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MULTIPLE SOLUTIONS FOR A NONHOMOGENEOUS SCHRÖDINGER-POISSON SYSTEM WITH CONCAVE AND CONVEX NONLINEARITIES*

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Abstract In this paper, we consider the following nonhomogeneous Schrödinger-Poisson equation

$$(*)\begin{cases}-\Delta u+V(x)u+\phi(x)u=-k(x)|u|^{q-2}u+h(x)|u|^{p-2}u+g(x), & x\in\mathbb{R}^3,\\ -\Delta\phi=u^2, \quad \lim_{|x|\to+\infty}\phi(x)=0, & x\in\mathbb{R}^3,\end{cases}$$

where $1<q<2, 4<p<6$. Under some suitable assumptions on $V(x), k(x), h(x)$ and $g(x)$, the existence of multiple solutions is proved by using the Ekeland's variational principle and the Mountain Pass Theorem in critical point theory.

Keywords Schrödinger-Poisson systems, concave and convex nonlinearities, variational methods, Ekeland's variational principle, Mountain Pass Theorem.

MSC(2010) 35B33, 35J65,35Q55.

1. Introduction

The system

$$\begin{cases}-\Delta u+V(x)u+K(x)\phi(x)u=f(x,u)+g(x), & x\in\mathbb{R}^3,\\ -\Delta\phi=K(x)u^2, \quad \lim\limits_{|x|\to+\infty}\phi(x)=0, & x\in\mathbb{R}^3,\end{cases}\tag{1.1}$$

arises from several interesting physical contexts. It is well known that (1.1) has a strong physical meaning since it appears in quantum mechanics models (see [8,22]) and in semiconductor theory (see [7, 23, 24]). From the point view of quantum mechanics, the system (1.1) describes the mutual interactions of many particles [30]. Indeed, if the terms $f(x,u)$ and $g(x)$ are replaced with 0, then problem (1.1) becomes the Schrödinger-Poisson system. In some recent works (see [2,4,9,14,17,35,39,47]), different nonlinearities have added to Schrödinger-Poisson equation, giving rise to

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the so-called nonlinear Schrödinger-Poisson system. These nonlinear terms have been traditionally used in the Schrödinger equation to model the interaction among particles.

Many mathematicians have been devoted to the study of (1.1) with virous nonlinearities $f(x,u)$. We recall some of them as follows.

The case of $g \equiv 0$, that is the homogeneous case, has been studied widely in [1,3,10,11,13,21,26,28,37,38,48]. Very recently, Cerami etc [9] study system (1.1) in the case of $f(x,u) = a(x)|u|^{p-2}u$ with $4 < p < 6$ and $a(x)$ being non-negative. They establish a global compactness lemma to overcome the lack of compactness of the embedding of $H^1(\mathbb{R}^3)$ into the Lebesgue space $L^s(\mathbb{R}^3), s \in [2,6)$, preventing from using the variational techniques in a standard way. They prove the existence of positive ground state and bound state solutions by minimizing the associated functional restricted to the Nehari manifold, where for the coefficient function $K(x)$ Cerami etc [9] assume that $K \in L^2(\mathbb{R}^3), \lim_{|x|\to\infty} K(x) = 0, K(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $K(x) \not\equiv 0$.

In 2012, the authors [34] consider another case, that is, $f(x,u) = a(x)\tilde{f}(u)$ where $\tilde{f}$ is asymptotically linear at infinity, i.e., $\tilde{f}(s)/s \to c$ as $c \to +\infty$ with a suitable constant c. They establish a compactness lemma different from that in [9] and prove the existence of ground state solutions. In [46], Ye and Tang study the existence and multiplicity of solutions for homogeneous system of (1.1) when the potential V may change sign and the nonlinear term f is superlinear or sublinear in u as $|u| \to \infty$. For the Schrödinger-Poisson system with sign-changing potential see [35].

Huang etc [17] consider the case that $f(x,u)$ is a combination of a superlinear term and a linear term. More precisely, $f(x,u) = k_1(x)|u|^{p-2}u + \mu h_1(x)u$, where $4 < p < 6$ and $\mu > 0, k_1 \in C(\mathbb{R}^3)$, k_1 changes sign in $\mathbb{R}^3$ and $\lim_{|x|\to+\infty} k_1(x) = k_\infty < 0$. They prove the existence of at least two positive solutions in the case that $\mu > \mu_1$ and near μ_1, where μ_1 is the first eigenvalue of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ and with weight function h. In another two papers [18,19], the authors consider the critical case of $p = 6$. Shen etc [31] consider the critical case of $p = 4$.

Sun etc [36] get infinitely many solutions for (1.1), where the nonlinearity $f(x,u) = k_2(x)|u|^{q-2}u - h_2(x)|u|^{l-2}u, 1 < q < 2 < l < \infty$, i.e. the nonlinearity involve a combination of concave and convex terms. For more results on the effect of concave and convex terms of elliptic equations see [43,44] and the reference therein.

Next, we consider the nonhomogeneous case of (1.1), that is $g \not\equiv 0$. The existence of radially symmetric solutions is obtained for above nonhomogeneous system in [29]. Chen etc [12] obtain two solutions for the nonhomogeneous system with $f(x,u)$ satisfying Ambrosetti-Rabinowitz type condition and V being nonradially symmetric. In [15,16], the system with asymptotically linear and 3-linear nonlinearity is considered. For more results on the nonhomogeneous case see [20,45] and the reference therein. Other nonhomogeneous equations with sign-changing potential see [40,41]. Variational approach to other equations see [5,27]. There is a natural question, whether we can get the multiple solutions for nonhomogeneous Schrödinger-Poisson system with a combination of concave and convex terms.

Motivated by the works mentioned above, in the present paper, we consider the following nonhomogeneous Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = -k(x)|u|^{q-2}u + h(x)|u|^{p-2}u + g(x), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, \quad \lim\limits_{|x|\to+\infty} \phi(x) = 0, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $1 < q < 2, 4 < p < 6$, i.e. the nonlinearity of this problem may involve a combination of concave and convex terms. To our best knowledge, this is the first result on the existence of multiple solutions to problem (1.2).

We assume that $V(x), k(x), h(x)$ and $g(x)$ are measurable functions satisfying the following conditions:

- (V_0) $V(x) \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ satisfies $inf_{x\in\mathbb{R}^3} V(x) = a_1 > 0$.
- (V) for any $M > 0$, $meas\{x \in \mathbb{R}^3 : V(x) < M\} < +\infty$, where $meas$ denotes the Lebesgue measures.
- (K) $k(x) \in L^{6/(6-q)}(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3)$ and $k(x) \geq 0$ is not identically zero for a.e. $x \in \mathbb{R}^3$.
- (H) $h(x) \in L^{\infty}(\mathbb{R}^3)$ and $h(x) > 0$ for a.e. $x \in \mathbb{R}^3$.
- (G) $g(x) \in L^2(\mathbb{R}^3)$ and $g(x) > 0$ for a.e. $x \in \mathbb{R}^3$.

Before stating our main result, we give several notations. Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard scalar and norm

$$(u,v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)dx; \qquad \|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx\right)^{1/2}.$$

$D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^{\infty}(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D := \|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{1/2}.$$

Let

$$E := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx < \infty\right\}.$$

Then E is a Hilbert space with the inner product

$$(u,v)_E = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)\, dx$$

and the norm $\|u\|_E = (u,u)_E^{1/2}$. Obviously, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, for any $s \in [2, 2^*]$, where $2^* = 6$.

Now we state our main result:

Theorem 1.1. *Let* $1 < q < 2, 4 < p < 6, (V_0), (V)$, (K), (H) *and* (G) *hold, then there exists a constant* $m_0 > 0$ *such that problem (1.2) admits at least two different solutions* $u_0, \widetilde{u}_0$ *in* E *satisfying* $I(u_0) < 0$ *and* $I(\widetilde{u}_0) > 0$ *if* $\|g\|_2 < m_0$.

Remark 1.1. The condition in (V), which implies the compactness of embedding of the working space E and contains the coercivity condition: $V(x) \to \infty$ as $|x| \to \infty$, is first introduced by Bartsch and Wang in [6] to overcome the lack of compactness. We are not sure whether Theorem 1.1 is hold without the condition (V).

Remark 1.2. Salvatore [29] obtain the existence of multiple radially symmetric solutions on $\mathbb{R}^3$ for the homogeneous and the nonhomogeneous system (1.1). Since the potential V may be not radially symmetric in Theorem 1.1, we get the multiple non-radially symmetrical solutions for system (1.2) with the concave and convex nonlinearities.

Remark 1.3. Our proof is variational. The main difficulty is the loss of compactness of the Sobolev embedding $H^1(\mathbb{R}^3)$ into $L^s(\mathbb{R}^3)$, $s \in [2,6]$ since this problem is set on $\mathbb{R}^3$. To recover this difficulty, some of the papers use special function space, such as the radially symmetric function space, which possesses compact embedding, see [32]. In this paper, the integrability of k and the main assumption $1 < q < 2$ to ensure the space E is compactly embedding in the weighted Lebesgue space (see Lemma 2.1). Although the methods are used before, we need to study carefully some properties of the term $\phi(x)u$ and the effect of the sublinear therm.

Remark 1.4. To the best of our knowledge, it seems that Theorem 1.1 is the first result about the existence of multiple solutions for the nonhomogeneous Schrödinger-Poisson equations on $\mathbb{R}^3$ with concave and convex terms. In [36], the authors get the infinitely many solutions by using the variant fountain theorem established by Zou [50]. However, since the nonhomogeneous term $g(x) > 0$, we only get two solutions for the nonhomogeneous Schrödinger-Poisson equations on $\mathbb{R}^3$ with concave and convex terms. If $k(x) = h(x) \equiv 0$, Theorem 1.1 is the result of [12]. So our result generalized the results of [12].

The paper is organized as follows. In Section 2, we will introduce the variational setting for the problem. In Section 3, we give the proof of Theorem 1.1. Throughout this paper, the letters a, a_i, C denote various positive constants. The norm on $L^s = L^s(\mathbb{R}^3)$ with $1 < s < \infty$ is given by $\|u\|_s^s = \int_{\mathbb{R}^3} |u|^s dx$.

2. Variational setting and preliminaries

In this section, we give the variational setting of the problem.

It is known that problem (1.2) can be reduced to a single equation see [14]. In fact, for every $u \in E$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u = u^2, \quad u \in \mathbb{R}^3, \tag{2.1}$$

with

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy.$$

By (2.1), the Hölder inequality and the Sobolev inequality, we get

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx \le \|u\|_{12/5}^2 \|\phi_u\|_6 \le C\|u\|_{12/5}^2 \|\phi_u\|_D,$$

then

$$\|\phi_u\|_D \le C\|u\|_{12/5}^2,$$

and

$$\int_{\mathbb{R}^3} \phi_u u^2 dx \le C\|u\|_{12/5}^4 \le C\|u\|_E^4. \tag{2.2}$$

Therefore, problem (1.2) can be reduced to the following equation:

$$-\Delta u + V(x)u + \phi_u u = -k(x)|u|^{q-2}u + h(x)|u|^{p-2}u + g(x), \qquad x \in \mathbb{R}^3.$$

In this paper, we will apply the variational methods to prove our theorem. First, we recall some results.

Lemma 2.1 (Lemma 3.4, [49])**.** *Under assumption* (V_0) *and* (V)*, the embedding* $E \hookrightarrow L^s(\mathbb{R}^3)$ *is compact for any* $s \in [2, 2^*)$*.*

We introduce the functional $I : E \to \mathbb{R}$ defined by

$$\begin{aligned} I(u) =& \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx + \frac{1}{q}\int_{\mathbb{R}^3}k(x)|u|^q dx \\ & - \frac{1}{p}\int_{\mathbb{R}^3}h(x)|u|^p dx - \int_{\mathbb{R}^3}g(x)udx. \end{aligned} \tag{2.3}$$

By (2.2) and the conditions of Theorem 1.1, all the integrals in (2.3) are well-defined and in $C^1(E, \mathbb{R})$. Now, it is easy to verify that the weak solutions of (1.2) correspond to the critical points of $I : E \to \mathbb{R}$ with derivative given by

$$\begin{aligned} \langle I'(u), v\rangle = \int_{\mathbb{R}^3}[&\nabla u \nabla v + V(x)uv + \phi_u uv + k(x)|u|^{q-2}uv \\ &- h(x)|u|^{p-2}uv - g(x)v]dx. \end{aligned}$$

Lemma 2.2. *Let* $g \in L^2(\mathbb{R}^3)$*. Suppose* (V_0) *and* (V) *hold. Then there exist some constants* $\rho, \alpha, m_0 > 0$ *such that* $I(u)|_{\|u\|_E=\rho} \geq \alpha$ *for all* g *satisfying* $\|g\|_2 < m_0$*.*

Proof. Since $\phi_u \geq 0, k(x) \geq 0$, using the Hölder inequality and $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3), s \in [2, 6]$,

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|_E^2 - \frac{|h|_\infty}{p}\|u\|_p^p - \|g\|_{L^2}\|u\|_2 \\ &\geq \frac{1}{2}\|u\|_E^2 - a_2\|u\|_E^p - \frac{1}{\sqrt{a_1}}\|g\|_{L^2}\|u\|_E \\ &= \|u\|_E\left(\frac{1}{2}\|u\|_E - a_2\|u\|_E^{p-1} - \frac{1}{\sqrt{a_1}}\|g\|_2\right), \end{aligned}$$

where a_1 is a lower bound of the potential V from (V_0) and $a_2 > 0$ is a constant.

Setting

$$g(t) = \frac{1}{2}t - a_2 t^{p-1}, \quad t \geq 0,$$

we see that there exists a constant $\rho > 0$ such that $max_{t\geq 0}g(t) = g(\rho) > 0$. Taking $m_0 := \frac{1}{2}\sqrt{a_1}g(\rho)$, then it follows that there exists a constant $\alpha > 0$ such that $I(u)|_{\|u\|_E=\rho} \geq \alpha$ for all g satisfying $\|g\|_2 < m_0$. The proof is complete. □

Lemma 2.3. *Suppose that* (V_0) *and* $1 < q < 2, 4 < p < 6$ *hold, then there exists a function* $v \in E$ *with* $\|v\|_E > \rho$ *such that* $I(v) < 0$*, where* ρ *is given in Lemma 2.2.*

Proof. Since $1 < q < 2, 4 < p < 6$, $h(x) \geq 0$, we have

$$\begin{aligned} I(tu) &= \frac{t^2}{2}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{tu}(tu)^2 dx + \frac{t^q}{q}\int_{\mathbb{R}^3}k(x)|u|^q dx \\ &\quad - \frac{t^p}{p}\int_{\mathbb{R}^3}h(x)|u|^p dx - t\int_{\mathbb{R}^3}g(x)udx \\ &\leq \frac{t^2}{2}\|u\|_E^2 + \frac{t^4}{4}\|u\|_E^4 + \frac{t^q}{q}\int_{\mathbb{R}^3}k(x)|u|^q dx \end{aligned}$$

$$
\begin{aligned}
&- \frac{t^p}{p}\int_{\mathbb{R}^3} h(x)|u|^p dx - t\int_{\mathbb{R}^3} g(x)u dx\\
&\to -\infty, \quad \text{as } t \to +\infty,
\end{aligned}
$$

for $u \in E, u \neq 0$. The lemma is proved by taking $v = t_0 u$ with $t_0 > 0$ large enough and $u \neq 0$. The proof is complete. $\Box$

Lemma 2.4. *Assume that* $(V_0), (V)$, $(K), (H), (G)$ *hold, and* $\{u_n\} \subset E$ *is a bounded (PS) sequence of* I*, then* $\{u_n\}$ *has a strongly convergent subsequence in* E*.*

Proof. Consider a sequence $\{u_n\}$ in E which satisfies

$$
I(u_n) \to c, \quad I'(u_n) \to 0, \quad \sup_n \|u_n\|_E < +\infty.
$$

Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E. In view of Lemma 2.1, $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 2^*)$. By the derivative of I, we easily obtain

$$
\begin{aligned}
\|u_n - u\|_E^2 =&\langle I'(u_n) - I'(u), u_n - u\rangle - \int_{\mathbb{R}^3} k(x)(|u_n|^{q-1} - |u|^{q-1})(u_n - u)dx\\
&+ \int_{\mathbb{R}^3} h(x)(|u_n|^{p-1} - |u|^{p-1})(u_n - u)dx - \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)dx.
\end{aligned}
$$

It is clear that

$$
\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty. \tag{2.4}
$$

By the Hölder inequality and the Sobolev inequality, we have

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3} \phi_{u_n} u_n (u_n - u)dx\right| &\le \|\phi_{u_n}\|_6 \|u_n\|_{12/5} \|u_n - u\|_{12/5}\\
&\le C_1 \|\phi_{u_n}\|_D \|u_n\|_{12/5} \|u_n - u\|_{12/5}\\
&\le C_2 \|u_n\|_{12/5}^3 \|u_n - u\|_{12/5} \to 0,
\end{aligned}
$$

since $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 2^*)$. We obtain

$$
\int_{\mathbb{R}^3} \phi_{u_n} u_n (u_n - u)dx \to 0 \quad \text{as } n \to \infty. \tag{2.5}
$$

Similarly we can also obtain

$$
\int_{\mathbb{R}^3} \phi_u u (u_n - u)dx \to 0 \quad \text{as } n \to \infty. \tag{2.6}
$$

By $4 < p < 6$, (H) and the Hölder inequality, one has

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3} h(x)(|u_n|^{p-1} - |u|^{p-1})(u_n - u)dx\right| &\le |h|_\infty (\|u_n\|_p^{p-1} + \|u\|_p^{p-1})\|u_n - u\|_p\\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{2.7}
$$

By $1 < q < 2$, (K) and the Hölder inequality, one has

$$
\int_{\mathbb{R}^3} k(x)|u_n|^{q-1}(u_n - u)dx = \int_{\mathbb{R}^3} k(x)^{\frac{q-1}{q}} k(x)^{\frac{1}{q}} |u_n|^{q-1}(u_n - u)dx
$$

$$\begin{aligned}
&\le |k|_\infty^{1-\frac{1}{q}} \left[\int_{\mathbb{R}^3} \left(k(x)^{\frac{1}{q}} |u_n|^{q-1}\right)^{\frac{6}{6-q}} dx\right]^{\frac{6-q}{6}} \left(\int_{\mathbb{R}^3} (u_n - u)^{\frac{6}{q}} dx\right)^{\frac{q}{6}} \\
&\le |k|_\infty^{1-\frac{1}{q}} \left(\int_{\mathbb{R}^3} k(x)^{\frac{6}{6-q}} dx\right)^{\frac{6-q}{6}\cdot\frac{1}{q}} \left(\int_{\mathbb{R}^3} |u_n|^{\frac{6q}{6-q}} dx\right)^{\frac{6-q}{6q}(q-1)} \|u_n - u\|_{\frac{6}{q}} \qquad (2.8) \\
&= |k|_\infty^{1-\frac{1}{q}} \|k(x)\|_{\frac{6}{6-q}}^{\frac{1}{q}} \|u_n\|_{\frac{6q}{6-q}}^{q-1} \|u_n - u\|_{\frac{6}{q}} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

since $3 < \frac{6}{q} < 6$, $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 2^*)$.

Similarly, we also obtain

$$\int_{\mathbb{R}^3} k(x)|u|^{q-1}(u_n - u)dx \to 0 \quad \text{as } n \to \infty. \qquad (2.9)$$

Therefore, by (2.5)-(2.9), we get $\|u_n - u\|_E \to 0$. The proof is complete. □

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into two steps.

Step 1 There exists a function $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) < 0$.

Since $g \in L^2(\mathbb{R}^3)$ and $g > 0$, we can choose a function $\psi \in E$ such that

$$\int_{\mathbb{R}^3} g(x)\psi(x)dx > 0.$$

Hence, we have

$$\begin{aligned}
I(t\psi) \le \frac{1}{2}t^2\|\psi\|_E^2 + \frac{t^4}{4}\|\psi\|_E^4 + \frac{t^q}{q}\int_{\mathbb{R}^3} k(x)|\psi|^q dx - \frac{t^p}{p}\int_{\mathbb{R}^3} h(x)|\psi|^p dx - t\int_{\mathbb{R}^3} g(x)\psi dx \\
< 0 \quad \text{for} \quad t > 0 \quad \text{small enough.}
\end{aligned}$$

Thus, we obtain

$$c_0 = \inf\{I(u) : u \in \overline{B}_\rho\} < 0,$$

where $\rho > 0$ is given by Lemma 2.2, $B_\rho = \{u \in E : \|u\|_E < \rho\}$. By the Ekeland's variational principle [25, 42], there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \le I(u_n) < c_0 + \frac{1}{n},$$

and

$$I(w) \ge I(u_n) - \frac{1}{n}\|w - u_n\|_E$$

for all $w \in \overline{B}_\rho$. Then by a standard procedure, we can show that $\{u_n\}$ is a bounded Palais-Smale sequence of I. Therefore Lemma 2.4 implies that there exists a function $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$.

Step 2 There exists a function $\widetilde{u}_0 \in E$ such that $I'(\widetilde{u}_0) = 0$ and $I(\widetilde{u}_0) > 0$.

From Lemmas 2.2, 2.3 and the Mountain Pass Theorem, there is a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to \widetilde{c_0} > 0, \quad \text{and} \quad I'(u_n) \to 0.$$

In view of Lemma 2.4, we only need to check that $\{u_n\}$ is bounded in E.

$$\begin{aligned}\widetilde{c_0}+1+\|u\|_E &\geq I(u_n)-\frac{1}{4}\langle I'(u_n),u_n\rangle\\ &=\frac{1}{4}\|u_n\|_E^2+(\frac{1}{q}-\frac{1}{4})\int_{\mathbb{R}^3}k(x)|u_n|^q dx+(\frac{1}{4}-\frac{1}{p})\int_{\mathbb{R}^3}h(x)|u_n|^p dx\\ &\quad-\frac{3}{4}\int_{\mathbb{R}^3}g(x)u_n dx\\ &\geq\frac{1}{4}\|u_n\|_E^2-\frac{3}{4}\|g\|_2\|u_n\|_2\end{aligned}$$

for n large enough. Since $\|g\|_2 < m_0$, it follows from $1 < q < 2, 4 < p < 6$ that $\{u_n\}$ is bounded in E. The proof is complete. □

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