in \mathbb{R}$ with $|\beta| < \varepsilon$ we have*

$$h(0) = 1 \quad \textit{and} \;\; h(\beta)(u+\beta v) \in \mathcal{N}_\lambda^- \;\; (\textit{respectively } \mathcal{N}_\lambda^+).$$

Proof. We introduce the function $\psi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ define by:

$$\psi(t,\beta) = at^{p(m+1)+\alpha-1}\|(u+\beta v)\|^{p(m+1)}\,\mathrm{d}x - \int_\Omega \left(|u+\beta v|^{1-\alpha} - \lambda t^{\alpha+q}|u+\beta v|^{1+q}\right)\mathrm{d}x.$$

Hence,

$$\psi_t(t,\beta) =a(p(m+1)+\alpha-1)t^{p(m+1)+\alpha-2}\|(u+\beta v)\|^{p(m+1)}\,\mathrm{d}x$$

$$- \lambda(\alpha+q)t^{\alpha+q-1}\int_\Omega |u+\beta v|^{1+q}\,\mathrm{d}x,$$

is continuous on $\mathbb{R}\times\mathbb{R}$. Since $u\in\mathcal{N}_\lambda^-\subset\mathcal{N}_\lambda$, we have $\psi(1,0)=0$ and

$$\psi_t(1,0)=a(p(m+1)+\alpha-1)\|u\|^{p(m+1)}\,\mathrm{d}x-\lambda(\alpha+q)\int_\Omega |u|^{1+q}\,\mathrm{d}x<0.$$

Therefore, applying the implicit function theorem to the function ψ at the point $(1,0)$ we obtain a $\delta>0$ and a positive continuous function h satisfying

$$h(0)=1,\ h(s)(u+\beta v)\in\mathcal{N}_\lambda,\ \forall\,\beta\in\mathbb{R},\ |\beta|<\delta.$$

Hence, taking $\varepsilon>0$ possibly smaller enough, we get

$$h(\beta)(u+\beta v)\in\mathcal{N}_\lambda^-,\ \forall\beta\in\mathbb{R},\ |\beta|<\varepsilon.$$

The case $u\in\mathcal{N}_\lambda^+$. may be obtained in the same way. This completes the proof of the Lemma 3.4. □

4. Proof of the main result

For any $u\in\mathcal{N}_\lambda$, we have $J_\lambda(u)=J_\lambda(|u|)$, then we can assume without loose any generalities that all functions of the set $\mathcal{N}_\lambda$ are non negative. Using Lemma 3.2 and 3.3, we can denote

$$m^+=\inf_{u\in\mathcal{N}_\lambda^+}J_\lambda(u)\ \text{ and } m^-=\inf_{u\in\mathcal{N}_\lambda^-}J_\lambda(u). \tag{4.1}$$

Proposition 4.1. *Let $\lambda\in(0,T_{q,\alpha})$. Then, $m^+=\inf_{u\in\mathcal{N}_\lambda^+}J_\lambda(u)<0$.*

Proof. Recall that for $u\in\mathcal{N}_\lambda^+$, we have

$$a(p(m+1)+\alpha-1)\|u\|^{p(m+1)}-\lambda(q+\alpha)\int_\Omega |u(x)|^{1+q}\,\mathrm{d}x>0.$$

Due to the fact that $0<\alpha<1$ and $p(m+1)<q+1$, we obtain

$$\begin{aligned}J_\lambda(u)&=\frac{a}{p(m+1)}\|u\|^{p(m+1)}\,\mathrm{d}x-\frac{q+\alpha}{(1-\alpha)(1+q)}\int_\Omega |u(x)|^{1-\alpha}dx\\&<a\frac{1+\alpha-p(m+1)}{p(m+1)(1-\alpha)}\|u\|^{p(m+1)}+a\frac{p(m+1)+\alpha-1}{(q+1)(1+\alpha)}\|u\|^{p(m+1)}\\&=a\Big(\frac{1+\alpha-p(m+1)}{(1-\alpha)}\Big)\Big(\frac{1}{q+1}-\frac{1}{p(m+1)}\Big)\|u\|^{p(m+1)}<0.\end{aligned}$$

Consequently,

$$m^+=\inf_{u\in\mathcal{N}_\lambda^+}J_\lambda(u)<0,\ \text{ for all }\lambda\in(0,T_{q,\alpha}).$$

□

We divide two steps to complete the proof of Theorem 1.1.

First Step Applying Ekeland's variational principle to this minimization problem. There exists a sequence $(u_n)\subset\mathcal{N}_\lambda^+$ satisfying these properties:

i) $J_\lambda(u_n) < m^+ + \frac{1}{n}$.
ii) $m^+ \geq J_\lambda(u_n) - \frac{1}{n}\|u - u_n\|$, for all $u \in \mathcal{N}_\lambda^+$.

Due to the fact that $J_\lambda(u) = J_\lambda(|u|)$, we can assume that $u_n(x) \geq 0$. Using the coercivity of the functional J_λ on $\mathcal{N}_\lambda$, we deduce that $\{u_n\}$ is a bounded sequence in E. There exists a sub-sequence denoted again by $\{u_n\}$, there exists $u_0 \geq 0$ such that $u_n \rightharpoonup u_0$, weakly in E, $u_n \to u_0$, strongly in $L^q(\Omega)$, for $1 \leq q < p^*$, and $u_n(x) \to u_0(x)$, a.e. in Ω, as $n \to \infty$. Now, from (4.1) and using the weak lower semi-continuity of the norm, we get $J_\lambda(u_0) \leq \liminf J_\lambda(u_n) = \inf\limits_{\mathcal{N}_\lambda{}^+} J_\lambda$, we see that $u_0 \not\equiv 0$ in Ω. Moreover, we have

Proposition 4.2. $u_0(x) > 0$, a.e. *in* Ω.

Proof. Applying Hölder's inequality, we obtain, as $n \to \infty$,

$$\begin{aligned}\int_\Omega u_n^{1-\alpha}\mathrm{d}x &\leq \int_\Omega u_0^{1-\alpha}\mathrm{d}x + \int_\Omega \mid u_n - u_0 \mid^{1-\alpha} \mathrm{d}x\\ &\leq \int_\Omega u_0^{1-\alpha}\mathrm{d}x + C \parallel u_n - u_0 \parallel_{L^2(\Omega)}^{1-\alpha}\\ &= \int_\Omega u_0^{1-\alpha}\mathrm{d}x + o(1).\end{aligned}$$

In addition,

$$\begin{aligned}\int_\Omega u_0^{1-\alpha}\mathrm{d}x &\leq \int_\Omega u_n^{1-\alpha}\mathrm{d}x + \int_\Omega \mid u_n - u_0 \mid^{1-\alpha} \mathrm{d}x\\ &\leq \int_\Omega u_0^{1-\alpha}\mathrm{d}x + C \parallel u_n - u_0 \parallel_{L^2(\Omega)}^{1-\alpha}\\ &= \int_\Omega u_n^{1-\alpha}\mathrm{d}x + o(1).\end{aligned}$$

Consequently,

$$\int_\Omega u_n^{1-\alpha}\mathrm{d}x = \int_\Omega u_0^{1-\alpha}\mathrm{d}x + o(1). \tag{4.2}$$

Similarly, we have

$$\int_\Omega u_n^{1+q}\mathrm{d}x = \int_\Omega u_0^{1+q}\mathrm{d}x + o(1). \tag{4.3}$$

Due to (4.2), (4.3) and the weakly lower semi continuity of the norm, we get

$$\begin{aligned}J_\lambda(u_0) &= \frac{a}{p(m+1)}\|u_0\|^{p(m+1)} - \frac{1}{1-\alpha}\int_\Omega |u_0(x)|^{1-\alpha}dx - \frac{\lambda}{1+q}\int_\Omega |u_0(x)|^{q+1}\mathrm{d}x\\ &\leq \liminf_{n\to\infty}\Big[\frac{a}{p(m+1)}\|u_n\|^{p(m+1)} - \frac{1}{1-\alpha}\int_\Omega |u_n(x)|^{1-\alpha}dx\\ &\quad - \frac{\lambda}{1+q}\int_\Omega |u_n(x)|^{q+1}dx\Big]\\ &= \liminf_{n\to\infty} J_\lambda(u_n) < 0.\end{aligned}$$

Hence, $u_0 \neq 0$ on Ω. In the sequel we prove that $u_0 > 0$.

Since $u_n \in \mathcal{N}_\lambda^+$, we get

$$a(p(m+1) + \alpha - 1)\|u_n\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |u_n(x)|^{1+q}\,\mathrm{d}x > 0. \tag{4.4}$$

Thus we can claim that up to a subsequence u_n (still denoted by u_n), there exists a constant $c_1 > 0$ such that

$$a(p(m+1)+\alpha-1)\|u_n\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |u_n(x)|^{1+q}\,\mathrm{d}x \geq c_1. \tag{4.5}$$

That is

$$(p(m+1)+\alpha-1)\int_\Omega |u_n(x)|^{1-\alpha}\,\mathrm{d}x - \lambda(q+1-p(m+1))\int_\Omega |u_n(x)|^{1+q}\,\mathrm{d}x \geq 0. \tag{4.6}$$

Passing to the limit as $n \to \infty$, we obtain

$$(p(m+1)+\alpha-1)\int_\Omega |u_0(x)|^{1-\alpha}\,\mathrm{d}x - \lambda(q+1-p(m+1))\int_\Omega |u_0(x)|^{1+q}\,\mathrm{d}x \geq 0. \tag{4.7}$$

Next, we argue by contradiction that

$$(p(m+1)+\alpha-1)\int_\Omega |u_0(x)|^{1-\alpha}\,\mathrm{d}x - \lambda(q+1-p(m+1))\int_\Omega |u_0(x)|^{1+q}\,\mathrm{d}x > 0. \tag{4.8}$$

To prove that assume

$$(p(m+1)+\alpha-1)\int_\Omega |u_0(x)|^{1-\alpha}\,\mathrm{d}x - \lambda(q+1-p(m+1))\int_\Omega |u_0(x)|^{1+q}\,\mathrm{d}x = 0. \tag{4.9}$$

Since,

$$a\|u_n\|^{p(m+1)} - \lambda\int_\Omega |u_n(x)|^{1+q}\,\mathrm{d}x - \int_\Omega |u_n(x)|^{1-\alpha}\,\mathrm{d}x = 0 \tag{4.10}$$

and using the weakly semi-continuity of the norm, we get

$$\begin{aligned}
0 &\geq a\|u_0\|^{p(m+1)} - \lambda\int_\Omega |u_0(x)|^{1+q}\,\mathrm{d}x - \int_\Omega |u_0(x)|^{1-\alpha}\,\mathrm{d}x\\
&= a\|u_0\|^{p(m+1)} - \frac{q+\alpha}{q+1-p(m+1)}\int_\Omega |u_0(x)|^{1-\alpha}dx\\
&= a\|u_0\|^{p(m+1)} - \lambda\frac{q+\alpha}{p(m+1)+\alpha-1}\int_\Omega |u_n(x)|^{1+q}\,\mathrm{d}x.
\end{aligned}$$

In view of (3.3) and (3.6)

$$\begin{aligned}
0 <& A.B.a^{\frac{\alpha+q}{p(m+1)+\alpha-1}}\left(\int_\Omega |u_0(x)|^{1-\alpha}\,\mathrm{d}x\right)^{\frac{q+1-p(m+1)}{p(m+1)+\alpha-1}} - \lambda\int_\Omega |u_0(x)|^{1+q}\,dx\\
=& AB.a^{\frac{\alpha+q}{p(m+1)+\alpha-1}}\frac{\|u_0\|^{\frac{p(m+1)(\alpha+q)}{p(m+1)+\alpha-1}}}{\left(\frac{a(q+1-p(m+1))}{\alpha+q}\|u_0\|^{p(m+1)}\right)^{\frac{q+1-p(m+1)}{p(m+1)+\alpha-1}}}\\
&- \frac{a(p(m+1)+\alpha-1)}{(q+\alpha)}\|u_0\|^{p(m+1)} = 0,
\end{aligned}$$

which is a contradiction. Therefore, our claim holds true.

Now, let us consider the function $\psi \in E$, with $\psi \geq 0$. From Lemma 3.4 with $u = u_n$, there exits a sequence of continuous functions $h_n = h_n(t)$ such that $h_n(t)(u_n + t\psi) \in \mathcal{N}_\lambda^+$ and $h_n(0) = 1$. Hence,

$$ah_n^{p(m+1)}(t)\|u_n + t\psi\|^{p(m+1)} - \lambda h_n^{q+1}(t)\int_\Omega |u_n(x) + t\psi|^{1+q}\,\mathrm{d}x$$

$$- h_n^{1-\alpha}(t)\int_\Omega |u_n(x)+t\psi|^{1-\alpha}\,\mathrm{d}x = 0.$$

Due to (4.10), we obtain

$$\begin{aligned}
0=&a(h_n^{p(m+1)}(t)-1)\|u_n+t\psi\|^{p(m+1)} + a(\|u_n+t\psi\|^{p(m+1)} - \|u_n\|^{p(m+1)})\\
&-\lambda(h_n^{q+1}(t)-1)\int_\Omega |u_n(x)+t\psi|^{1+q}\,\mathrm{d}x - (h_n^{1-\alpha}(t)-1)\int_\Omega |u_n(x)+t\psi|^{1-\alpha}\,\mathrm{d}x\\
&-\int_\Omega (|u_n(x)+t\psi|^{1-\alpha}-|u_n(x)|^{1-\alpha})dx - \lambda\int_\Omega (|u_n(x)+t\psi|^{1+q}-|u_n(x)|^{1+q})dx\\
\le& a(h_n^{p(m+1)}(t)-1)\|u_n+t\psi\|^{p(m+1)} + a(\|u_n+t\psi\|^{p(m+1)} - \|u_n\|^{p(m+1)})\\
&-\lambda(h_n^{q+1}(t)-1)\int_\Omega (|u_n(x)+t\psi|^{1+q}-|u_n(x)|^{1+q})\,\mathrm{d}x\\
&-(h_n^{1-\alpha}(t)-1)\int_\Omega (u_n(x)+t\psi)^{1-\alpha}.
\end{aligned}$$

Dividing the above quantity by $t>0$ and passing to the limit as $t\to 0$, we get

$$\begin{aligned}
0 \le& h_n'(0)\Big[ap(m+1)\|u_n\|^{p(m+1)} - (1-\alpha)\int_\Omega (u_n(x))^{1-\alpha}dx - \lambda(1+q)\int_\Omega |u_n(x)|^{1+q}dx\Big]\\
&+ p(m+1)\int_{\mathbb{R}^{2n}} \frac{|u_n(x)-u_n(y)|^{p-2}((u_n(x)-u_n(y))(\psi(x)-\psi(y)))}{|x-y|^{n+sp}}dxdy.\\
=& h_n'(0)\Big[a(p(m+1)-q-1)\|u_n\|^{p(m+1)} + (q+\alpha)\int_\Omega (u_n(x))^{1-\alpha}dx\Big]\\
&+ p(m+1)\int_{\mathbb{R}^{2n}} \frac{|u_n(x)-u_n(y)|^{p-2}((u_n(x)-u_n(y))(\psi(x)-\psi(y)))}{|x-y|^{n+sp}}dxdy,
\end{aligned}$$

where $h_n'(0)\in[-\infty,\infty]$ denotes the right derivative of $h_n(t)$ at zero and since $u_n\in\mathcal{N}^+$, $h_n'(0)\neq-\infty$. To simplify, we suppose that the right derivative of h_n at $t=0$ exists. Moreover, from (4.9) $h_n'(0)$ is uniformly bounded from below. Now, using the condition (ii), we get

$$\begin{aligned}
&|h_n(t)-1|\frac{1}{n}||u_n|| + th_n(t)\frac{\|\psi\|}{n} \ge J_\lambda(u_n) - J_\lambda\left(h_n(t)(u_n+t\psi)\right)\\
=& -a\frac{p(m+1)+\alpha-1}{p(m+1)(1-\alpha)}\|u_n\|^{p(m+1)} + \lambda\frac{q+\alpha}{(1-\alpha)(q+1)}\int_\Omega |u_n(x)|^{1+q}\,\mathrm{d}x\\
&+ a\frac{p(m+1)+\alpha-1}{p(m+1)(1-\alpha)}h_n^{p(m+1)}(t)\|u_n+t\psi\|^{p(m+1)}\\
&- \frac{\lambda(q+\alpha)}{(1-\alpha)(q+1)}h_n^{q+1}(t)\int_\Omega |u_n(x)+t\psi|^{1+q}\,\mathrm{d}x\\
=& a\frac{p(m+1)+\alpha-1}{p(m+1)(1-\alpha)}\Big[\|u_n+t\psi\|^{p(m+1)} - \|u_n\|^{p(m+1)} + (h_n^{p(m+1)}(t)-1)\|u_n+t\psi\|^{p(m+1)}\Big]\\
&-\lambda\frac{q+\alpha}{(1-\alpha)(q+1)}\Big[\int_\Omega (|u_n(x)+t\psi|^{1+q}-|u_n(x)|^{1+q}) + (h_n^{q+1}(t)-1)\int_\Omega |u_n(x)+t\psi|^{1+q}\Big].
\end{aligned}$$

If we divide the last inequality by $t>0$ and we pass to limit as $t\to 0$, we get

$$\frac{|h_n'(0)|\|u_n(x)\| + \|\psi\|}{n}$$

$$\geq \frac{h_n'(0)}{1-\alpha}\Big[\frac{q+\alpha}{(q+1)}\int_\Omega (u_n(x))^{1-\alpha}dx - a\frac{p(m+1)+\alpha-1}{(1-\alpha)}\|u_n\|^{p(m+1)}\Big]$$
$$+ap\frac{p(m+1)+\alpha-1}{(1-\alpha)}\int_{\mathbb{R}^{2n}}\frac{|u_n(x)-u_n(y)|^{p-2}((u_n(x)-u_n(y))(\psi(x)-\psi(y)))}{|x-y|^{n+sp}}dxdy$$
$$-\lambda\frac{q+\alpha}{(1-\alpha)(q+1)}\int_\Omega |u_n(x)|^{1+q}\,\mathrm{d}x.$$

Due to (4.9), there exists a positive constant $C>0$ such that

$$\frac{(q+1-p(m+1))}{\alpha-1}|u_n(x)|^{p(m+1)} - \frac{(\alpha+q)}{\alpha-1}\int_\Omega (u_n(x))^{1-\alpha}dx - \frac{\|u_n(x)\|}{n} \geq C > 0. \tag{4.11}$$

Combining (4.9) and (4.11), $h_n'(0)$ is uniformly bounded from above. Consequently,

$$h_n'(0)\quad \text{is uniformly bounded for } n \text{ large enough.} \tag{4.12}$$

Hence condition (ii) implies that for $t>0$ small enough,

$$J_\lambda(u_n) \leq J_\lambda(h_n(t)(u_n+t\psi)) + \frac{1}{n}\|h_n(t)(u_n+t\psi)-u_n\|. \tag{4.13}$$

Therefore,

$$\frac{1}{n}\left(|h_n(t)-1|\|u_n\| + th_n(t)\|\psi\|\right)$$
$$\geq \frac{1}{n}\|h_n(t)(u_n+t\psi)-u_n\| \geq J_\lambda(u_n) - J_\lambda\left(h_n(t)(u_n+t\psi)\right)$$
$$= -\frac{a(h_n^{p(m+1)}(t)-1)}{p(m+1)}\|u_n\|^{p(m+1)} + \frac{h_n^{1-\alpha}(t)-1}{1-\alpha}\int_\Omega u_n^{1-\alpha}dx$$
$$+\frac{h_n^{1-\alpha}(t)}{1-\alpha}\int_\Omega \left((u_n+t\psi)^{1-\alpha} - u_n^{1-\alpha}\right)dx$$
$$+\frac{h_n^{p(m+1)}(t)}{p(m+1)}\left(\int_\Omega (\|u_n\|^{p(m+1)} - \|u_n+t\psi\|^{p(m+1)})dx\right)$$
$$+\frac{\lambda}{1+q}h_n^{1+q}(t)\int_\Omega (u_n+t\psi)^{1+q} - u_n^{1+q}dx + \frac{\lambda}{1+q}(h_n^{1+q}(t)-1)\int_\Omega u_n^{1+q}dx.$$

Passing to the limit $t\to 0$ after dividing by $t>0$, we get

$$\frac{1}{n}\left(|h_n'(0)|\|u_n\| + \|\psi\|\right)$$
$$\geq -ah_n'(0)\left[\|u_n\|^{p(m+1)} + \int_\Omega u_n^{1-\alpha}dx + \lambda\int_\Omega u_n^{1+q}dx\right]$$
$$+\lambda\int_\Omega u_n^q\psi dx - a\|u_n\|^{pm}\int_{\mathbb{R}^{2n}}\frac{|u_n(x)-u_n(y)|^{p-2}((u_n(x)-u_n(y))(\psi(x)-\psi(y)))}{|x-y|^{n+sp}}dxdy$$
$$+\liminf_{t\to 0^+}\frac{1}{1-\alpha}\int_\Omega\left(\frac{(u_n+t\psi)^{1-\alpha}-u_n^{1-\alpha}}{t}\,\mathrm{d}x\right)$$

and so

$$\frac{1}{n}\left(|h_n'(0)|\|u_n\| + \|\psi\|\right)$$

$$
\begin{aligned}
\geq&\lambda\int_\Omega u_n^q\psi dx - a\|u_n\|^{pm}\int_{\mathbb{R}^{2n}}\frac{|u_n(x)-u_n(y)|^{p-2}((u_n(x)-u_n(y))(\psi(x)-\psi(y)))}{|x-y|^{n+sp}}dxdy\\
&+\liminf_{t\to0^+}\frac{1}{1-\alpha}\int_\Omega\left(\frac{(u_n+t\psi)^{1-\alpha}-u_n^{1-\alpha}}{t}\,\mathrm{d}x\right).
\end{aligned}\tag{4.14}
$$

Consequently,

$$
\begin{aligned}
&\liminf_{t\to0^+}\frac{1}{1-\alpha}\int_\Omega\left(\frac{(u_n+t\psi)^{1-\alpha}-u_n^{1-\alpha}}{t}\,\mathrm{d}x\right)\\
\leq&\frac{1}{n}\left(|h_n'(0)|\|u_n\|+\|\psi\|\right)-\lambda\int_\Omega u_n^q\psi dx\\
&+a\|u_n\|^{pm}\int_{\mathbb{R}^{2n}}\frac{|u_n(x)-u_n(y)|^{p-2}((u_n(x)-u_n(y))(\psi(x)-\psi(y)))}{|x-y|^{n+sp}}dxdy.
\end{aligned}\tag{4.15}
$$

Using the fact that

$$[(u_n+t\psi)^{1-\alpha}-u_n^{1-\alpha}]\geq0,\quad \forall\, x\in\Omega,\ \forall\, t>0$$

and applying Fatou's Lemma we get

$$\int_\Omega u_n^{-\alpha}\psi\,\mathrm{d}x\leq\liminf_{t\to0^+}\frac{1}{1-\alpha}\int_\Omega\left(\frac{(u_n+t\psi)^{1-\alpha}-u_n^{1-\alpha}}{t}\right)\mathrm{d}x.$$

In view of (4.15), we obtain

$$
\begin{aligned}
\int_\Omega u_n^{-\alpha}\psi\,\mathrm{d}x\leq&a\|u_n\|^{pm}\int_{\mathbb{R}^{2n}}\frac{|u_n(x)-u_n(y)|^{p-2}((u_n(x)-u_n(y))(\psi(x)-\psi(y)))}{|x-y|^{n+sp}}dxdy\\
&-\lambda\int_\Omega u_n^q\psi dx+h_n'(0)\frac{\|u_n\|+\|\psi\|}{n},\quad\text{for } n \text{ large.}
\end{aligned}\tag{4.16}
$$

Using (4.12) and applying Fatou's Lemma again, to conclude that $u_0(x)>0$ a.e. in Ω. In addition

$$
\begin{aligned}
&a\|u_0\|^{pm}\int_{\mathbb{R}^{2n}}\frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))(\psi(x)-\psi(y)))}{|x-y|^{n+sp}}dxdy\\
\geq&\int_\Omega u_0^{-\alpha}\psi\,\mathrm{d}x+\lambda\int_\Omega u_0^q\psi dx,
\end{aligned}\tag{4.17}
$$

for all $\psi\in E$, with $\psi\geq0$. Now, we prove that $u_0\in\mathcal{N}_\lambda^+$ for all $\lambda\in(0,T_{q,\alpha})$. Then, choosing $\psi=u_0$ in (4.16), we obtain

$$a\|u_0\|^{p(m+1)}\geq\int_\Omega u_0^{1-\alpha}\,\mathrm{d}x+\lambda\int_\Omega u_0^{1+q}dx.$$

On the other hand, from (4.10) we get

$$a\|u_0\|^{p(m+1)}\leq\int_\Omega u_0^{-\alpha}\psi\,\mathrm{d}x+\lambda\int_\Omega u_0^q\psi dx.$$

Hence, for $\psi=u_0$ we obtain

$$a\|u_0\|^{p(m+1)}=\int_\Omega u_0^{1-\alpha}\,\mathrm{d}x+\lambda\int_\Omega u_0^{1+q}dx.\tag{4.18}$$

Therefore, $u_0^+ \in \mathcal{N}_\lambda^+$ as required. In addition, we have

$$a \lim_{n\to\infty} \|u_n\|^{p(m+1)} = \int_\Omega (u_0^+)^{1-\alpha}\,\mathrm{d}x + \lambda \int_\Omega (u_0^+)^{1+q} dx.$$

Due to (4.10) and (4.18), we get by the limit as $n \to \infty$

$$a(p(m+1)+\alpha-1)\|u_0\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |u_0(x)|^{1+q}\,\mathrm{d}x > 0.$$

It's the required inequality, proving that $u_0 \in \mathcal{N}_\lambda^+$.

Second step. u_0 is a solution of problem (P_λ). Our proof is inspired by SUN and Sw [31]. Let $\phi \in E$ and $\epsilon > 0$. We define $\psi \in E$ by $\psi := (u_0 + \epsilon\phi)^+$, where $(u_0+\epsilon\phi)^+ = \max\{u_0+\epsilon\phi, 0\}$. It follows by using equation (4.17)

$$\begin{aligned}
0 \leq & a\|u_n\|^{pm} \int_{\mathbb{R}^{2n}} \frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))(\psi(x)-\psi(y)))}{|x-y|^{n+sp}} \\
& - \int_\Omega u_0^{-\alpha}\psi\,\mathrm{d}x - \lambda\int_\Omega u_0^q\psi dx \\
= & a\|u_n\|^{pm} \int_K \frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))((u_0+\epsilon\phi)(x)-(u_0+\epsilon\phi)(y)))}{|x-y|^{n+sp}} \\
& - \int_\Omega u_0^{-\alpha}(u_0+\epsilon\phi)\,\mathrm{d}x - \lambda\int_\Omega u_0^q(u_0+\epsilon\phi)dx,
\end{aligned}$$

where $K = \{(x,y), u_0+\epsilon\phi > 0\}$;

$$\begin{aligned}
= & a\|u_n\|^{pm} \int_{\mathbb{R}^{2n}} \frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))((u_0+\epsilon\phi)(x)-(u_0+\epsilon\phi)(y)))}{|x-y|^{n+sp}} \\
& - \int_\Omega u_0^{-\alpha}(u_0+\epsilon\phi)\,\mathrm{d}x - \lambda\int_\Omega u_0^q(u_0+\epsilon\phi)dx \\
= & a\|u_n\|^{p(m+1)} - \int_\Omega u_0^{1-\alpha}dx - \lambda\int_\Omega u_0^{q+1}dx - \int_\Omega (u_0^{-\alpha}\phi + \lambda u_0^q\phi)dx \\
& + \epsilon a\|u_n\|^{pm} \int_{\mathbb{R}^{2n}} \frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))(\phi(x)-\phi(y)))}{|x-y|^{n+sp}} dxdy \\
& - a\|u_n\|^{pm} \int_H \frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))((u_0+\epsilon\phi)(x)-(u_0+\epsilon\phi)(y)))}{|x-y|^{n+sp}} dxdy \\
& - \int_{\{(x,y),u_0+\epsilon\phi\leq 0\}} u_0^{-\alpha}(u_0+\epsilon\phi) + \lambda u_0^q(u_0+\epsilon\phi))dx,
\end{aligned}$$

where $H = \{(x,y), u_0+\epsilon\phi \leq 0\}$;

$$\begin{aligned}
= & \epsilon a\|u_n\|^{pm} \int_{\mathbb{R}^{2n}} \frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))(\phi(x)-\phi(y)))}{|x-y|^{n+sp}} dxdy \\
& - \epsilon\int_\Omega (u_0^{-\alpha}\phi + \lambda u_0^q\phi)dx \\
& - a\|u_n\|^{pm} \int_H \frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))((u_0+\epsilon\phi)(x)-(u_0+\epsilon\phi)(y)))}{|x-y|^{n+sp}} dxdy
\end{aligned}$$

$$- \epsilon \int_{\{x, u_0+\epsilon\phi\le 0\}} u_0^{-\alpha}(u_0+\epsilon\phi)\,\mathrm{d}x - \lambda \int_\Omega u_0^q(u_0+\epsilon\phi)dx.$$

On the other hand, the measure of the set $\{x : u_0 + \epsilon\phi < 0\}$ tends to zero as $\epsilon \to 0^+$. Then if $\epsilon \to 0^+$, we get

$$\int_H \frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))((u_0+\epsilon\phi)(x)-(u_0+\epsilon\phi)(y)))}{|x-y|^{n+sp}} dxdy \to 0.$$

Passing to the limit as $\epsilon \to 0^+$ after dividing by $\epsilon > 0$, we obtain

$$\begin{aligned} &a\|u_n\|^{pm} \int_{\mathbb{R}^{2n}} \frac{|u_0(x)-u_0(y)|^{p-2}((u_0(x)-u_0(y))(\phi(x)-\phi(y)))}{|x-y|^{n+sp}} \\ &- \int_\Omega u_0^{-\alpha}\phi\,\mathrm{d}x - \lambda \int_\Omega u_0^q \phi dx \ge 0. \end{aligned}$$

Note that the equality holds true if we replace ϕ by $-\phi$ implying that u_0 is a positive solution of problem (P_λ).

Step 2: In this step, we show that problem (P_λ) possesses a positive solution in $\mathcal{N}_\lambda^-$. The needed tool is similar to the first step. Precisely, we apply Ekeland's variational principle to the minimization problem $m^- = \inf_{w\in\mathcal{N}_\lambda^-} J_\lambda(w)$ there exists a sequence $\{w_n\} \subset \mathcal{N}_\lambda^-$ satisfying

i) $J_\lambda(w_n) < m^+ + \frac{1}{n}$,

ii) $J_\lambda(w) \ge J_\lambda(w_n) - \frac{1}{n}\|w - w_n\|$, for all $w \in \mathcal{N}_\lambda{}^-$.

Since $J_\lambda(w) = J_\lambda(|w|)$, we may assume that $w_n(x) \ge 0$. Consequently, as J_λ is coercive on $\mathcal{N}_\lambda$, $\{w_n\}$ is a bounded sequence in E, going to a sub-sequence denoted by $\{w_n\}$, and $w_0 \ge 0$ such that $w_n \rightharpoonup w_0$, weakly in E, $w_n \to w_0$, strongly in $L^{1-\alpha}(\Omega)$, and $L^p(\Omega)$, for $1 \le p < p^*$, and $w_n(x) \to w_0(x)$, a.e. in Ω, as $n \to \infty$. Now, from (4.1) and using the weak lower semi-continuity of norm $J_\lambda(w_0) \le \liminf J_\lambda(w_n) = \inf_{\mathcal{N}^-} J_\lambda$, we see that $w_0 \not\equiv 0$ in Ω. Now, we prove that $w_0(x) > 0$ a.e. in Ω. Similarly to the arguments in Claim 1, we start by observing that, since $w_n \in \mathcal{N}_\lambda^-$, one has

$$a(p(m+1)+\alpha-1)\|w_n\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |w_n(x)|^{1+q}\,\mathrm{d}x < 0.$$

Analogous Calculation gives us

$$a(p(m+1)+\alpha-1)\|w_0\|^{p(m+1)} - \lambda(q+\alpha)\int_\Omega |w_0(x)|^{1+q}\,\mathrm{d}x < 0.$$

After that we consider the function $\psi \in E$, with $\psi \ge 0$ and using Lemma 3.4 with $w = w_n$, there exits a sequence of continuous functions $h_n = h_n(t)$ such that $h_n(t)(w_n + t\psi) \in \mathcal{N}_\lambda^+$ and $h_n(0) = 1$. Repeating the same argument as in the claim 1, we obtain first that $h'(0)$ is uniformly bounded for n large enough and we get

$$\begin{aligned} 0 \le &a\|u_n\|^{pm} \int_{\mathbb{R}^{2n}} \frac{|w_0(x)-w_0(y)|^{p-2}((w_0(x)-w_0(y))(\phi(x)-\phi(y)))}{|x-y|^{n+sp}} \\ &- \int_\Omega w_0^{-\alpha}\phi\mathrm{d}x - \lambda\int_\Omega w_0^q\phi dx, \end{aligned}$$

for all function $\phi \in E$. The last step is similar to the second step, in which we prove that $w_0 \in \mathcal{N}_\lambda^-$ is also a positive solution to the problem (P_λ). Therefore, we obtain at least two positive weak solutions as required. □

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