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EFFECT OF HERD SHAPE IN A DIFFUSIVE PREDATOR-PREY MODEL WITH TIME DELAY

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Abstract In this paper, we deal with the effect of the shape of herd behavior on the interaction between predator and prey. The model analysis was studied in three parts. The first, The analysis of the system in the absence of spatial diffusion and the time delay, where the local stability of the equilibrium states, the existence of Hopf bifurcation have been investigated. For the second part, the spatiotemporal dynamics introduce by self diffusion was determined, where the existence of Hopf bifurcation, Turing driven instability, Turing-Hopf bifurcation point have been proved. Further, the order of Hopf bifurcation points and regions of the stability of the non trivial equilibrium state was given. In the last part of the paper, we studied the delay effect on the stability of the non trivial equilibrium, where we proved that the delay can lead to the instability of interior equilibrium state, and also the existence of Hopf bifurcation. A numerical simulation was carried out to insure the theoretical results.

Keywords Predator-prey model, herd behavior, spatial diffusion, time delay, quadratic mortality, Turing instability, Turing-Hopf bifurcation.

MSC(2010) 34D23, 35B40, 35F10, 92D25.

1. Introduction

After the pioneering works of Venturino [30], a new behavior of the prey has been introduced into the interface, which called by herd behavior. This behavior can be seen in the real world when the prey gather in a huge group. This social behavior can be a useful strategy for defending of the prey from the predator. The smartness of the prey in using this behavior can bee seen in the partition of the prey pack for better defending against the predator. The strongest prey will situated on the boundary of the herd for the role defending and weakest in the center of the herd. This smartness of behavior can make a lot of difficulty for the predator. The task of the predator of hunting prey is not easy as in the other case where it puts its focus on the weakest one. For this case of interaction the predator can not hunt the inside herd prey. In other word, the predator hunts only on the boundary of the group. Obviously, the hunted prey by a predator will be proportional to the number of the prey on the bounders of the pack. In the previous work such as [1, 5, 8, 12, 29–31] the main assumption is to consider that the prey make a group in $\mathbb{R}^2$ with a regular form such as circle or a square. In this case the number of the consumed prey by

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the predator will be proportional to the square root of the prey population density. Obviously, the prey can not make a regular form only on $\mathbb{R}^2$. For instance can make it in $\mathbb{R}^3$. For better information we give a simple example of the case when the prey make a cube, this case can be seen in dynamic of birds or fiche, so on. The number of the consumed prey (denoted by R) is proportional to $R^{\frac{2}{3}}$, for not losing generality we will assume that the consumed prey is proportional to R^α where $0 < \alpha < 1$ (see Venturino and Bulai [3]) is the rate of the shape of prey herd. This case was introduced in the first time by [34] and studied further by Venturino and Bulai [3]. Also the presence of spatial diffusion was investigated in the paper [38] where the existence and non existence of constant steady states. In the present work, we will investigate in this case with the quadratic mortality of the predator. The proposed system is the following one:

$$\begin{cases} R_t = rR(1 - \dfrac{R}{k}) - aR^\alpha F, \\ F_t = \beta(-\mu F^2 + aR^\alpha F), \end{cases} \tag{1.1}$$

where $R(t)$ and $F(t)$ represents the prey and predator densities at the time t, k is the carrying capacity of space for the prey population, r is the reproduction rate of the prey, β is the conversion rate of prey to predators where $0 < \beta < 1$, $\beta\mu$ is the mortality rate. βa is the predation rate of the predator.

In the actual world, the prey and predator are always in movement. Mathematically can be modeled by presence of self diffusion. The spatial diffusion has been the subject of many works such as the papers [2,4,9,23,25–27,32,33,36]. On the other hand, It is well known that the reproduction of the predator is not instantaneous, which means that the new born predator can not become an adult predator in the same time of borne. In other word, the new born predator take a time τ to become an adult which can be modeled by the presence of time delay in the interaction functional for more examples see the papers [20,24]. The system (1.1) becomes in the presence of self-diffusion and time delay:

$$\begin{cases} \dfrac{dR(x,t)}{dt} - d_1 \dfrac{d^2R(x,t)}{dx^2} = rR(x,t)(1 - \dfrac{R(x,t)}{k}) - aR(x,t)^\alpha F(x,t), \ x \in [0,\pi] \\ \dfrac{dF(x,t)}{dt} - d_2 \dfrac{d^2F(x,t)}{dx^2} = \beta(-\mu F(x,t)^2 + aR(x,t-\tau)^\alpha F(x,t)), \ t \geq 0 \\ \dfrac{dR}{dx}(0,t) = \dfrac{dF}{dx}(0,t) = \dfrac{dR}{dx}(\pi,t) = \dfrac{dF}{dx}(\pi,t) = 0 \qquad \forall t \geq 0, \\ R(x,t) = \phi(x,t) \geq 0 \qquad F(x,t) = \psi(x,t) \geq 0 \qquad (x,t) \in [0,\pi] \times [-\tau, 0]. \end{cases} \tag{1.2}$$

Where d_1, d_2 are the diffusions coefficient for the prey and predator, respectively. x represents the distance covered by the prey or the predator. The homogeneous Neumann boundary condition means that the prey and predator are in an isolated patches. The steady of our system can implies many works, for instance for $\tau = 0$ and $\alpha = \frac{1}{2}$ the system (1.2) becomes the system presented by Z.Xu and Y.Song [35] and for $\tau > 0, \alpha = \frac{1}{2}$ and if we replace the quadratic mortality by a linear mortality we obtain the work presented by Song et al [28].

However, according to the best of our acquaintance there are no results on dynamics of the systems (1.1), (1.2). For the subject of the study of the proposed systems we organize the paper as follows

• In the next section, we will give some definitions and notations which are going to be useful for the systems analysis.

• Section 3 is devoted to prove the existence and the uniqness of the positive equilibrium.

• In section 4, the stability of the equilibrium states for the system (1.1) has been studied. Further, the existence of Hopf bifurcation has been shown.

• Sections 5, the study of the diffusion effect on system (1.1) where the presence of Hopf bifurcation, diffusion driven instability, Turing-Hopf bifurcation have been shown.

• In the last section, the study of the delay effect on the stability of the positive equilibrium has been investigated, where we assumed that the non trivial equilibrium is locally asymptotically stable in the presence of self-diffusion only and we proved that the delay can lead to the presence of Hopf bifurcation.

• In the end of the paper we will give some graphical representations for verifying the theoretical results.

2. Preliminaries

The corresponding space to the boundary condition is $C([-\tau, 0], X)$ where

$$X = \{R, F \in H^2(0,\pi)/R_x(0,t) = R_x(\pi,t) = F_x(0,t) = F_x(\pi,t) = 0\}.$$

For $U_1, U_2 \in X$, defining the usual inner product $\langle U_1, U_2\rangle = \int_0^\pi (R_1R_2 + F_1F_2)dx$ and the associated Hilbertian norm of X denoted by $\|\cdot\|_{2,2}$.

The associated eigenvalue problem is given by:

$$\begin{cases} -\Theta'' = \mu\Theta \quad x \in (0,\pi), \\ \quad \Theta'(0) = \Theta'(\pi) = 0, \end{cases} \tag{2.1}$$

it is well known that $\mu_n = n^2$ and $\cos(nx)$ $(n = 0, 1....)$ are the eigenvalues and the eigenfunctions of the problem (2.1) on X, respectively.

Definition 2.1 (diffusion driven instability)**.** If we assume that a equilibrium is locally asymptotically stable in the absence of diffusion under some condition on model parameters, and becomes unstable in the presence of diffusion. This behavior called by Turing instability or diffusion driven instability.

Putting the characteristic equation for an equilibrium state in the variable λ as the form

$$\lambda^2 + Tr_n\lambda + Det_n = 0.$$

Definition 2.2 (Turing-Hopf bifurcation)**.** If there exist two integer n,m such that n$\neq m$ and Hopf bifurcation curve defined by the equation $\mathrm{Tr}_n = 0$. intersect Turing instability (T_n) curve defined by the equation $Det_m = 0$ then we can say we have the existence of Turing-Hopf bifurcation.

3. The existence and the uniqueness of the positive equilibrium

For this section, we will prove the existence and the uniqness of the non trivial equilibrium. Obviously, the equilibrium states of the system (1.1) are also equilibrium states for (1.2). It is easy to verify that the system (1.1) has two boundary equilibrium, the semi trivial equilibrium (k,0), and the trivial equilibrium (0,0). Now, lets focus on proving the existence of the interior equilibrium (R^*, F^*). Clearly, the nontrivial equilibrium is the solution of the following system

$$\begin{cases} R^{1-\alpha}(1-\dfrac{R}{k}) - aF = 0, \\ -\mu F + aR^{\alpha} = 0. \end{cases} \tag{3.1}$$

Theorem 3.1. *Assume that $\alpha \geq \frac{1}{2}$the system (1.1) has a unique nontrivial equilibrium (R^*, F^*) verifying $R^* < k$.*

Proof. The system (3.1) is equivalent to the following system

$$\begin{cases} F_1(R) = \dfrac{r}{a}R^{1-\alpha}(1-\dfrac{R}{k}), \\ F_2(R) = \dfrac{a}{\mu}R^{\alpha}. \end{cases} \tag{3.2}$$

It is easy to see that the intersection point between the curves defined by the functions $F_1(R), F_2(R)$ in the $R-F$ plan is coordinates of the non trivial equilibrium. Also, for $R > k$ there no intersection between the two curves. It is easy to check that the functionals of the system (3.2) verify the following conditions

$$F_1(0) = F_1(k) = 0,$$

$$F_1'(R) = \frac{r}{a}(1-\alpha)R^{-\alpha} - \frac{r}{ak}(2-\alpha)R^{1-\alpha},$$

$$F_1''(R) = -\frac{r}{a}(1-\alpha)\alpha R^{-\alpha-1} - \frac{r}{ak}(2-\alpha)(1-\alpha)R^{-\alpha} < 0,$$

which means that the functional $F_1(R)$ is a concave functional and has a maximum at $R_* = \frac{k(1-\alpha)}{(2-\alpha)} > 0$, and for the functional F_1 we have

$$F_2(0) = 0, \quad F_2(k) = \frac{a}{\mu}k^{\alpha} > 0,$$

$$F_2'(R) = \frac{a\alpha}{\mu}R^{\alpha-1} > 0,$$

$$F_2''(R) = \frac{a\alpha}{\mu}(\alpha-1)R^{\alpha-2} < 0,$$

leads to say that F_2 is strictly increasing concave functional. Now putting $F_1 = F_2$ which equivalent to

$$-\frac{\mu r R^*}{a^2 k} - R^{2\alpha-1} + \frac{\mu r}{a^2} = 0. \tag{3.3}$$

The existence and the uniqness of a positive root of the equation (3.3) can be easily deduced under the condition $\alpha \geq \frac{1}{2}$. In other word, the two curves intersect in two points (0,0) and (R^*, F^*) verifying $R^* < k$ (see Figure 1). Which completes the prove. $\square$

Remark 3.1. If $\alpha < \frac{1}{2}$ the system can have two interior equilibrium states. The system can undergoes saddle-node bifurcation. the value of bifurcation point can be determined using the equation (3.3).

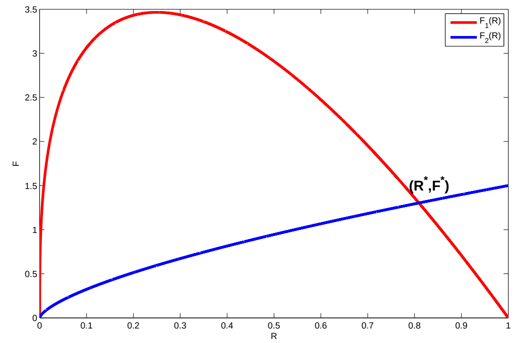

Figure 1. the existence of the nontrivial equilibrium $(R^*, F^*) = (0.8112, 1.305)$ for the values k=1;r=1.1;a=0.15;$\mu = 0.1$; $\alpha = \frac{2}{3}$.

4. Dynamics induced by the absence of delay and diffusion

This section is devoted to study of the system (1.1) where the local stability of the semi trivial equilibrium and the interior equilibrium, the existence of Hopf bifurcation have been investigated.

The Jacobian matrix of the system (1.1) is given as follows

$$J(R,F) = \begin{pmatrix} r - \dfrac{2r}{k}R - a\alpha R^{\alpha-1}F & aR^{\alpha} \\ \beta a\alpha R^{\alpha-1}F & \beta(-2\mu F + \alpha R^{\alpha}) \end{pmatrix}. \tag{4.1}$$

It is easy to verify that the boundary equilibrium is always unstable. The Jacobian matrix at (R^*, F^*) is given as follows

$$J(R^*,F^*) = \begin{pmatrix} r - \dfrac{2r}{k}R^* - \dfrac{a^2\alpha}{\mu}R^{*2\alpha-1} & -aR^{*\alpha} \\ \beta\frac{a^2\alpha}{\mu}R^{*2\alpha-1} & -\beta aR^{*\alpha} \end{pmatrix}. \tag{4.2}$$

The eigenvalue of the the jacobian matrix (4.2) is the solution of the following equation in λ

$$\begin{cases} \Lambda \triangleq \lambda^2 + Tr_0\lambda + Det_0 = 0, \\ Tr_0 = -r + \dfrac{2r}{k}R^* + \dfrac{a^2\alpha}{\mu}R^{*2\alpha-1} + \beta aR^{*\alpha}, \\ Det_0 = \beta aR^{*\alpha}[-r + \dfrac{2r}{k}R^* + \dfrac{a^2\alpha}{\mu}R^{*2\alpha-1}(1+\alpha)]. \end{cases} \tag{4.3}$$

Obviously, Hopf bifurcation occurs when $Tr_0 = 0$ and $Det_0 > 0$. Choosing β as bifurcation parameter. Now putting

$$T = -r + \frac{2r}{k}R^* + \frac{a^2\alpha}{\mu}R^{*2\alpha-1}, \tag{4.4}$$

for the positivity of T we draw the following lemma.

Lemma 4.1. *Assume that* $R_0^* = \dfrac{1-\alpha}{2-\alpha}k$ *then* $T < 0$ *for* $R^* < R_0^*$ *and* $T > 0$ *for* $R^* > R_0^*$.

Proof. From the proof of **Theorem 3.1** we have $F_1(R^*) = F_2(R^*)$ which is equivalent to

$$\frac{r}{a}R^{*1-\alpha}(1-\frac{R^*}{k}) = \frac{a}{\mu}R^{*\alpha},$$

which gives

$$a^2 = \mu r R^{*1-2\alpha}(1-\frac{R^*}{k}). \tag{4.5}$$

Replacing a^2 defined by (4.5) in (4.4) we obtain

$$T = r(-1+\alpha+\frac{R^*}{k}(2-\alpha)), \tag{4.6}$$

which means that $T < 0$ for $R^* < R_0^*$ and $T > 0$ for $R^* > R_0^*$. The proof of lemma is completed. □

Obviously $Det_0 = \beta a R^{*\alpha} D^*$ where

$$D^* = -r + \frac{2r}{k}R^* + \frac{a^2\alpha}{\mu}R^{*2\alpha-1}(1+\alpha) \tag{4.7}$$

and $sign(Det_0) = sign(D^*)$. The following lemma illustrate the condition for the positivity of D^*.

Lemma 4.2. *Assume that the condition of the **Theorem 3.1** holds and define* $R_1^* = \dfrac{-\alpha^2-\alpha+1}{(2+\alpha)(1-\alpha)}k$ *then we have*

(i) if $\alpha \in (\frac{1}{2}, \frac{\sqrt{5}-1}{2})$ *then* $D^* > 0$,

(ii) if $\alpha \in (\frac{\sqrt{5}-1}{2}, 1)$ *and* $R^* > R_1^*$ *then* $D^* > 0$,

(iii) if $\alpha \in (\frac{\sqrt{5}-1}{2}, 1)$ *and* $R^* < R_1^*$ *then* $D^* < 0$.

Proof. we replace a^2 defined by (4.5) in (4.7) we get

$$D^* = r(\alpha^2+\alpha-1) + \frac{rR^*}{k}(2+\alpha)(1-\alpha).$$

It is easy to verify the condition (i),(ii),(iii). The proof is completed. □

Obviously, for having Hopf bifurcation we need to have $Tr_0 = 0$ and $Det_0 > 0$ and if $T > 0$ or $D^* < 0$ we can not have Hopf bifurcation. The following assumption are for verifying $T < 0$ and $D^* > 0$. In other word, the following condition are a necessary but not sufficient condition for having Hopf bifurcation.

(H_1): $R^* < R_0^*$, $\alpha \in (\frac{1}{2}, \frac{\sqrt{5}-1}{2})$.

(H_2): $\alpha \in (\frac{\sqrt{5}-1}{2}, 1)$, $R_1^* < R^* < R_0^*$.

Under (H_1) or (H_2) we have $T < 0$ and $D^* > 0$ which means that $Det_0 > 0$ and Hopf bifurcation occurs if $Tr_0 = 0$ and equivalent to

$$\beta = \beta_0 = \frac{-T}{aR^{*\alpha}} > 0. \tag{4.8}$$

Letting $\lambda = \delta(\beta) \pm iw(\beta)$ be the solution of the characteristic equation Λ verifying $\delta(\beta_0) = 0$ and $w(\beta_0) = \sqrt{Det_0}$ and

$$\frac{d}{d\beta}\delta(\beta)\Big|_{\beta=\beta_0} = -\frac{1}{2}aR^{*\alpha} < 0,$$

leads to the existence of of Hopf bifurcation at $\beta = \beta_0$.

5. The effect of the self-diffusion and time delay on the proposed system

In this section, we will investigate with the presence of spatial diffusion and time delay. Now, assume that the condition of the **Theorem 3.1** holds next to the condition (H_1) or (H_2). The linearized system of (1.2) at (R^*, F^*) is as follows:

$$\begin{cases} \dfrac{dR(x,t)}{dt} - d_1\dfrac{d^2R(x,t)}{dx^2} = -TR(x,t) - aR^{*\alpha}R(x,t), \\ \dfrac{dF(x,t)}{dt} - d_2\dfrac{d^2F(x,t)}{dx^2} = -a\beta R^{*\alpha}F(x,t) + \frac{\beta a^2\alpha}{\mu}R^{*2\alpha-1}R(x,t-\tau). \end{cases} \tag{5.1}$$

The characteristic equation of (5.1) is

$$\begin{aligned} &\lambda^2 + \lambda(T + a\beta R^{*\alpha} + n^2(d_1+d_2)) + d_1d_2n^4 + n^2(d_2T + d_1a\beta R^{*\alpha}) \\ &+ a\beta R^{*\alpha}T + \frac{\beta a^3\alpha R^{*3\alpha-1}}{\mu}e^{-\lambda\tau} = 0. \end{aligned} \tag{5.2}$$

For $\tau = 0$ the characteristic equation (5.2) becomes

$$\lambda^2 + Tr_n\lambda + Det_n = 0, \tag{5.3}$$

where

$$\begin{aligned} Tr_n &= T + a\beta R^{*\alpha} + n^2(d_1+d_2) = n^2(d_1+d_2) + Tr_0, \\ Det_n &= d_1d_2n^4 + n^2(d_2T + d_1a\beta R^{*\alpha}) + Det_0, \end{aligned} \tag{5.4}$$

where Tr_0 and Det_0 are defined in (4.3).

5.1. Dynamics deduced by spatial diffusion

In this subsection, we will study the spatiotemporal dynamics in the absence of time delay, which means we let $\tau = 0$. It is well known that Hopf bifurcation occurs if $Tr_n = 0$ and $Det_n > 0$. In first, the following theorem gives the condition for not having Hopf bifurcation.

Theorem 5.1. *If $k > R^* > R_0^*$ then the system (1.2) has no Hopf bifurcation.*

Proof. Obviously, if $k > R^* > R_0^*$ the non trivial equilibrium (R^*, F^*) exists. Under this condition we have $T > 0$ leads to deduce that $Tr_n > 0$ which means we can not have $Tr_n = 0$, then there is no Hopf bifurcation, which completes the proof. □

Now assume that the condition (H$_1$) and (H$_2$) holds. $Tr_n = 0$ is equivalent to

$$\beta = \beta_n^{(H)} = \frac{-T - (d_1 + d_2)n^2}{aR^{*\alpha}} > 0. \tag{5.5}$$

$\beta_n^{(H)} > 0$ if and only if $n < N_1$ where $N_1 = \max\{n \in \mathbb{N}/ -T - (d_1 + d_2)n^2 > 0\}$. $\beta_n^{(H)}$ are the eventual bifurcation points. It is well known that Hopf bifurcation occurs for $Tr_n = 0$ and $Det_n > 0$ and obviously $Det_0 > 0$. Det_n can be written as a quadratic polynomial in n^2 which means:

$$Det_n = Det(n^2) = d_1 d_2 \left(n^2\right)^2 + n^2(d_2 T + d_1 a\beta R^{*\alpha}) + Det_0.$$

$Det(0) > 0$ leads to the existence of an positive integer n_* such that $Det(n^2) > 0$ for $n < n^*$ and $Det(n^2) < 0$ for $n > n^*$. Choosing $N^* = \min\{N_1, n_*\}$ and for $n < N^*$ and $\beta = \beta_n^{(H)}$ we have $Tr_n = 0$ and $Det_n > 0$ which means that the characteristic equation (5.3) has a purely imaginary roots. Now letting $\lambda = \delta(\beta) \pm iw(\beta)$ be the solution of the characteristic equation (5.3) verifying $\delta(\beta_0) = 0$ and $w(\beta_n^{(H)}) = \sqrt{Det_0}$ and

$$\frac{d}{d\beta}\delta(\beta)\bigg|_{\beta=\beta_n^{(H)}} = -\frac{1}{2}Tr'_n(\beta_n^{(H)}) = -\frac{1}{2}aR^{*\alpha} < 0,$$

which means that the system undergoes Hopf bifurcation at at $\beta = \beta_n^{(H)}$. Now lets put our main focus on studying the order of Hopf bifurcation points, then we have the following lemma.

Lemma 5.1. *Assume that (H$_1$) or (H$_2$) holds and $n < N^*$ then we have the following estimation*

$$\beta_{N^*}^{(H)} < \beta_{N^*-1}^{(H)} < ... < \beta_n^{(H)} < \beta_{n-1}^{(H)} < ... < \beta_1^{(H)} < \beta_0^{(H)}. \tag{5.6}$$

Proof. From the characteristic equation we have $Tr_n = 0$ is equivalent to

$$Tr_0(\beta) = -(d_1 + d_2)n^2, \tag{5.7}$$

where Tr_0 defined in (4.3) and under (H$_1$) and (H$_2$) we have $Tr_0(0) = T < 0$ and $Tr'_0(\beta) > 0$. Which means $Tr_0(\beta)$ is strictly increasing in β and intersect the horizontal axis at β_0. The equation (5.7) posses solution if and only if $-(d_1+d_2)n^2 > Tr_0(0) = T$ which leads to $n < N^* \le N_1$. It is clear that $-(d_1+d_2)n^2$ is strictly decreasing in n, which leads to the order (5.6) see **Figure 2**. The proof is completed. □

Under the above analysis the Hopf bifurcation induced by the presence of spatial diffusion can be reduced by following theorem.

Theorem 5.2. *Assume that the condition of **Theorem 3.1**, and (H$_1$) or (H$_2$) holds and $\beta_n^{(H)}$ defined in (5.5) we have the following results:*
(i) The positive equilibrium state (R^*, F^*) is locally asymptotically stable for $1 > \beta > \beta_0^{(H)}$ and unstable for $\beta < \beta_0^{(H)}$.
(ii) System (1.2) undergoes Hopf bifurcation near (R^*, F^*) for $\beta = \beta_n^{(H)}$ and spatially homogeneous periodic solution occur for $n = 0$, and spatially non homogeneous periodic solution for $n = 1, ...n^*$.

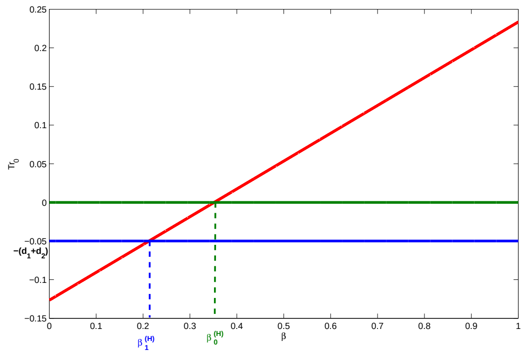

Figure 2. The order of Hopf bifurcation points for the values $d_1 = 0.02$, $d_2 = 0.03$, $k = 10$, $R^* = 3.72$, $r = 1.1$, $a = 0.15$, $\mu = 0.05$, $\alpha = \frac{2}{3}$ and $T = -0.1267$, $N_1 = 1$, $R^* > R_0^* = 2.5$ and $R_1^* = -0.25 < R^*$.

Now, lets put our main focus on studying the diffusion effect, which can elaborated by the presence of diffusion driven instability known by Turing instability.

Theorem 5.3. *Assume that* $R^* < R_0^*$ *and* $\beta > \beta^* = \dfrac{-d_2 T}{d_1 a R^*} > 0$ *holds, then we can not have Turing driven instability.*

Proof. For $R^* < R_0^*$ and $\beta > \beta^* = \dfrac{-d_2 T}{d_1 a R^*} > 0$ we have $Det_n > 0$ which means that Turing instability does not exists. □

Theorem 5.4. *Assume that* $R^* < R_0^*$ *and* $\beta < \beta_* = \min\{\beta^*, \beta^{**}\}$, *where* $\beta^{**} = \dfrac{-d_2 T}{d_1 a D^*}$ *the system can undergoes diffusion driven instability.*

Proof. Solving $Det_n = 0$ for d_1 we obtain

$$d_1(\beta, n) = -\frac{d_2 T n^2 + a\beta R^{*\alpha} D^*}{n^2(d_2 n^2 + a\beta R^{*\alpha})}, \quad n \neq 0. \tag{5.8}$$

It is easy to check that d_1 is strictly decreasing in β where $0 < \beta < \beta^*$. Also we can deduce that

$$d_1(\beta, n) = \begin{cases} > 0 \text{ for } 0 < \beta < \beta_n^* = \dfrac{-d_2 T n^2}{a R^{*\alpha} D^*}, \\ < 0 \text{ for } 0 < \beta < \beta_n^* = \dfrac{-d_2 T}{a R^{*\alpha} D^*}, \end{cases} \quad n = 1, 2, ...$$

and the sequence β_n^* is strictly increasing in n which means for $\beta_n^* > \beta_1^* = \beta_*$ we have $d_1(\beta, n) > 0$ and for $0 < \beta < \beta_*$ we have $d_1(\beta, n) > 0$ for any integer $n \neq 0$. In other word, for $0 < \beta < \beta_*, n < N^*$ and $d_1 < d_1(\beta, n)$ we have $Det_n < 0$ and remind that $Tr_0 > 0$ and $Det_0 > 0$ (see region G_3 as example in Figure 3). □

Now focusing on proving the existence of Turing Hopf bifurcation. The main interest on studying the existence of Turing-Hopf bifurcation is for deducing the region of the stability of the non trivial equilibrium. Using **Definition 2.1** we assume that the conditions $R^* < k$, $(H_1), (H_2)$ holds. Putting n=0, and solving $Tr_0(\beta) = 0$ in β we find

$$(H_0) : \beta = \beta_0 = \frac{-T}{a R^{*\alpha}} \tag{5.9}$$

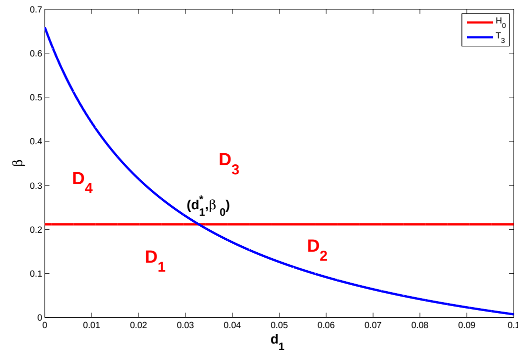

Figure 3. The existence of Turing-Hopf bifurcation point for the values the order of Hopf bifurcation points for the values $d_2 = 0.03$, $k = 10$, $R^* = 0.44$, $r = 1.1$, $a = 1.15$, $\mu = 0.5$, $\alpha = \frac{2}{3}$, $n_* = 2$ and $T = -0.4201$, $D^* = 0.15$, $n_* = 2$. The region D_1, D_2 the case when the non trivial equilibrium is unstable, D_3 represent the region of Turing instability, D_4 is the region of the stability of E^*.

and for $n \neq 0$ solving $Det_n = 0$ in β we obtain

$$(T_n) : \beta = \beta_T(d_1, n) = \frac{d_2 n^2(-d_1 n^2 - T)}{aR^{*\alpha}(d_1 n^2 + D^*)}. \tag{5.10}$$

Defining the following functional

$$h(x) = \frac{d_2 n^2(-xn^2 - T)}{aR^{*\alpha}(xn^2 + R^{*\alpha}aD^*)}.$$

It is easy to verify that the functional h is strictly decreasing and $h(0) = \frac{-d_2 n^2 T}{aR^{*\alpha}D^*} > 0$ which means for having an intersection point between the curves (H_0) and (T_n) in the $d_1 - \beta$ plan we need to have $h(0) > \beta_0$ which leads to $N^* > n > \left[\sqrt{\frac{D^*}{d_2}}\right]$ choosing the minimum integer verifying this inequality.denoted by n_* where

$$n_* = \begin{cases} \left[\sqrt{\frac{D^*}{d_2}}\right] + 1 & \text{if } \left[\sqrt{\frac{D^*}{d_2}}\right] + 1 < N^*, \\ N^* & \text{if } \left[\sqrt{\frac{D^*}{d_2}}\right] + 1 > N^*. \end{cases}$$

Now we can deduce that the Hopf bifurcation curve (H_0) intersect Turing instability curve (T_{n_*}) at the point $(d_1^*, \dfrac{-T}{aR^{*\alpha}})$ where $d_1^* = \dfrac{-T(d_2 n_*^2 - D^*)}{n_*^2(d_2 n_*^2 - T)}$. For more detail see figure 3.

5.2. Hopf bifurcation deduced by the presence of time delay

The main interest in this subsection is to study the delay effect on the stability of the nontrivial equilibrium (R^*, F^*) for the system (1.2) by analyzing the characteristic equation (5.2). Now, lets assume that the equilibrium (R^*, F^*) is stable for in the absence of time delay, where we assume that the conditions $R^* < k$, (H_1) or (H_2) holds and $1 > \beta > \beta_0^H$. However, we will prove the possibility of having Hopf bifurcation, where we will investigate with the existence of purely imaginary roots

$\lambda = i\omega(\omega > 0)$ for the characteristic equation (5.2), the equation (5.2) becomes

$$\begin{aligned} &-\omega^2 + (T + a\beta R^{*\alpha} + n^2(d_1 + d_2))i\omega + d_1 d_2 n^4 + n^2(d_2 T + d_1 a\beta R^{*\alpha}) \\ &+a\beta R^{*\alpha} T + \tfrac{\beta a^3 \alpha}{\mu} R^{*3\alpha-1} e^{-i\omega\tau} = 0, \end{aligned} \tag{5.11}$$

which equivalent to the following system

$$\begin{cases} -\omega^2 + A_n + B\cos(\omega\tau) = 0, \\ Tr_n\omega - B\sin(\omega\tau) = 0, \end{cases} \tag{5.12}$$

where Tr_n is defined in (5.4) and

$$\begin{cases} A_n = d_1 d_2 n^4 + n^2(d_2 T + d_1 a\beta R^{*\alpha}) + a\beta R^{*\alpha} T, \\ B = \frac{\beta a^3 \alpha}{\mu} R^{*3\alpha-1}. \end{cases}$$

It is easy to rewrite the system (5.12) as a quadratic equation in ω^2

$$\omega^4 + P_n\omega^2 + Q_n = 0, \tag{5.13}$$

where

$$\begin{cases} P_n = Tr_n^2 - 2A_n = (T + d_1 n^2)^2 + (a\beta R^{*\alpha} + d_2 n^2)^2 > 0, \\ Q_n = A_n^2 - B^2 = Det_n \left[d_1 d_2 n^4 + n^2(d_2 T + d_1 a\beta R^{*\alpha}) + aT\beta R^{*\alpha} - \frac{\beta a^3 \alpha}{\mu} R^{*3\alpha-1} \right]. \end{cases} \tag{5.14}$$

It is well know that for having positive solution for the equation (5.13) it must be $Q_n < 0$, we define the following functional

$$L(n^2) = d_1 d_2 n^4 - n^2(d_2 T + d_1 a\beta R^{*\alpha}) + aT\beta R^{*\alpha} - a^3\alpha^2\beta R^{*3\alpha-1}.$$

It is easy to see that $L(n^2)$ has the same sign with Q_n since we have $Det_n > 0$. indeed, $L(0) < 0$ then we can deduce the existence of an integer N_1 such that

$$L(n^2) < 0, \text{ for } n = 0, 1, ...N_1, \text{ and } L(n^2) > 0 \text{ for } n > N_1 \tag{5.15}$$

under the condition (5.15) we can deduce the existence of a unique positive solution of the equation (5.13) which is

$$\omega_n = \sqrt{\frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2}}, \tag{5.16}$$

when $n = 0, 1, ...N_1$, and we have the following results

Lemma 5.2. *The equation (5.2) posses a purely imaginary roots $i\omega_n$ for each $n \in \{0, 1, ..., N_1\}$, and has no purely imaginary roots for $n > N_1$.*

Using (5.12) we have for $n \in \{0, 1, ..., N_1\}$

$$\tau_{nj} = \tau_{n0} + \frac{2\pi j}{\omega_n}, \text{ and } \tau_{n0} = \frac{1}{\omega_n^+} \arccos\left(\frac{\omega^2 - A_n}{B}\right). \tag{5.17}$$

Lemma 5.3. *The Hopf bifurcation points verify the following estimation*

$$\tau_{0j} < \tau_{1j} < \tau_{2j} < ... < \tau_{(N_1-2)j} < \tau_{(N_1-1)j}. \qquad j = 0, 1, ... \tag{5.18}$$

Proof.

$$\omega_n^2 = \frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2} = \frac{2}{\frac{P_n}{-Q_n} + \sqrt{\left(\frac{P_n}{-Q_n}\right)^2 + \frac{4}{-Q_n}}}.$$

It is well known that P_n, Q_n are strictly increasing in n, which mean that $\frac{P_n}{-Q_n}, \frac{1}{-Q_n}$ are also strictly increasing in n which leads to deduce that ω_n^2 is decreasing in n, which gives $\frac{\omega^2 - A_n}{B}$ is decreasing in n. All the above results leads to say that τ_{n0} is increasing in n and that gives the estimation (5.18). The proof is completed. □

It is easy to see that

$$\tau_{\min} = \tau_{00} = \min_{n\in\{0,1,...,N_1\} \quad j\in\mathbb{N}} \{\tau_{nj}\}. \tag{5.19}$$

For the transversality condition we let $\lambda(\tau) = \varkappa(\tau) \pm i\upsilon(\tau)$ are the roots of the characteristic equation (5.2) near $\tau = \tau_{nj}$ satisfying $\varkappa(\tau_{nj}) = 0$ and $\upsilon(\tau_{nj}) = i\omega_n$, then we have

Lemma 5.4.

$$\left.\frac{dRe(\lambda(\tau))}{d\tau}\right|_{\tau=\tau_{nj}} > 0 \tag{5.20}$$

for $n \in \{0, 1, ..., N_1\} \quad j \in \mathbb{N}$.

Proof. It is easy to verify that (by differentiating the two sides of the equation (5.2) for τ)

$$\frac{1}{\lambda'(\tau)} = \frac{(2\lambda + Tr_n)e^{\lambda\tau}}{B\lambda} - \frac{\tau}{\lambda} \tag{5.21}$$

leads to

$$\begin{aligned} Re\left(\frac{1}{\lambda'(\tau)}\right)\Big|_{\tau=\tau_{nj}} &= Re\left(\frac{(2\lambda + Tr_n)e^{\lambda\tau}}{B\lambda} - \frac{\tau}{\lambda}\right)\Big|_{\tau=\tau_{nj}} \\ &= Re\left(\frac{(2i\omega_n + Tr_n)e^{\lambda\tau}}{Bi\omega_n} - \frac{\tau}{i\omega_n}\right) \\ &= \frac{2\omega B\cos\omega_n\tau_{nj} + BTr_n \sin\omega_n^+\tau_{nj}}{\omega_n B^2} \\ &= \frac{2\omega_n^{+2} + Tr_n^2 - 2A_n}{B^2} \\ &= \frac{2\omega_n^{+2} + P_n}{B^2} > 0. \end{aligned}$$

Which completes the proof. □

The above results illustrate the following theorem.

Theorem 5.5. *Assume that* $R^* < k$, (H_1) *or* $(H_2) > 0$ *holds and,* $1 > \beta > \beta_0$ *and* τ_{00} *defined by (5.19) then we have the following results:*
(i) If $\tau \in [0, \tau_m)$ *then the nontrivial equilibrium* (R^*, F^*) *is asymptotically stable.*

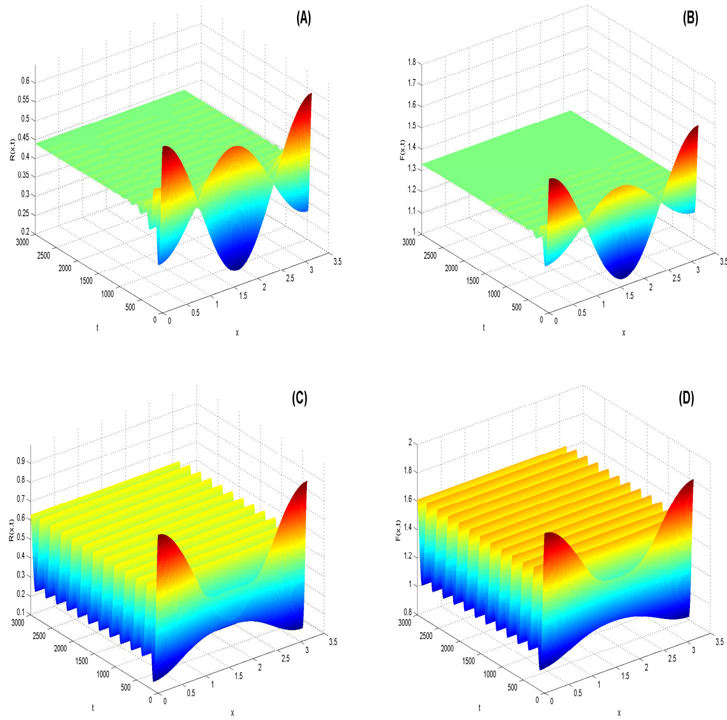

Figure 4. Numerical simulation shows the dynamic near Hopf bifurcation point in system (1.2) and $\tau = 0$ for the values in Figure 2. For (A) and (B) we have $(d_1, \beta) = (0.01, 0.3)$ (which means (d_1, β) in D_4) the non trivial equilibrium is asymptotically stable and the initial data $(\phi, \psi) = (R^* + 0.3\cos 2x, P^* + 0.3\cos 2x)$. For (C) and (D) we have $(d_1, \beta) = (0.01, 0.2)$ (which means (d_1, β) in D_1) the non trivial equilibrium is unstable and the initial data $(\phi, \psi) = (R^* + 0.3\cos 2x, P^* + 0.3\cos 2x)$.

(ii) If $\tau > \tau_m$ then the nontrivial equilibrium (R^*, F^*) is unstable.
(iii) The system (1.2) undergoes Hopf bifurcation near (R^*, F^*) at $\tau = \tau_{nj}$ for $n \in \{0, 1, ..., N_1\}$ and $j \in \mathbb{N}$. Further, when $n = 0$ the periodic solution are homogeneous, and for $n = 1, ..., N_1$ are non homogeneous.

6. Conclusion

We dealt in this paper with the effect of the shape of herd behavior in the interaction between the prey and the predator. The shape can control the number of the hunted prey by a predator which is going to be proportional to the number of the prey on the boundary of the pack. The spatiotemporal dynamics was successfully studied in different cases. In the second section, we analyzed the local stability of the constant equilibrium states. For the effect of diffusion on the behavior of solution we applied a local bifurcation theory to study the existence of Hopf bifurcation for some values of the system parameters, also the presence of diffusion can lead to multi existence of Hopf bifurcation points where we tried to give the order of this bifurcation points which can be seen easily in figure 2. Based on the result introduced by Turing, the existence of diffusion driven instability has been proved.

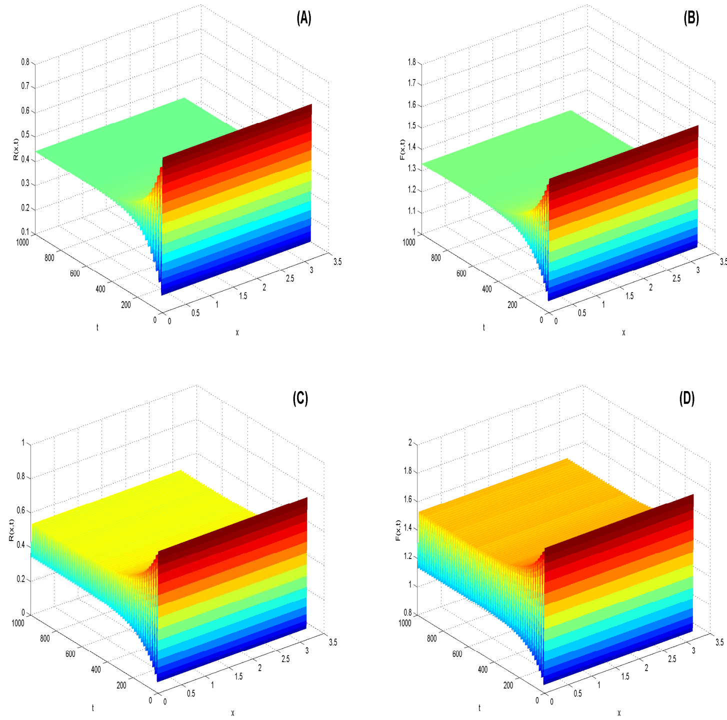

Figure 5. Dynamics introduced by the presence of time delay for the system (1.2) where we assumed that (d_1, β) in D_4 for the same value of Figure 2 and 4 ((A) and (B)). For (A) and (B) we have $\tau = 2 < \tau_{00} = 2.8601$ and the initial conditions $(\phi, \psi) = (R^* + 0.5, P^* + 0.5)$. For (C) and (D) we have $\tau = 3 > \tau_{00} = 2.8601$ and the initial conditions $(\phi, \psi) = (R^* + 0.5, P^* + 0.5)$.

On the other hand, we have investigated the existence of both Hopf bifurcation and Turing driven instability which was called by Turing-Hopf bifurcation point. The importance of calculation of Turing-Hopf bifurcation point is for determining the region of the stability of the non trivial equilibrium, where in figure 3 we gave in detail the region of the stability of the positive equilibrium, for instance in G_1 we have $Tr_0 > 0$ and $Det_1 < 0$ which means that E_* is unstable, in G_2 we have $Tr_0 > 0$ and $Det_1 > 0$ which means that E_* is unstable, in G_3 we have $Tr_0 > 0$ and $Det_1 < 0$ which means that E_* is unstable (this region is called region of Turing instability), in G_4 we have $Tr_0 < 0$ and $Det_1 > 0$ which means that E_* is stable. In the last part of the paper we investigated with the delay effect on the stability of the non trivial equilibrium, where the Theorem 5.3 has been used to verify the stability of the positive equilibrium in the presence of diffusion only. We obtained that the delay can lead to instability and even Hopf bifurcation. Finally, Figure 4 has been used to verify some results obtained by the presence of self diffusion only, more precisely the stability of E_*, in the region G_4 in Figure 3 and the instability of E_*, in the region G_1 in Figure 3 and the existence of a stable homogeneous periodic solution. For Figure 5, the local stability of the nontrivial equilibrium in the presence of time delay for $\tau < \tau_{00}$ and instability for $\tau > \tau_{00}$ have been shown.

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References

[1] P. A. Braza, *Predator-prey dynamics with square root functional responses*, Nonlin Anal Real World Appl, 2012, 13, 1837–43.

[2] M. Baurmanna, T. Gross and U. Feudel, *Instabilities in spatially extended predator-prey systems: Spatio-temporal patterns in the neighborhood of Turing-Hopf bifurcations* , Journal of Theoretical Biology, 2007, 245, 220–229.

[3] I. M. Bulai and E. Venturino, *Shape effects on herd behavior in ecological interacting population models*, Mathematics and Computers in Simulation, 2017, 141, 40–55.

[4] I. Boudjema and S. Djilali, *Turing-Hopf bifurcation in Gauss-type model with cross diffusion and its application*, Nonlinear Studies, 2018, 25(3), 665–687.

[5] E. Cagliero and E. Venturino, *Ecoepidemics with infected prey in herd defense: the harmless and toxic cases*, Int. J. Comput. Math., 2016, 93, 108–127.

[6] J. Carr, *Applications of Center Manifold Theory*, New York, SpringerVerlag, 1981.

[7] M. Cavani and M. Farkas, *Bifurcations in a predator-prey model with memory and diffusion. I: Andronov-hopf bifurcations*, Acta Math Hungar, 1994, 63, 213–29.

[8] S. Djilali, *Herd behavior in a predator-prey model with spatial diffusion: bifurcation analysis and Turing instability*, Journal of Applied Mathematics and Computing, 2018, 58, 125–149.

[9] S.Djilali, *Impact of prey herd shape on the predator-prey interaction*, Chaos, Solitons and Fractals, 2019, 120, 139–148.

[10] S. Djilali, T. M. Touaoula and S. E-H.Miri, *A heroin epidemic model: very general non linear incidence, treat-age, and global stability*, Acta Applicandae Mathematicae, 2017, 152(1), 171–194.

[11] T. Faria, *Stability and Bifurcation for a Delayed Predator-Prey Model and the Effect of Diffusion*, Applied Mathematics and Computation, 2001, 254, 433–463.

[12] J. Luo and Y. Zhao, *Stability and bifurcation analysis in a predator-prey system with constant harvesting and prey group defense*, International Journal of Bifurcation and Chaos, 2017, 27, 1750179.

[13] X. Liu, T. Zhang, X. Meng and T. Zhang, *Turing-Hopf bifurcations in a predator-prey model with herd behavior, quadratic mortality and prey-taxis*, Physica A, 2018, 496, 446–460.

[14] B. Liu, R. Wu and L. Chen, *Patterns induced by super cross-diffusion in a predator-prey system with Michaelis-Menten type harvesting*, Mathematical Biosiences,2018, 298, 71–79.

[15] C. V. Pao, *Convergence of solutions of reaction-diffusion systems whith time delays*, Nonlinear Analysis, 2002, 48, 349–362.

[16] F. Rao, C. Chavez, Y. Kang, *Dynamics of a diffusion reaction prey-predator model with delay in prey: Effecs of delay and spatial components*, Journal of Mathematical Analysis and Applications, 2018, 461(2), 1177–1214.

[17] Y. Song and X. Zou, *Bifurcation analysis of a diffusive ratio-dependent predator-prey model*, Nonlinear Dynamics, 2017, 78, 49–70.

[18] Y. Song and X. Zou, *Spatiotemporal dynamics in a diffusive Ratio-dependent predator-prey model near a Hopf-Turing bifurcation point*, Computers and Mathematics with Applications, 2014, 67, 1978–1967.

[19] Y. Song, T. Zhang and Y. Peng, *Turing-Hopf bifurcation in the reaction diffusion equations and its applications*, Commun Nonlinear Sci NumerSimulat, 2015, 33, 229–258.

[20] Y. Song, Y. Peng and X. Zou, *Persisstence, stability and Hopf bifurcation in a diffusive Ratio-Dependent predator-prey model with delay*, International Journal of Bifurcation and Chaos, 2014, 24, 1450093.

[21] Y. Song, H. Jiang, Q. X. Liu and Y. Yuan, *Spatiotemporal dynamics of the diffusive Mussel-Algae model near Turing-Hopf bifurcation*, SIAM Journal of Applied Dynamical Systems, 2017, 16, 2030–2062.

[22] Y. Song and X. Tang, *Stability, Steady-State Bifurcations, and Turing Patterns in a Predator- Prey Model with Herd Behavior and Prey-taxis*, Stud. Appl. Math., 2017, 139, 371–404.

[23] M. Sambath, K. Balachandran and L. N. Guin, *Spatiotemporal patterns in a predator-prey model with cross-diffusion effect*, International Journal of Bifurcation and Chaos, 2018, 28, 1830004.

[24] X. Tang, H. Jiang, Z. Deng and T. Yu, *Delay induced subcritical Hopf bifurcation in a diffusive predator-prey model with herd behavior and hyperbolic mortality*, Journal of Applied Analysis and Computation, 2017, 7, 1385–1401.

[25] X. Tang and Y. Song, *Bifurcation analysis and Turing instability in a diffusive predator prey model with herd behavior and hyperbolic mortality*, Chaos, Solitons and Fractals, 2015, 81, 303–3014.

[26] X. Tang and Y. Song, *Cross-diffusion induced spatiotemporal patterns in a predator-prey model with herd behavior*, Nonlinear Analysis: Real World Applications, 2015, 24, 36–49.

[27] X. Tang and Y. Song and T. Zhang, *Turing-Hopf bifurcation analysis of a predator-prey model with herd behavior and cross diffusion*, Nonlinear Dynamics, 2016, 86, 73-89.

[28] X. Tang and Y. Song, *Stability, Hopf bifurcations and spatial patterns in a delayed diffusive predator-prey model with herd behavior*, Applied Mathematics and Computation, 2015, 254, 375–391.

[29] E. Venturino, *A minimal model for ecoepidemics with group defense*, J. Biol. Syst., 2011, 19, 763–785.

[30] E. Venturino, *Modeling herd behavior in population systems*, Nonlinear Analysis: Real World Applications, 2011, 12, 2319–2338.

[31] E. Venturino and S. Petrovskii, *Spatiotemporal behavior of a prey-predator system with a group defense for prey*, Ecological Complexity, 2013, 14, 37–47.

[32] R. Wu, M.Chen, B. Liu and L. Chen, *Hopf bifurcation and Turing instability in a predator-prey model with Michaelis-Menten functional reponse*, Nonlinear Dynamics, 2018, 91(3), 2033–2047.

[33] C. Wang and S. Qi, *Spatial dynamics of a predator-prey system with cross diffuusion*, Chaos, Solitons and Fractals, 2018, 107, 55–60.

[34] C. Xu, C. Yuan and T. Zhang, *Global dynamics of a predator-prey model with defence mechanism for prey*, Applied Mathematics Letters, 2016, 62, 42–48.

[35] Z. Xu and Y. Song, *Bifurcation analysis of a diffusive predator-prey system with a herd behavior and quadratic mortality*, Math. Meth. Appl. Sci., 2015, 38(4), 2994–3006.

[36] R. Yang and Y. Song, *Spatial resonance and Turing-Hopf bifurcations in the Gierer-Meinhardt model*, Nonlinear Analysis: Real World Applications, 2016, 31, 356–387.

[37] W. Yang, *Analysis on existence of bifurcation solutions for a predator-prey model with herd behavior*, Applied mathematical Modelling, 2017, 53, 433–446.

[38] H. Zhu and X. Zhang, *Dynamics and Patterns of a Diffusive Prey-Predator System with a Group Defense for Prey*, Discrete Dynamics in Nature and Society, 2018. DIO: 10.1155/2018/6519696.