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INFINITELY MANY SOLUTIONS FOR NON-AUTONOMOUS SECOND-ORDER SYSTEMS WITH IMPULSIVE EFFECTS*

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Abstract In this paper, we establish the existence of infinitely many solutions for a class of non-autonomous second-order systems with impulsive effects. Our technique is based on the Fountain Theorem of Bartsch and the Symmetric Mountain Pass Lemma due to Kajikiya.

Keywords Second-order systems, impulsive effects, Fountain Theorem, Symmetric Mountain Pass Lemma.

MSC(2010) 34B15, 34B37, 58E30.

1. Introduction and main results

We consider the existence of infinitely many periodic solutions for the following second-order Hamiltonian systems with impulsive effects

$$\begin{cases} \ddot{u}(t) + V(t)u(t) + \nabla W(t, u(t)) = 0, & a.e.\ t \in [0, T], \\ \Delta(\dot{u}^i(t_j)) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) = I_{ij}(u^i(t_j)), \ i = 1, 2, \cdots, N, j = 1, 2, \cdots, l, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $V(t)$ is an $N \times N$ symmetric matrix, continuous and T-periodic in t. $u(t) = (u^1(t), u^2(t), \cdots, u^N(t))$, t_j, $j = 1, 2, \cdots, l$, are the instants where the impulses occur and $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T$, $I_{i,j} : \mathbb{R} \to \mathbb{R}\,(i = 1, 2, \cdots, N, j = 1, 2, \cdots, l)$ are continuous and $W(t, x) : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in t for all $x \in \mathbb{R}^N$ and satisfies the following assumption:

(A) $W(t, x)$ is measurable in t for each $x \in \mathbb{R}^N$ and continuously differential in x for $a.e.\, t \in [0, T]$ and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|W(t, x)| \le a(|x|)b(t), \quad |\nabla W(t, x)| \le a(|x|)b(t)$$

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for all $x \in \mathbb{R}^N$ and $a.e. \, t \in [0, T]$, where $\nabla W(t, x)$ denotes the gradient of $W(t, x)$ in x.

As a special case of dynamical systems, Hamiltonian systems are very important in the various applications in mechanics, electronics, economics and so on. In 1978, Rabinowitz [20] published his pioneer paper for the existence of periodic solutions for Hamiltonian systems via the critical point theory. From then on, there has been a vast literature on the study of existence and multiplicity of periodic solutions for Hamiltonian systems, see [4, 8, 13, 14, 21, 24, 25] and the references therein.

In recent years, some classical methods and techniques such as the coincidence degree theory, upper and lower solutions method, iterative technique, fixed point theory have been applied to study the impulsive differential equations by many researchers, due to its widely application in various problems of technology and science, see [3, 7, 9, 10, 12, 16, 17, 22, 23, 26, 28–30] and the references therein. Especially, Nieto and O'Regan [16] presented a new approach via variational methods and critical point theory to study the existence of solutions to impulsive problems.

In the present paper, we are interested in the existence of infinitely many solutions of problem (1.1) under some new conditions. With the aid of variational methods, we get the multiplicity results for both superquadratic and subquadratic cases, which generalize and sharply improve the results in [7, 23]. Moreover, our proofs are much simpler.

1.1. The superquadratic case

By applying a variant of Fountain Theorem (see [31]), Sun, Chen and Nieto [23] proved the existence of infinitely many solutions for system (1.1), where $W(t, x)$ is even in x. The following theorem was obtained.

Theorem 1.1 (Theorem 1.1, [23]). *Assume the following conditions hold:*

(V_1) $V(t) = (v_{ij}(t))$ *is a symmetric matrix with* $v_{ij} \in L^\infty([0, T])$ *for every* $t \in [0, T]$.

(V_2) There exists a positive constant ν *such that* $V(t)u \cdot u \geq \nu |u|^2$ *for every* $u \in \mathbb{R}^N$ *and a.e. in* $[0, T]$.

($\mathcal{H}_0$) I_{ij} $(i = 1, 2, \cdots, N, j = 1, 2, \cdots, l)$ *are odd and satisfy*

$$\int_0^s I_{ij}(t)dt \geq 0 \quad \text{for all } s \in \mathbb{R}.$$

($\mathcal{H}_1$)

$$2\int_0^s I_{ij}(t)dt - I_{ij}(s)s \geq 0 \quad \text{for all } s \in \mathbb{R}.$$

($\mathcal{H}_2$) There exist constants a_{ij}, $b_{ij} > 0$ *and* $\gamma_{ij} \in [0, 1)$ *such that*

$$|I_{ij}(s)| \leq a_{ij} + b_{ij}|s|^{\gamma_{ij}} \quad \text{for } s \in \mathbb{R}.$$

($\mathcal{H}_3$) $\lim\limits_{|x| \to +\infty} \frac{W(t,x)}{|x|^2} = +\infty$ *uniformly for* $t \in [0, T]$.

($\mathcal{H}_4$) $W(t, 0) \equiv 0$, $0 \leq W(t, x) = o(|x|^2)$ *as* $|x| \to 0$ *uniformly for* $t \in [0, T]$.

($\mathcal{H}_5$) There exist constants $\alpha > 1$, $1 < \beta < 1 + \frac{\alpha - 1}{\alpha}$, c_1, $c_2 > 0$ *and* $L > 0$ *such that*

$$(\nabla W(t, x), x) - 2W(t, x) \geq c_1|x|^\alpha, \quad |\nabla W(t, x)| \leq c_2|x|^\beta$$

for every $t \in [0, T]$ *and* $x \in \mathbb{R}^N$ *with* $|x| \geq L$.

($\mathcal{H}_6$) $W(t,-x) = W(t,x)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Then problem (1.1) has infinitely many solutions.

In [7], Chen and He established the following result for system (1.1), which improves Theorem 1.1.

Theorem 1.2 (Theorem 1.4, [7])**.** *Suppose that (V_1), ($\mathcal{H}_0$), ($\mathcal{H}_2$), ($\mathcal{H}_3$), ($\mathcal{H}_6$) hold. Assume the following conditions hold:*

(V_2') There exists $\nu' \geq 0$ such that $V(t)u \cdot u \geq -\nu'|u|^2$ for every $u \in \mathbb{R}^N$ and a.e. in $[0,T]$.

($\mathcal{H}_1'$) There exists $L_0 > 0$ such that

$$2\int_0^s I_{ij}(t)dt - I_{ij}(s)s \geq 0 \quad \textit{for} \quad |s| \geq L_0.$$

($\mathcal{H}_5'$) There exist constants $\lambda > 2$, $\mu > \lambda - 2$ such that

$$\liminf_{|x|\to+\infty} \frac{(\nabla W(t,x),x) - 2W(t,x)}{|x|^\mu} > 0$$

and

$$\limsup_{|x|\to+\infty} \frac{W(t,x)}{|x|^\lambda} < +\infty$$

uniformly for a.e. $t \in [0,T]$.

Then problem (1.1) has infinitely many solutions.

Here, applying the Fountain Theorem due to Bartsch (see [2, Theorem 2.5] and [27, Theorem 3.6]), we obtain the existence of infinitely many solutions for problem (1.1) with some more general conditions, which generalizes and improves upon the results mentioned above. The following theorem is established.

Theorem 1.3. *Assume that conditions ($\mathcal{H}_0$), ($\mathcal{H}_1'$), ($\mathcal{H}_2$), ($\mathcal{H}_3$), ($\mathcal{H}_6$) hold and $W(t,x)$ satisfies the following condition:*

($\mathcal{H}_7$) There exist constants $a_1 > 0$ and $L_1 > L_0$ such that

$$(\nabla W(t,x),x) - 2W(t,x) \geq \frac{a_1}{|x|^2}W(t,x)$$

for all $x \in \mathbb{R}^N$ with $|x| \geq L_1$ and a.e. $t \in [0,T]$.

Then problem (1.1) has infinitely many solutions.

Remark 1.1. Clearly, condition ($\mathcal{H}_7$) is weaker than ($\mathcal{H}_5$) and ($\mathcal{H}_5'$). Moreover, conditions (V_1), (V_2), (V_2') and ($\mathcal{H}_4$) are removed. So, our Theorem 1.3 extends Theorem 1.1 and Theorem 1.2. There are functions $W(t,x)$ satisfying our assumptions of Theorem 1.3 and not satisfying the conditions of Theorem 1.1 and Theorem 1.2. For example, set

$$W(t,x) = |x|^2 \ln(1+|x|^2) + sin(|x|^2) - \ln(1+|x|^2)$$

for all $x \in \mathbb{R}^N$ and $a.e.$ $t \in [0,T]$. Then, W does not satisfy the result mentioned above. But W satisfies Theorem 1.3 with $a_1 = 1$.

1.2. The subquadratic case

Sun, Chen and Nieto studied systems (1.1) that are asymptotically case and subquadratic case. Applying the minimax technique, they obtained the two following theorems.

Theorem 1.4 (Theorem 1.2, [23]). *Assume that the following conditions are satisfied:*

($\mathcal{S}_1$) *$I_{ij}(i = 1, 2, \cdots, N, j = 1, 2, \cdots, l)$ are odd and satisfy $I_{ij}(s)s \geq 0$ for all $s \in \mathbb{R}$.*

($\mathcal{S}_2$) *There exist constant $b_{ij} > 0$ and $\gamma_{ij} \in [0, 1)$ such that*

$$|I_{ij}(s)| \leq b_{ij}|s|^{\gamma_{ij}} \quad \text{for } s \in \mathbb{R}.$$

($\mathcal{S}_3$) *$W(t, x) = \frac{\rho}{2}|x|^2 + F(t, x)$, where $F(t, x)$ is even in x, ρ is a positive constant and not a spectrum point of $-\frac{d^2}{dt^2} + V$ and $F(t, 0) \equiv 0$.*

($\mathcal{S}_4$) *There exist δ_1, $\delta_2 \in [1, 2)$ with $\delta_1 < \min_{1\leq i\leq N,\, 1\leq j\leq l}\{\gamma_{ij}\} + 1$ and d_1, $d_2 > 0$ such that*

$$d_1|x|^{\delta_1} \leq F(t, x), \quad |\nabla F(t, x)| \leq d_2|u|^{\delta_2 - 1}$$

for all $(t, u) \in [0, T] \times \mathbb{R}^N$.

Then problem (1.1) has infinitely many solutions.

Theorem 1.5 (Theorem 1.3, [23]). *Suppose that ($\mathcal{S}_1$), ($\mathcal{S}_2$) and ($\mathcal{H}_6$) hold. Assume that W satisfies (**A**) and the following conditions:*

($\mathcal{J}_1$) *$W(t, 0) \equiv 0$ for any $t \in [0, T]$.*

($\mathcal{J}_2$) *There are constants $k_1 > 0$ and $\zeta_1 \in [1, 2)$ with $\zeta_1 < \min_{1\leq i\leq N,\, 1\leq j\leq l}\{\gamma_{ij}\} + 1$ such that*

$$W(t, x) \geq k_1|x|^{\zeta_1} \quad \text{for any } (t, x) \in [0, T] \times \mathbb{R}^N.$$

($\mathcal{J}_3$) *There exist constants $k_2 > 0$ and $\zeta_2 \in [1, 2)$ such that*

$$|\nabla W(t, x)| \leq k_2|x|^{\zeta_2 - 1} \quad \text{for any } (t, x) \in [0, T] \times \mathbb{R}^N.$$

Then problem (1.1) has infinitely many solutions.

Here, by means of the new version of the Symmetric Mountain Pass Lemma established in [11], we obtain the following theorem, which unifies and significantly improves upon Theorems 1.4 and 1.5.

Theorem 1.6. *Assume that the following conditions hold:*

($\mathcal{S}_2'$) *There exist constants $\delta_3 > 0$, $b_{ij} > 0$ and $1 < \gamma^* = \min_{1\leq i\leq N, 1\leq j\leq l}\{\gamma_{ij}\}$ such that*

$$|I_{ij}(s)| \leq b_{ij}|s|^{\gamma_{ij}} \quad \text{and} \quad I_{ij}(-s) = -I_{ij}(s)$$

for $|s| \leq \delta_3$, $i = 1, 2, \cdots, N, j = 1, 2, \cdots, l$.

($\mathcal{H}_8$) *$W(t, -x) = W(t, x)$ for a.e. $t \in [0, T]$ and $x \in \mathbb{R}^N$ with $|x| \leq \delta_3$.*

($\mathcal{H}_9$) *$W(t, 0) \equiv 0$ for a.e. $t \in [0, T]$ and*

$$\lim_{|x|\to 0} \frac{W(t, x)}{|x|^2} = +\infty \quad \text{uniformly for a.e. } t \in [0, T].$$

Then problem (1.1) possesses infinitely many solutions.

2. Proof of main results

Let us consider the functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \frac{1}{2}\int_0^T (V(t)u, u)dt - \int_0^T W(t, u)dt + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt$$

for any $u \in H_T^1$, where

$H_T^1 = \{u : [0, T] \to \mathbb{R}^N \mid u$ is absolutely continuous, $u(0) = u(T),\ \dot{u} \in L^2(0, T; \mathbb{R}^N)\}$

is a Hilbert space with the norm defined by

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}.$$

It follows from assumption (**A**) that the functional φ is continuously differentiable on H_T^1. Moreover, one has

$$\langle \varphi'(u), v\rangle = \int_0^T ((\dot{u}, \dot{v}) - (V(t)u, v) - (\nabla W(t, u), v))dt + \sum_{j=1}^{l}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j)$$

for all $u, v \in H_T^1$. It is well known that the solutions of problem (1.1) correspond to the critical points of φ. Since the embedding $H_T^1 \hookrightarrow C(0, T; \mathbb{R}^N)$ is compact, which implies that

$$\|u\|_\infty \leq C\|u\| \tag{2.1}$$

for some $C > 0$ and all $u \in H_T^1$, where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$ (see [14]). Letting

$$q(u) = \int_0^T (1/2)(|\dot{u}|^2 - (V(t)u, u))dt,$$

one has that

$$\begin{aligned} q(u) =& (1/2)\|u\|^2 - (1/2)\int_0^T ((V(t) + I)u, u)dt \\ =& (1/2)((I - K)u, u), \end{aligned}$$

where $K : H_T^1 \to H_T^1$ is the linear self-adjoint operator defined, applying Riesz representation theorem, by

$$\int_0^T ((V(t) + I)u, v)dt = (Ku, v), \qquad \forall u, v \in H_T^1.$$

The compact imbedding of H_T^1 into $C(0, T; \mathbb{R}^N)$ (see [14]) implies that K is compact. Using classical spectral theory, the following decomposition holds

$$H_T^1 = H^- \oplus H^0 \oplus H^+,$$

where $H^0 = N(I - K)$, H^- and H^+ are such that, for some $\delta > 0$,

$$\begin{aligned} q(u) &\leq -(\delta/2)\|u\|^2 \quad \text{if } u \in H^-, \\ q(u) &\geq (\delta/2)\|u\|^2 \quad \text{if } u \in H^+. \end{aligned} \tag{2.2}$$

Let X be a reflexive and separable Banach space. It is well known that there exist $\{w_n\}_{n\in\mathbb{N}} \subset X, \{\psi_n\}_{n\in\mathbb{N}} \subset X^*$ such that

(1) $\langle \psi_n, w_m \rangle = \chi_{n,m}$, where $\chi_{n,m} = 1$ for $n = m$ and $\chi_{n,m} = 0$ for $n \neq m$.

(2) $\overline{span}\{w_n | n \in \mathbb{N}\} = X$, $\overline{span}^{\omega^*}\{\psi_n | n \in \mathbb{N}\} = X^*$.

Let $X_j = \mathbb{R} w_j$, then $X = \overline{\oplus_{j \geq 1} X_j}$. We define

$$Y_k = \oplus_{j=1}^{k} X_j, \ Z_k = \overline{\oplus_{j=k}^{\infty} X_j}.$$

Definition 2.1 (Palais-Smale condition, [14, 15, 18]). Let X be a Bananch space, $\varphi \in C^1(X, \mathbb{R})$. The function φ satisfies the Palais-Smale condition if any sequence $\{u_n\} \subset X$ such that

$$\{\varphi(u_n)\} \text{ is bounded}, \quad \varphi'(u_n) \to 0 \quad \text{as } n \to \infty,$$

has a convergent subsequence.

Definition 2.2 (Cerami condition, [6, 15, 19]). Let X be a Bananch space, $\varphi \in C^1(X, \mathbb{R})$. The function φ satisfies the Cerami condition if any sequence $\{u_n\} \subset X$ such that

$$\{\varphi(u_n)\} \text{ is bounded}, \quad \varphi'(u_n)(1 + \|u_n\|) \to 0 \quad \text{as } n \to \infty,$$

has a convergent subsequence.

Definition 2.3 ($(PS)_c$ condition, [5, 14, 15, 27]). Let X be a Bananch space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. The function φ satisfies the $(PS)_c$ condition if any sequence $\{u_n\} \subset X$ such that

$$\varphi(u_n) \to c, \ \varphi'(u_n) \to 0 \quad \text{as } n \to \infty,$$

has a convergent subsequence.

Palais-Smale condition (PS-condition) was introduced by Palais and Smale in [18]. This condition and some of its variants have been essential in the development of critical point theory on Banach spaces or Banach manifolds, and are referred as Palais-Smale-type conditions. It is clear that the PS-condition implies the $(PS)_c$ condition for each $c \in \mathbb{R}$. Cerami condition (condition (C)) was given by Cerami [6]. It is well known that condition (C) is weaker than PS-condition. In [2], Bartsch established the Fountain Theorem (Theorem 2.5 in [2], Theorem 3.6 in [27]) under the $(PS)_c$ condition. As shown in [1, 15, 19], the deformation lemma can be proved with the weaker condition (C) replacing the usual PS-condition, and it turns out that the Fountain Theorem is true under the condition (C). So, we have the following Fountain Theorem.

Theorem 2.1 (Fountain Theorem, [2, 27]). *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the condition (C), $\varphi(-u) = \varphi(u)$. If for almost every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(A_1)

$$a_k := \max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \leq 0,$$

(A_2)

$$b_k := \inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \to \infty, \ k \to \infty,$$

then φ has an unbounded sequence of critical values.

To prove Theorem 1.6, we need the following Symmetric Mountain Pass Theorem due to Kajikiya (see [11]). Before stating it, we first recall the definition of genus. Let X be a Banach space and let A be a subset of X. A is said to be symmetric if $u \in A$ implies that $-u \in A$. For a closed symmetric set A, which does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \backslash \{0\}$. If such a k does not exist, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$. For the convenience of the readers, we summarize the property of genus that will be used in the proof of Theorem 1.6. We refer the readers to [21, proposition 7.5] for the proof of the next proposition.

Proposition 2.1. *Let A and B be closed symmetric subsets of X that do not contain the origin. Then the following hold.*

(i) If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$.

(ii) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.

(iii) If A is compact, then $\gamma(A) < +\infty$ and $\gamma(N_\delta(A)) = \gamma(A)$ for $\delta > 0$ small enough, where $N_\delta(A) = \{x \in X : \|x - A\| \leq \delta\}$.

(iv) The n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk-Ulam theorem.

We now state the Symmetric Mountain Pass Lemma.

Theorem 2.2 (Theorem 1, [11])**.** *Let X be an infinite-dimensional Banach space and let $\varphi \in C^1(X, \mathbb{R})$ satisfy the following conditions.*

(1) $\varphi(u)$ is even, bounded from below, $\varphi(0) = 0$ and φ satisfies the PS-condition.

(2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} \varphi(u) < 0$.

Then, φ possesses a sequence of critical points $\{u_k\}$ such that $\varphi(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

Now, we can demonstrate the proof of our results.

Proof of Theorem 1.3. First of all, we will prove that φ satisfies condition (C). Let $\{u_n\}$ be a sequence in H_T^1 such that

$$\{\varphi(u_n)\} \text{ is bounded and } \ (1 + \|u_n\|)\|\varphi'(u_n)\| \to 0 \ \text{ as } n \to \infty.$$

Then there exists a positive constant M_0 such that

$$|\varphi(u_n)| \leq M_0, \qquad (1 + \|u_n\|)\|\varphi'(u_n)\| \leq M_0. \tag{2.3}$$

By a standard argument, we only need to prove that $\{u_n\}$ is a bounded sequence in H_T^1. Otherwise, we can assume that $\|u_n\| \to +\infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, one then has $\|v_n\| = 1$. Going if necessary to a subsequence, we can suppose that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{weakly in } H_T^1, \\ &v_n \to v \quad \text{strongly in } C(0, T; \mathbb{R}^N) \end{aligned} \tag{2.4}$$

as $n \to \infty$.

By $(\mathcal{H}_2)$, (2.3) and (2.4), we obtain

$$\begin{aligned}
&\left|\int_0^T \frac{W(t,u_n)}{\|u_n\|^2}dt-\frac{1}{2}\right| \\
\le&\frac{|\varphi(u_n)|}{\|u_n\|^2}+\frac{1}{2}\int_0^T((V(t)+I)v_n,v_n)dt+\frac{\sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)}|I_{ij}(t)|dt}{\|u_n\|^2} \\
\le&\frac{M_0}{\|u_n\|^2}+\frac{1}{2}(B_0+1)T\|v_n\|_\infty^2+\frac{\sum_{j=1}^{l}\sum_{i=1}^{N}(a_{ij}\|u_n\|_\infty+b_{ij}\|u_n\|_\infty^{\gamma_{ij}+1})}{\|u_n\|^2},
\end{aligned} \tag{2.5}$$

where B_0 is a constant such that

$$|(V(t)x,x)|\le B_0|x|^2 \quad \forall x\in\mathbb{R}^N. \tag{2.6}$$

Here, we claim that $v\not\equiv 0$. Otherwise, if $v\equiv 0$, by (2.5), we have

$$\lim_{n\to\infty}\int_0^T \frac{W(t,u_n)}{\|u_n\|^2}dt=\frac{1}{2}. \tag{2.7}$$

From $(\mathcal{H}_3)$, there exists a positive constant $L_2>L_1$ such that

$$W(t,x)\ge 0 \tag{2.8}$$

for all $x\in\mathbb{R}^N$ with $|x|\ge L_2$ and *a.e.* $t\in[0,T]$. By assumption (**A**), one has

$$|W(t,x)|\le a_2 b(t), \qquad |\nabla W(t,x)|\le a_2 b(t) \tag{2.9}$$

for all $x\in\mathbb{R}^N$ with $|x|\le L_2$ and *a.e.* $t\in[0,T]$, where $a_2=\max_{0\le s\le L_2}a(s)$. Hence, we get from (2.8) and (2.9) that

$$W(t,x)\ge -a_2 b(t) \tag{2.10}$$

for all $x\in\mathbb{R}^N$ and *a.e.* $t\in[0,T]$.

It follows from $(\mathcal{H}_1')$ that there exist positive constant B_1 such that

$$2\int_0^s I_{ij}(t)dt-I_{ij}(s)s\ge -B_1 \quad \text{for} \quad s\in\mathbb{R}.$$

We deduce from $(\mathcal{H}_7)$ and (2.9) that

$$\begin{aligned}
&\int_{\{t||u_n(t)|\ge L_1\}}\frac{|W(t,u_n)|}{|u_n|^2}dt \\
\le&a_1^{-1}\int_{\{t||u_n(t)|\ge L_1\}}((\nabla W(t,u_n),u_n)-2W(t,u_n))dt \\
=&a_1^{-1}\int_0^T((\nabla W(t,u_n),u_n)-2W(t,u_n))dt \\
&-a_1^{-1}\int_{\{t||u_n(t)|<L_1\}}((\nabla W(t,u_n),u_n)-2W(t,u_n))dt
\end{aligned}$$

$$
\begin{aligned}
&\le a_1^{-1}(2\varphi(u_n) - \langle \varphi'(u_n), u_n\rangle) + a_1^{-1}(L_1+2)\int_{\{t||u_n(t)|<L_1\}} a_2 b(t)dt \\
&\quad + \sum_{j=1}^{l}\sum_{i=1}^{N}\left(I_{ij}(u^i(t_j))u^i(t_j) - 2\int_0^{u^i(t_j)} I_{ij}(t)dt\right) \\
&\le a_1^{-1}(2\varphi(u_n) - \langle \varphi'(u_n), u_n\rangle) + a_1^{-1}(L_1+2)a_2\|b\|_{L^1} + lNB_1 \\
&\le 3a_1^{-1}M_0 + a_1^{-1}(L_1+2)a_2\|b\|_{L^1} + lNB_1.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\left|\int_0^T \frac{W(t,u_n)}{\|u_n\|^2}dt\right| &\le \int_0^T \frac{|W(t,u_n)|}{\|u_n\|^2}dt \\
&= \int_{\{t||u_n(t)|\ge L_1\}} \frac{|W(t,u_n)|}{\|u_n\|^2}dt + \int_{\{t||u_n(t)|<L_1\}} \frac{|W(t,u_n)|}{\|u_n\|^2}dt \\
&\le \int_{\{t||u_n(t)|\ge L_1\}} \frac{|W(t,u_n)|}{|u_n|^2}|v_n|^2dt + \frac{a_2\|b\|_{L^1}}{\|u_n\|^2} \\
&\le \|v_n\|_\infty^2 \int_{\{t||u_n(t)|\ge L_1\}} \frac{|W(t,u_n)|}{|u_n|^2}dt + \frac{a_2\|b\|_{L^1}}{\|u_n\|^2} \\
&\le \|v_n\|_\infty^2 (3a_1^{-1}M_0 + a_1^{-1}(L_1+2)a_2\|b\|_{L^1} + lNB_1) + \frac{a_2\|b\|_{L^1}}{\|u_n\|^2} \\
&\to 0
\end{aligned}
$$

as $n \to \infty$, which is a contradiction to (2.7). So, $v \not\equiv 0$. Now, letting $\Omega = \{t \in [0,T] : |v(t)| > 0\}$, one has $|\Omega| > 0$. Since $\|u_n\| \to +\infty$, one gets $|u_n| \to +\infty$ as $n \to \infty$ for $a.e.$ $t \in \Omega$. From $(\mathcal{H}_3)$, one sees

$$
\lim_{n\to+\infty} \frac{W(t,u_n(t))}{|u_n(t)|^2} = +\infty \quad a.e. \text{ on } \Omega.
$$

We conclude from (2.10) and Fatou Lemma, one has

$$
\begin{aligned}
&\liminf_{n\to\infty} \int_0^T \frac{W(t,u_n)}{\|u_n\|^2}dt \\
\ge &\liminf_{n\to\infty} \left(\int_\Omega \frac{W(t,u_n)}{|u_n|^2}|v_n|^2dt - \frac{a_2}{\|u_n\|^2}\int_{[0,T]\setminus\Omega} b(t)dt\right) \\
\ge &\liminf_{n\to\infty} \left(\int_\Omega \frac{W(t,u_n)}{|u_n|^2}|v_n|^2dt - \frac{a_2\|b\|_{L^1}}{\|u_n\|^2}\right) \\
= &\liminf_{n\to\infty} \int_\Omega \frac{W(t,u_n)}{|u_n|^2}|v_n|^2dt \\
= &+\infty,
\end{aligned}
$$

which contradicts to (2.5). So, $\|u_n\|$ is bounded. And, the condition (C) holds.

Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for H_T^1 and define Y_k and Z_k as in Theorem 2.1. Since $dim(Y_k) < \infty$, all the norms are equivalent. For each $u \in Y_k$, there exists constant $C_k > 0$ such that

$$
\|u\| \le C_k\|u\|_{L^2}. \tag{2.11}
$$

From condition $(\mathcal{H}_3)$, there exists $L_3 > 0$ such that

$$W(t,x) \geq (1+B_0)C_k^2|x|^2 \tag{2.12}$$

for all $|x| \geq L_3$ and *a.e.* $t \in [0,T]$. From assumption (**A**), one gets

$$|W(t,x)| \leq a_3 b(t) \tag{2.13}$$

for all $x \in \mathbb{R}^N$ with $|x| \leq L_3$ and *a.e.* $t \in [0,T]$, where $a_3 = \max\limits_{0\leq s\leq L_3} a(s)$. Hence, we obtain from (2.12) and (2.13) that

$$W(t,x) \geq (1+B_0)C_k^2(|x|^2 - L_3^2) - a_3 b(t) \tag{2.14}$$

for all $x \in \mathbb{R}^N$ and *a.e.* $t \in [0,T]$.

Then, for $u \in Y_k$, it follows from (2.6), (2.11) and (2.14) that

$$\begin{aligned}
\varphi(u) =& \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \frac{1}{2}\int_0^T (V(t)u,u)dt - \int_0^T W(t,u)dt \\
&+ \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt \\
\leq& \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \frac{B_0}{2}\|u\|_{L^2}^2 - (1+B_0)C_k^2(\|u\|_{L^2}^2 - L_3^2 T) + a_3\|b\|_{L^1} \\
&+ \sum_{j=1}^{l}\sum_{i=1}^{N}(a_{ij}\|u\|_\infty + b_{ij}\|u\|_\infty^{\gamma_{ij}+1}) \\
\leq& \frac{1+B_0}{2}\|u\|^2 - (1+B_0)C_k^2\|u\|_{L^2}^2 + (1+B_0)C_k^2 L_3^2 T + a_3\|b\|_{L^1} \\
&+ \sum_{j=1}^{l}\sum_{i=1}^{N}(a_{ij}\|u\|_\infty + b_{ij}\|u\|_\infty^{\gamma_{ij}+1}) \\
\leq& -\frac{1+B_0}{2}\|u\|^2 + (1+B_0)C_k^2 L_3^2 T + a_3\|b\|_{L^1} \\
&+ \sum_{j=1}^{l}\sum_{i=1}^{N}(a_{ij}C\|u\| + b_{ij}C^{\gamma_{ij}+1}\|u\|^{\gamma_{ij}+1}),
\end{aligned}$$

which implies that

$$\varphi(u) \to -\infty \quad \text{as} \quad \|u\| \to \infty, \quad \text{in} \quad Y_k.$$

So, (A_1) of Theorem 2.1 is satisfied for every $\rho_k > 0$ large enough.

Let us define

$$\beta_k = \sup_{u\in Z_k, \|u\|=1} \|u\|_\infty,$$

then

$$\beta_k \to 0 \qquad \text{as} \quad k \to \infty.$$

Obviously, one has $0 \leq \beta_{k+1} \leq \beta_k$, which yields $\beta_k \to \beta^*$ as $k \to \infty$. For each $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_\infty > \frac{\beta_k}{2}$. Since $u_k \in Z_k$, one sees

$u_k \rightharpoonup 0$ in H_T^1. By (2.1), we obtain $u_k \to 0$ in $C([0,T],\mathbb{R}^N)$. Hence, it follows that $\beta^* = 0$.

Set $r_k = \beta_k^{-1}$, one has

$$r_k \to \infty \qquad \text{as} \quad k \to \infty. \tag{2.15}$$

Thus, for k large enough such that $Z_k \subset H^+$ and $r_k^2 \geq 4\delta^{-1} a_4 \|b\|_{L^1}$, where $a_4 = \max_{0 \leq s \leq 1} a(s)$. Then, for $u \in Z_k$ with $\|u\| = r_k$, one sees $\|u\|_\infty \leq 1$. So, by $(\mathcal{H}_0)$ and (2.2), we have

$$\begin{aligned}
\varphi(u) =& \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \frac{1}{2}\int_0^T (V(t)u,u)dt - \int_0^T W(t,u)dt \\
&+ \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt \\
\geq& \frac{\delta\|u\|^2}{2} - \int_0^T W(t,u)dt \\
\geq& \frac{\delta\|u\|^2}{2} - a_4\|b\|_{L^1} \\
\geq& \frac{\delta r_k^2}{4}.
\end{aligned}$$

Therefore, it follows from (2.15) and the above expression that

$$\inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \to \infty, \ k \to \infty.$$

And, relation (A_2) is proved. Hence, the proof is completed by using the Fountain theorem. $\square$

Proof of Theorem 1.6. We consider the following truncated functional

$$\begin{aligned}
\psi(u) =& \frac{1}{2}\|u\|^2 - h(\|u\|)\bigg(\frac{1}{2}\int_0^T |u|^2 dt + \frac{1}{2}\int_0^T (V(t)u,u)dt + \int_0^T W(t,u)dt \\
&- \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt\bigg)
\end{aligned}$$

for any $u \in H_T^1$, where $h : \mathbb{R}^+ \to [0,1]$ is a non-increasing C^1 function such that

$$h(t) = \begin{cases} 1, & 0 \leq s \leq \delta_3/(2C), \\ 0, & s \geq \delta_3/C. \end{cases}$$

Obviously, $\psi \in C^1(H_T^1, \mathbb{R})$ and $\psi(0) = 0$. Since

$$\varphi(u) = \psi(u) \qquad \text{for} \quad \|u\| \leq \delta_3/(2C).$$

Hence, if we can get that ψ possesses a sequence of critical points $\{u_k\}$ such that

$$\psi(u_k) \leq 0, \quad u_k \neq 0 \quad \text{and} \quad u_k \to 0 \quad \text{as} \quad k \to \infty,$$

then for the k large enough, the critical points of ψ satisfying $\|u_k\| \le \delta_3/(2C)$ are just critical points of φ. So, the conclusion of Theorem 1.6 holds.

By (2.1), if $\|u\| \le \delta_3/C$, we have

$$|u(t)| \le \|u\|_\infty \le C\|u\| \le \delta_3 \qquad \forall t \in [0,T].$$

It follows from $(\mathcal{S}_2')$ and $(\mathcal{H}_8)$ that

$$\psi(-u) = \psi(u) \tag{2.16}$$

for $u \in H_T^1$ with $\|u\| \le \delta_3/C$. When $\|u\| \ge \delta_3/C$, one has

$$\psi(u) = \frac{1}{2}\|u\|^2. \tag{2.17}$$

Combining this with (2.16) implies

$$\psi(-u) = \psi(u) \qquad \text{for all} \quad u \in H_T^1.$$

On the other hand, expression (2.17) yields

$$\psi(u) \to +\infty \qquad \text{as} \quad \|u\| \to \infty.$$

Hence, ψ is bounded from below and satisfies the PS-condition.

From condition $(\mathcal{H}_9)$, there exists $\delta_4 > 0$ such that

$$W(t,x) \ge (1+B_0)C_k^2|x|^2 \tag{2.18}$$

for all $|x| \le \delta_4$ and $a.e.$ $t \in [0,T]$, where B_0 refers to (2.6). So, for $u \in Y_k$ with

$$\|u\| = \theta_k := \frac{1}{2}\min\left\{1, \delta_3/(2C), \delta_4/C, \left(\frac{1+B_0}{4\sum_{j=1}^{l}\sum_{i=1}^{N} b_{ij}C^{\gamma_{ij}+1}}\right)^{\gamma^*-1}\right\},$$

it follows from $(\mathcal{S}_2')$ and (2.18) that

$$\begin{aligned}
\varphi(u) =& \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \frac{1}{2}\int_0^T (V(t)u,u)dt - \int_0^T W(t,u)dt \\
&+ \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt \\
\le& \frac{1+B_0}{2}\|u\|^2 - (1+B_0)C_k^2\|u\|_{L^2}^2 + \sum_{j=1}^{l}\sum_{i=1}^{N} b_{ij}\|u\|_\infty^{\gamma_{ij}+1} \\
\le& -\frac{1+B_0}{2}\|u\|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N} b_{ij}C^{\gamma_{ij}+1}\|u\|^{\gamma_{ij}+1} \\
\le& -\frac{1+B_0}{2}\|u\|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N} b_{ij}C^{\gamma_{ij}+1}\|u\|^{\gamma^*+1} \\
\le& -\frac{1+B_0}{4}\|u\|^2,
\end{aligned}$$

which implies

$$\{u \in Y_k \mid \|u\| = \theta_k\} \subset \left\{ u \in H_T^1 \;\middle|\; \psi(u) \le -\frac{1+B_0}{4}\theta_k^2 \right\}.$$

Now, taking

$$A_k = \left\{ u \in H_T^1 : \psi(u) \le -\frac{1+B_0}{4}\theta_k^2 \right\},$$

by Proposition 2.1 one sees that

$$\gamma(A_k) \ge \gamma(\{u \in Y_k | \|u\| = \theta_k\}) \ge k.$$

So, we get $A_k \in \Gamma_k$ and

$$\sup_{u \in A_k} \psi(u) \le -\frac{1+B_0}{4}\theta_k^2 < 0.$$

Then, Theorem 1.6 follows from Theorem 2.2 and the proof is complete. □

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