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A WEAK GALERKIN METHOD FOR SECOND ORDER ELLIPTIC PROBLEMS WITH POLYNOMIAL REDUCTION

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Abstract The second order elliptic equation, which is also know as the diffusion-convection equation, is of great interest in many branches of physics and industry. In this paper, we use the weak Galerkin finite element method to study the general second order elliptic equation. A weak Galerkin finite element method is proposed and analyzed. This scheme features piecewise polynomials of degree $k \geq 1$ on each element and piecewise polynomials of degree $k-1 \geq 0$ on each edge or face of the element. Error estimates of optimal order of convergence rate are established in both discrete H^1 and standard L^2 norm. The paper also presents some numerical experiments to verify the efficiency of the method.

Keywords Galerkin finite element methods, second-order elliptic problems, discrete gradient, mixed finite element methods.

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1. Introduction

In the past few years, many researchers have investigated Galerkin methods utilizing fully discontinuous approximating spaces. Weak Galerkin (WG) finite element method is one of these methods. Wang and Ye introduced and analyzed the weak Galerkin method [19] for the second-order elliptic problems in 2013. Since then the weak Galerkin method has been widely applied to different equations, such as the Stokes equations [21], Helmholtz equations [11], Maxwell equations [14] and biharmonic equations [12, 13, 15, 22], etc. Weak Galerkin refers to finite element techniques for solving partial differential equations where the differential operators are approximated by weak forms as distributions. The main idea is to use weak functions and their weak derivatives. The continuity is regained by the stabilizer term.

We consider the second-order elliptic equations with Dirichlet boundary condition, which seeks an unknown function $u = u(x)$ satisfying,

$$-\nabla \cdot (a\nabla u) + \mathbf{b} \cdot \nabla u + cu = f, \text{ in } \Omega, \tag{1.1}$$

$$u = g, \text{ on } \partial\Omega, \tag{1.2}$$

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where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$, $a = (a_{ij}(x))_{d\times d} \in [L^\infty(\Omega)]^{d^2}$ is a symmetric matrix-valued function, $\mathbf{b} = (b_i(x))_{d\times 1}$ is a vector-valued function, and $c = c(x)$ is a scalar function on Ω. Assume that the matrix a satisfies the following property: there exists a constant $\alpha > 0$ such that

$$\alpha\xi^T\xi \le \xi^T a\xi, \quad \forall\xi \in \mathbb{R}^d.$$

We concentrate on two-dimensional problems only (i.e., $d = 2$). An extension to higher-dimensional problems is straightforward.

The general second order elliptic equation, which is also known as the convection-diffusion equation, is a fundamental equation that has been widely used in many areas of science and engineering [3,6,18]. In general, the diffusion term is dominated by the convection term, which presents a challenging numerical computational tasks.

The general second order elliptic equation (1.1) has been studied using various finite element methods. The standard Galerkin methods [4,7,9] and various interior penalty type discontinuous methods [1, 2, 5, 16, 17] are few examples. The elliptic equation (1.1) has also been studied in [19] using the weak Galerkin finite element method. However, the formulation is limited to classical finite element partitions of triangles and tetrahedral and cannot be applied to a general mesh. In [10], a new weak Galerkin method has been developed to overcome this but with a as the only non-zero coefficient.

In this paper, we are extending the results of [10] to include the coefficients $\mathbf{b}$ and c. We study the elliptic equation (1.1) with all coefficients $a, \mathbf{b}$, and c being non-zero. We use weak functions of the form $v = \{v_0, v_b\}$, where the function v takes the value v_0 inside each element and takes the value v_b on the boundary of each element. Another objective of this paper is to study the reliability, flexibility and the accuracy of the suggested weak Galerkin method through numerical tests for both homogeneous and non-homogeneous boundary conditions. The corresponding WG solution converges to the exact solution of (1.1) with rate of $O(h^k)$ in discrete H^1 norm and $O(h^{k+1})$ in standard L^2 norm, provided that the exact solution of the original problem is sufficiently smooth. In the numerical analysis, both v_0 and v_b are approximated by polynomials in $P_1(T)$ and $P_0(e)$ respectively, where T represents an element and e represents an edge of T.

This paper is organized as follows. In section 2, we describe the WG schemes and review the definition of the weak gradient operator. In section 3, we present some technical estimates that will be used later. Section 4 is dedicated to the error analysis: in 4.1 we derive the error equation, and in 4.2 we present coercivity, existence and uniqueness, and the optimal order error estimates in both discrete H^1 and standard L^2 norms. Finally in section 5, we provide some numerical results that confirm the theoretical results.

2. Weak Galerkin Finite Element Schemes

Let $\mathcal{T}_h$ be a shape regular (see [8,20]) triangulation of Ω with elements T and edges e. Let h_T be the diameter of T and $h = \max_{T\in\mathcal{T}_h} h_T$.

Our weak formulation will use the following vector spaces of functions on Ω:

$$V_h = \{v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b|_e \in P_{k-1}(e), e \in \partial T, T \in \mathcal{T}_h\},$$
$$V_h^0 = \{v \in V_h : v_b|_e = 0 \text{ for } e \in \partial\Omega\}.$$

The notation $e \in \partial\Omega$ means that e is an edge on the boundary of Ω. Also note that any function $v \in V_h$ has a single value v_b on each edge e.

The projection operators Q_0 and Q_b, defined piecewise on the interior and boundary of each of the elements of $\mathcal{T}_h$, are the L^2-projections onto $P_k(T)$ and onto $P_k(e)$, respectively. Let R_h be the projection operator onto $[P_{k-1}(T)]^2$ whose components are each the L^2-projection onto $P_{k-1}(T)$.

The discrete weak gradient operator, denoted by $\nabla_w v$ is defined as the unique polynomial $\nabla_w v|_T \in [P_{k-1}(T)]^2$ satisfying,

$$(\nabla_w v, \mathbf{q})_T = -(v_0, \nabla\cdot\mathbf{q})_T + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}, \quad \forall \mathbf{q} \in [P_{k-1}(T)]^2. \tag{2.1}$$

Now we introduce three new forms on V_h as follows. For all $v, w \in V_h$,

$$\begin{aligned} a(v,w) &= \sum_{T\in\mathcal{T}_h} (a\nabla_w v, \nabla_w w)_T - \sum_{T\in\mathcal{T}_h} (\mathbf{b}v, \nabla_w w)_T + \sum_{T\in\mathcal{T}_h} (cv, w)_T, \\ s(v,w) &= \sum_{T\in\mathcal{T}_h} h_T^{-1} \langle Q_b v_0 - v_b, Q_b w_0 - w_b\rangle_{\partial T}, \\ a_s(v,w) &= a(v,w) + s(v,w). \end{aligned} \tag{2.2}$$

Weak Galerkin Algorithm 1. The weak Galerkin formulation for our boundary value problem can now be given as: Find $u_h \in V_h$ such that $u_b = Q_b g$ for $e \in \partial\Omega$, and

$$a_s(u_h, v) = (f, v), \quad \forall v \in V_h^0. \tag{2.3}$$

The $|||v|||$ is defined as,

$$|||v|||^2 = \sum_{T\in\mathcal{T}_h} (a\nabla_w v, \nabla_w v)_T + \sum_{T\in\mathcal{T}_h} (cv, v)_T + \sum_{T\in\mathcal{T}_h} h_T^{-1} \langle Q_b v_0 - v_b, Q_b v_0 - v_b\rangle_{\partial T}.$$

The fact that $|||.|||$ defines a norm in V_h^0 can be easily verified.

3. Some Estimates

In this section we are going to present some technical results that are used in later sections. In what follows C denotes a generic constant independent of the mesh size h.

Let T be an element with e as an edge. For any function $v \in H^1(T)$, the following trace inequality holds true, [20],

$$\|v\|_e^2 \le C(h_T^{-1}\|v\|_T^2 + h_T\|\nabla v\|_T^2). \tag{3.1}$$

Another useful result is a commutative property for projection operators. The proof of lemma 3.1 can be found in [10].

Lemma 3.1. *Let Q_h and R_h be the L^2 projection operators defined in previous section. Then, on each element $T \in \mathcal{T}_h$ we have the following commutative property,*

$$\nabla_w Q_h v = R_h \nabla v, \quad \forall v \in H^1(T). \tag{3.2}$$

Lemma 3.2. *Let $v \in V_h$. Then for any $w \in H^{k+1}(\Omega)$, we have*

$$\sum_{T\in\mathcal{T}_h} (\nabla v_0, a\nabla w)_T = \sum_{T\in\mathcal{T}_h} (\nabla_w v, a\nabla_w Q_h w)_T + \sum_{T\in\mathcal{T}_h} \langle v_0 - v_b, aR_h\nabla w \cdot \mathbf{n}\rangle_{\partial T}. \quad (3.3)$$

Proof. Using (3.2), the definition of the discrete weak gradient, integration by parts and properties of projections gives rise to,

$$\begin{aligned}(\nabla_w v, a\nabla_w Q_h w)_T =&(\nabla_w v, aR_h\nabla w)_T\\ =&-(v_0, \nabla\cdot(aR_h\nabla w))_T + \langle v_b, aR_h\nabla w\cdot\mathbf{n}\rangle_{\partial T}\\ =&(\nabla v_0, aR_h\nabla w)_T - \langle v_0 - v_b, aR_h\nabla w\cdot\mathbf{n}\rangle_{\partial T}\\ =&(\nabla v_0, a\nabla w)_T - \langle v_0 - v_b, aR_h\nabla w\cdot\mathbf{n}\rangle_{\partial T},\end{aligned}$$

which gives us

$$\sum_{T\in\mathcal{T}_h} (\nabla v_0, a\nabla w)_T = \sum_{T\in\mathcal{T}_h} (\nabla_w v, a\nabla_w Q_h w)_T + \sum_{T\in\mathcal{T}_h} \langle v_0 - v_b, aR_h\nabla w \cdot \mathbf{n}\rangle_{\partial T}.$$

This concludes the proof. □

Lemma 3.3. *Let $v \in V_h$. Then for any $w \in H^{k+1}(\Omega)$, we have*

$$\sum_{T\in\mathcal{T}_h} (\nabla v_0, \mathbf{b}w)_T = \sum_{T\in\mathcal{T}_h} (\nabla_w v, \mathbf{b}Q_h w)_T + \sum_{T\in\mathcal{T}_h} \langle v_0 - v_b, \mathbf{b}Q_h w \cdot \mathbf{n}\rangle_{\partial T}. \quad (3.4)$$

Proof. From the definition of the discrete weak gradient, integration by parts and properties of projections, we get

$$\begin{aligned}(\nabla_w v, \mathbf{b}Q_h w)_T =&-(v_0, \nabla\cdot\mathbf{b}Q_h w)_T + \langle v_b, \mathbf{b}Q_h w\cdot\mathbf{n}\rangle_{\partial T}\\ =&(\nabla v_0, \mathbf{b}Q_h w)_T - \langle v_0 - v_b, \mathbf{b}Q_h w\cdot\mathbf{n}\rangle_{\partial T}\\ =&(\nabla v_0, \mathbf{b}w)_T - \langle v_0 - v_b, \mathbf{b}Q_h w\cdot\mathbf{n}\rangle_{\partial T},\end{aligned}$$

which gives rise to

$$\sum_{T\in\mathcal{T}_h} (\nabla v_0, \mathbf{b}w)_T = \sum_{T\in\mathcal{T}_h} (\nabla_w v, \mathbf{b}Q_h w)_T + \sum_{T\in\mathcal{T}_h} \langle v_0 - v_b, \mathbf{b}Q_h w \cdot \mathbf{n}\rangle_{\partial T}.$$

This concludes the proof. □

The proofs for lemma 3.4 and 3.5 can be found in [10].

Lemma 3.4. *Let $\mathcal{T}_h$ be a finite element partition of Ω that is shape regular. For all $v \in H^{k+1}(\Omega)$, we have*

$$\left(\sum_{T\in\mathcal{T}_h} \|Q_0 v - v\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|\nabla(Q_0 v - v)\|_T^2\right)^{\frac{1}{2}} \leq \quad Ch^{k+1}\|v\|_{k+1}, \quad (3.5)$$

$$\left(\sum_{T\in\mathcal{T}_h} \|a(R_h\nabla v - \nabla v)\|_T^2\right)^{\frac{1}{2}} \leq \quad Ch^{k}\|v\|_{k+1}. \quad (3.6)$$

Lemma 3.5. *For all $v \in V_h$, we have*

$$\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0 - v_b\|_{\partial T}^2\right)^{\frac{1}{2}} \leq \left(\sum_{T\in\mathcal{T}_h} \{\|\nabla v_0\|_T^2 + h_T^{-1}\|Q_b v_0 - v_b\|_{\partial T}^2\}\right)^{\frac{1}{2}}. \quad (3.7)$$

We introduce a discrete H^1 semi-norm on V_h as follows:

$$\|v\|_{1,h} = \left(\sum_{T\in\mathcal{T}_h} (\|\nabla v_0\|_T^2 + h_T^{-1}\|Q_b v_0 - v_b\|_{\partial T}^2) \right)^{\frac{1}{2}}. \tag{3.8}$$

The following lemma indicates that $\|\cdot\|_{1,h}$ is equivalent to $|||\cdot|||$

Lemma 3.6. *There exists two positive constants c_1 and c_2 such that for any $v = \{v_0, v_b\} \in V_h$, we have*

$$C_1\|v\|_{1,h} \le |||v||| \le C_2\|v\|_{1,h}. \tag{3.9}$$

Proof. Let $v = \{v_0, v_b\} \in V_h$. From the definition of weak gradient and the properties of projections,

$$\begin{aligned}(\nabla_w v, \mathbf{q})_T =& -(v_0, \nabla\cdot\mathbf{q})_T + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \\ =&(\nabla v_0, \mathbf{q})_T - \langle v_0, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \\ =&(\nabla v_0, \mathbf{q})_T - \langle Q_b v_0, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \\ =&(\nabla v_0, \mathbf{q})_T - \langle Q_b v_0 - v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned} \tag{3.10}$$

Let $\mathbf{q} = \nabla_w v$ in (3.10), then

$$(\nabla_w v, \nabla_w v)_T = (\nabla v_0, \nabla_w v)_T - \langle Q_b v_0 - v_b, \nabla_w v\cdot\mathbf{n}\rangle_{\partial T}.$$

From the trace inequality (3.1) and the inverse inequality, we get

$$\begin{aligned}\|\nabla_w v\|_T^2 \le& \|\nabla v_0\|_T\|\nabla_w v\|_T + \|Q_b v_0 - v_b\|_{\partial T}\|\nabla_w v\|_{\partial T} \\ \le& \|\nabla v_0\|_T\|\nabla_w v\|_T + ch^{-\frac{1}{2}}\|Q_b v_0 - v_b\|_{\partial T}\|\nabla_w v\|_T.\end{aligned}$$

Therefore,

$$\|\nabla_w v\|_T \le C\left(\|\nabla v_0\|_T^2 + h_T^{-1}\|Q_b v_0 - v_b\|_{\partial T}^2\right)^{\frac{1}{2}},$$

which verifies the upper bound for the $|||v|||$.

As for the lower bound, let $\mathbf{q} = \nabla v_0$ in (3.10) to obtain,

$$(\nabla_w v, \nabla v_0)_T = (\nabla v_0, \nabla v_0)_T + \langle Q_b v_0 - v_b, \nabla v_0\cdot\mathbf{n}\rangle_{\partial T}.$$

From the trace inequality (3.1) and the inverse inequality,

$$\|\nabla v_0\|_T^2 \le \|\nabla_w v\|_T\|\nabla v_0\|_T + ch^{-\frac{1}{2}}\|Q_b v_0 - v_b\|_{\partial T}\|\nabla v_0\|_T.$$

Therefore,

$$\|\nabla v_0\|_T \le C\left(\|\nabla_w v\|_T^2 + ch^{-1}\|Q_b v_0 - v_b\|_{\partial T}^2\right)^{\frac{1}{2}},$$

which verifies the lower bound for $|||v|||$. Together they complete the proof. □

Lemma 3.7. *Assume that $\mathcal{T}_h$ is shape regular. Then for any $w \in H^{k+1}(\Omega)$ and $v = \{v_0, v_b\} \in V_h$, we have*

$$\left|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b(Q_0 w) - Q_b w, Q_b v_0 - v_b\rangle_{\partial T}\right| \le Ch^k\|w\|_{k+1}|||v|||, \tag{3.11}$$

$$\left| \sum_{T \in \mathcal{T}_h} \langle a(R_h \nabla w - \nabla w) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \right| \leq C h^k \|w\|_{k+1} |||v|||, \tag{3.12}$$

$$\left| \sum_{T \in \mathcal{T}_h} \langle \mathbf{b}(Q_h w - w) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \right| \leq C h^k \|w\|_{k+1} |||v|||. \tag{3.13}$$

Proof. The proof for the first inequality can be found in [10]. For the second inequality, using (3.6), (3.7), and (3.9), we get

$$\begin{aligned}
&\left| \sum_{T \in \mathcal{T}_h} \langle a(R_h \nabla w - \nabla w) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \right| \\
\leq &C \left(\sum_{T \in \mathcal{T}_h} h_T \|a(R_h \nabla w - \nabla w)\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
\leq &C h^k \|w\|_{k+1} \left(\sum_{T \in \mathcal{T}_h} (\|\nabla v_0\|_T^2 + h_T^{-1} \|Q_b v_0 - v_b\|_{\partial T}^2) \right)^{\frac{1}{2}} \\
\leq &C h^k \|w\|_{k+1} |||v|||.
\end{aligned}$$

Using (3.5), (3.7), and (3.9), we get

$$\begin{aligned}
&\left| \sum_{T \in \mathcal{T}_h} \langle \mathbf{b}(Q_h w - w) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \right| \\
\leq &C \left(\sum_{T \in \mathcal{T}_h} h_T \|\mathbf{b}(Q_h w - w)\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
\leq &C h^k \|w\|_{k+1} \left(\sum_{T \in \mathcal{T}_h} (\|\nabla v_0\|_T^2 + h_T^{-1} \|Q_b v_0 - v_b\|_{\partial T}^2) \right)^{\frac{1}{2}} \\
\leq &C h^k \|w\|_{k+1} |||v|||.
\end{aligned}$$

This concludes the lemma. □

4. Error Analysis

4.1. Error Equation

Let $u_h = \{u_0, u_b\} \in V_h$ be the weak Galerkin finite element solution arising from 2.3 and assume that the exact solution of 1.1 is given by u. The L^2 projection of u on to the finite element space V_h can be given as

$$Q_h u = \{Q_0 u, Q_b u\}.$$

Let

$$e_h = \{e_0, e_b\} = \{Q_0 u - u_0, Q_b u - u_b\}$$

be the error between the weak Galerkin finite element solution and L^2 projection of the exact solution.

Theorem 4.1. *Let e_h be the error of the weak Galerkin finite element solution arising from (2.3). Then for any $v \in V_h^0$, we have*

$$a_s(e_h, v) = \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b(Q_0 u) - Q_b u, Q_b v_0 - v_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h}(a(R_h\nabla u - \nabla u)\cdot\mathbf{n}, \\ v_0 - v_b)_{\partial T} + \sum_{T\in\mathcal{T}_h}(\mathbf{b}(Q_h u - u)\cdot\mathbf{n}, v_0 - v_b)_{\partial T}.$$

Proof. Testing (1.1) by v_0 and using integration by parts, we get

$$\sum_{T\in\mathcal{T}_h}(a\nabla u, \nabla v_0)_T - \sum_{T\in\mathcal{T}_h}\langle a\nabla u\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h}(\mathbf{b}u, \nabla v_0)_T \\ + \sum_{T\in\mathcal{T}_h}\langle \mathbf{b}u\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h}(cu, v_0)_T = (f, v_0), \tag{4.1}$$

where we have used the fact that $\sum_{T\in\mathcal{T}_h}(a\nabla u\cdot\mathbf{n}, v_b)_{\partial T} = \sum_{T\in\mathcal{T}_h}(\mathbf{b}u\cdot\mathbf{n}, v_b)_{\partial T} = 0$.

Using (3.3), (3.4) and properties of projections, we get

$$a(Q_h u, v) = (f, v_0) - \sum_{T\in\mathcal{T}_h}(a(R_h\nabla u - \nabla u)\cdot\mathbf{n}, v_0 - v_b)_{\partial T} \\ + \sum_{T\in\mathcal{T}_h}(\mathbf{b}(Q_h u - u)\cdot\mathbf{n}, v_0 - v_b)_{\partial T}.$$

Adding the term $s(Q_h u, v)$ to both sides of the above equation gives rise to

$$a_s(Q_h u, v) = \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b(Q_0 u) - Q_b u, Q_b v_0 - v_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h}(a(R_h\nabla u - \nabla u)\cdot\mathbf{n}, \\ v_0 - v_b)_{\partial T} + \sum_{T\in\mathcal{T}_h}(\mathbf{b}(Q_h u - u)\cdot\mathbf{n}, v_0 - v_b)_{\partial T} + (f, v_0). \tag{4.2}$$

Subtracting (2.3) by (4.2) yields the following error equation,

$$a_s(e_h, v) = \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b(Q_0 u) - Q_b u, Q_b v_0 - v_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h}(a(R_h\nabla u - \nabla u)\cdot\mathbf{n}, \\ v_0 - v_b)_{\partial T} + \sum_{T\in\mathcal{T}_h}(\mathbf{b}(Q_h u - u)\cdot\mathbf{n}, v_0 - v_b)_{\partial T}, \quad \forall v \in V_h.$$

This concludes the proof. □

4.2. Error Estimates

Define the dual problem:

$$-\nabla\cdot(a\nabla w) + \mathbf{b}\cdot\nabla w + cw = e_0, \text{ in } \Omega, \\ w = 0, \text{ on } \partial\Omega, \tag{4.3}$$

with the regularity assumption $\|w\|_2 \le C\|e_0\|$.

Theorem 4.2. *Let $u_h \in V_h$ be the weak Galerkin finite element solution of the problem (1.1) arising from (2.3). Assume the exact solution $u \in H^{k+1}(\Omega)$. In addition, assume that the dual problem (4.3) has the usual H^2-regularity. Then, there exists a constant C such that*

$$\|e_0\| \le Ch|||e_0|||. \tag{4.4}$$

Proof. Testing (4.3) with e_0, we get

$$\|e_0\|^2 = (-\nabla\cdot(a\nabla w), e_0) + (\mathbf{b}\cdot\nabla w, e_0) + (cw, e_0).$$

From integration by parts, we get

$$\begin{aligned}\|e_0\|^2 = &\sum_{T\in\mathcal{T}_h}(a\nabla w, \nabla e_0)_T - \sum_{T\in\mathcal{T}_h}\langle a\nabla w\cdot\mathbf{n}, e_0\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h}(\mathbf{b}w, \nabla e_0)_T \\ &+ \sum_{T\in\mathcal{T}_h}\langle \mathbf{b}w\cdot\mathbf{n}, e_0\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h}(cw, e_0)_T.\end{aligned}$$

Since $\sum_{T\in\mathcal{T}_h}\langle a\nabla w\cdot\mathbf{n}, e_b\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\langle \mathbf{b}w\cdot\mathbf{n}, e_b\rangle_{\partial T} = 0$, we can rewrite the above expression as,

$$\begin{aligned}\|e_0\|^2 = &\sum_{T\in\mathcal{T}_h}(a\nabla w, \nabla e_0)_T - \sum_{T\in\mathcal{T}_h}\langle a\nabla w\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h}(\mathbf{b}w, \nabla e_0)_T \\ &+ \sum_{T\in\mathcal{T}_h}\langle \mathbf{b}w\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h}(cw, e_0)_T. \qquad (4.5)\end{aligned}$$

Using (3.3) and (3.4) together with equation (4.5) gives us,

$$\begin{aligned}\|e_0\|^2 = &a(Q_h w, e_h) + \sum_{T\in\mathcal{T}_h}\langle a(R_h\nabla w - \nabla w)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T} \\ &- \sum_{T\in\mathcal{T}_h}\langle \mathbf{b}(Q_h w - w)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T},\end{aligned}$$

adding and subtracting the term $s(Q_h w, e_h)$, we get

$$\begin{aligned}\|e_0\|^2 = &a_s(Q_h w, e_h) + \sum_{T\in\mathcal{T}_h}\langle a(R_h\nabla w - \nabla w)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T} \\ &- \sum_{T\in\mathcal{T}_h}\langle \mathbf{b}(Q_h w - w)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T} \\ &- \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b(Q_0 w) - Q_b w, Q_b e_0 - e_b\rangle_{\partial T},\end{aligned}$$

using theorem 4.1 together with (3.11), (3.12) and (3.13), we get

$$\|e_0\|^2 \leq Ch\|w\|_2 |||e_0|||.$$

This together with the regularity assumption $\|w\|_2 \leq C\|e_0\|$, provides the required result. □

Theorem 4.3 (Coercivity)**.** *Let $e_h = \{e_0, e_b\} = \{Q_0 u - u_0, Q_b u - u_b\}$. Then,*

$$|||e_h|||^2 \leq a_s(e_h, e_h).$$

Proof. Using the definition of the $|||e_h|||$ and (4.2) in (2.2), we get

$$\begin{aligned}a_s(e_h, e_h) = &(a\nabla_w e_h, \nabla_w e_h) + (\mathbf{b}e_h, \nabla_w e_h) + (ce_h, e_h) + s(e_h, e_h) \\ = &|||e_0|||^2 + (\mathbf{b}e_h, \nabla_w e_h)\end{aligned}$$

$$\begin{aligned}&\geq \|\!|e_h|\!\|^2 - C\|e_h\|\|\nabla_w e_h\|\\&\geq (C_1 - C_2 h)\|\!|e_h|\!\|^2\\&\geq C\|\!|e_h|\!\|^2,\end{aligned}$$

when h is sufficiently small. □

Using the coercivity, now we can prove that scheme (2.3) has a unique solution.

Theorem 4.4. *The weak Galerkin finite element scheme (2.3) has a unique solution.*

Proof. Suppose that $u_h^{(1)}$ and $u_h^{(2)}$ are two solutions of (2.3). Then $e_h = u_h^{(1)} - u_h^{(2)}$ would satisfy the equation

$$a_s(e_h, v) = 0, \quad \forall v \in V_h.$$

Note that $e_h \in V_h^0$. Letting $v = e_h$, we get

$$a_s(e_h, e_h) = 0, \quad \forall v \in V_h.$$

From theorem 4.3, we get

$$\|\!|e_h|\!\| \leq a_s(e_h, e_h) = 0.$$

The fact that $\|\!|\cdot|\!\|$ is a norm in V_h^0 implies $e_h = 0$, hence $u_h^{(1)} = u_h^{(2)}$. This concludes the proof. □

Theorem 4.5 (H1 error)**.** *Let $u_h \in V_h$ be the weak Galerkin finite element solution of the problem (1.1) arising from (2.3). Assume the exact solution $u \in H^{k+1}(\Omega)$. Then, there exists a constant C such that*

$$\|\!|u_h - Q_h u|\!\| \leq Ch^k \|u\|_{k+1}.$$

Proof. From theorem 4.3, theorem 4.1, and equations (3.11), (3.12) and (3.13), we get

$$\begin{aligned}\|\!|e_h|\!\|^2 \leq& a_s(e_h, e_h)\\=&\sum_{T\in\mathcal{T}_h} \langle \mathbf{b}(Q_h u - u)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle a(R_h \nabla u - \nabla u)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T}\\&-\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b(Q_0 u) - Q_b u, Q_b e_0 - e_b\rangle_{\partial T}\\\leq& Ch^k\|u\|_{k+1}\|\!|e_h|\!\|,\end{aligned}$$

which concludes the proof. □

Theorem 4.6 (L2 error)**.** *Let $u_h \in V_h$ be the weak Galerkin finite element solution of the problem (1.1) arising from (2.3). Assume the exact solution $u \in H^{k+1}(\Omega)$. In addition, assume that the dual problem (4.3) has the usual H^2-regularity. Then, there exists a constant C such that*

$$\|u - u_0\| \leq Ch^{k+1}\|u\|_{k+1}.$$

Proof. From theorems 4.2, and 4.5, we get

$$\|e_0\| \leq Ch\|\!|e_0|\!\| \leq Ch^{k+1}\|u\|_{k+1},$$

which concludes the proof. □

5. Numerical Experiments

In this section, we provide some numerical examples using scheme (2.3) to verify the results in theorems 4.5, and 4.6.

For simplicity, we define our domain as a rectangle $\Omega = [0,1] \times [0,1]$. Then we construct the triangular mesh by uniformly partitioning the square domain $\Omega = [0,1] \times [0,1]$ into $N \times N$ sub-squares, and then dividing each square element into two triangles by using the diagonal with a positive slope. Also, we consider $a = 1, 0.01, 0.001$, $b = (1,1)^T$ and $c = 1$.

Consider different mesh sizes $h = \frac{1}{N}(N = 2, 4, 8, 16, 32, 64, 128)$ for different triangular meshes. The following examples use these triangulations of Ω to find a solution $u_h = \{u_0, u_b\}$ where $u_0|_T \in P_1(T)$, and $u_b|_e \in P_0(e)$.

5.1. Homogeneous Boundary Conditions

Consider the elliptic problem

$$\begin{aligned} -\nabla \cdot (a\nabla u) + \mathbf{b} \cdot \nabla u + cu =& f, \text{ in } \Omega \\ u =& 0, \text{ on } \partial\Omega \end{aligned} \tag{5.1}$$

where a is a unit matrix, $\mathbf{b} = (1,1)$ and $c = 1$. The source term $f(x)$ is choosen according to the corresponding analytical solution of the given example.

Example 5.1. The analytical solution to (5.1) is

$$u = \sin(\pi x)\sin(\pi y).$$

The finite element scheme (2.3), with different mesh sizes h is applied, and the corresponding discrete H^1-norm and L^2-norm errors and convergence rates are listed in Tables 1 and 2. Figure 5.1 represents the approximation of weak Galerkin solution on the left and the exact solution on the right for example (5.1).

Table 1. H^1 norm errors and their corresponding convergence rates for Example 5.1.

h	$\|\|\|u - u_h\|\|\|$	order	$\|\|\|u - u_h\|\|\|$	order	$\|\|\|u - u_h\|\|\|$	order
	when a=1		when a=0.01		when a=0.001	
1/2	7.29E-01		1.41E+00		2.02E+00	
1/4	2.44E-01	1.58	7.21E-01	0.97	1.44E+00	0.49
1/8	9.68E-02	1.34	4.01E-01	0.84	9.50E-01	0.60
1/16	4.45E-02	1.12	2.22E-01	0.85	5.52E-01	0.78
1/32	2.17E-02	1.03	1.15E-01	0.95	3.24E-01	0.77
1/64	1.08E-02	1.01	5.78E-02	0.99	1.77E-01	0.87
1/128	5.39E-03	1.00	2.89E-02	1.00	9.08E-02	0.96

Table 2. L^2 norm errors and their corresponding convergence rates for Example 5.1.

h	$\|u - u_h\|$	order	$\|u - u_h\|$	order	$\|u - u_h\|$	order
	when a=1		when a=0.01		when a=0.001	
1/2	6.31E-01		1.12E+00		1.94E+00	
1/4	1.67E-01	1.92	4.05E-01	1.46	1.21E+00	0.68
1/8	4.24E-02	1.98	1.57E-01	1.36	5.90E-01	1.03
1/16	1.06E-02	1.99	4.93E-02	1.67	2.53E-01	1.22
1/32	2.66E-03	2.00	1.31E-02	1.91	1.02E-01	1.31
1/64	6.66E-04	2.00	3.34E-03	1.97	3.01E-02	1.76
1/128	1.67E-04	2.00	8.40E-04	1.99	7.85E-03	1.94

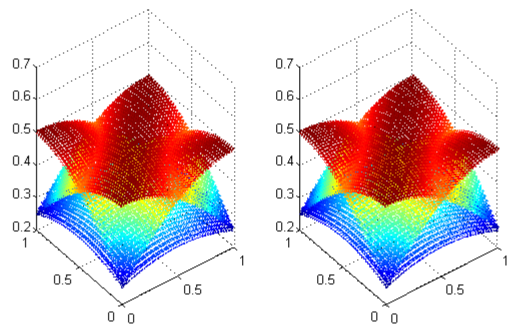

Figure 1. Weak Galerkin Solution vs Exact Solution

Example 5.2. The analytical solution to (5.1) is

$$u = x(1-x)y(1-y)\exp(x-y).$$

Numerical errors and convergence rates are listed in Table 3 and 4. Figure 5.2 represents the approximation of weak Galerkin solution on the left and the exact solution on the right for example (5.2).

Table 3. H^1 norm errors and their corresponding convergence rates for Example 5.2.

h	$\|\|\|u - u_h\|\|\|$	order	$\|\|\|u - u_h\|\|\|$	order	$\|\|\|u - u_h\|\|\|$	order
	when a=1		when a=0.01		when a=0.001	
1/2	7.23E-02		8.59E-02		1.23E-01	
1/4	3.05E-02	1.24	5.21E-02	0.72	1.04E-01	0.24
1/8	1.45E-02	1.07	3.01E-02	0.79	7.15E-02	0.54
1/16	7.16E-03	1.02	1.69E-02	0.84	4.18E-02	0.77
1/32	3.57E-03	1.00	8.77E-03	0.94	2.44E-02	0.78
1/64	1.78E-03	1.00	4.42E-03	0.99	1.34E-02	0.86
1/128	8.91E-04	1.00	2.21E-03	1.00	6.93E-03	0.96

Table 4. L^2 norm errors and their corresponding convergence rates for Example 5.2.

h	$\|u-u_h\|$	order	$\|u-u_h\|$	order	$\|u-u_h\|$	order
	when a=1		when a=0.01		when a=0.001	
1/2	5.86E-02		6.83E-02		1.18E-01	
1/4	1.68E-02	1.80	2.93E-02	1.22	8.72E-02	0.68
1/8	4.37E-03	1.94	1.23E-02	1.26	4.41E-02	1.03
1/16	1.10E-03	1.99	4.29E-03	1.52	1.87E-02	1.22
1/32	2.77E-04	2.00	1.20E-03	1.83	8.08E-03	1.31
1/64	6.92E-05	2.00	3.11E-04	1.95	2.62E-03	1.76
1/128	1.73E-05	2.00	7.85E-05	1.99	7.08E-04	1.94

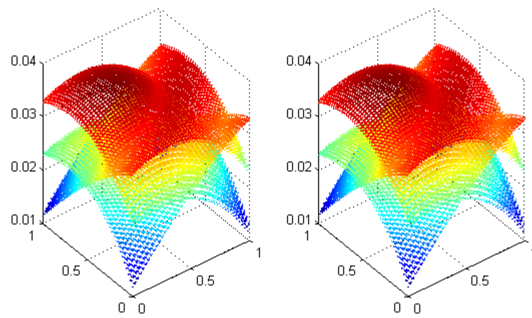

Figure 2. Weak Galerkin Solution vs Exact Solution

The numerical examples given in this section gives the first order convergence rate in discrete H^1 norm and second order convergence rate in the standard L^2 norm. The numerical results are in good agreement with the theoretical results, which shows that the WG finite element scheme (2.3) is stable and have the optimal order convergence for the homogeneous boundary case.

5.2. Non-homogeneous Boundary Conditions

Now let's consider some examples of the general elliptic problem with non-homogeneous boundary conditions. Consider the following elliptic problem

$$\begin{aligned} -\nabla \cdot (a\nabla u) + \mathbf{b} \cdot \nabla u + cu =& f, \text{ in } \Omega, \\ u =& g, \text{ on } \partial\Omega, \end{aligned} \tag{5.2}$$

where a is a unit matrix, $\mathbf{b} = (1, 1)$ and $c = 1$. The source term $f(x)$ is choosen according to the corresponding analytical solution of the given example.

Example 5.3. The analytical solution to (5.2) is

$$u = \sin(\pi x)\sin(\pi y) + x.$$

The finite element scheme (2.3), with the different mesh sizes h is applied, and the corresponding discrete H^1 and L^2 norm errors and convergence rates are listed in Table 5 and 6. Figure 5.3 represents the approximation of weak Galerkin solution on the left and the exact solution on the right for example (5.3).

Table 5. H^1 norm errors and their corresponding convergence rates for Example 5.3.

h	$\|\|\|u-u_h\|\|\|$	order	$\|\|\|u-u_h\|\|\|$	order	$\|\|\|u-u_h\|\|\|$	order
	when a=1		when a=0.01		when a=0.001	
1/2	9.96E-02		1.47E+00		2.06E+00	
1/4	4.05E-02	1.30	7.68E-01	0.93	1.49E+00	0.46
1/8	1.90E-02	1.09	4.35E-01	0.82	1.01E+00	0.57
1/16	9.35E-03	1.02	2.42E-01	0.84	5.97E-01	0.75
1/32	4.66E-03	1.00	1.25E-01	0.95	3.54E-01	0.76
1/64	2.33E-03	1.00	6.31E-02	0.99	1.94E-01	0.87
1/128	1.16E-03	1.00	3.16E-02	1.00	9.95E-02	0.96

Table 6. L^2 norm errors and their corresponding convergence rates for Example 5.3.

h	$\|u-u_h\|$	order	$\|u-u_h\|$	order	$\|u-u_h\|$	order
	when a=1		when a=0.01		when a=0.001	
1/2	8.44E-02		1.21E+00		1.98E+00	
1/4	2.24E-02	1.91	4.60E-01	1.40	1.27E+00	0.64
1/8	5.70E-03	1.97	1.78E-01	1.37	6.54E-01	0.96
1/16	1.43E-03	1.99	5.53E-02	1.68	2.90E-01	1.17
1/32	3.59E-04	2.00	1.47E-02	1.91	1.15E-01	1.33
1/64	8.97E-05	2.00	3.75E-03	1.98	3.39E-02	1.77
1/128	2.24E-05	2.00	9.41E-04	1.99	8.82E-03	1.94

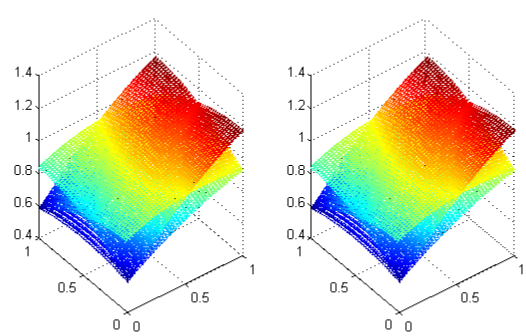

Figure 3. Weak Galerkin Solution vs Exact Solution

Example 5.4. The analytical solution to (5.2) is

$$u = x(1-x)y(1-y)\exp(x-y) + x.$$

Numerical errors and convergence rates are listed in Table 7 and 8. Figure 5.4 represents the approximation of weak Galerkin solution on the left and the exact solution on the right for example (5.4).

Table 7. H^1 norm errors and their corresponding convergence rates for Example 5.4.

h	$\|\|\|u-u_h\|\|\|$	order	$\|\|\|u-u_h\|\|\|$	order	$\|\|\|u-u_h\|\|\|$	order
	when a=1		when a=0.01		when a=0.001	
1/2	1.13E-01		4.43E-01		5.84E-01	
1/4	4.70E-02	1.27	2.93E-01	0.60	5.21E-01	0.17
1/8	2.21E-02	1.09	1.79E-01	0.71	3.89E-01	0.42
1/16	1.09E-02	1.02	1.00E-01	0.83	2.48E-01	0.65
1/32	5.42E-03	1.00	5.21E-02	0.95	1.49E-01	0.74
1/64	2.71E-03	1.00	2.63E-02	0.99	8.08E-02	0.88
1/128	1.36E-03	1.00	1.32E-02	1.00	4.14E-02	0.96

Table 8. L^2 norm errors and their corresponding convergence rates for Example 5.4.

h	$\|u-u_h\|$	order	$\|u-u_h\|$	order	$\|u-u_h\|$	order
	when a=1		when a=0.01		when a=0.001	
1/2	9.49E-02		3.33E-01		5.60E-01	
1/4	2.60E-02	1.87	1.50E-01	1.15	4.44E-01	0.33
1/8	6.67E-03	1.96	5.91E-02	1.35	2.50E-01	0.83
1/16	1.68E-03	1.99	1.84E-02	1.69	1.10E-01	1.19
1/32	4.21E-04	2.00	4.89E-03	1.91	4.14E-02	1.41
1/64	1.05E-04	2.00	1.24E-03	1.98	1.20E-02	1.79
1/128	2.63E-05	2.00	3.12E-04	1.99	3.12E-03	1.94

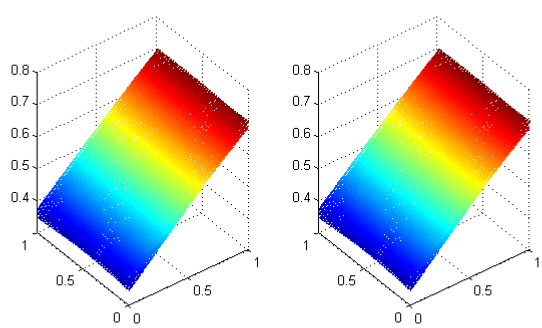

Figure 4. Weak Galerkin Solution vs Exact Solution

The numerical examples given in this section gives the first order convergence rate in discrete H^1 norm and second order convergence rate in the standard L^2 norm. The numerical results are in good agreement with the theoretical results, which shows that the WG finite element scheme (2.3) is stable and have the optimal order convergence for the non-homogeneous boundary case.

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