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THE EXISTENCE OF PERIODIC SOLUTIONS FOR THREE-ORDER NEUTRAL DIFFERENTIAL EQUATIONS*

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Abstract In this paper, by using Kranoselskii fixed point point theorem and Mawhin's continuation theorem, we establish two existence theorem on the periodic solutions for a class of three-order neutral differential equations.

Keywords Periodic solutions, multiple deviating arguments, neutral differential equation.

MSC(2010) 39A23.

1. Introduction

We consider the three-order neutral differential equations with multiple deviating arguments of the form

$$\begin{aligned} p(t) = {} & x'''(t) + cx'''(t-\tau) + a_2(t)x''(t) + a_1(t)x'(t) + a_0(t)x(t) \\ & + \sum_{i=1}^{n} \beta_i(t) g_i(x(t-\tau_i(t))). \end{aligned} \tag{1.1}$$

where $|c| < 1$, τ is a constant, $a_2(t)$, $a_1(t)$, $a_0(t)$, $\tau_i(t)$, $\beta_i(t)$ $(i = 1, 2, ..., n)$ and $p(t)$ are real continuous functions defined on $\mathbb{R}$ with positive period T and $g_i(x)$ $(i = 1, 2, ..., n)$ are real continuous functions defined on $\mathbb{R}$.

Recently, the existence of periodic solutions for differential equations have arouse extensive attention. Most of the results obtained in literature are about the periodic solutions on delay differential equations. Only a small portion of the results [1, 2, 5, 8, 18, 19, 21] concern the periodic solutions on neutral differential equations. For the detailed basic theory, we would like to recommend interested readers to refer to [3, 6, 10–15, 17, 20, 22–24]. This note is inspired by [7] and [9] which discuss the existence of multiple periodic solutions for neutral differential equations with one and two order, and now we study the existence of periodic solutions for neutral differential equations (1.1) by applying two diverse methods.

The following note will be described in these aspects. In Section 2, using Kranoselskii fixed point theorem to reveal that (1.1) exists periodic solutions. In Section 3, we state that Mawhin's continuation theorem is used for proving the existence of periodic solutions for (1.1).

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2. Existence of periodic solutions (I)

For the integrity of this note, we first suggest that the reader refer to appendix about Theorem A (Kranoselskii fixed point theorem) and Theorem B (Mawhin's continuation theorem)(see [4]).

Throughout this paper, to establish our main result, we assume the following conditions hold.

(V_1) $M_k = \max_{t\in[0,T]} a_k(t) \geq m_k = \min_{t\in[0,T]} a_k(t) > 0$ $(k = 0, 1, 2)$;

(V_2) $\beta_i^{(2)} = \max_{t\in[0,T]} \beta_i(t) \geq \beta_i(t) \geq \beta_i^{(1)} = \inf_{t\in[0,T]} \beta_i(t) > 0$ $(i = 1, 2, ..., n)$;

(V_3) $M < (\frac{\pi}{T})^2$;

(V_4) $2T\sigma M_0M_1 - C^* > 0$ and $(2T\sigma M_0M_1 - C^*)A^* - B^*D^* > 0$, where

$$A^* = 1 - |c| - \frac{TM_2}{2} - M_1(\frac{T}{2})^2,\ B^* = \frac{TM_2}{2} + (M_1 - m_1)(\frac{T}{2})^2 + |c|,$$

$$C^* = (M_1 + T\sigma M_0)[M_1 + T\sigma(M_0 - m_0 + \gamma)],$$

$$D^* = (M_1 + T\sigma M_0)[M_1 + T\sigma(M_0 + m_0 + \gamma)]$$

and $\gamma = \sum_{i=1}^{n} \beta_i^{(2)} r_i$ will be given in Theorem 3.1.

For the sake of convenience, let

$$h_1 = \frac{2M_1 + T\sigma M_0}{M_1^2},\ h_2 = \frac{M_0 + T\sigma M_1}{M_1^2}$$

and

$$h_3 = \max_{\{t\in[0,T],|x|\leq K_0\}} \sum_{i=1}^{n} \beta_i^{(2)}|g_i(x)| + \max_{t\in[0,T]} |p(t)|,$$

where $\kappa = \frac{M_1T}{2\sin\frac{\sqrt{M_1T}}{2}}, \sigma = \frac{e^{\frac{M_0}{M_1}T}}{e^{\frac{M_0}{M_1}T} - 1}$ and K_0 will be given in Theorem 2.1.

Theorem 2.1. *Let assumptions $(V_1) - (V_3)$ be satisfied. Assume there exists a constant $K_0 > 0$ such that*

$$h_2[M_2K_2 + (M_1 - m_1)K_1 + h_3] \leq [1 - 2|c| - h_2(M_0 - m_0) - |c|\kappa h_2M_1]K_0, \quad (2.1)$$

where

$$K_1 = \frac{h_1d_4M_2 + d_1d_3}{d_3(1 - 2|c| - h_1(M_1 - m_1)) - h_1d_2M_2},$$

$$K_2 = \frac{d_2(h_1d_4M_2 + d_1d_3)}{d_3(d_3(1 - 2|c| - h_1(M_1 - m_1)) - h_1d_2M_2)} + \frac{d_4}{d_3},$$

$$K_3 = \frac{(d_2M_2 + d_3M_1)(h_1d_4M_2 + d_1d_3)}{d_3(1-|c|)[d_3(1-2|c|-h_1(M_1-m_1))-h_1d_2M_2]} + \frac{d_3M_0K_0+h_3d_3+d_4M_2}{(1 - |c|)d_3},$$

here

$$d_1 = h_1[(M_0 - m_0) + |c|M_1\kappa]K_0 + h_1h_3,$$

$$d_2 = \frac{(M_0h_1 + \kappa)(M_1 - m_1) + M_0}{M_1},$$

$$d_3 = \frac{M_1 - 2|c|M_1 - M_2(h_1M_0 + \kappa)}{M_1}$$

and

$$d_4 = \frac{[h_1(M_0 - m_0) + \kappa]M_0 + |c|\kappa(h_1M_0M_1 + M_1^{\frac{3}{2}})}{M_1}K_0 + \frac{h_1M_0 + \kappa}{M_1}h_3.$$

Then (1.1) *has a nontrivial T-periodic solution.*

For the sake of testifying Theorem 2.1, we first need to assume

$$X: \ = \ \{x|x \in C^3(\mathbb{R}, \mathbb{R}), x(t+T) = x(t), \text{for each } t \in \mathbb{R}\}$$

and $x^{(0)}(t) = x(t)$ and define the following norm on X.

$$||x|| = \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |x'(t)| + \max_{t\in[0,T]} |x''(t)| + \max_{t\in[0,T]} |x'''(t)|,$$

and set

$$Y: \ = \ \{y|y \in C(\mathbb{R}, \mathbb{R}), y(t+T) = y(t), \text{for each } t \in \mathbb{R}\}.$$

We define the norm on Y as follow $\|y\|_0 = \max_{t\in[0,T]} |y(t)|$. Therefore both $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are Banach spaces.

Meanwhile if $x \in X$, then $x^{(i)}(0) = x^{(i)}(T)$ $(i = 0, 1, 2)$. For the convenience of our proof , the following Lemma (see [17]) is used by us.

Lemma 2.1. *Let M be a positive number with $0 < M < (\frac{\pi}{T})^2$. Then for any function φ defined in $[0,T]$, the following equation*

$$\begin{cases} x''(t) + Mx(t) = \varphi(t), \\ x(0) = x(T), \ x'(0) = x'(T) \end{cases}$$

has a unique solution

$$x(t) = \int_0^T G(t,s)\varphi(s)ds,$$

where

$$G(t,s) = \begin{cases} \omega(t-s), \quad (k-1)T \le s \le t \le kT \\ \omega(T+t-s), \quad (k-1)T \le t \le s \le kT, \ (k \in \mathbb{N}), \end{cases}$$

$$\omega(t) = \frac{\cos\alpha(t - \frac{T}{2})}{2\alpha \sin\frac{\alpha T}{2}},$$

$\alpha = \sqrt{M}$ *and*

$$\max_{t\in[0,T]} \int_0^T |G(t,s)|ds = \frac{1}{M}.$$

From (1.1) and Lemma 2.1, we have

$$\begin{aligned} x'(t) = \int_0^T \Psi(t,s)[&-cx'''(s-\tau) - a_2(s)x''(s) - a_0(s)x(s) + p(s) \\ &- \sum_{i=1}^{n} \beta_i(s) g_i(x(s-\tau_i(s))) + (M - a_1(s))x'(s)]ds, \end{aligned} \tag{2.2}$$

where

$$\Psi(t,t_1) = \begin{cases} w_1(t-t_1), & (k-1)T \le t_1 \le t \le kT, \\ w_1(T+t-t_1), & (k-1)T \le t \le t_1 \le kT, \ (k \in \mathbb{N}), \end{cases} \tag{2.3}$$

$$w_1(t) = \frac{\cos \alpha_1 (t - \frac{T}{2})}{2\alpha_1 \sin \frac{\alpha_1 T}{2}},$$

$\alpha_1 = \sqrt{M_1}$ and

$$\max_{s\in[0,T]} \int_0^T |\Psi(s,t_1)| dt_1 = \frac{1}{M_1}. \tag{2.4}$$

From (2.2), by applying a method of constant variation we get

$$\begin{aligned} x(t) = (e^{\frac{M_0}{M_1}T} - 1) \int_0^t \Phi(t,s) \int_0^T &\Psi(s,t_1)[p(t_1) + (M_1 - a_1(t_1))x'(t_1) \\ &- a_2(t_1)x''(t_1) - cx'''(t_1-\tau) - \sum_{i=1}^{n} \beta_i(t_1) g_i(x(t_1 - \tau_i(t_1)))]dt_1 ds \\ &+ (e^{\frac{M_0}{M_1}T} - 1) \int_0^t \Phi(t,s)[\frac{M_0}{M_1}x(s) - \int_0^T \Psi(s,t_1) a_0(t_1) x(t_1) dt_1] ds \\ &+ \int_0^T \Phi(t,s) \int_0^T \Psi(s,t_1)[p(t_1) + (M_1 - a_1(t_1))x'(t_1) - a_2(t_1)x''(t_1) \\ &- cx'''(t_1-\tau) - \sum_{i=1}^{n} \beta_i(t_1) g_i(x(t_1 - \tau_i(t_1)))]dt_1 ds \\ &+ \int_0^T \Phi(t,s)[\frac{M_0}{M_1}x(s) - \int_0^T \Psi(s,t_1) a_0(t_1) x(t_1) dt_1] ds, \end{aligned} \tag{2.5}$$

here

$$\Phi(t,s) = \frac{e^{\frac{M_0}{M_1}(s-t)}}{e^{\frac{M_0}{M_1}T} - 1}, \tag{2.6}$$

$$\Phi(t,s) \le \Phi(t,t+T) = \frac{e^{\frac{M_0}{M_1}T}}{e^{\frac{M_0}{M_1}T} - 1} = \sigma$$

and

$$\int_0^t \Phi(t,s) ds = \frac{M_1(1 - e^{-\frac{M_0}{M_1}t})}{M_0(e^{\frac{M_0}{M_1}T} - 1)} \le \frac{M_1}{M_0(e^{\frac{M_0}{M_1}T} - 1)}.$$

Now we give the proof of Theorem 2.1.

Proof. For each $x \in X$, define the operators $U : X \longrightarrow X$ and $S : X \longrightarrow X$ as follows:

$$(Ux)(t) = -cx(t-\tau) \tag{2.7}$$

and

$$\begin{aligned}(Sx)(t) = (e^{\frac{M_0}{M_1}T} - 1)&\int_0^t \Phi(t,s)\int_0^T \Psi(s,t_1)[p(t_1) + (M_1 - a_1(t_1))x'(t_1)\\
&- a_2(t_1)x''(t_1) - cx'''(t_1-\tau) - \sum_{i=1}^{n}\beta_i(t_1)g_i(x(t_1-\tau_i(t_1)))]dt_1 ds\\
&+ (e^{\frac{M_0}{M_1}T} - 1)\int_0^t \Phi(t,s)[\frac{M_0}{M_1}x(s) - \int_0^T \Psi(s,t_1)a_0(t_1)x(t_1)dt_1]ds\\
&+ \int_0^T \Phi(t,s)\int_0^T \Psi(s,t_1)[p(t_1) + (M_1 - a_1(t_1))x'(t_1)\\
&- a_2(t_1)x''(t_1) - cx'''(t_1-\tau) - \sum_{i=1}^{n}\beta_i(t_1)g_i(x(t_1-\tau_i(t_1)))]dt_1 ds\\
&+ \int_0^T \Phi(t,s)[\frac{M_0}{M_1}x(s) - \int_0^T \Psi(s,t_1)a_0(t_1)x(t_1)dt_1]ds + cx(t-\tau).\end{aligned} \tag{2.8}$$

Thus the fixed point of $U + S$ is a T-periodic solution of (1.1).

To prove that U and S satisfy the conditions of Theorem A. Set

$$G = \{x \in X : |x(t)| \le K_0,\ |x'(t)| \le K_1,\ |x''(t)| \le K_2,\ |x'''(t)| \le K_3\},$$

here K_i $(i = 0,1,2,3)$ are as in the statement of Theorem 2.1. Then G is a bounded, convex and closed subset of X.

(1) For every $x, y \in G$, we need to show that

$$|Uy + Sx| \le K_0, \tag{2.9}$$

$$|\frac{d}{dt}[(Uy)(t) + (Sx)(t)]| \le K_1, \tag{2.10}$$

$$|\frac{d^2[(Uy)(t) + (Sx)(t)]}{dt^2}| \le K_2 \tag{2.11}$$

and

$$|\frac{d^3[(Uy)(t) + (Sx)(t)]}{dt^3}| \le K_3. \tag{2.12}$$

It follows form (2.7) that

$$\frac{d}{dt}[(Ux)(t)] = -cx'(t-\tau), \tag{2.13}$$

$$\frac{d^2[(Ux)(t)]}{dt^2} = -cx''(t-\tau) \tag{2.14}$$

and

$$\frac{d^3[(Ux)(t)]}{dt^3} = -cx'''(t-\tau). \tag{2.15}$$

Note that

$$\int_0^T \Psi(t,s)x'''(s-\tau)ds = M_1 \int_0^T \Psi_t(t,s)x(s-\tau)ds, \tag{2.16}$$

where

$$\Psi_t(t,s) = \begin{cases} \widetilde{\omega}(t-s), & (k-1)T \le s \le t \le kT, \\ \widetilde{\omega}(T+t-s), & (k-1)T \le t \le s \le kT, \ (k \in \mathbb{N}), \end{cases}$$

and

$$\widetilde{\omega} = -\frac{\sin\alpha_1(t-\frac{T}{2})}{2\sin\frac{\alpha_1 T}{2}}.$$

It follows from Lemma 2.1 that

$$\begin{aligned}
&\int_0^T \Psi(t,s)x'''(s-\tau)ds \\
&= \int_0^t \frac{\cos\alpha_1(t-s-\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} d[x''(s-\tau)] + \int_t^T \frac{\cos\alpha_1(t-s+\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} d[x''(s-\tau)] \\
&= \frac{\cos\alpha_1(t-s-\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} x''(s-\tau)|_0^t - \alpha_1 \int_0^t \frac{\sin\alpha_1(t-s-\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} d[x'(s-\tau)] \\
&\quad + \frac{\cos\alpha_1(t-s+\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} x''(s-\tau)|_t^T - \alpha_1 \int_t^T \frac{\sin\alpha_1(t-s+\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} d[x'(s-\tau)] \\
&= -\alpha_1[\frac{\sin\alpha_1(t-s-\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} x'(s-\tau)|_0^t + \frac{\sin\alpha_1(t-s+\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} x'(s-\tau)|_t^T] \\
&\quad + \alpha_1^2\{\int_0^t \frac{\cos\alpha_1(t-s-\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} d[x(s-\tau)] + \int_t^T \frac{\cos\alpha_1(t-s+\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} d[x(s-\tau)]\} \\
&= \alpha_1^2\{\frac{\cos\alpha_1(t-s-\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} x(s-\tau)|_0^t - \alpha_1 \int_0^t \frac{\sin\alpha_1(t-s-\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} x(s-\tau)ds\} \\
&\quad + \alpha_1^2\{\frac{\cos\alpha_1(t-s+\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} x(s-\tau)|_t^T - \alpha_1 \int_t^T \frac{\sin\alpha_1(t-s+\frac{T}{2})}{2\alpha_1\sin\frac{T\alpha_1}{2}} x(s-\tau)ds\} \\
&= M_1 \int_0^T \Psi_t(t,s)x(s-\tau)ds.
\end{aligned} \tag{2.17}$$

Thus (2.16) holds.

From (2.1), (2.7), (2.8) and (2.16), we have

$$\begin{aligned}
|(Uy)(t) + (Sx)(t)| &\le 2|c|K_0 + (\frac{M_0}{M_1} + T\sigma)[\frac{1}{M_1}(M_2K_2 + \|p\|_0 + \|g\|_0 \\
&\quad + (M_1 - m_1)K_1) + (\frac{M_0 - m_0}{M_1} + |c|\kappa)K_0] \\
&\le K_0, \ x, y \in X.
\end{aligned} \tag{2.18}$$

According to Lemma 2.1 and (2.8), we can have

$$\begin{aligned}
&\frac{d[(Sx)(t)]}{dt}\\
&= (2+\frac{M_0}{M_1}T\sigma)\int_0^T \Psi(s,t_1)[-a_2(t_1)x''(t_1)+(M_1-a_1(t_1))x'(t_1)\\
&\quad + p(t_1)-\sum_{i=1}^{n}\beta_i(t_1)g_i(x(t_1-\tau_i(t_1)))]ds-\int_0^T \Psi(s,t_1)a_0(t_1)x(t_1)ds\\
&\quad + \frac{M_0}{M_1}x(t_1)-|c|\int_0^T \Psi(t_1,s)x'''(s-\tau)ds+|c|x'(t-\tau),
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
&\frac{d^2[(Sx)(t)]}{dt^2}\\
&= (\frac{2M_0}{M_1}+\frac{M_0^2}{M_1^2}T\sigma)\int_0^T \Psi(s,t_1)[-a_2(t_1)x''(t_1)+(M_1-a_1(t_1))x'(t_1)\\
&\quad + p(t_1)-\sum_{i=1}^{n}\beta_i(t_1)g_i(x(t_1-\tau_i(t_1)))]ds-\int_0^T \Psi(s,t_1)a_0(t_1)x(t_1)ds\\
&\quad + \frac{M_0}{M_1}x(t_1)-|c|\int_0^T \Psi(t_1,s)x'''(s-\tau)ds+\int_0^T \Psi_t(t,s)[-a_2(t_1)x''(t_1))\\
&\quad + (M_1-m_1)x'(t_1)+p(t_1)-\sum_{i=1}^{n}\beta_i(t_1)g_i(x(t_1-\tau_i(t_1)))]\\
&\quad - \int_0^T \Psi_t(t,s)a_0(t_1)x(t_1)+\frac{M_0}{M_1}x'(t_1)-|c|M_1\int_0^T \Psi_{tt}(t_1,s)x(s-\tau)ds\\
&\quad + |c|x''(t-\tau)
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
&\frac{d^3[(Sx)(t)]}{dt_3}\\
&= p(t)-cx'''(t-\tau)-a_2(t)x''(t)-a_1(t)x'(t)-a_0(t)x(t)\\
&\quad -\sum_{i=1}^{n}\beta_i(t)g_i(x(t-\tau_i(t))),
\end{aligned} \tag{2.21}$$

where

$$\Psi_{tt}(t,s)=\begin{cases}\widetilde{\omega}'(t-s), & (k-1)T\le s\le t\le kT,\\ \widetilde{\omega}'(T+t-s), & (k-1)T\le t\le s\le kT,\ (k\in\mathbb{N}),\end{cases}$$

and

$$\widetilde{\omega}'=-\frac{\alpha_1\cos\alpha_1(t-\frac{T}{2})}{2\sin\frac{\alpha_1 T}{2}}.$$

Through (2.1), (2.10) – (2.12) and (2.19) – (2.21), we get

$$\begin{aligned}
&\Big|\frac{d}{dt}[(Uy)(t)+(Sx)(t)]\Big| \\
&\leq 2|c|K_1+(2+\frac{M_0}{M_1}T\sigma)[\frac{1}{M_1}(M_2K_2+(M_1-m_1)K_1 \\
&\quad +(M_0-m_0)K_0+\|p\|_0+\|g\|_0)]+(2+\frac{M_0}{M_1}T\sigma)|c|\kappa K_0 \\
&\leq K_1,
\end{aligned}\tag{2.22}$$

$$\begin{aligned}
&|\frac{d^2[(Uy)(t)+(Sx)(t)]}{dt^2}| \\
&\leq 2|c|K_2+(\frac{2M_0}{M_1}+\frac{M_0^2}{M_1^2}T\sigma)[\frac{1}{M_1}(M_2K_2+(M_1-m_1)K_1 \\
&\quad +(M_0-m_0)K_0+\|p\|_0+\|g\|_0)+|c|\kappa K_0]+\frac{\kappa}{M_1}[M_2K_2 \\
&\quad +(M_1-m_1)K_1+\|g\|_0+\|p\|_0]+\frac{\kappa M_0}{M_1}K_0+\frac{M_0}{M_1}K_1+|c|\kappa\sqrt{M_1}K_0 \\
&\leq K_2
\end{aligned}\tag{2.23}$$

and

$$\begin{aligned}
|\frac{d^3[(Uy)(t)+(Sx)(t)]}{dt^3}| &\leq |c|K_3+M_2K_2+M_1K_1+M_0K_0+\|g\|_0+\|p\|_0 \\
&\leq K_3.
\end{aligned}\tag{2.24}$$

From (2.18) and (2.18) – (2.24), we have $Ux+Sy\in G$ if $x,y\in G$.

(2) U is a contraction mapping.

Let $x,y\in G$. It follows from (2.7) that

$$\begin{aligned}
&\|Ux-Uy\| \\
&=\max_{t\in[0,T]}|cx(t-\tau)-cy(t-\tau)|+\max_{t\in[0,T]}|cx'(t-\tau)-cy'(t-\tau)| \\
&\quad +\max_{t\in[0,T]}|cx''(t-\tau)-cy''(t-\tau)|+\max_{t\in[0,T]}|cx'''(t-\tau)-cy'''(t-\tau)| \\
&=|c|[\max_{t\in[0,T]}|x(t-\tau)-y(t-\tau)|+\max_{t\in[0,T]}|x'(t-\tau)-y'(t-\tau)| \\
&\quad +\max_{t\in[0,T]}|x''(t-\tau)-y''(t-\tau)|+\max_{t\in[0,T]}|x'''(t-\tau)-y'''(t-\tau)|] \\
&=|c|\|x-y\|.
\end{aligned}$$

Thus U is a contraction mapping for $|c|<1$.

(3) S is completely continuous.

From the continuity of $a(t),\ p(t)$ and $g(t,x(t-\tau_1(t)),\ x(t-\tau_2(t))...,\ x(t-\tau_n(t)))$ for $t\in[0,T],\ x\in G$, we can gain the continuity of S. Actually, if $x_k\in G$ and $\|x_k-s\|\longrightarrow 0$ as $k\longrightarrow+\infty$, then $x\in G$ since G is closed convex subset of X.

Thus

$$
\begin{aligned}
&|Sx_k - Sx| \\
=& \Big| \int_0^T \Phi(t,s) \int_0^T \Psi(s,t_1)\{-a_2(t_1)(x_k''(t_1) - x''(t_1)) + (M_1 \\
&- x'(t_1))(x_k'(t_1) - x'(t_1)) - c[x_k'''(t_1-\tau) - x'''(t_1-\tau)] \\
&- [\sum_{i=1}^n \beta_i(t_1) g_i(x_k(t_1 - \tau_i(t_1))) - \sum_{i=1}^n \beta_i(t_1) g_i(x(t_1 - \tau_i(t_1)))]\} dt_1 ds \\
&+ \int_0^T \Phi(t,s)[\frac{M_0}{M_1}(x_k(s) - x(s)) - \int_0^T \Psi(s,t_1) a_0(t_1)(x_k(t_1) \\
&- x(t_1)) dt_1] ds + (e^{\frac{M_0}{M_1}T} - 1) \int_0^t \Phi(t,s) \int_0^T \Psi(s,t_1)\{-a_2(t_1)(x_k''(t_1) \\
&- x''(t_1)) + (M_1 - a_1(t_1))(x_k'(t_1) - x'(t_1)) - c[x_k'''(t_1 - \tau) \\
&- x'''(t_1-\tau)] - [\sum_{i=1}^n \beta_i(t_1) g_i(x_k(t_1 - \tau_i(t_1))) - \sum_{i=1}^n \beta_i(t_1) g_i(x(t_1 \\
&- \tau_i(t_1)))]\} dt_1 ds + (e^{\frac{M_0}{M_1}T} - 1) \int_0^t \Phi(t,s)[\frac{M_0}{M_1}(x_k(s) - x(s)) \\
&- \int_0^T \Psi(s,t_1) a_0(t_1)(x_k(t_1) - x(t_1)) dt_1] ds + c[x_k(t-\tau) - x(t-\tau)] \Big|.
\end{aligned}
\tag{2.25}
$$

Combing Lebesgue dominated convergence theorem, with $(2.19)-(2.21)$ and (2.25), we have

$$\lim_{k \to +\infty} \|Sx_k - Sx\| = 0.$$

Then S is continuous.

We first demonstrate that Sx is relatively compact. It suffices to show that the family of functions $\{Sx : x \in G\}$ is uniformly bounded and equicontinuous on $[0, T]$. From (2.8) and $(2.19) - (2.21)$, it is easy to see that $\{Sx : x \in G\}$ is uniformly bounded and equicontinuous. S is completely continuous since S is not only continuous but also relatively compact. We get a fixed point x of $U + S$ based on Theorem A (Kranoselskii fixed point theorem). Thus a T-periodic solution of (1.1) is x. □

3. Existence of periodic solutions (II)

Theorem 3.1. *Suppose that* $(V_1) - (V_4)$ *hold and let*

$$\lim_{|x| \to \infty} \sup |\frac{g_i(x)}{x}| \le r_i \ (i = 1, 2, ..., n) \tag{3.1}$$

and

$$\lim_{|x| \to \infty} sgn(x) g_i(x) = +\infty. \tag{3.2}$$

Then (1.1) *possesses at least one* T*-periodic solution.*

Now, let

$$Z: \;=\; \{x|x \in C^3(\mathbb{R},\mathbb{R}),\ x(t+T)=x(t),\ \text{for each } t\in\mathbb{R}\}$$

and $x^{(0)} = x(t)$ and define the following norm on Z

$$\|x\| = \max\{\max_{t\in[0,T]}|x(t)|, \max_{t\in[0,T]}|x'(t)|, \max_{t\in[0,T]}|x''(t)|, \max_{t\in[0,T]}|x'''(t)|\}.$$

Let

$$Y: \;=\; \{y|y\in C(\mathbb{R},\mathbb{R}),\ y(t+T)=y(t),\ \text{for each } t\in\mathbb{R}\}.$$

We define the norm on Y as $\|y\|_0 = \max_{t\in[0,T]}|y(t)|$. Both $(Z,\|\cdot\|)$ and $(Y,\|\cdot\|_0)$ are Banach spaces. Moreover, if $x\in Z$, then $x^{(i)}(0)=x^{(i)}(T)(i=0,1,2)$.

Define the operators $L:\ Z\longrightarrow Y$ and $N: Z\longrightarrow Y$ as following

$$Lx(t) = x'''(t),\ t\in\mathbb{R} \tag{3.3}$$

and

$$\begin{aligned} Nx(t) = &-cx'''(t-\tau) - a_2(t)x''(t) - a_1(t)x'(t) - a_0(t)x(t)\\ &-\sum_{i=1}^{n}\beta_i(t)g_i(x(t-\tau_i(t))) + p(t),\ t\in\mathbb{R}. \end{aligned} \tag{3.4}$$

Then

$$\mathrm{Ker}L = \{x\in Z: x(t)=c\in\mathbb{R}\} \tag{3.5}$$

and

$$\mathrm{Im}L = \{y\in Y: \int_0^T y(t)dt = 0\} \tag{3.6}$$

is closed in Y. So L is a Fredholm mapping of index zero.

Consider $P:\ Z\longrightarrow Z$ and $Q:\ Y\longrightarrow Y/\mathrm{Im}(L)$ with

$$Px(t) = \frac{1}{T}\int_0^T x(t)dt = x(0),\ t\in\mathbb{R}, \tag{3.7}$$

for $x = x(t)\in X$ and

$$Qy(t) = \frac{1}{T}\int_0^T y(t)dt,\ t\in\mathbb{R}, \tag{3.8}$$

for $y=y(t)\in Y$. Then $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q)$. It follows that $L|_{\mathrm{Dom}\ L\cap\mathrm{Ker}\ P}: (I-P)Z\longrightarrow \mathrm{Im}L$ has an inverse which will be denoted by K_p.

Let Ω be an open and bounded subset of Z, we have $QN(\overline{\Omega})$ is bounded and $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact. Then the mapping N is L-compact on $\overline{\Omega}$. We have the following result.

Lemma 3.1. *Let L, N, P and Q be definded by* (3.3), (3.4), (3.7) *and* (3.8) *respectively. Then L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$, where Ω is any open and bounded subset of Z.*

In order to prove Theorem 3.1, we need the following Lemma (see [16]).

Lemma 3.2. *Let* $x(t) \in C^{(n)}(\mathbb{R},\mathbb{R}) \cap C_T$. *Then*

$$\|x^{(i)}\|_0 \leq \frac{1}{2}\int_0^T |x^{(i+1)}(s)|ds,\ \ i=1,2,...,n-1,$$

where $n \geq 2$ *and* $C_T :\ = \{x|x \in C(\mathbb{R},\mathbb{R}),\ x(t+T)=x(t),\ \textit{for each } t \in \mathbb{R}\}$.

Now, consider the following equation

$$\begin{aligned} x'''(t) = \lambda[&-cx'''(t-\tau) - a_2(t)x''(t) - a_1(t)x'(t) - a_0(t)x(t) \\ &- \sum_{i=1}^{n} \beta_i(t) g_i(x(t-\tau_i(t))) + p(t)], \end{aligned} \tag{3.9}$$

where $0 < \lambda < 1$.

Lemma 3.3. *Suppose that conditions of Theorem 3.1 are satisfied. If* $x(t)$ *is a* T*-periodic solution of* (3.9), *then there are positive constants* D_j $(j=0,1)$, *which are independent of* λ, *such that*

$$\|x^{(j)}\|_0 \leq D_j,\ \ t \in [0,T],\ \ j=0,1. \tag{3.10}$$

Proof. If $x(t)$ is a T-periodic solution of (3.9), set

$$\varepsilon = \frac{1}{2}[\frac{M_1 - T\sigma(M_0 - m_0)}{T\sigma} - \gamma]. \tag{3.11}$$

By (3.1), there exists a $\overline{M} > 0$ such that

$$|g_i(x(t-\tau_i(t)))| \leq (r_i + \frac{\varepsilon}{n\beta_2^{(2)}})|x(t-\tau_i(t))|,\ \ |x(t)| > \overline{M}\ \ (i=1,2,...,n). \tag{3.12}$$

Denote that

$$E_1^{(i)} = \{t | t \in [0,T],\ \ |x(t)| > \overline{M}\}, \tag{3.13}$$

$$E_2^{(i)} = [0,T] \backslash E_1^{(i)} \tag{3.14}$$

and

$$\rho_i = \max_{|x| \leq \overline{M}} |g_i(x)| \tag{3.15}$$

From (3.12) – (3.15) and Lemma 3.1, we get

$$\begin{aligned} &\max_{t\in[0,T]} |x'''(t)| \\ \leq &\max_{t\in[0,T]} \{\lambda|cx'''(t-\tau)| + \lambda|a_2(t)x''(t)| + \lambda|a_1(t)x'(t)| \\ &+ \lambda|a_0(t)x(t)| + \lambda|\sum_{i=1}^{n} \beta_i(t) g_i(x(t-\tau_i(t)))| + \lambda|p(t)|\} \\ \leq &|c|\max_{t\in[0,T]}|x'''(t)| + M_2 \max_{t\in[0,T]}|x''(t)| + M_1 \max_{t\in[0,T]}|x'(t)| \\ &+ M_0 \max_{t\in[0,T]}|x(t)| + \max_{t\in[0,T]}|p(t)| + \max_{x\in E_1^{(1)}}|\beta_1(t) g_1(x(t-\tau_1(t)))| \\ &+ \max_{x\in E_2^{(1)}}|\beta_1(t) g_1(x(t-\tau_1(t)))| + ... + \max_{x\in E_1^{(n)}}|\beta_n(t) g_n(x(t-\tau_n(t)))| \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
&+ \max_{x\in E_2^{(n)}} |\beta_n(t)g_n(x(t-\tau_n(t)))| \\
&\le |c| \max_{t\in[0,T]} |x'''(t)| + M_2 \max_{t\in[0,T]} |x''(t)| + M_1\|x'\|_0 + M_0\|x\|_0 \\
&+ (\gamma+\varepsilon)\|x\|_0 + \rho + \|p\|_0 \\
&\le |c| \max_{t\in[0,T]} |x'''(t)| + \frac{TM_2}{2} \max_{t\in[0,T]} |x'''(t)| + M_1(\frac{T}{2})^2 \max_{t\in[0,T]} |x'''(t)| \\
&+ \frac{M_1 + T\sigma(M_0+m_0+\gamma)}{2T\sigma}\|x\|_0 + \overline{C},
\end{aligned}
$$

where $\overline{C} = \rho + \|p\|_0$ and $\rho = \sum_{i=1}^{n} \rho_i$. Furthermore, for $t \in [0,T]$, we obtain that

$$\max_{t\in[0,T]} |x'''(t)| \le (A^*)^{-1}[\frac{M_1 + T\sigma(M_0+m_0+\gamma)}{2T\sigma}\|x\|_0 + \overline{C}]. \tag{3.17}$$

On the other hand, it follows from (3.9) and Lemma 2.1 that

$$
\begin{aligned}
x'(t) = &\int_0^T G_1(t,t_1)\lambda[(M_1 - a_1(t_1))x'(t_1) + p(t_1) - cx'''(t_1-\tau) \\
&- \sum_{i=1}^{n} \beta_i(t_1)g_i(x(t_1-\tau_i(t_1))) - a_0(t_1)x(t_1)]dt_1,
\end{aligned}
\tag{3.18}
$$

where

$$G_1(t,t_1) = \begin{cases} \omega_1(t-t_1), & (k-1)T \le t_1 \le t \le kT \\ \omega_1(T+t-t_1), & (k-1)T \le t \le t_1 \le kT,\ (k\in\mathbb{N}), \end{cases} \tag{3.19}$$

$$\omega_1(t) = \frac{\cos\alpha_1(t-\frac{T}{2})}{2\alpha_1 \sin\frac{\alpha_1 T}{2}}, \tag{3.20}$$

$\alpha_1 = \sqrt{\lambda M_1}$ and

$$\max_{t\in[0,T]} \int_0^T |G_1(t,t_1)|dt_1 = \frac{1}{\lambda M_1}. \tag{3.21}$$

From (3.18), we have

$$
\begin{aligned}
&x(t) \\
&= (e^{\frac{M_0}{M_1}T} - 1)\int_0^t \lambda\Phi(t,s)\int_0^T G_1(s,t_1)[p(t_1) + (M_1 - a_1(t_1))x'(t_1) \\
&- a_2(t_1)x''(t_1) - cx'''(t_1-\tau) - \sum_{i=1}^{n} \beta_i(t_1)g_i(x(t_1-\tau_i(t_1)))]dt_1 ds \\
&+ (e^{\frac{M_0}{M_1}T} - 1)\int_0^t \Phi(t,s)[\frac{M_0}{M_1}x(s) - \int_0^T \lambda G_1(s,t_1)a_0(t_1)x(t_1)dt_1]ds \\
&+ \int_0^T \lambda\Phi(t,s)\int_0^T G_1(s,t-1)[p(t_1) + (M_1 - a_1(t_1))x'(t_1) \\
&- a_2(t_1)x''(t_1) - cx'''(t_1-\tau) - \sum_{i=1}^{n} \beta_i(t_1)g_i(x(t_1-\tau_i(t_1)))]dt_1 ds
\end{aligned}
\tag{3.22}
$$

$$+\int_0^T \Phi(t,s)[\frac{M_0}{M_1}x(s)-\int_0^T \lambda G_1(s,t_1)a_0(t_1)x(t_1)dt_1]ds,$$

here $\Phi(t,s)$ is as in the statement of (2.6).

According to (3.22) and Lemma 3.2, we have

$$\begin{aligned}
&\|x\|_0\\
&\leq (\frac{T\sigma}{M_1}+\frac{1}{M_0})\{\|p\|_0+|c|\max_{t\in[0,T]}|x'''(t)|+M_2\|x''\|_0+(M_1-m_1)\|x'\|_0\\
&\quad+(M_0-m_0)\|x\|_0+\max_{t_1\in[0,T]}[\int_{E_1^{(1)}}|\beta_1(t_1)g_1(x(t_1-\tau_1(t_1)))|dt_1\\
&\quad+\int_{E_2^{(1)}}|\beta_1(t_1)g_1(x(t_1-\tau_1(t_1)))|d_1]+...+\max_{t_1\in[0,T]}[\int_{E_1^{(n)}}|\beta_n(t_1)g_n(x(t_1\\
&\quad-\tau_n(t_1)))|dt_1+\int_{E_2^{(n)}}|\beta_n(t_1)g_n(x(t_1-\tau_n(t_1)))|d_1]\}\\
&\leq (\frac{T\sigma}{M_1}+\frac{1}{M_0})\{|c|\max_{t\in[0,T]}|x'''(t)|+M_2(\frac{T}{2})\max_{t\in[0,T]}|x'''(t)|+(M_1\\
&\quad-m_1)(\frac{T}{2})^2\max_{t\in[0,T]}|x'''(t)|+(M_0-m_0)\|x\|_0+(\gamma+\varepsilon)\|x\|_0+\overline{C}\}.
\end{aligned}\tag{3.23}$$

By (3.23), we obtain

$$\|x\|_0\leq\frac{2T\sigma(T\sigma M_0+M_1)[B^*\max_{t\in[0,T]}|x'''(t)|+\overline{C}]}{2T\sigma M_0M_1-(T\sigma M_0+M_1)[M_1+T\sigma(M_0-m_0+\gamma)]}.\tag{3.24}$$

Combining (3.17) and (3.24), we have

$$\begin{aligned}\max_{t\in[0,T]}|x'''(t)|&\leq\frac{(2T\sigma M_0M_1-C^*+D^*)\overline{C}}{(2T\sigma M_0M_1-C^*)A^*-B^*D^*}\\&=D_0\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}\|x\|_0&\leq\frac{2T\sigma(T\sigma M_0+M_1)}{2T\sigma M_0M_1-C^*}[\frac{(2T\sigma M_0M_1-C^*+D^*)B^*\overline{C}}{(2T\sigma M_0M_1-C^*)A^*-B^*D^*}+\overline{C}]\\&=D_1.\end{aligned}\tag{3.26}$$

According to Lemma 3.2, (3.25) and (3.26) we get

$$\|x^{(i)}\|_0\leq D_i\ (i=0,1).\tag{3.27}$$

The proof of Lemma 3.3 is complete. □

Now we give the proof of Theorem 3.1.

Proof. Let $x(t)$ be a T-periodic solution of (3.9). Then by Lemma 3.3, we have positive constants D_i $(i=0,1,2,3)$ which are independent of λ such that (3.10) is true. By (3.2), we know that exist a $\widehat{M}>0$, such that

$$\mathrm{sgn}(x)g_i(x)>\frac{1}{n\beta_i^{(1)}}\|p\|_0,\ |x(t)|>\widehat{M}.\tag{3.28}$$

For any positive constant $\overline{D} > \max_{0\le i\le 3}\{D_i\} + \widehat{M}$.

Let

$$\Omega: \ = \{x \in X,\ \|x\| < \overline{D}\}.$$

Then L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$ (see [2]). In terms of valuation of bounds of period solutions in Lemma 3.3 for any $x \in \partial\Omega \cap \mathrm{Dom}(L)$ and $\lambda \in (0,1)$, $Lx \neq \lambda Nx$. For any $x \in \partial\Omega \cap \mathrm{Ker}(L)$, $x = \overline{D}$ or $x = -\overline{D}$. It follows from (3.2), (3.4), (3.8) and (3.28), we have

$$\begin{aligned}(QNx) = \frac{1}{T}\int_0^T &[p(t) - cx'''(t-\tau) - a_2(t)x''(t) - a_1(t)x'(t)\\ &- a_0(t)x(t) - \sum_{i=1}^n \beta_i(t)g_i(x(t-\tau_i(t)))]dt\\ \neq 0.&\end{aligned}$$

In particular, we have

$$-\frac{1}{T}\int_0^T [a_0(t)\overline{D} + \sum_{i=1}^n \beta_i(t)g_i(\overline{D}) - p(t)]dt < 0$$

and

$$\frac{1}{T}\int_0^T [a_0(t)\overline{D} - \sum_{i=1}^n \beta_i(t)g_i(-\overline{D}) + p(t)]dt > 0.$$

Thus, for each $x \in \mathrm{Ker}L \cap \partial\Omega$ and $\eta \in [0,1]$, we get

$$\begin{aligned}&xH(x,\eta)\\ &= -\eta x^2 - \frac{x}{T}(1-\eta)\int_0^T [-p(t) + cx'''(t-\tau) + a_2(t)x''(t) + a_1(t)x'(t)\\ &\quad + a_0(t)x(t) + \sum_{i=1}^n \beta_i(t)g_i(x(t-\tau_i(t)))]dt\\ &\neq 0.\end{aligned}$$

Hence, $H(x,\eta)$ is a homotopy. This shows that

$$\begin{aligned}&\deg\{QN, \Omega\cap \mathrm{Ker}(L), 0\}\\ &= \deg\{-\frac{1}{T}\int_0^T [-p(t) + cx'''(t-\tau) + a_2(t)x''(t) + a_1(t)x'(t)\\ &\quad + a_0(t)x(t) + \sum_{i=1}^n \beta_i(t)g_i(x(t-\tau_i(t)))]dt, \Omega\cap \mathrm{Ker}(L), 0\}\\ &\neq 0.\end{aligned}$$

By Theorem B, the equation $Lx = Nx$ has at least a solution in $\mathrm{Dom}(L)\cap\overline{\Omega}$, then there exists a T-periodic solution of (1.1). This completes the proof. □

Similarly, we can prove Theorem 3.2. Details are so omitted here and left to the readers.

Theorem 3.2. *Suppose* $(V_1) - (V_4)$ *hold and also assume*

$$\lim_{|x|\to 0^+} sup|\frac{g_i(x)}{x}| \le r_i \ (i = 1, 2, ..., n) \tag{3.29}$$

and

$$\lim_{|x|\to 0^+} sgn(x)g_i(x) = 0. \tag{3.30}$$

Then (1.1) *possesses at least one* T*-periodic solution.*

Appendix

Theorem A (Kranoselskii fixed point theorem). *Let* Ω *be a Banach space and* X *be a bounded, convex and closed subset of* Ω*. If* $U, S : X \longrightarrow \Omega$ *satisfy the following conditions:*

(1) $Ux + Sy \in X$ *for every* $x, y \in X$

(2) U *is a contraction mapping*

(3) S *is completely continuous,*

then $U + S$ *has a fixed point in* X*.*

Theorem B (Mawhin's continuation theorem). *Let* L *be a Fredholm mapping of index zero, and let* N *be* L*-compact on* $\overline{\Omega}$*. Suppose that*

(1) $Lx \neq \lambda Nx$*, where* $\lambda \in (0, 1)$ *and* $x \in \partial\Omega$*, and*

(2) $QNx \neq 0$ *and* $deg(QN, \Omega \cap Ker(L), 0) \neq 0$*, where* $x \in \partial\Omega \cap Ker(L)$*,*

then the equation $Lx = Nx$ *has at least one solution in* $\overline{\Omega} \cap D(L)$*.*

Notation. Let X and Y be two Banach spaces. Suppose that $L : \mathrm{Dom}L \subset X \longrightarrow Y$ is a linear mapping and $N : X \longrightarrow Y$ is a continuous mapping. Then the mapping L is said to be a Fredholm mapping of index zero if $\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L < +\infty$, and $\mathrm{Im}L$ is closed in Y.

Let L is a Fredholm mapping of index zero. Then there exist continuous projectors $P : X \longrightarrow X$ and $Q : Y \longrightarrow Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$. Thus $L|_{\mathrm{Dom}\ L \cap \mathrm{Ker}\ P} : (I - P)X \longrightarrow \mathrm{Im}L$ has an inverse which will be denoted by K_p. Suppose that Ω is an open and bounded subset of X. Then the mapping N is said to be L-compact on Ω if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N(\overline{\Omega})$ is compact. For $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \longrightarrow \mathrm{Ker}L$.

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