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INVERSE PROBLEMS IN MAGNETO-ELECTROSCANNING (IN ENCEPHALOGRAPHY, FOR MAGNETIC MICROSCOPES, ETC.)*

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Dedicated to my friend Claude Brauner

Abstract Contrary to the prevailing opinion about the incorrectness of the inverse MEEG-problem, we prove its unique solvability in the framework of the system of Maxwell's equations [3]. The solution of this problem is the distribution of $\mathbf{y} \mapsto \mathbf{q}(\mathbf{y})$ current dipoles of brain neurons that occupies the region $Y \subset \mathbb{R}^3$. It is uniquely determined by the non-invasive measurements of the electric and magnetic fields induced by the current dipoles of neurons on the patient's head. The solution can be represented in the form $\mathbf{q} = \mathbf{q}_0 + \mathbf{p}_0\delta\Big|_{\partial Y}$, where $\mathbf{q}_0$ is the usual function defined in Y, and $\mathbf{p}_0\delta\Big|_{\partial Y}$ is a δ-function on the boundary of the domain Y with a certain density $\mathbf{p}_0$. It is essential that $\mathbf{p}_0$ and $\mathbf{q}_0$ are interrelated. This ensures the correctness of the inverse MEEG-problem. However, the components of the required 3-dimensional distribution $\mathbf{q}$ must turn out to be linearly dependent if only the magnetic field $\mathbf{B}$ is taken into account. This question is considered in detail in a flat model of the situation.

Keywords Inverse problems, integral equations, pseudo-differential operators, magneto-electroscanning.

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1. Formulation of the inverse MEEG-problem

The inverse MEEG-problem is the problem of finding the distribution of dipoles $\mathbf{q} : Y \to \mathbb{R}^3$ (current dipole moment) in the neurons of the brain, which occupies a domain $Y \subset \mathbb{R}^3$, according to the electric $\mathcal{D} = \varepsilon\mathcal{E}$, as well as the magnetic induction $\mathcal{B} = \mu\mathcal{H}$, measured on the surface X, which is the internal part of the helmet, with the SQUID sensors (Superconducting quantum Interference device) [9, 13]. The fields $\mathcal{E}$ and $\mathcal{H}$ are called the electric and magnetic field strengths. The parameters μ and $\varepsilon = \varepsilon(\mathbf{x}) > 0$ are magnetic and dielectric permeabilities.

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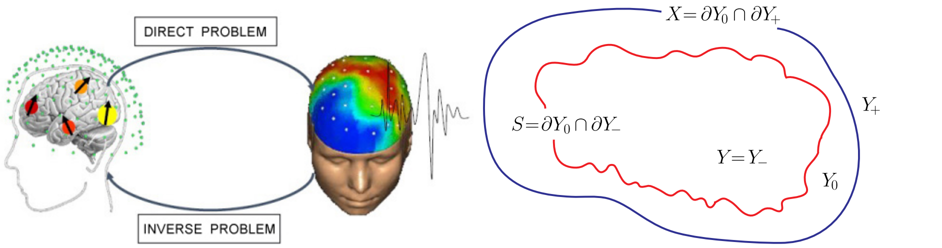

The figure on the left shows that the magnetic field $\mathcal{B}$ and the electric field $\mathcal{E}$ induced by electric dipoles $\mathbf{q}$ in the neurons of the brain can be registered on the head surface. The problem is to find a dipole distribution $\mathbf{q}$ from $\mathcal{B}$ and $\mathcal{E}$. This problem is called *inverse*, in contrast to the *direct* problem, in which the magnetic and electric fields are calculated from the given distribution of dipoles using formulas of the Biot–Savar type. In the figure on the right $Y_- = Y$, Y_0 and Y_+ are the regions of the brain, the skull and the air surrounding the head.

We shall start from the Maxwell equations

$$\begin{aligned} &\mu\partial_t\mathcal{H}(\mathbf{x},t) + \operatorname{rot}\mathcal{E}(\mathbf{x},t) = 0\,, && \operatorname{div}\mathcal{B}(\mathbf{x},t) = 0\,, \\ &-\varepsilon(\mathbf{x})\partial_t\mathcal{E}(\mathbf{x},t) + \operatorname{rot}\mathcal{H}(\mathbf{x},t) = \mathbf{J}^v(\mathbf{x}) + \mathbf{J}^p(\mathbf{x})\,, && \operatorname{div}\mathcal{D}(\mathbf{x},t) = \rho\,. \end{aligned} \tag{1.1}$$

Here $\mathbf{J}^v = \sigma\mathcal{E}$ is the so-called volumetric or, as they say, ohmic current (more precisely, its density), because it satisfies Ohm's law associated with the coefficient of electrical conductivity $\sigma = \sigma(\mathbf{x}) \geq 0$, which is assumed to be independent of t. We note that such conditions are physically justified:

$$\sigma_+ = 0 \quad \text{on} \quad Y_+, \qquad \sigma_0 > 0 \quad \text{on} \quad Y_0, \qquad \sigma_- > \sigma_0 \quad \text{on} \quad Y_-. \tag{1.2}$$

The volume current is the result of the action of a macroscopic electric field on the charge carriers in the conducting medium of the brain. Neuronal same activity causes the so-called primary (principal) current $\mathbf{J}^p$. It arises as a result of dielectric polarization and it represents a movement of charges inside or near the cell. The volume density of these charges is denoted by ρ. Particles possessing these charges are part of the molecules. They are displaced from their equilibrium positions under the action of an external electric field, without leaving the molecule into which they enter.

Essential is the circumstance, especially noted in the fundamental work [9] (on page 426). It is related to the frequency ratio ω of the oscillations of the electromagnetic field $\mathcal{H}(\mathbf{x},t) = \mathbf{H}(\mathbf{x})e^{i\omega t}$, $\mathcal{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})e^{i\omega t}$ and the frequency of electrical oscillations in brain cells. The analysis in [9] shows that for the system (1.1) the quasistatic approximation corresponding to the leading term of the asymptotics as $\omega \to 0$ is justified. There, on the same page, is additionally noted: "A current dipole $\mathbf{q}$, approximating a localized primary current, is a widely used concept in neuromagnetism . . . . In EEG and MEG applications, a current dipole is used as an equivalent source for the unidirectional primary current that may extend over several square centimeters of cortex." Such a conclusion is valid not only for cell biophysics, but also for a number of other problems, including scanning magnetic microscopes [12]. As a result, we arrive at the following equations

$$\operatorname{rot}\mathbf{E} = 0, \qquad \operatorname{rot}\mathbf{B} = \mu(\sigma\mathbf{E} + \mathbf{q}), \qquad \operatorname{div}\mathbf{B} = 0, \qquad \operatorname{div}\mathbf{D} = \rho. \tag{1.3}$$

2. Integral equation and the formula for solving the inverse MEEG problem

1. As is known,

$$\mathrm{rot}\mathbf{E} = 0 \quad \Leftrightarrow \quad \mathbf{E} = -\nabla\Phi\,, \qquad\qquad \mathrm{div}\mathbf{B} = 0 \quad \Leftrightarrow \quad \mathbf{B} = \mathrm{rot}\mathbf{A}\,. \tag{2.1}$$

Since $\mathrm{div}(\varepsilon\mathbf{E}) = \rho$, then

$$-\varepsilon\Delta\Phi - \nabla\varepsilon\nabla\Phi = \rho. \tag{2.2}$$

According to physical representations, the field potential $\Phi \overset{(2.2)}{=} \Phi_\rho$ at infinity is a constant, which can be considered equal to zero. For similar reasons, the vector potential $\mathbf{A}$ of field $\mathbf{B} = \mathrm{rot}\mathbf{A}$ is also chosen to be zero at infinity.

Since $\mathrm{rot}(\mathrm{rot}\mathbf{A}) = \nabla\,\mathrm{div}\mathbf{A} - \Delta\mathbf{A}$, then $\Delta\mathbf{A} = -\mathrm{rot}\mathbf{B} + \nabla\,\mathrm{div}\mathbf{A}$. But* $\mathrm{rot}\mathbf{B} = \sigma\mathbf{E} + \mathbf{q}$, and $\mathbf{E} = -\nabla\Phi$. Thus

$$\Delta\mathbf{A}(\mathbf{x}) = -\mathbf{q}(\mathbf{x}) + \nabla\big[\sigma(\mathbf{x})\Phi(\mathbf{x}) + \mathrm{div}\mathbf{A}(\mathbf{x})\big] - \Phi(\mathbf{x})\nabla\sigma(\mathbf{x})\,. \tag{2.3}$$

The vector potential $\mathbf{A}$ is determined up to a potential field. Indeed, we have: $\mathrm{rot}(\mathbf{A} - \mathbf{A}^*) = 0 \overset{(2.1)}{\Leftrightarrow} \mathbf{A} - \mathbf{A}^* = \nabla\varphi$, ie $\mathbf{A} = \mathbf{A}^* + \nabla\varphi$, where φ is a function.

Taking as φ solution of the equation $\Delta\varphi = -\mathrm{div}\mathbf{A}^* - \sigma\Phi$, subjected to condition $\varphi\big|_\infty = 0$ (because $\mathbf{A}^*\big|_\infty = 0$, $\Phi\big|_\infty = 0$), we obtain

$$\sigma(\mathbf{x})\Phi(\mathbf{x}) + \mathrm{div}\mathbf{A}(\mathbf{x}) = 0\,, \tag{2.4}$$

and therefore from (2.3) we get

$$\Delta\mathbf{A}(\mathbf{x}) = -\mathbf{F}(\mathbf{x})\,, \qquad \text{where} \qquad \mathbf{F}(\mathbf{x}) = \mathbf{q}(\mathbf{x}) + \Phi_\rho(\mathbf{x})\nabla\sigma(\mathbf{x})\,. \tag{2.5}$$

We emphasize that the equations (2.4) and (2.5) are not independent. They are equivalent to (1.3) and are therefore related by an implicit relation between $\mathbf{A}$, Φ, ρ and $\mathbf{q}$, given by the equations (1.3).

The equation (2.5) is equivalent to formula

$$\mathbf{q}(\mathbf{x}) = -\Delta\mathbf{A}(\mathbf{x}) - \Phi(\mathbf{x})\nabla\sigma(\mathbf{x})\,. \tag{2.6}$$

This formula gives the required solution of the inverse problem, but only if the potentials $\mathbf{A}$ and Φ are known in Y. However, only the fact is known a priori about them (in addition to the fact that they are zero at infinity) that there are data on the measurements of the fields $\mathbf{B} = \mathrm{rot}\mathbf{A}$ and $\mathbf{E} = -\nabla\Phi$ in the finite collection points $\mathbf{x}_k \in X$ (see the right side of the figure). Nevertheless, as shown in item 4 below, these data and the results of item 3 still allow us to find the "essentially" different† approximations of the electromagnetic field defined in the entire (!) space $\mathbb{R}^3 \supset Y$. They correspond to a priori possible "essentially" different solutions of the inverse problem.

*For the bio-medium $\mu \approx \mu_0$ magnetic permeability of the vacuum. Therefore in this section we will assume that $\mu = 1$.

†cf. A.S. Demidov and V.V. Savelev, *Essentially different distributions of current in the inverse problem for the Grad-Shafranov equation*, Russian J. Math. Ph., 2010, 17(1), 56–65.

2. We show that, along the normal to $S = \partial Y$, the components of the potential $\mathbf{A}$ and Φ, subject to the equation (2.5), have, in general, a kink[‡] on S and therefore $\Delta\mathbf{A}$, the solution $\mathbf{q}$ contains a δ-function on the boundary of the domain Y with some density $\mathbf{p}_0$.

Assuming $\mathbf{a} = (a_1, a_2, a_3)$, where $\Delta a_j(\mathbf{x}) = \delta(\mathbf{x})$, $a_j(\infty) = 0$, ie $a_j(\mathbf{x}) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}$, we obtain

$$\Delta\mathbf{A}(\mathbf{x}) \overset{(2.5)}{=} -\int \mathbf{F}(\mathbf{y})\Delta\mathbf{a}(\mathbf{x}-\mathbf{y})\,d\mathbf{y} = \Delta\Big[-\int \mathbf{F}(\mathbf{y})\mathbf{a}(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\Big].$$

From here

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int \mathbf{F}(\mathbf{y})\frac{1}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y} \overset{(2.5)}{=} \frac{1}{4\pi}\int \Big(\mathbf{q}(\mathbf{y}) + \Phi(\mathbf{y})\nabla\sigma(\mathbf{y})\Big)\frac{1}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y},$$

since the Laplace equation has a unique solution that vanishes at infinity (as already noted, $\mathbf{A}\big|_\infty = 0$). As a result, we obtain an integral equation of the I-kind

$$\int_Y \frac{\mathbf{q}(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in Y, \tag{2.7}$$

whose right-hand side, given by the formula

$$\mathbf{f}(\mathbf{x}) = 4\pi\mathbf{A}(\mathbf{x}) - \int_Y \frac{\Phi(\mathbf{y})\nabla\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y}. \tag{2.8}$$

If the function σ, subject to the condition (1.2), is piecewise constant, then

$$\mathbf{f}(\mathbf{x}) \overset{(2.8)}{=} 4\pi\mathbf{A}(\mathbf{x}) - (\sigma_0-\sigma_+)\mathbf{n}_X\int_X \Phi(\mathbf{y}_X)\frac{d\mathbf{y}_X}{|\mathbf{x}-\mathbf{y}_X|} - (\sigma_- - \sigma_0)\mathbf{n}_S\int_S \Phi(\mathbf{y}_S)\frac{d\mathbf{y}_S}{|\mathbf{x}-\mathbf{y}_S|},$$

where $\mathbf{n}_X$ and $\mathbf{n}_S$ are the external normals to $X = \partial Y_0 \cap \partial Y_+ = \partial Y_+$ and $S = \partial Y_0 \cap \partial Y_- = \partial Y$ (see Fig.).

3. The theorem given below gives the key to solving the inverse MEEG-problem. The proof of this theorem is by no means easy [4]. But after its formulation, we give an explanation of why this is correct.

Theorem 2.1. *Let Y be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\Gamma = \partial Y$, and let $\mathbf{f} \in C^\infty(\overline{Y})$. Then the integral equation of the I-kind*

$$\int_Y \frac{\mathbf{q}(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in Y \tag{2.9}$$

uniquely solvable, and its solution has the form

$$\mathbf{q}(\mathbf{x}) = \mathbf{q}_0(\mathbf{x}) + \mathbf{p}_0(\mathbf{y}')\delta\Big|_{\partial Y}, \tag{2.10}$$

where $\delta\big|_{\partial Y}$ is the δ-function on ∂Y, and $\mathbf{q}_0 \in C^\infty(\overline{Y})$, $\mathbf{p}_0 \in C^\infty(\partial Y)$ if $\mathbf{f} \in C^\infty(\overline{Y})$.

[‡]By virtue of (2.2) and (2.4), the potential of Φ has a similar singularity.

Remark 2.1. For a less smoothness of the function $\mathbf{f}$ the components $\mathbf{q}_0$ and $\mathbf{p}_0$ of the solution are also less smooth. Namely, $\mathbf{q}_0 \in H^{s-2}(Y)$, $\mathbf{p}_0 \in H^{s-1}(\partial Y)$ if $\mathbf{f} \in H^s(Y)$, where $s > 3/2$. Moreover, as was proved in [4]

$$\mathbf{q}_0(\mathbf{x}) + \mathbf{p}_0(\mathbf{y}')\delta\Big|_{\partial Y} \in H^{s-2}_{(\varkappa)}(Y),$$

where the so-called factorization[§] index $\varkappa$ equals -1. As for space $H^{s-2}_{(-1)}(Y)$, the structure of its elements near the locally "straightened" boundary ∂Y is given by the formulas (2.14) (see below). These formulas reflect the relationship between the components $\mathbf{q}_0$ and $\mathbf{p}_0$ of the solution (2.9). This is extremely important, even to a greater extent than the fact that the solution contains a δ-function on the boundary of the domain Y, since this entails implicit, but essential restrictions on the function $\mathbf{q}_0$. This is what guarantees the correctness of the inverse MEEG problem, in particular, its uniqueness. Unfortunately, this went past the understanding by biophysicists in their numerous works on the inverse MEEG-problem[¶]

The proof of the theorem 2.1 is given in [4]. It is also proved there that the operator

$$\mathfrak{I} : H^{s-2}_{(-1)}(Y) \ni \mathbf{q} \mapsto \mathfrak{I}\mathbf{q} = \int_Y \frac{\mathbf{q}(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \in H^s(Y)$$

realizes an isomorphism. Here we briefly explain why the solution of the equation (2.7) is representable by the formula (2.10), and its components $\mathbf{q}_0$ and $\mathbf{p}_0$ are interrelated.

We first consider a somewhat different integral equation of the first kind, namely,

$$\int_{\mathbb{R}^3_+} \frac{e^{-2\pi\lambda|x-y|}u(y)dy}{|x-y|} = f(x)\,, \quad \text{where} \quad \lambda > 0\,, \quad x \in \mathbb{R}^3_+\,, \tag{2.11}$$

[§]If $\varkappa \in \mathbb{Z}$, $r > \max(\varkappa, 0) - 1/2$, and $u \in H^r_{(\varkappa)}(Y)$, then for $\varkappa \geq 0$ the function u satisfies the $\varkappa$ zero-Dirichlet data. And for $\varkappa < 0$, the function $u \in H^r_{(\varkappa)}(Y)$ has the form $u(\mathbf{x}) = u_0(\mathbf{x}) + \sum_{j=0}^{|\varkappa|-1} p_j(\mathbf{y}')\delta^{(j)}\Big|_{\partial Y}$; while u_0 and p_j are interrelated.

[¶]Since the 80s of the XX century, when they began to actively study the inverse MEEG-problem, the opinion about the incorrectness of this inverse problem began to spread widely in the scientific and popular science literature (see, for example, [14, 15] and the literature cited there). Often this view was reinforced by references to Helmholtz's authority, to his article [10], in which, allegedly, there is a corresponding statement. However, the authors of this type of reference either did not open this Helmholtz article (as in the case of [15]), or misinterpreted it. There is nothing close to this opinion in Helmholtz's paper [10]. Until now, biophysics are purely intuitive (and, unfortunately, mistakenly) believe that the solution of the inverse MEEG-problem is a certain set of point sources at the required points with the desired coefficients. The one or other multidimensional problem of linear algebra that arises in this huge stream of work on this subject turned out to be incorrect, because it is initially considered in functional spaces that are inadequate to the problem. Numerous attempts to correct the ill-posed problems arising in this way with the help of various techniques, as a rule, by means of the so-called "Tikhonov regularization" of course, can not always give a reliable result, because these techniques do not disclose the true causes of incorrectness, but only obscure them, thus hindering the real solution of problems. The situation is similar to the following. Let $F : \mathbb{R}^2 = X \times Y \ni (x, y) \mapsto F(x, y) = x$. Then for each $a \in \mathbb{R}$ the equation $F\big|_X = a$, i.e. the equation $F(x, 0) = a$ is absolutely correct, and the equation $F\big|_Y = a$ is solvable only for $a = 0$ and in this case the solution of this equation $F(0, y) = a$ is not unique. This simple example shows that the question of the correctness of the equation (including the inverse of the MEEG problem) can not be considered outside the framework of the choice of functional spaces that are adequate to this equation.

in which the kernel of the corresponding operator has an additional factor $e^{-2\pi\lambda|x-y|}$ with $\lambda > 0$. We take a continuous extension $\Phi f = f_+ + f_-$ of the function f and taking into account that $f_+\big|_{\mathbb{R}^3_+} = f$ and $f_-\big|_{\mathbb{R}^3_+} = 0$, note that the equation (2.11) can be written in the following form

$$Op\Big(\frac{1}{|\xi|^2+\lambda^2}\Big)u_+ = \Phi f\,, \quad u_+(y) = \begin{cases} u\,, \text{ if } y \in \mathbb{R}^3_+ \\ 0\,, \text{ otherwise.} \end{cases} \tag{2.12}$$

Indeed, (2.12) $\Leftrightarrow \big(-\Delta + (2\pi\lambda)^2\big)\Phi f = 4\pi^2 u_+$, where Δ is the Laplace operator. The solution Φf of the last equation can be represented as the convolution $4\pi^2 G * u_+$ of the function u_+ with the fundamental solution $G(x) = \exp(-2\pi\lambda|x|)/4\pi|x|$ of the operator $-\Delta + (2\pi\lambda)^2$ (see [16]). Let $|\xi'|^2_\lambda \overset{def}{=} \xi_1^2 + \xi_2^2 + (2\pi\lambda)^2$, and $\theta(y)$ is a characteristic function of the half-space $\mathbb{R}^3_+$. Then the solution of the equation (2.12) is given by the formula‖

$$u_+(x) = Op\big(i\xi_3 + |\xi'|_\lambda\big)\theta(x)Op\big(-i\xi_3 + |\xi'|_\lambda\big)\Phi f\,. \tag{2.13}$$

And since $Op\big(i\xi_3+|\xi'|_\lambda\big)Op\big(-i\xi_3+|\xi'|_\lambda\big) = \frac{1}{(2\pi)^2}\big(-\Delta+(2\pi\lambda)^2\big)$ and $Op\big(i\xi_3\big) = \frac{1}{2\pi}\frac{\partial}{\partial x_3}$, then $u_+(x) = u_0(x) + \rho_0(x')\delta(x')$, where formulas

$$u_0(x)=\theta(x)\big(-\frac{1}{(2\pi)^2}\Delta+\lambda^2\big)f, \rho_0(x')=-\frac{1}{4\pi^2}\frac{\partial}{\partial x_3}f(x_1,x_2,x_3)\Big|_{x_3=0}+\frac{1}{2\pi}Op(|\xi'|_\lambda)f(x_1,x_2,0), \tag{2.14}$$

representing components u_+, reflect their interrelation.

The condition $\lambda > 0$ in the equations (2.11) and (2.12) is significant (since for $\lambda = 0$ the corresponding operators are not defined). However, if the domain Y is bounded, then using the partition of unity and applying the general elliptic theory, including the theorem on the stability of the index of elliptic operators, it is possible [4] to establish the formula (2.10).

We also note that there is a connection between the solution $\mathbf{q}$ of the integral equation (2.7) and the solution $\mathbf{u}$ of an integral equation of the second kind

$$\eta^2\mathbf{u}(\mathbf{x}) + \int_Y \frac{\mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in Y\,, \quad \eta > 0. \tag{2.15}$$

In [5] it is proved

Theorem 2.2. *The solution of the equation* (2.15) *is representable in the form*

$$\mathbf{u}(\mathbf{x}) = \mathbf{q}_0(\mathbf{x}) + \frac{1}{\eta}\mathbf{p}_0(\mathbf{y}')\varphi\, e^{-\mathbf{y}_n/\eta} + r_0(\mathbf{x},\eta), \tag{2.16}$$

where $\|r_0\|_{L^2} \le C\sqrt{\eta}$, $\mathbf{y}_n$ *is the distance along the normal from* $\mathbf{x}$ *to* $\mathbf{y}' \in \Gamma$, *and* $\varphi \in C^\infty(\overline{Y})$, $\varphi \equiv 1$ *in a small neighborhood* ∂Y *and* $\varphi \equiv 0$ *outside a slightly larger neighborhood.*

4. For simplicity, we assume here that the permittivity is constant ($\varepsilon = \text{const}$). In this case, not only the scalar components of the equation (2.5) for the potential $\mathbf{A}$, but also the equation (2.2) for the potential Φ have the form $\Delta u = g$. In this case, by specifying the function g, we simulate the assignment of the test values ρ and $\mathbf{q}$. It corresponds to the solution of the MEEG problem, represented (via the

‖According to the Paley-Wiener theorem, $\theta(x)Op\big(-i\xi_n + |\xi'|_\lambda\big)f_- = 0$ and therefore $u_+(x)$ does not depend on f_-.

function u) by the potentials $\mathbf{A}_g = \mathbf{A}(\rho, \mathbf{q})$ and $\Phi_g = \Phi(\rho, \mathbf{q})$. Thus, the inverse MEEG problem reduces to minimizing the functional

$$G(\rho, \mathbf{q}) \stackrel{def}{=} \sum_k \left(\|\mathbf{B}(\mathbf{x}_k) - \mathrm{rot}\mathbf{A}_g\Big|_{\mathbf{x}=\mathbf{x}_k}\|^2 + \|\mathbf{E}(\mathbf{x}_k) + \nabla\Phi_g\Big|_{\mathbf{x}=\mathbf{x}_k}\|^2 \right), \qquad (2.17)$$

where $\mathbf{B}(\mathbf{x}_k)$ and $\mathbf{E}(\mathbf{x}_k)$ are the known measurements of the electromagnetic field at the points $\mathbf{x}_k \in X$. As for the a priori of the unknown potentials $\mathbf{A}_g$ and Φ_g, they can be modeled, taking into account their singularity noted at the beginning of the third paragraph (a break on $S = \partial Y$ along the normal to S).

We confine ourselves here to the consideration of the spherical model, when the regions $Y = Y_-$ and Y_+ appearing in (1.2) are such that Y is the ball $|x| < R$, and $Y_+ = \mathbb{R}^3 \setminus Y$. Thus (see the right-hand side of the figure), $X = \partial Y = \partial Y_+$, and $Y_0 = \varnothing$. We introduce the spherical coordinates (r, θ, φ). Let

$$Y_n^{\pm} : (\theta, \varphi) \mapsto \sum_{m=0}^{n} \left[A_{nm}^{\pm} \cos(m\varphi) + B_{nm}^{\pm} \sin(m\varphi) \right] P_n^{(m)}(\cos\theta)$$

the so-called spherical functions**, parametrized by the coefficients $A_{nm}^{\pm}$ and $B_{nm}^{\pm}$, and

$$u(r, \theta, \varphi) = \frac{1}{R^2} \begin{cases} \sum\limits_{n \geq 0} \Big[\sum\limits_{k \geq 2} C_k^- \big(r/R\big)^{n+k} + D_n^- \big(r/R\big)^n \Big] Y_n^-(\theta, \varphi) \text{ in } Y_- \\ \\ \sum\limits_{n \geq 1} \Big[\sum\limits_{k \geq 2} C_k^+ \big(R/r\big)^{n+k} + D_n^+ \big(R/r\big)^n \Big] Y_n^+(\theta, \varphi) \text{ in } Y_+ , \end{cases} \qquad (2.18)$$

at that

$$\sum_{n \geq 0} \Big[\sum_{k \geq s2} C_k^- + D_n^- \Big] = \sum_{n \geq 1} \Big[\sum_{k \geq 2} C_k^+ + D_n^+ \Big], \qquad (2.19)$$

which is a condition for the continuity of the function u.

Then the function $g = \Delta u$ is given by the formula

$$g(r, \theta, \varphi) = \frac{1}{R^2} \begin{cases} \sum\limits_{n \geq 0} \sum\limits_{k \geq 2} g_{kn}^-(r) \big(r/R\big)^{n+k-2} Y_n^-(\theta, \varphi) \text{ in } Y_- \\ \\ \sum\limits_{n \geq 1} \sum\limits_{k \geq 2} g_{kn}^+(r) \big(R/r\big)^{n+k+2} Y_n^+(\theta, \varphi) \text{ in } Y_+ = \mathbb{R}^3 \setminus Y_- , \end{cases} \qquad (2.20)$$

where $g_{kn}^{\pm}(r) = C_k^{\pm}(n+k)(n+k+1) - n(n+1)r^2$ are interrelated†† by the relation (2.19).

The function (2.18), depending on the family of numerical parameters $\mathcal{N} = \{A_{nm}^{\pm}, B_{nm}^{\pm}, C_{nm}^{\pm}, D_{nm}^{\pm}\}$, represents, as was said above, the potentials $\mathbf{A} = \mathbf{A}_{\mathcal{N}}$ and $\Phi = \Phi_{\mathcal{N}}$. And they, in turn, give approximations $\mathrm{rot}\mathbf{A}_{\mathcal{N}}$ and $\nabla\Phi_{\mathcal{N}}$ of the a

** $P_n^{(m)}$ denotes the associated Legendre functions, i.e.

$$P_n^{(m)}(t) = (1-t^2)^{\frac{m}{2}} \frac{d^m}{dt^m} P_n(t), \qquad P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} \left[(t^2-1)^m \right].$$

†† This reflects the above relationship of the components solution (2.7) of the equation (2.10).

priori of the undefined $\mathrm{rot}\mathbf{A}_g$ and $\nabla\Phi_g$, which are part of the (2.17). Thus, the functional (2.17) is approximated by the functional

$$H(\mathcal{N}) \overset{def}{=} \sum_k \Big(\|\mathbf{B}(\mathbf{x}_k) - \mathrm{rot}\mathbf{A}_{\mathcal{N}}\Big|_{\mathbf{x}=\mathbf{x}_k}\|^2 + \|\mathbf{E}(\mathbf{x}_k) + \nabla\Phi_{\mathcal{N}}\Big|_{\mathbf{x}=\mathbf{x}_k}\|^2 \Big),$$

whose minimization of which on the elements $\mathcal{N}^*$ reveals a priori possible "essentially" different solutions of the inverse MEEG problem in the spherical case since for the potentials $\mathbf{A} = \mathbf{A}_{\mathcal{N}^*}$ and $\Phi = \Phi_{\mathcal{N}^*}$ the formula (2.6) allows us to compute in the domain Y the component $\mathbf{q}_0$ of the desired solution $\mathbf{q}(\mathbf{x}) \overset{(2.10)}{=} \mathbf{q}_0(\mathbf{x}) + \mathbf{p}_0(\mathbf{y}')\delta\Big|_{\partial Y}$. Knowing this component $\mathbf{q}_0$ makes it possible to effectively find the solution (2.16) of the equation (2.15), and therefore the density $\mathbf{p}_0$.

In the general case, spherical functions must be replaced by a multi-parameter set of functions corresponding to the domains Y^-, Y_0, Y_+.

Let's note one more circumstance. If the electric field data is not specified, i.e. the data is a priori arbitrary, then the right-hand side of the equation (2.7) is defined, as noted above, no more than up to $\nabla\varphi$, where φ is subject to the condition: $\Delta\varphi \in H^{s-1}(\mathbb{R}^3)$. According to the theorem 2.1, in this case the components (q_1, q_2, q_3) of any solution (2.10) are also linearly dependent and therefore there is infinite-dimensional ambiguity in the choice of solution. In the case $Y = \mathbb{R}^2$ this fact was established by different ways in [2,6] (see also the section 3).

3. Flat model of the inverse MEG-problem

This is not a MEEG problem, since there is no data on the electric field. However, this problem is of particular interest, since it has a direct relationship to *scanning magnetic microscopes*. These tool [12] make it possible to record magnetic fields, for example, in integrated circuits, in magnetotactic bacteria. They are used in materials science, mineralogy, paleomagnetic analysis [1,17].

In this model case X is the plane $\mathbb{R}^2 \ni \mathbf{x} = (x_1,\ x_2)$, and $Y = \{\mathbf{y} = (\mathbf{x}, -1)\}$ is a plane parallel to it that stay away from X at distance 1. X is a surface at the points $\mathbf{x}$ of which the magnetic field $\mathbf{B}(\mathbf{x}) = (B_1(\mathbf{x}), B_2(\mathbf{x}), B_3(\mathbf{x}))$ is measured, and Y is the set, in which we have the distribution of the electric dipoles $\mathbf{Q} : Y \ni \mathbf{y} = (\mathbf{x}, -1) \mapsto \mathbf{Q}(\mathbf{y}) = (Q_1(\mathbf{y}), Q_2(\mathbf{y}), Q_3(\mathbf{y}))$. In what follows we assume that $\mu = 4\pi$.

Lemma 3.1 (Integral version of Biot–Sawar law)**.**

$$\int_Y \mathbf{K}(\mathbf{x}-\mathbf{y})\,\mathbf{Q}(\mathbf{y})\,d\mathbf{y} = \mathbf{B}(\mathbf{x}). \tag{3.1}$$

Here $\mathbf{K}(\mathbf{x}-\mathbf{y})\,\mathbf{Q}(\mathbf{y}) = \frac{\mathbf{Q}(\mathbf{y})\times(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}$, $\mathbf{a}\times\mathbf{b}$ *is cross product* $\mathbf{a}$ *and* $\mathbf{b}$. *There by*

$$\mathbf{K}(\mathbf{t}) = \begin{bmatrix} 0 & K_{12}(\mathbf{t}) & -K_{31}(\mathbf{t}) \\ -K_{12}(\mathbf{t}) & 0 & K_{23}(\mathbf{t}) \\ K_{31}(\mathbf{t}) & -K_{23}(\mathbf{t}) & 0 \end{bmatrix},$$

$$K_{12}(\mathbf{t}) = \frac{1}{|\mathbf{t}|^3}, \qquad K_{31}(\mathbf{t}) = \frac{t_2}{|\mathbf{t}|^3}, \qquad K_{23}(\mathbf{t}) = \frac{t_1}{|\mathbf{t}|^3}, \tag{3.2}$$

$$|\mathbf{t}| = \sqrt{t_1^2 + t_2^2 + 1}, \qquad \mathbf{t} = (t_1, t_2, 1) \in \mathbb{R}^3.$$

Let us rewrite the equation (3.1) in the pseudo-differential equation

$$Op\big(\widetilde{\mathbf{K}}(\xi)\big)\mathbf{Q} = \mathbf{B} \;\Leftrightarrow\; \widetilde{\mathbf{K}}(\xi)\widetilde{\mathbf{Q}}(\xi) = \widetilde{\mathbf{B}}(\xi), \qquad Op\big(\widetilde{\mathbf{K}}(\xi)\big) \overset{def}{=} \mathbf{F}^{-1}_{\xi\to\mathbf{x}}\big(\widetilde{\mathbf{K}}(\xi)\big)\mathbf{F}_{\mathbf{y}\to\xi},$$

where

$$\widetilde{\mathbf{K}}(\xi) = \mathbf{F}_{\mathbf{s}\to\xi}\,\mathbf{K}(\mathbf{s}) \overset{def}{=} \int_{\mathbb{R}^2} e^{-\mathring{i}\mathbf{s}\xi}\mathbf{K}(\mathbf{s})d\mathbf{s}\,, \qquad \mathring{i} \overset{def}{=} 2\pi i\,.$$

Lemma 3.2. [‡‡] *The following relations are valid:*

$$\widetilde{\mathbf{K}}(\xi) = \begin{bmatrix} 0 & 1 & i\frac{\xi_2}{|\xi|} \\ -1 & 0 & -i\frac{\xi_1}{|\xi|} \\ -i\frac{\xi_2}{|\xi|} & i\frac{\xi_1}{|\xi|} & 0 \end{bmatrix} E(\xi), \qquad E(\xi) = 2\pi e^{-2\pi|\xi|}.$$

Proof. Let $\bar{1} = \{23\}$, $\bar{2} = \{31\}$, $\bar{3} = \{12\}$, i.e. $\bar{m}$ — are those two of the three digits $\{1,2,3\}$, that complement the index m when acting cyclic permutation: $\{1,2,3\} \to \{2,3,1\} \to \{3,1,2\}$. Note that

$$K_{\bar{m}}(\mathbf{s}) \overset{(3.2)}{=} -\frac{\partial}{\partial s_m}\frac{1}{|\mathbf{s}|} \quad \text{where} \quad m \neq 3\,.$$

Thus, for $m \neq 3$ we have

$$\widetilde{K}_{\bar{m}}(\xi) = -\lim_{N\to\infty}\int_{s_1^2+s_2^2\le N^2} e^{-\mathring{i}(s_1\xi_1+s_2\xi_2)}\frac{\partial}{\partial s_m}\frac{1}{|\mathbf{s}|}\,ds_1\,ds_2,$$

and

$$\widetilde{K}_{\bar{3}}(\xi) \overset{\text{ie}}{=} \widetilde{K}_{12}(\xi) \overset{(3.2)}{=} \int_{\mathbb{R}^2}\frac{e^{-\mathring{i}(s_1\xi_1+s_2\xi_2)}\,ds_1 ds_2}{[\,s_1^2+s_2^2+1]^{3/2}}\,.$$

Assuming $re^{i\phi} = s_1 + is_2$, $\rho e^{i\psi} = \xi_1 + i\xi_2$, $|\xi| = \sqrt{\xi_1^2+\xi_2^2}$, rewrite $\widetilde{K}_{\bar{3}}(\xi)$, using the following Hankel formula, also called the Fourier–Bessel transform:

$$\int_{r=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{e^{-\mathring{i}r|\xi|\cos(\phi-\psi)}}{[r^2+1]^{3/2}}r\,drd\phi = 2\pi\int_0^\infty\frac{rJ_0(2\pi|\xi|r)}{[r^2+1]^{3/2}}dr\,,$$

where $J_0(\zeta) = \frac{1}{2\pi}\int_0^{2\pi}e^{i\zeta\cos\theta}\,d\theta$ — is the zero-order Bessel function. Similarly, for $m \neq 3$ we have

$$\begin{aligned} &-\lim_{N\to\infty}\int_{s_1^2+s_2^2\le N^2} e^{-\mathring{i}(s_1\xi_1+s_2\xi_2)}\frac{\partial}{\partial s_m}\frac{1}{\sqrt{s_1^2+s_2^2+1}}\,ds_1ds_2 \\ &= -\mathring{i}\xi_m\lim_{N\to\infty}\int_{r=\sqrt{s_1^2+s_2^2}\le N}\frac{e^{-\mathring{i}(s_1\xi_1+s_2\xi_2)}\,ds_1ds_2}{[r^2(s_1,s_2)+1]^{1/2}} \\ &= -\mathring{i}\xi_m\lim_{N\to\infty}\int_0^N\left(\int_0^{2\pi}\frac{e^{-\mathring{i}r\rho\cos\phi}}{[r^2+1]^{1/2}}d\phi\right)rdr = -2\pi\mathring{i}\xi_m\int_0^\infty\frac{rJ_0(2\pi|\xi|r)}{[r^2+1]^{1/2}}dr. \end{aligned}$$

[‡‡]Proven with A.S. Kochurov's participation.

It is known (see, for example [8], formulas 6.554 (1 and 4)) that

$$\int_0^\infty \frac{rJ_0(qr)\,dr}{(r^2+a^2)^{3/2}} = \frac{1}{a}e^{-aq}\Big|_{q>0}, \qquad \int_0^\infty \frac{rJ_0(qr)\,dr}{(r^2+a^2)^{1/2}} = \frac{1}{q}e^{-aq}\Big|_{q>0}.$$

Thus,

$$\widetilde{K}_{\bar{3}}(\xi) = 2\pi e^{-2\pi|\xi|}, \quad \widetilde{K}_{\bar{m}}(\xi)\Big|_{m\neq 3} = -2\pi i\frac{\xi_m}{|\xi|}e^{-2\pi|\xi|}.$$

□

According to Lemma 3.2, the coordinate-wise form of the equation $\widetilde{\mathbf{K}}(\xi)\widetilde{\mathbf{Q}}(\xi) = \widetilde{\mathbf{B}}(\xi)$ is as follows*:

$$\begin{aligned} &\Big[\widetilde{Q}_2(\xi) + i\frac{\xi_2}{|\xi|}\widetilde{Q}_3(\xi)\Big]E(\xi) = \widetilde{B}_1(\xi),\\ &-\Big[\widetilde{Q}_1(\xi) + i\frac{\xi_1}{|\xi|}\widetilde{Q}_3(\xi)\Big]E(\xi) = \widetilde{B}_2(\xi),\\ &\Big[-i\frac{\xi_2}{|\xi|}\widetilde{Q}_1(\xi) + i\frac{\xi_1}{|\xi|}\widetilde{Q}_2(\xi)\Big]E(\xi) = \widetilde{B}_3(\xi). \end{aligned} \tag{3.3}$$

Lemma 3.3. *The following relations hold*

$$\widetilde{Q}_1(\xi) = -\frac{\widetilde{B}_2(\xi)}{E(\xi)} - i\frac{\xi_1}{|\xi|}\widetilde{Q}_3(\xi), \qquad \widetilde{Q}_2(\xi) = \frac{\widetilde{B}_1(\xi)}{E(\xi)} - i\frac{\xi_2}{|\xi|}\widetilde{Q}_3(\xi), \tag{3.4}$$

$$\xi_1\widetilde{B}_1(\xi) + \xi_2\widetilde{B}_2(\xi) + i|\xi|\widetilde{B}_3(\xi) = 0. \tag{3.5}$$

Proof. Formulas (3.4) immediately follow from (3.3), and we get (3.5), substituting $\widetilde{Q}_1(\xi)$ and $\widetilde{Q}_2(\xi)$ from (3.4) in

$$\Big[-i\frac{\xi_2}{|\xi|}\widetilde{Q}_1(\xi) + i\frac{\xi_1}{|\xi|}\widetilde{Q}_2(\xi)\Big]E(\xi) \overset{(3.3)}{=} \widetilde{B}_3(\xi)\,.$$

□

Directly from Lemma 3.3 follows

Theorem 3.1. *Let $\widetilde{B}_k/E \in L^1$, $k = 1,2$. The general solution of*

$$\int_Y \mathbf{K}(\mathbf{x}-\mathbf{y})\,\mathbf{Q}(\mathbf{y})\,d\mathbf{y} = \mathbf{B}(\mathbf{x})$$

is representable in the form $\mathbf{Q} = \mathbf{Q}^B + \mathbf{Q}^0$. Here $\mathbf{Q}^0 = (Q_1^0, Q_2^0, Q_3^0)$, where $Q_3^0 \in L^2$,

$$Q_1^0 = -Op\left(i\frac{\xi_1}{|\xi|}\right)Q_3^0, \qquad Q_2^0 = -Op\left(i\frac{\xi_2}{|\xi|}\right)Q_3^0,$$

and $\mathbf{Q}^B = (A_1(\mathbf{y})\,,\ A_2(\mathbf{y})\,,\ 0)$, where

$$A_1(\mathbf{y}) = \mathbf{F}^{-1}_{\xi\to\mathbf{y}}\Big(-\frac{\widetilde{B}_2(\xi)}{E(\xi)}\Big), \qquad A_2(\mathbf{y}) = \mathbf{F}^{-1}_{\xi\to\mathbf{y}}\Big(\frac{\widetilde{B}_1(\xi)}{E(\xi)}\Big)\,.$$

A similar result concerning the problem of measuring the magnetic field by scanning magnetic microscope was obtained in work [2]. In the next section we strengthen theorem 3.1, by taking into account that the vector $\mathbf{B}$, according to its physical meaning, is real and is given in a finite number of points $\mathbf{x}_k$.

*These formulas, as well as the lemma 3.3 and the theorem 3.1 were obtained jointly with M.A. Galchenkova.

4. Formulas for numerical calculations

The strengthening of theorem 3.1 is that in addition

1) The functions $\mathbf{F}^{-1}_{\xi\to\mathbf{x}}\widetilde{B}_j(\xi)$ are real (this imposes restrictions on the real and imaginary parts of the functions $\widetilde{B}_j$) for each j.

2) Vector $\big(\widetilde{B}_1(\xi),\widetilde{B}_2(\xi),\widetilde{B}_3(\xi)\big)\Big|_{\widetilde{B}_3(\xi)=\frac{i}{|\xi|}\left(\xi_1\widetilde{B}_1(\xi)+\xi_2\widetilde{B}_2(\xi)\right)}$ delivers a minimum of functional

$$\Phi(\widetilde{\mathbf{B}})=\sum_{j=1}^{3}\sum_{k=(k_1,k_2)}\Big|\mathbf{F}^{-1}_{\xi\to\mathbf{x}_k}\widetilde{B}_j(\xi)-B_j(\mathbf{x}_k)\Big|^2. \tag{4.1}$$

For analyze these requirements and numerical realization, the following two statements are useful.

Proposition 4.1. *Let* $x_1=r\cos 2\pi\theta,\ x_2=r\sin 2\pi\theta,$

$$D(r,\theta)\overset{def}{=}d(x_1,x_2)=\sum_{m\in\mathbb{Z}}D_m(r)e^{\mathring{i}m\theta},\quad D_m(r)\in\mathbb{C}\,,\quad \mathring{i}\overset{def}{=}2\pi i.$$

Then

$$\mathbf{F}_{\mathbf{x}\to\xi}\,d(x)=\sum_{n\in\mathbb{Z}}(-i)^n e^{\mathring{i}\omega n}\int_0^\infty rD_n(r)J_n(2\pi|\xi|r)\,dr\,, \tag{4.2}$$

where $\mathbf{x}=(x_1,x_2)$, $\xi=(\xi_1,\xi_2)$ *and* $\xi_1=|\xi|\cos 2\pi\omega,\ \xi_2=|\xi|\sin 2\pi\omega.$

Proof. We have $\mathbf{F}_{\mathbf{x}\to\xi}\,d(x)=\int_0^\infty r\Big(\int_0^1 D(r,\theta)e^{-\mathring{i}|\xi|r\cos 2\pi(\theta-\omega)}d\theta\Big)dr$, and †

$$e^{-\mathring{i}|\xi|r\cos 2\pi(\theta-\omega)}=\sum_{n\in\mathbb{Z}}J_n(-2\pi|\xi|r)i^n e^{\mathring{i}n(\theta-\omega)}\,. \tag{4.3}$$

Next,

$$\int_0^1 e^{\mathring{i}(n-m)\theta}d\theta=\begin{cases}0 \text{ for } m\neq n\\ 1 \text{ for } m=n,\end{cases}\quad\text{and } J_{-n}(-a)=J_n(a)\overset{def}{=}\frac{1}{\pi}\int_0^\pi\cos(nt-a\sin t)dt.$$

Hence we obtain (4.2). □

Proposition 4.2. *Let* $\xi=|\xi|e^{\mathring{i}\omega}$, *and* $\widetilde{C}(|\xi|,\omega)\overset{def}{=}\widetilde{c}(\xi_1,\xi_2)=\sum_{m\in\mathbb{Z}}\widetilde{C}_m(|\xi|)e^{-\mathring{i}m\omega}$, $\widetilde{C}_m(\rho)\in\mathbb{C}$. *Then*

$$\mathbf{F}^{-1}_{\xi\to\mathbf{y}}\widetilde{c}(\xi)=\sum_{n\in\mathbb{Z}}i^n e^{-\mathring{i}\phi n}\int_0^\infty|\xi|\widetilde{C}_n(|\xi|)J_n(2\pi|\xi|\rho)\,d|\xi|.$$

Proof. We have $\mathbf{F}^{-1}_{\xi\to\mathbf{y}}\widetilde{c}(\xi)=\int_0^\infty|\xi|\Big(\int_0^1\widetilde{C}(|\xi|,\omega)e^{\mathring{i}\rho|\xi|\cos 2\pi(\omega-\phi)}d\omega\Big)d|\xi|$ and (cf. with (4.3)) $e^{\mathring{i}\rho|\xi|\cos 2\pi(\omega-\phi)}=\sum_{n\in\mathbb{Z}}J_n(2\pi\rho|\xi|)i^n e^{\mathring{i}n(\omega-\phi)}$. □

Galchenkova applied these propositions and the corollary $\mathbf{F}^{-1}_{\xi\to\mathbf{x}}\frac{1}{|\xi|}=\frac{1}{2\pi|\mathbf{x}|}$ in numerical calculations [7].

† *Generating function* for $J_n(\mu)$, i.e. the formal power series $\sum_{n\in\mathbb{Z}}J_n(\mu)\,t^n$, is $e^{\frac{\mu}{2}\left(t-\frac{1}{t}\right)}$ (see [11]). Setting $t=ie^{\mathring{i}(\theta-\omega)}$, we obtain (4.3).

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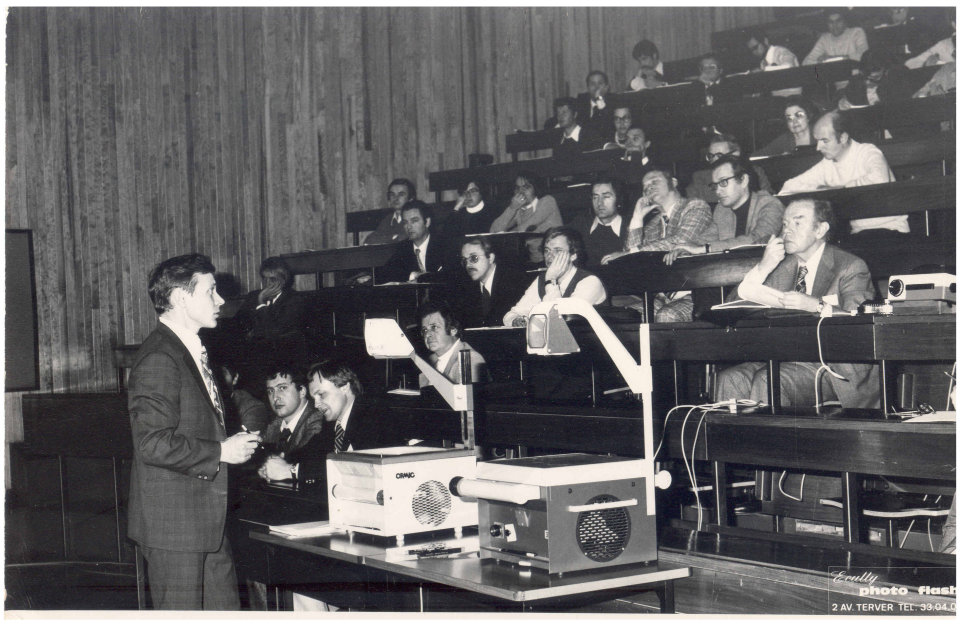

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