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WELL-POSEDNESS FOR THE COUPLED BBM SYSTEMS

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Abstract Consideration is given to initial value problem for systems of two evolution equations of generalized BBM-type coupled through nonlinearity described in (1.3). It is shown that the problem is always locally well-posed in the L_2-based Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 0$. Under exact conditions on $A, \cdots, F$, the local well-posedness theory extends globally, and bounds for the growth in time of relevant norms of solutions corresponding to very general auxiliary data are derived.

Keywords Regularized long wave, coupled BBM-BBM equations, nonlinear dispersive wave equations.

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1. Introduction

It is well known that the regularized long wave of small amplitude equation, aka, BBM equation

$$\eta_t + \eta_x - \eta_{xxt} + \eta\eta_x = 0, \tag{1.1}$$

where $\eta = \eta(x,t)$ is a real-valued function defined for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, is globally well posed if the initial condition $\eta(x,0)$ lies in Sobolev space $H^s = H^s(\mathbb{R})$ for any $s \geq 0$, see Benjamin etc [1], Bona and Tzvetkov [8], Bona and Chen [2], Bona etc [3], [4] and Chen [9]. However, if η is considered to be complex-valued function, write it in real part and imaginary part as $\eta = u + iv$, then (1.1) can be represented as the following system of equations,

$$\begin{cases} u_t + u_x - u_{xxt} + (\frac{1}{2}u^2 - \frac{1}{2}v^2)_x = 0, \\ v_t + v_x - v_{xxt} + (uv)_x = 0. \end{cases} \tag{1.2}$$

Our preliminary numerical results show that solutions of (1.2) blow up in finite time very quickly for small Gaussian initial data. Of course, if the minus sign $-$ in front $\frac{1}{2}v^2$ is replaced with plus $+$, it is two BBM equations of dependent variables $u + v$ and $u - v$. In other words, the system can be decoupled. Hence the problem

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is globally well posed in $H^s \times H^s$ for all $s \geq 0$. This leads us to a more general system of equations

$$\begin{cases} u_t + u_x - u_{xxt} + (Au^2 + Buv + Cv^2)_x = 0, \\ v_t + v_x - v_{xxt} + (Du^2 + Euv + Fv^2)_x = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \end{cases} \tag{1.3}$$

for consideration, where $A, B, C.D, E$ and F are real numbers. The aim of the current work is to understand a wide range of condition on $A, \cdots, F$ which makes the system globally well posed.

It is worth pointing out that the KdV-KdV system

$$\begin{cases} u_t + u_{xxx} + (Au^2 + Buv + Cv^2)_x = 0, \\ v_t + v_{xxx} + (Du^2 + Euv + Fv^2)_x = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \end{cases} \tag{1.4}$$

was introduced and studied by Bona, Cohen and Wang [6]. They show that for any $s > -\frac{3}{4}$, if the initial data

$$u_0(x),\, v_0(x) \in H^s(\mathbb{R}), \tag{1.5}$$

then (1.4) is well-posed locally in time. That is to say, (1.4) has a unique solution $(u, v) \in C\big([0,T]; H^s(\mathbb{R}) \times H^s(\mathbb{R})\big)$ for some finite number $T > 0$. Moreover, they show that system (1.4) is globally well-posed if the system of linear equations

$$\begin{cases} 2Ba + (E - 2A)b - 4Dc = 0, \\ 4Ca + (2F - B)b - 2Ec = 0 \end{cases} \tag{1.6}$$

has a nontrivial solution (a, b, c) such that $4ac - b^2 > 0$, i.e. the solution $u, v \in C([0,\infty); H^s(\mathbb{R}) \times H^s\mathbb{R}))$. In more recent paper [5], Bona, Chen and Karakashian pointed out that global well-posedness holds true as well when $4ac - b^2 = 0$.

In this current work, we like to bring the theory for (1.3) closely into line with that appearing in Bona etc [6]. What we show is that system (1.3) is locally well-posed if $u_0, v_0 \in H^s(\mathbb{R})$ for any $s \geq 0$. Moreover, we show that the length of time T only depends on $L_2(\mathbb{R})$-norms of the initial data $\|u_0\|_{L_2(\mathbb{R})}$ and $\|v_0\|_{L_2(\mathbb{R})}$. If (1.6) has one non-trivial solution (a, b, c) with property $4ac - b^2 \geq 0$, then the time interval for the solution can be extended to $[0, \infty)$. Moreover, we have bound growth in time of $\|u(\cdot, t)\|_{H^s(\mathbb{R})}, \|v(\cdot, t)\|_{H^s(\mathbb{R})}$ for any $s \geq 0$.

Unlike a single KdV equation, system (1.3) does not have infinite number of invariants, hence, when global well-posedness occurs, except $s = 1$, we do not expect Sobolev norms $\|(u(\cdot,t), v(\cdot,t))\|_{H^s(\mathbb{R}^s)\times H^s(\mathbb{R})}$ of solutions (u, v) to be uniformly bounded for $t \in [0, \infty)$.

Here is our main results in a rough statement.

Theorem 1.1. *If algebraic equations* (1.6) *has a non zero solution* (a, b, c) *such that* $4ac - b^2 > 0$, *then* (1.3) *is globally well-posed in* $H^s \times H^s$ *for any* $s \geq 0$. *However, the bounds structure is different as follows.*

$$\|(u(\cdot,t), v(\cdot,t))\|_{H^s\times H^s} \leq c_s (1+t)^{\frac{2}{3}(s-1)+\frac{1}{3}(s-\lfloor s\rfloor)} \quad \textit{if} \quad s \geq 1,$$

$$\|(u(\cdot,t), v(\cdot,t))\|_{H^s\times H^s} \leq c_{s,1} e^{c_s t} \quad \textit{if} \quad \frac{1}{4} < s < 1,$$

$$\|(u(\cdot,t), v(\cdot,t))\|_{H^s\times H^s} \leq c_{s,1} e^{c_s t^2} \quad \textit{if} \quad 0 \leq s \leq \frac{1}{4},$$

where c_s and $c_{s,1}$ are constants which depend only on the corresponding Sobolev norms of the initial data.

Theorem 1.2. *If system of algebraic equations* (1.6) *has a non zero solution* (a,b,c) *such that* $4ac-b^2=0$, *then* (1.3) *is globally well-posed in* $H^s\times H^s$ *for any* $s\ge 0$. *Here is the bound structure based on segment where* s *lies:*

$$\|(u(\cdot,t),v(\cdot,t))\|_{H^s\times H^s}\le c_{s,1}e^{c_{s,2}t}\quad \text{if}\quad s\ge 1,$$
$$\|(u(\cdot,t),v(\cdot,t))\|_{H^s\times H^s}\le c_{s,1}e^{e^{c_{s,2}t}}\quad \text{if}\quad \frac{1}{4}<s<1,$$
$$\|(u(\cdot,t),v(\cdot,t))\|_{H^s\times H^s}\le c_{s,1}e^{e^{c_{s,2}t^2}}\quad \text{if}\quad 0\le s\le \frac{1}{4},$$

where $c_{s,1},c_{s,2}$ are constants dependent only on the corresponding Sobolev norms of the initial data $\|u_0\|_{H^s},\|v_0\|_{H^s}$.

Remark. For $s<0$, the problem is not expected to be locally well posed since a single BBM equation was shown not to be well posed, see Bona and Tzvetkov [8] and Bona and Dai [7].

Remark. The single KdV-equation has infinite number of invariants, so the upper bounds for norms of the solution in Sobolev spaces H^s can be shown to be uniformly bounded in time for $s\ge 0$ being integers. However, the single BBM-equation has only three invariants, one of them is useful to show H^1 norm of the solution to be uniformly bounded. For $s>1$, the current techniques cannot show the uniform boundedness of H^s norm of the solution. On the other hand, numerical evidences suggested that H^s norm of the solution is uniformly bounded. For time being, it is not expected that the results for system (1.3) better than that of the single BBM-equation.

The plan of the remainder of the paper is as follows. In Section 3, the initial value problem (1.3) is converted into an equivalent system of integral equations. Then local well-posedness is first deduced for the $L_2\times L_2$ case by applying the contraction-mapping principle and then extended to $H^s(\mathbb{R})\times H^s(\mathbb{R})$ for $s\ge 0$ by bootstrapping. Section 4 is concerned with global well-posedness. Conserved quantities are introduced at the beginning of the section and the conditions on the coefficients $A,B,\cdots,F$ under which the invariants exist are determined. These will be used to derive *a priori* bounds that lead to global well-posedness. Following the idea in Chen [9], the growth bounds of relative norms of solutions in time are obtained for the case $H^s\times H^s$ separately when $s\ge 1$ and $0\le s<1$, which also conclude the proof of the global well-posedness for these cases as well.

2. Notation

The notational conventions and function-space designations used in this paper are set out here. $C_b=C_b(\mathbb{R})$ is a Banach space of uniformly bounded and continuous functions defined on the real number line $\mathbb{R}$ with the standard norm. For $1\le p<\infty$, $L_p=L_p(\mathbb{R})$ connotes the p^{th}-power Lebesque-integrable functions with the usual modification for the case $p=\infty$. The norm of a function $f\in L_r$ with $1\le r\le\infty$ is written $|f|_r$ while the $L_r\times L_r$-norm of a pair (f,g) of such functions is written $|(f,g)|_{L_r\times L_r}=|f|_r+|g|_r$. In general, if X and Y are Banach spaces, then their Cartesian product $X \times Y$ is a Banach space with a product norm defined by $\|(f,g)\|_{X\times Y} = \|f\|_X + \|g\|_Y$.

The Sobolev class $H^s = H^s(\mathbb{R})$ for $s \geq 0$ is the class of measurable functions f whose Fourier transform $\widehat{f}(\xi)$ is a measurable function, square integrable with respect to the measure $(1+\xi^2)^{\frac{1}{2}s}d\xi$, where

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

We will simply use H^s rather than $H^s(\mathbb{R})$ unless emphasis on the domain of definition of the functions is needed. A pair of functions (u,v) or $\begin{pmatrix} u \\ v \end{pmatrix}$ is denoted a bold-faced letter $\mathbf{v}$ some time. When $\mathbf{v} \in H^s \times H^s$, its norm is defined as $\|\mathbf{v}\|_{s\times s} = \|(u,v)\|_{H^s\times H^s} = \|u\|_s + \|v\|_s$. In special case $s = 0$, so $H^s = L_2$, we simply write the corresponding norm as $\|\mathbf{v}\| = \|(u,v)\| = \|u\| + \|v\|$. If X is any Banach space and $T > 0$ given, $C(0,T;X)$ is the class of continuous maps from $[0,T]$ into X with its usual norm

$$\|u\|_{C(0,T;X)} = \sup_{t\in[0,T]} \|u(t)\|_X.$$

The subspace $C^1(0,T;X)$ of the elements of $C(0,T;X)$ for which the limit

$$u'(t) = \lim_{h\to 0} \frac{u(t+h)-u(t)}{h}$$

exists in $C(0,T;X)$, is also a Banach space with the obvious norm. For $k \in \mathbb{N}$ the spaces $C^k(0,T;X)$ are defined inductively and by analogy. For convenience and when there couldn't be any confusion created,

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \text{is replaced by} \quad \int f(x)\,dx.$$

3. Local Well-posedness

The analysis for (1.3) begins with local well-posedness in a reasonably broad set of functional classes. To accomplish this, the given system is converted to an equivalent system of integral equations.

Let the bold faced letter $\mathbf{v}$ denote the vector

$$\mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix}$$

of dependent variables u and v, $\mathbf{v}_0$ the vector

$$\mathbf{v_0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

of initial data and $\mathbf{G}$ the vector of non-linearities

$$\mathbf{G} = \mathbf{G}(u,v) = \mathbf{G}(\mathbf{v}) = \begin{pmatrix} P(u,v) \\ Q(u,v) \end{pmatrix} = \begin{pmatrix} Au^2 + Buv + Cv^2 \\ Du^2 + Euv + Fv^2 \end{pmatrix}.$$

Then the system (1.3) can be written as

$$\begin{cases} \mathbf{v}_t + \mathbf{v}_x - \mathbf{v}_{xxt} + \mathbf{G}_x = 0, \\ \mathbf{v}(x,0) = \mathbf{v}_0(x). \end{cases} \tag{3.1}$$

Rearranging the equation as

$$\mathbf{v}_t - \mathbf{v}_{xxt} = -\mathbf{v}_x - \mathbf{G}_x,$$

or

$$(I - \partial_x^2)\mathbf{v}_t = -\partial_x(\mathbf{v} + \mathbf{G}).$$

Inverting the operator $(I - \partial_x^2)$ subject to boundness at $\pm\infty$ leads to

$$\begin{aligned} \mathbf{v}_t(x,t) &= -(I - \partial_x^2)^{-1}\partial_x\big[\mathbf{v}(x,t) + \mathbf{G}(\mathbf{v}(x,t))\big] \\ &= \int K(x-y)\big[\mathbf{v}(y,t) + \mathbf{G}(\mathbf{v}(y,t))\big]\,dy, \end{aligned} \tag{3.2}$$

where the kernel K is applied to the vectors $\mathbf{v}$ and $\mathbf{G}(\mathbf{v})$ componentwise and

$$K(x) = \frac{1}{2}\mathrm{sgn}(x)e^{-|x|} \quad \text{whose Fourier transform is} \quad \widehat{K}(\xi) = \frac{-i\xi}{1+\xi^2}. \tag{3.3}$$

In (3.2), integrating with respect to t leads to the integral equations

$$\mathbf{v}(x,t) = \mathbf{v}_0(x) + \int_0^t \int K(x-y)\Big[\mathbf{v}(y,t) + \mathbf{G}\big(\mathbf{v}(y,t)\big)\Big]\,dydt. \tag{3.4}$$

Define an operator $\mathcal{K}$ as

$$\mathcal{K}f(x) = K * f(x) = \int K(x-y)f(y)\,dy, \tag{3.5}$$

then (3.4) can be written as,

$$\mathbf{v}(x,t) = \mathbf{v}_0(x) + \int_0^t \mathcal{K}\Big[\mathbf{v} + \mathbf{G}\big(\mathbf{v}\big)\Big](x,t)\,dt. \tag{3.6}$$

Or, what is the same,

$$\begin{cases} u(x,t) = u_0(x) + \displaystyle\int_0^t \mathcal{K}\big(u + Au^2 + Buv + Cv^2\big)(x,t)\,dt \\ v(x,t) = v_0(x) + \displaystyle\int_0^t \mathcal{K}\big(v + Du^2 + Euv + Fv^2\big)(x,t)\,dt. \end{cases} \tag{3.7}$$

Write the integral expression in terms of operator form,

$$\mathbf{v}(x,t) = \mathcal{A}\mathbf{v}(x,t) := \mathbf{v}_0(x) + \int_0^t \mathcal{K}\big[\mathbf{v} + \mathbf{G}(\mathbf{v})\big](x,t)dt, \tag{3.8}$$

a solution to system (3.7) becomes a fixed-point of the operator $\mathcal{A}$. Hence, it is sufficient to show that $\mathcal{A}$ is a contraction mapping on a complete metric space.

Proposition 3.1. *The operator $\mathcal{K}$ maps H^s to H^{s+1} continuously. However, for $f, g \in H^s$, if $0 \le s \le \frac{1}{4}$, then $\mathcal{K}(fg) \in H^{s+\frac{1}{2}-\epsilon} \cap C_b$ for any $\epsilon > 0$; if $\frac{1}{4} < s \le \frac{1}{2}$, then $\mathcal{K}(fg) \in H^1$.*

Proof. That the operator $\mathcal{K}$ maps H^s to H^{s+1} continuously is obvious since

$$\|\mathcal{K}f\|_{s+1}^2 = \int (1+\xi^2)^{s+1} \Big| \frac{-i\xi}{1+\xi^2} \widehat{f}(\xi) \Big|^2 \, d\xi \le \|f\|_s^2. \tag{3.9}$$

If $0 \le s \le \frac{1}{4}$, H^s is not an algebra, when $f, g \in H^s$, the product fg is not necessarily in H^s, so $\mathcal{K}(fg)$ is not expected to be in H^{s+1}. However, it is smoother than both f and g. Let $r < \frac{1}{2} + s$, then

$$\begin{aligned} \int (1+\xi^2)^r |\widehat{\mathcal{K}fg}(\xi)|^2 \, d\xi &= \int (1+\xi^2)^r \Big| \frac{i\xi}{1+\xi^2} \Big|^2 |\widehat{f} * \widehat{g}(\xi)|^2 \, d\xi. \\ &= \int (1+\xi^2)^{r-2} |\xi|^{2-2s} |\xi|^{2s} \, |\widehat{f} * \widehat{g}(\xi)|^2 \, d\xi. \end{aligned}$$

Since $|\xi|^{2s} \le |\xi - \eta|^{2s} + |\eta|^{2s}$,

$$\begin{aligned} |\xi|^{2s} \, |\widehat{f} * \widehat{g}(\xi)|^2 &= |\xi|^{2s} \int |\widehat{f}(\xi-\eta)\widehat{g}(\eta)|^2 \, d\eta \\ &\le \int |\xi-\eta|^{2s} |\widehat{f}(\xi-\eta)\widehat{g}(\eta)|^2 \, d\eta + \int |\eta|^{2s} |\widehat{f}(\xi-\eta)\widehat{g}(\eta)|^2 \, d\eta \\ &\le \|f\|_s^2 \|g\|^2 + \|f\|^2 \|g\|_s^2. \end{aligned}$$

It follows that

$$\int (1+\xi^2)^r |\widehat{\mathcal{K}fg}(\xi)|^2 \, d\xi \le \int (1+\xi^2)^{r-2} |\xi|^{2-2s} \, d\xi \, \Big(\|f\|_s^2 \|g\|^2 + \|f\|^2 \|g\|_s^2 \Big).$$

The fact that $r < \frac{1}{2} + s$ implies that the integral $c = \int (1+\xi^2)^{r-2} |\xi|^{2-2s} \, d\xi < \infty$, that is to say, $\mathcal{K}(fg) \in H^{\frac{1}{2}+s-\epsilon} \subset H^s$ for any $\epsilon > 0$ with

$$\|\mathcal{K}(fg)\|_{\frac{1}{2}+s-\epsilon} \le \Big(\int (1+\xi^2)^{s-\frac{3}{2}-\epsilon} |\xi|^{2-2s} \, d\xi \Big)^{\frac{1}{2}} \Big(\|f\|_s \|g\| + \|f\| \|g\|_s \Big). \tag{3.10}$$

By Young's inequality,

$$|\mathcal{K}(fg)|_\infty = |K * (fg)|_\infty \le |K|_\infty |fg|_1 \le \|f\| \|g\|, \tag{3.11}$$

it implies that $\mathcal{K}(fg) \in C_b$. Hence, the second part of the proposition is established. It remains to show the last part.

If $\frac{1}{4} < s < 1$, $f, g \in H^s$ implies $fg \in L_2$ due to H^s is embedded in L_4 continuously, hence $\mathcal{K}(fg) \in H^1$ with

$$\|\mathcal{K}(fg)\|_1 \le \|fg\| \le \kappa \|f\|_s \|g\|_s \tag{3.12}$$

where κ is an embedding constant from H^s to L_4.

The proof is complete.

□

Theorem 3.1. *(Local well posedness in $L_2 \times L_2$) If $\mathbf{v_0} \in L_2 \times L_2$, then there is a positive number $T = T(\|\mathbf{v_0}\|)$ such that the operator $\mathcal{A}$ has a fixed point, $\mathbf{v}$ say, in $C(0,T;L_2) \times C(0;T;L_2)$. Moreover,*

$$\mathbf{w} = \mathbf{v} - \mathbf{v}_0 \tag{3.13}$$

lies in $C(0,T;H^r \cap C_b) \cap C(0,T;H^r \times C_b)$ for any $r < \frac{1}{2}$ and

$$\mathbf{z} = \mathbf{v} - \mathbf{v}_0 - tK * \Big(\mathbf{v}_0 + \mathbf{G}(\mathbf{v}_0)\Big) \tag{3.14}$$

lies in $C(0,T;H^1) \times C(0,T;H^1)$.

Proof. From the last proposition regarding properties of the operator $\mathcal{K}$, if $\mathbf{v} \in L_2 \times L_2$, then $\mathcal{K}\mathbf{v} \in H^1 \times H^1$ and $\mathcal{K}(\mathbf{G}(\mathbf{v})) \in H^{\frac{1}{2}-\epsilon} \times H^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$ with $\|\mathcal{K}(\mathbf{G}(\mathbf{v}))\|_{\frac{1}{2}-\epsilon} \le c\|\mathbf{v}\|$ for some constant c. Hence, the operator $\mathcal{A}$ maps $C(0,T;L_2) \times C(0,T;L_2)$ to itself for any $T > 0$. Choose

$$T = \frac{1}{2(1 + \frac{1}{4}\max\{|A|, \frac{1}{2}|B|, \frac{1}{2}|C|, \frac{1}{2}|D|, \frac{1}{2}|E|, |F|\})\,\|\mathbf{v_0}\|}, \tag{3.15}$$

an elementary calculation shows that $\mathcal{A}$ is a contraction mapping on the complete metric space

$$X_T = \{(u,v) : u, v \in C(0,T;L_2), \max_{0\le t\le T} \|(u(\cdot,t), v(\cdot,t))\| \le 2\|\mathbf{v_0}\|\}.$$

The same is to say that integral equation (3.4), (3.6), (3.7) or (3.8) has a unique solution $\mathbf{v}$ in X_T. The first part of the theorem is established.

To show the second part, reorganize equation (3.8) as follows,

$$\mathbf{v}(x,t) - \mathbf{v}_0(x) = \int_0^t \mathcal{K}\Big[\mathbf{v} + \mathbf{G}\big(\mathbf{v}\big)\Big](x,t)\,dt.$$

Since $\mathbf{v} \in C(0,T;L_2) \times C(0,T;L_2)$, and the two components of $\mathbf{G}(\mathbf{v})$ are quadratic form of $\mathbf{v}$, the proposition 3.1 shows that

$$\mathcal{K}\Big[\mathbf{v} + \mathbf{G}\big(\mathbf{v}\big)\Big] \in C(0,T;H^r \cap C_b) \times C(0,T;H^r \cap C_b)$$

for any $r < \frac{1}{2}$. Hence, $\mathbf{w} = \mathbf{v} - \mathbf{v}_0$ lies in $C(0,T;H^r \cap C_b) \times C(0,T;H^r \cap C_b)$. Estimate (3.13) is established.

Substitute $\mathbf{v}(x,t)$ with $\mathbf{w}(x,t) + \mathbf{v}_0(x)$ in (3.6), it obtains

$$\begin{aligned}\mathbf{w}(x,t) =& tK * \Big(\mathbf{v}_0(x) + \mathbf{G}(\mathbf{v}_0(x))\Big) \\ &+ \int_0^t K * \Big[\mathbf{w} + \begin{pmatrix} P_u(u_0,v_0) & P_v(u_0,v_0) \\ Q_u(u_0,v_0) & Q_v(u_0,v_0)\end{pmatrix} \mathbf{w} + \mathbf{G}\big(\mathbf{w}\big)\Big](x,t)\,dt.\end{aligned} \tag{3.16}$$

Since $P_u(u_0,v_0)$, $P_v(u_0,v_0)$, $Q_u(u_0,v_0)$ and $Q_v(u_0,v_0)$ are linear combinations of u_0 and v_0, the fact that $\mathbf{w}(\cdot,t) \in H^r \times H^r \cap C_b \times C_b$ implies that the terms in the square bracket belongs to $L_2 \times L_2$, hence the integrant is a member of $C(0,T;H^1) \times C(0,T;H^1)$. Namely the expression in (3.14) lies in $C(0,T;H^1) \times C(0,T;H^1)$.

The theorem is established. □

Theorem 3.2. *(Local well posedness in $H^s \times H^s$ for any $s \geq 0$.) For any given $s \geq 0$, if the initial condition $\mathbf{v_0} = (u_0, v_0) \in H^s \times H^s$, the solution $\mathbf{v} = (u, v)$ obtained in Theorem 3.1 lies in $C(0,T;H^s) \times C(0,T;H^s)$ where T only depends on $\|\mathbf{v_0}\|$.*

Proof. We commence with the result of Theorem 3.1 that system (1.3) has a unique distributional solution $\mathbf{v} \in C(0,T;L_2) \times C(0,T;L_2)$ where T is given in (3.15) and

$$\mathbf{v} - \mathbf{v}_0 - tK * \Big(\mathbf{v}_0 + \mathbf{G}(\mathbf{v}_0)\Big) \in C(0,T;H^1) \times C(0,T;H^1).$$

What is same, in terms of its two components,

$$u - u_0 - t\mathcal{K}\Big(u_0 + P(u_0, v_0)\Big) \in C(0,T;H^1),$$

$$v - v_0 - t\mathcal{K}\Big(v_0 + Q(u_0, v_0)\Big) \in C(0,T;H^1).$$

When $0 \leq s \leq \frac{1}{4}$, by Proposition 3.1,

$$\mathcal{K}\Big(u_0 + P(u_0, v_0)\Big), \quad \mathcal{K}\Big(v_0 + Q(u_0, v_0)\Big) \in C(0,T;H^{s+\frac{1}{2}-\epsilon})$$

for any small $\epsilon > 0$. It follows immediately that

$$u - u_0 = \Big\{u - u_0 - t\mathcal{K}\big(u_0 + P(u_0, v_0)\big)\Big\} + t\mathcal{K}\big(u_0 + P(u_0, v_0)\big)$$

and

$$v - v_0 = \Big\{v - v_0 - t\mathcal{K}\big(v_0 + Q(u_0, v_0)\big)\Big\} + t\mathcal{K}\big(u_0 + P(u_0, v_0)\big)$$

belong to $C(0,T;H^{s+\frac{1}{2}-\epsilon})$. Therefore,

$$\mathbf{v} = (\mathbf{v} - \mathbf{v}_0) + \mathbf{v_0} \in C(0,T;H^s) \times C(0,T;H^s).$$

When $\frac{1}{4} < s < 1$, $\mathbf{G}(\mathbf{v}_0) \in L_2 \times L_2$ due to $H^s \subset L_4$ continuously. The smoothing property of $\mathcal{K}$, stated in Proposition 3.1, asserts that $\mathcal{K}\Big(\mathbf{v}_0 + \mathbf{G}(\mathbf{v}_0)\Big) \in H^1 \times H^1$. Hence $\mathbf{v} - \mathbf{v}_0 \in C(0,T;H^1 \times H^1)$, therefore,

$$\mathbf{v} = (\mathbf{v} - \mathbf{v}_0) + \mathbf{v}_0 \in C(0,T;H^s) \times C(0,T;H^s).$$

If $s \geq 1$, then $\mathbf{v} = (\mathbf{v} - \mathbf{v}_0) + \mathbf{v}_0 \in H^1 \times H^1$. So

$$\mathbf{v} - \mathbf{v}_0 = \int_0^t \mathcal{K}\Big(\mathbf{v} + \mathbf{G}(\mathbf{v})\Big)\, dt$$

lies in $C(0,T;H^2) \times C(0,T;H^2)$. Inductively,

$$\mathbf{v} - \mathbf{v}_0 = \int_0^t \mathcal{K}\Big(\mathbf{v} + \mathbf{G}(\mathbf{v})\Big)\, dt \in C(0,T;H^{\lfloor s \rfloor + 1}) \times C(0,T;H^{\lfloor s \rfloor + 1}).$$

Therefore,

$$\mathbf{v} = \mathbf{v}_0 + \int_0^t \mathcal{K}\Big(\mathbf{v} + \mathbf{G}(\mathbf{v})\Big)\, dt \in C(0,T;H^s) \times C(0,T;H^s).$$

The theorem is asserted. $\square$

4. Global well-posedness under $4ac - b^2 \geq 0$.

Throughout this section, it is assumed that (1.6) has a non-trivial solution (a, b, c) such that $4ac - b^2 \geq 0$.

4.1. Invariants

System (1.3) is locally well posed for all constant coefficients $A, \cdots, F$. To pass to a global theory, the technique here demands a priori information on growth bound in time of solutions in relevant spatial norms. Following steps in Bona etc [6], we look for sufficient conditions on $A, B, \cdots, F$ which provide helpful information in order to build global well-posedness.

In Bona etc [6], it is shown that (1.4) alway has a Hamiltonian structure, the similar calculation shows it is true for (1.3). Here are two important invariants

$$\Omega(u, v) = \int_{-\infty}^{\infty} (au^2 + buv + cv^2 + au_x^2 + bu_xv_x + cv_x^2)\, dx \tag{4.1}$$

and

$$\Theta(u, v) = \int_{-\infty}^{\infty} \Big(au_x^2 + bu_xv_x + cv_x^2 - R(u, v) \Big)\, dx, \tag{4.2}$$

where

$$R(u, v) = \frac{\alpha}{3}u^3 + \frac{1}{2}\beta u^2 v + \frac{1}{2}\gamma uv^2 + \frac{\delta}{3}v^3, \tag{4.3}$$

(a, b, c) is a non-trivial solution of (1.6) and, α, β, γ, δ are given by

$$\alpha = 2aA + bD,\ \beta = bA + 2cD,\ \gamma = 2aC + bF,\ \delta = bC + 2cF. \tag{4.4}$$

β and γ can be also represented in terms of B and E as

$$\beta = aB + \frac{1}{2}bE, \quad \gamma = \frac{1}{2}bB + cE. \tag{4.5}$$

Notice that,

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \mathbf{G}(\mathbf{v}) = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \begin{pmatrix} P(u, v) \\ Q(u, v) \end{pmatrix} = \nabla R(u, v). \tag{4.6}$$

4.2. Global well-posedness under $4ac - b^2 > 0$.

Theorem 4.1. *If system of algebraic equations* (1.6) *has a non zero solution* (a, b, c) *such that* $4ac - b^2 > 0$ *and the initial condition* $\mathbf{v_0} \in H^s \times H^s$ *where* $s \geq 1$, *then the distributional solution* $\mathbf{v} = (u, v)$ *in Theorem 3.2 has the following growth bound in time* t*:*

$$\|\mathbf{v}(\cdot, t)\|_{s\times s} \leq c_s(1 + t)^{\frac{2}{3}(s-1)+\frac{1}{3}(s-\lfloor s\rfloor)}, \tag{4.7}$$

where c_s *is a constant which depends on* $\|u_0\|_s$ *and* $\|v_0\|_s$. *Therefore,* (1.3) *is well posed globally in time,* $\mathbf{v}(\cdot, t) \in C(0, \infty; H^s \times H^s)$.

Proof. We begin with $s \geq 1$ being an integer. Without loss of generality, demand $a, c > 0$, hence the matrix

$$\mathcal{N} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \tag{4.8}$$

is positive definite. Multiply (3.1) by $\mathcal{N}$, it leads to that

$$\partial_t (I - \partial_{xx}) \mathcal{N} \mathbf{v} + \mathcal{N} \mathbf{v}_x + \partial_x \nabla R(u, v) = \mathbf{0}, \tag{4.9}$$

where R is defined in (4.3) and ∇ is gradient of the cubic polynomial R with resepect to u and v.

Starting with estimating $H^1 \times H^1$ norm of $\mathbf{v}$ by taking $L_2 \times L_2$ inner product of (4.9) with $\mathbf{v}$ as follows,

$$\langle \partial_t (I - \partial_{xx}) \mathcal{N} \mathbf{v}, \mathbf{v} \rangle + \langle \mathcal{N} \mathbf{v}_x, \mathbf{v} \rangle + \langle \partial_x \nabla R(u, v), \mathbf{v} \rangle = 0,$$

apparently, the last two inner products are equal to zero, that is to say,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\mathcal{N}^{\frac{1}{2}} \mathbf{v}\|^2 + \|\mathcal{N}^{\frac{1}{2}} \mathbf{v}_x\|^2) \\ =&\frac{d}{dt} \int (au^2 + buv + cv^2 + au_x^2 + bu_x v_x + cv_x^2)\, dx \\ =&0. \end{aligned} \tag{4.10}$$

It readily follows that

$$\|\mathcal{N}^{\frac{1}{2}} \mathbf{v}\|^2 + \|\mathcal{N}^{\frac{1}{2}} \mathbf{v}_x\|^2 = \|\mathcal{N}^{\frac{1}{2}} \mathbf{v}_0\|^2 + \|\mathcal{N}^{\frac{1}{2}} \mathbf{v_0}'\|^2. \tag{4.11}$$

The strict positive definite property of $\mathcal{N}$ implies

$$\|\mathbf{v}(\cdot, t)\|_1 = \|\mathcal{N}^{-\frac{1}{2}} \mathcal{N}^{\frac{1}{2}} \mathbf{v}\|_1 \leq \lambda^* \|\mathcal{N}^{\frac{1}{2}} \mathbf{v}\|_1 = \lambda^* \|\mathcal{N}^{\frac{1}{2}} \mathbf{v}_0\|_1, \tag{4.12}$$

and

$$\|\mathbf{v}(\cdot, t)\|_1 = \|\mathcal{N}^{-\frac{1}{2}} \mathcal{N}^{\frac{1}{2}} \mathbf{v}\|_1 \geq \lambda_* \|\mathcal{N}^{\frac{1}{2}} \mathbf{v}\|_1 = \lambda_* \|\mathcal{N}^{\frac{1}{2}} \mathbf{v}_0\|_1, \tag{4.13}$$

where $\lambda_*, \lambda^* > 0$ depend only on a, b and c. That is to say, the $H^1 \times H^1$-norm of the solution $\mathbf{v} = (u, v)$ is equivalent to that of $\mathcal{N}^{\frac{1}{2}} \mathbf{v}$, hence uniformly bounded. Estimate (4.7) is true for $s = 1$.

Take $L_2 \times L_2$ inner product of (4.9) with $-\mathbf{v_{xx}}$,

$$\langle \partial_t (I - \partial_{xx}) \mathcal{N} \mathbf{v}, -\mathbf{v_{xx}} \rangle + \langle \mathcal{N} \mathbf{v}_x, -\mathbf{v_{xx}} \rangle + \langle \partial_x \nabla R(u, v), -\mathbf{v_{xx}} \rangle = 0.$$

A simple calculation shows $< \mathcal{N} \mathbf{v}_x, -\mathbf{v_{xx}} >= 0$, whence,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\mathcal{N}^{\frac{1}{2}} \mathbf{v}_x\|^2 + \|\mathcal{N}^{\frac{1}{2}} \mathbf{v}_{xx}\|^2) \\ =&\frac{d}{dt} \int (au_x^2 + bu_x v_x + cv_x^2 + au_{xx}^2 + bu_{xx} v_{xx} + cv_{xx}^2)\, dx \\ =&\int \Big(u_{xx} \partial_x R_u(u, v) + v_{xx} \partial_x R_v(u, v) \Big)\, dx \\ =&\int \Big(u_{xx} (\alpha u^2 + \beta uv + \frac{1}{2} \gamma v^2)_x + v_{xx} (\frac{1}{2} \beta u^2 + \gamma uv + \delta v^2)_x \Big)\, dx. \end{aligned}$$

Following idea in Chen [9], integrations by parts a few times, upon simplification, it yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\mathcal{N}^{\frac{1}{2}}\mathbf{v}_x\|^2+\|\mathcal{N}^{\frac{1}{2}}\mathbf{v}_{xx}\|^2)\\
&=-\int\Big(\alpha u_x^3+\frac{3}{2}\beta u_x^2 v_x+\frac{3}{2}\gamma u_x v_x^2+\delta v_x^3\Big)\,dx\\
&=-3\int R(u_x,v_x)\,dx\\
&\le \max\Big\{|\alpha|,\frac{3}{2}|\beta|,\frac{3}{2}|\gamma|,|\delta|\Big\}(|u_x|_\infty+|v_x|_\infty)(\|u_x\|^2+\|v_x\|^2)\\
&\le \max\Big\{|\alpha|,\frac{3}{2}|\beta|,\frac{3}{2}|\gamma|,|\delta|\Big\}(\|u_x\|^2+\|v_x\|^2)^{\frac{3}{2}}(\|u_{xx}\|^{\frac{1}{2}}+\|v_{xx}\|^{\frac{1}{2}}).
\end{aligned}$$

This together with (4.12) and (4.13) implies that

$$\frac{d}{dt}(\|\mathcal{N}^{\frac{1}{2}}\mathbf{v}_x\|^2+\|\mathcal{N}^{\frac{1}{2}}\mathbf{v}_{xx}\|^2)\le c_1\,(\|\mathcal{N}^{\frac{1}{2}}\mathbf{v}_x\|^2+\|\mathcal{N}^{\frac{1}{2}}\mathbf{v}_{xx}\|^2)^{\frac{1}{4}},$$

where c_1 is a constant only dependent on $\|\mathbf{v}_0\|_1$. Solving this inequality yields,

$$(\|\mathcal{N}^{\frac{1}{2}}\mathbf{v}_x\|^2+\|\mathcal{N}^{\frac{1}{2}}\mathbf{v}_{xx}\|^2)^{\frac{1}{2}}\le\Big(\|\mathcal{N}^{\frac{1}{2}}\mathbf{v}_0'\|_1^{\frac{3}{2}}+c_1 t\Big)^{\frac{2}{3}}.$$

This with (4.12) and (4.13) indicates

$$\|\mathbf{v}(\cdot,t)\|_2\le c_2(1+t)^{\frac{2}{3}}. \tag{4.14}$$

Suppose that the estimate (4.7) is true for $j=3,4,\cdots,s-1$, i.e.

$$\|\mathbf{v}(\cdot,t)\|_j\le c_j(1+t)^{\frac{2}{3}(j-1)} \tag{4.15}$$

where constant c_j only depends on $\|\mathbf{v}_0\|_j$. Then, take $L_2\times L_2$ inner product of (4.9) with $(-1)^{s-1}\partial_x^{2(s-1)}\mathbf{v}(x,t)$, it obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\mathcal{N}^{\frac{1}{2}}\partial_x^{s-1}\mathbf{v}\|^2+\|\mathcal{N}^{\frac{1}{2}}\partial_x^{s}\mathbf{v}\|^2)\\
=&\frac{d}{dt}\int\Big(a(\partial_x^{s-1}u)^2+b\partial_x^{s-1}u\partial_x^{s-1}v+c(\partial_x^{s-1}v)^2+a(\partial_x^s u)^2+b\partial_x^s u\partial_x^s v+c(\partial_x^s v)^2\Big)\,dx\\
=&(-1)^s\int\Big(\partial_x^{2(s-1)}u\partial_x R_u(u,v)+\partial_x^{s-1}vR_v(u,v)\Big)\,dx\\
=&(-1)^s\int\Big(\alpha(u^2)_x\partial_x^{2(s-1)}u+\beta(uv)_x\partial_x^{2(s-1)}u+\frac{1}{2}\beta(u^2)_x\partial_x^{2(s-1)}v\\
&\qquad+\frac{1}{2}\gamma(v^2)_x\partial_x^{2(s-1)}u+\gamma(uv)_x\partial_x^{2(s-1)}v+\delta(v^2)_x\partial_x^{2(s-1)}v\Big)\,dx.
\end{aligned}$$

Integrations by parts $s-1$ times, it transpires that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\mathcal{N}^{\frac{1}{2}}\partial_x^{s-1}\mathbf{v}\|^2+\|\mathcal{N}^{\frac{1}{2}}\partial_x^{s}\mathbf{v}\|^2)\\
&=-\alpha\int\partial_x^s(u^2)\partial_x^{s-1}u\,dx-\frac{1}{2}\beta\int\Big(2\partial_x^s(uv)_x\partial_x^{s-1}u+\partial_x^s(u^2)\partial_x^{s-1}v\Big)\,dx\\
&\qquad-\frac{1}{2}\gamma\int\Big(\partial_x^s(v^2)\partial_x^{s-1}u+2\partial_x^s(uv)\partial_x^{s-1}v\Big)\,dx-\delta\int\partial_x^s(v^2)\partial_x^{s-1}v\,dx.
\end{aligned} \tag{4.16}$$

To analyze the α-term, apply the Leibnitz formula, it follows

$$\begin{aligned}&\int \partial_x^s(u^2)\partial_x^{s-1}u\,dx\\ =&\int 2u\partial_x^s u\partial_x^{s-1}u\,dx+\sum_{j=1}^{s-1}\binom{s}{j}\int \partial_x^j u\partial_x^{s-j}u\partial_x^{s-1}u\,dx\\ =&-\int u_x(\partial_x^{s-1}u)^2\,dx+\sum_{j=2}^{s-2}\binom{s}{j}\int \partial_x^j u\partial_x^{s-j}u\partial_x^{s-1}u\,dx+2s\int u_x(\partial_x^{s-1}u)^2\,dx\\ =&(2s-1)\int u_x(\partial_x^{s-1}u)^2\,dx+\sum_{j=2}^{s-2}\binom{s}{j}\int \partial_x^j u\partial_x^{s-j}u\partial_x^{s-1}u\,dx.\end{aligned}$$

We have

$$\begin{aligned}&|\int \partial_x^s(u^2)\partial_x^{s-1}u\,dx|\\ \le&(2s-1)|u_x(\cdot,t)|_\infty\|\partial_x^{s-1}u(\cdot,t)\|^2+\Sigma_{j=2}^{s-2}\binom{s}{j}|\partial_x^j u(\cdot,t)|_\infty\|\partial_x^{s-j}u(\cdot,t)\|\|\partial_x^{s-1}u(\cdot,t)\|\\ \le&(2s-1)\|u_x\|^{\frac12}\|u_{xx}\|^{\frac12}\|\partial_x^{s-1}u\|^2\\ &+\Sigma_{j=2}^{s-2}\binom{s}{j}\|\partial_x^j u\|^{\frac12}\|\partial_x^{j+1}u\|^{\frac12}\|\partial_x^{s-j}u(\cdot,t)\|\|\partial_x^{s-1}u(\cdot,t)\|.\end{aligned}$$

By inductive assumption (4.15), the first term in the last two lines is bounded by a constant multiplying $(1+t)^{\frac13}(1+t)^{\frac43(s-2)}=(1+t)^{\frac43 s-\frac73}$, and for $j=2,\cdots,s-2$, j-terms are bounded by a constant times $(1+t)^{\frac13(j-1)}(1+t)^{\frac13 j}(1+t)^{\frac23(s-1-j)}(1+t)^{\frac23(s-2)}=(1+t)^{\frac43 s-\frac73}$ as well, whence,

$$|\int \partial_x^s(u^2)\partial_x^{s-1}u\,dx|\le c(1+t)^{\frac43 s-\frac73}, \tag{4.17}$$

where c is constants independent of t.

For the β-term in (4.16),

$$\begin{aligned}&\int\Big(2\partial_x^s(uv)_x\partial_x^{s-1}u+\partial_x^s(u^2)\partial_x^{s-1}v\Big)\,dx\\ =&\sum_{j=0}^{s}\binom{s}{j}\int\Big(2\partial_x^j u\partial_x^{s-j}v\partial_x^{s-1}u+\partial_x^j u\partial_x^{s-j}u\partial_x^{s-1}v\Big)\,dx\\ =&\int\Big(2u\partial_x^s v\partial_x^{s-1}u+2v\partial_x^s u\partial_x^{s-1}u+2u\partial_x^s u\partial_x^{s-1}v\Big)\,dx\\ &+\sum_{j=1}^{s-1}\binom{s}{j}\int\Big(2\partial_x^j u\partial_x^{s-j}v\partial_x^{s-1}u+\partial_x^j u\partial_x^{s-j}u\partial_x^{s-1}v\Big)\,dx\\ =&-\int 2u_x\partial_x^{s-1}u\partial_x^{s-1}v\,dx+v_x(\partial_x^{s-1}u)^2\Big)\,dx,\\ &+\sum_{j=1}^{s-1}\binom{s}{j}\int\Big(2\partial_x^j u\partial_x^{s-j}v\partial_x^{s-1}u+\partial_x^j u\partial_x^{s-j}u\partial_x^{s-1}v\Big).\end{aligned}$$

Hence,

$$
\begin{aligned}
&|\int \Big(2\partial_x^s(uv)_x\partial_x^{s-1}u+\partial_x^s(u^2)\partial_x^{s-1}v\Big)\,dx|\\
\le&2|u_x|_\infty\|\partial_x^{s-1}u\|\|\partial_x^{s-1}v\|+|v_x|_\infty\|\partial_x^{s-1}u\|^2\\
&+\sum_{j=1}^{s-1}\binom{s}{j}\Big(2|\partial_x^j u|_\infty\|\partial_x^{s-j}v\|\|\partial_x^{s-1}u\|+|\partial_x^j u|_\infty\|\partial_x^{s-j}u\|\|\partial_x^{s-1}v\|\Big)\\
\le&2\|u_x\|^{\frac12}\|u_{xx}\|^{\frac12}\|\partial_x^{s-1}u\|\|\partial_x^{s-1}v\|+\|v_x\|^{\frac12}\|v_{xx}\|^{\frac12}\partial_x^{s-1}u\|^2\\
&+\sum_{j=1}^{s-1}\binom{s}{j}\Big(2\|\partial_x^j u\|^{\frac12}\|\partial_x^{j+1}u\|^{\frac12}\|\partial_x^{s-j}v\|\|\partial_x^{s-1}u\|+\|\partial_x^j u\|^{\frac12}\|\partial_x^{j+1}u\|^{\frac12}\|\partial_x^{s-j}u\|\|\partial_x^{s-1}v\|\Big).
\end{aligned}
$$

Again, by inductive assumption (4.15),

$$
|\int \Big(2\partial_x^s(uv)_x\partial_x^{s-1}u+\partial_x^s(u^2)\partial_x^{s-1}v\Big)\,dx|\le c(1+t)^{\frac43 s-\frac73}, \tag{4.18}
$$

where c is constants independent of t.

Similarly, for γ,δ-terms,

$$
\begin{aligned}
&|\int \Big(\partial_x^s(v^2)\partial_x^{s-1}u+2\partial_x^s(uv)\partial_x^{s-1}v\Big)\,dx|\\
\le&2\|v_x\|^{\frac12}\|v_{xx}\|^{\frac12}\|\partial_x^{s-1}v\|\|\partial_x^{s-1}u\|+\|u_x\|^{\frac12}\|u_{xx}\|^{\frac12}\|\partial_x^{s-1}v\|^2\\
&+\sum_{j=1}^{k}\binom{s}{j}\Big(2\|\partial_x^j v\|^{\frac12}\|\partial_x^{j+1}v\|^{\frac12}\|\partial_x^{s-j}u\|\|\partial_x^{s-1}v\|+\|\partial_x^j v\|^{\frac12}\|\partial_x^{j+1}v\|^{\frac12}\|\partial_x^{s-j}v\|\|\partial_x^{s-1}u\|\Big)\\
\le&c(1+t)^{\frac43 s-\frac73},
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
&|\int \partial_x^s(v^2)_x\partial_x^{s-1}v\,dx|\\
\le&(2s-1)\|v_x\|^{\frac12}\|v_{xx}\|^{\frac12}\|\partial_x^{s-1}v\|^2,\\
&+\Sigma_{j=2}^{s-2}\binom{s}{j}\|\partial_x^j v\|^{\frac12}\|\partial_x^{j+1}v\|^{\frac12}\|\partial_x^{s-j}v(\cdot,t)\|\|\partial_x^{s-1}v(\cdot,t)\|\\
\le&c(1+t)^{\frac43 s-\frac73}.
\end{aligned}
\tag{4.20}
$$

Combination of (4.16), (4.17) through (4.20) leads to

$$
\begin{aligned}
&\frac12\frac{d}{dt}(\|\mathcal{N}^{\frac12}\partial_x^{s-1}\mathbf{v}\|^2+\|\mathcal{N}^{\frac12}\partial_x^{s}\mathbf{v}\|^2)\\
\le&c_{s-1}(1+t)^{\frac13(j-1)+\frac13 j+\frac23(s-1-j)+\frac23(s-2)}\\
=&c_{s-1}(1+t)^{\frac43 s-\frac73}
\end{aligned}
\tag{4.21}
$$

where c_{s-1} is a constant only dependent on $\|\mathbf{v}_0\|_{s-1}$. Integrate with respect to t, it gives,

$$
\|\mathcal{N}^{\frac12}\partial_x^{s-1}\mathbf{v}\|^2+\|\mathcal{N}^{\frac12}\partial_x^{s}\mathbf{v}\|^2\le\|\mathcal{N}^{\frac12}\partial_x^{s-1}\mathbf{v}_0\|^2+\|\mathcal{N}^{\frac12}\partial_x^{s}\mathbf{v}_0\|^2+c_{s-1}(1+t)^{\frac43(s-1)}. \tag{4.22}
$$

This together with (4.11) shows that

$$\|\mathbf{v}(\cdot,t)\|_s \le c_s(1+t)^{\frac{2}{3}(s-1)}, \tag{4.23}$$

where c_s is a constant only dependent on $\|\mathbf{v}_0\|_s$. The theorem has been proved for s being a positive integer.

Consideration remains to the case that $s > 1$ is non-integer. Denote $n = \lfloor s \rfloor$ and $\sigma = s - n$, so that $n \ge 1$ is an integer and $0 < \sigma < 1$. Reorganize in (1.3) as follows,

$$\begin{cases} u_{xxt} = u_t + u_x + (Au^2 + Buv + Cv^2)_x, \\ v_{xxt} = v_t + v_x + (Du^2 + Euv + Fv^2)_x, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases}$$

integrate with respect to t, it obtains the following two relations

$$\partial_{xx}\big(u(x,t) - u_0(x)\big) = u(x,t) - u_0(x) + \int_0^t (u + Au^2 + Buv + Cv^2)_x\, dt,$$

$$\partial_{xx}\big(v(x,t) - v_0(x)\big) = v(x,t) - v_0(x) + \int_0^t (v + Du^2 + Euv + Fv^2)_x\, dt.$$

It is a simple observation that $u - u_0, v - v_0$ are one order smoother than u, v in spatial variable x. As we just showed that $u, v \in C(0,\infty; H^n)$, and $u - u_0$, $v - v_0 \in C(0,\infty; H^{n+1})$, take derivative with respect to x $n-1$ times, it yields following forms

$$\partial_x^{n+1}\big(u(x,t) - u_0(x)\big) = \partial_x^{n-1}\big(u(x,t) - u_0(x)\big) + \int_0^t \partial_x^n (u + Au^2 + Buv + Cv^2)\, dt,$$

$$\partial_x^{n+1}\big(v(x,t) - v_0(x)\big) = \partial_x^{n-1}\big(v(x,t) - v_0(x)\big) + \int_0^t \partial_x^n (v + Du^2 + Euv + Fv^2)\, dt.$$

We immediately have estimates below

$$\begin{aligned} \|\partial_x^{n+1}\big(u(\cdot,t) - u_0(\cdot)\big)\| &\le \|\partial_x^{n-1}\big(u(\cdot,t) - u_0(\cdot)\big)\| \\ &\quad + \int_0^t \|\partial_x^n (u + Au^2 + Buv + Cv^2)\|\, dt, \\ \|\partial_x^{n+1}\big(v(\cdot,t) - v_0(\cdot)\big)\| &\le \|\partial_x^{n-1}\big(v(\cdot,t) - v_0(\cdot)\big)\| \\ &\quad + \int_0^t \|\partial_x^n (v + Du^2 + Euv + Fv^2)\|\, dt. \end{aligned} \tag{4.24}$$

A little more detailed estimates on $\|\partial_x^n u^2\|, \|\partial_x^n (uv)\|$ and $\|\partial_x^n v^2\|$ are needed:

$$\begin{aligned} \|\partial_x^n (uv)\| =& \| \sum_{j=0}^n \binom{n}{j} \partial_x^j u \partial_x^{n-j} v\| \\ \le& \|u\partial_x^n v\| + \|v\partial_x^n\| + \sum_{j=1}^{n-1} \binom{n}{j} \|\partial_x^j u \partial_x^{n-j} v\| \\ \le& |u|_\infty \|\partial_x^n v\| + |v|_\infty \|\partial_x^n u\| + \sum_{j=1}^{n-1} \binom{n}{j} |\partial_x^j u|_\infty \|\partial_x^{n-j} v\| \\ \le& |u|_\infty \|\partial_x^n v\| + |v|_\infty \|\partial_x^n u\| + \sum_{j=1}^{n-1} \binom{n}{j} \|\partial_x^j u\|^{\frac{1}{2}} \|\partial_x^{j+1} u\|^{\frac{1}{2}} \|\partial_x^{n-j} v\|. \end{aligned}$$

Making use of (4.23), we have,

$$\|\partial_x^n(uv)\| \le c_n(1+t)^{\frac{2}{3}(n-1)} + c_n(1+t)^{\frac{2}{3}n-1} \le c_n(1+t)^{\frac{2}{3}(n-1)}.$$

Take $u = v$, we have

$$\|\partial_x^n(u^2)\| \le c_n(1+t)^{\frac{2}{3}(n-1)}$$

and

$$\|\partial_x^n(v^2)\| \le c_n(1+t)^{\frac{2}{3}(n-1)}.$$

Substitute these last three estimates into (4.24), it gets

$$\|\partial_x^{n+1}\big(u(\cdot,t) - u_0(\cdot)\big)\| \le c_n(1+t)^{\frac{2}{3}n+\frac{1}{3}}$$

and

$$\|\partial_x^{n+1}\big(v(\cdot,t) - v_0(\cdot)\big)\| \le c_n(1+t)^{\frac{2}{3}n+\frac{1}{3}}.$$

Since $\|u(\cdot,t)\|_1$ and $\|v(\cdot,t)\|_1$ are uniformly bounded for $t \ge 0$, we have

$$\|u(\cdot,t) - u_0(\cdot)\|_{n+1} \le c_n(1+t)^{\frac{2}{3}n+\frac{1}{3}}$$

and

$$\|v(\cdot,t) - v_0(\cdot)\|_{n+1} \le c_n(1+t)^{\frac{2}{3}n+\frac{1}{3}}.$$

By interpolation theorem,

$$\begin{aligned}
\|u(\cdot,t) - u_0(\cdot)\|_{n+\sigma} &\le \|u(\cdot,t) - u_0(\cdot)\|_n^{1-\sigma}\|u(\cdot,t) - u_0(\cdot)\|_{n+1}^{\sigma} \\
&\le c_n(1+t)^{\frac{2}{3}(1-\sigma)(n-1)}(1+t)^{\sigma(\frac{2}{3}n+\frac{1}{3})} \\
&= c_n(1+t)^{\frac{2}{3}(n+\sigma-1)+\frac{1}{3}\sigma} \\
&= c_n(1+t)^{\frac{2}{3}(s-1)+\frac{1}{3}(s-\lfloor s\rfloor)},
\end{aligned}$$

and similarly,

$$\|v(\cdot,t) - v_0(\cdot)\|_{n+\sigma} \le c_n(1+t)^{\frac{2}{3}(s-1)+\frac{1}{3}(s-\lfloor s\rfloor)}.$$

As $s = n+\sigma$, it follows immediately that

$$\|u(\cdot,t)\|_s \le \|u_0\|_s + \|u(\cdot,t) - u_0(\cdot)\|_s \le c_s(1+t)^{\frac{2}{3}(s-1)+\frac{1}{3}(s-\lfloor s\rfloor)}$$

and

$$\|v(\cdot,t)\|_s \le \|v_0\|_s + \|v(\cdot,t) - v_0(\cdot)\|_{n+1} \le c_s(1+t)^{\frac{2}{3}(s-1)+\frac{1}{3}(s-\lfloor s\rfloor)}.$$

These two estimates are exactly that of (4.7).

The theorem has been established. □

It remains to investigate global well-posedness for the case where $0 \le s < 1$, where regularity of data is below the energy level.

Theorem 4.2. *If system of algebraic equations* (1.6) *has a non zero solution* (a,b,c) *such that* $4ac-b^2 > 0$, *then* (1.3) *is globally well-posed in* $H^s \times H^s$ *for any* $0 \le s < 1$. *The growth bounds of relative norms of solutions in time is given below.*

$$\begin{aligned}
&\|\mathbf{v}(\cdot,t)\|_{s\times s} \le c_{s,1}e^{c_s t} \quad \text{if} \quad \frac{1}{4} < s < 1, \\
&\|\mathbf{v}(\cdot,t)\|_{s\times s} \le c_{s,1}e^{c_s t^2} \quad \text{if} \quad 0 \le s \le \frac{1}{4},
\end{aligned} \tag{4.25}$$

where $c_s, c_{s,1}$ *are constants only dependent on the corresponding Sobolev norms of the initial data* $\|\mathbf{v}_0\|_s$.

Proof. The local well-posedness is guaranteed by Theorem 3.2, and moreover, by Theroem 3.1,

$$\mathbf{w} = \mathbf{v} - \mathbf{v}_0 \in C(0,T;H^1) \times C(0,T;H^1) \quad \text{for} \quad \frac{1}{4} < s < 1,$$

and

$$\mathbf{z} = \mathbf{v} - \mathbf{v}_0 - tK * \Big(\mathbf{v}_0 + \mathbf{G}(\mathbf{v}_0)\Big) \in C(0,T;H^1) \times C(0,T;H^1) \quad \text{for} \quad 0 \le s \le \frac{1}{4},$$

where $T = T(\|\mathbf{v}_0\|)$ is only dependent on $L_2 \times L_2$-norm of the initial data. It is sufficient to establish a prior bounds (4.25).

For the case $\frac{1}{4} < s < 1$, $\mathbf{w} = \mathbf{v} - \mathbf{v}_0$ satisfies the integral equation (3.16), repeated here for readers' convenience:

$$\begin{aligned}\mathbf{w}(x,t) =& tK * \Big(\mathbf{v}_0 + \mathbf{G}(\mathbf{v}_0)\Big)(x) \\ &+ \int_0^t K * \Big[\mathbf{w} + \begin{pmatrix} P_u(u_0,v_0) & P_v(u_0,v_0) \\ Q_u(u_0,v_0) & Q_v(u_0,v_0) \end{pmatrix} \mathbf{w} + \mathbf{G}\big(\mathbf{w}\big)\Big](x,t)\,dt.\end{aligned}$$

As the matrix $\mathcal{N}$ defined in (4.8) is positive definite and remember the relation (4.6) between $\mathcal{N}$ and $\mathbf{G}$, (3.16) is equivalent to

$$\begin{aligned}\mathcal{N}\mathbf{w}(x,t) =& tK * \Big(\mathcal{N}\mathbf{v}_0 + \mathcal{N}\mathbf{G}(\mathbf{v}_0)\Big)(x) \\ &+ \int_0^t K * \Big[\mathcal{N}\mathbf{w} + \begin{pmatrix} R_{uu}(u_0,v_0) & R_{uv}(u_0,v_0) \\ R_{vu}(u_0,v_0) & R_{vv}(u_0,v_0) \end{pmatrix} \mathbf{w} + \nabla R\big(\mathbf{w}\big)\Big](x,t)\,dt,\end{aligned} \tag{4.26}$$

where R is a cubic polynomial function given in (4.3). It is same as

$$\begin{aligned}(1 - \partial_{xx})\mathcal{N}\mathbf{w}_t(x,t) = & -\partial_x\Big(\mathcal{N}\mathbf{v}_0(x) + \nabla R(\mathbf{v}_0(x))\Big) \\ & - \partial_x\Big[\mathcal{N}\mathbf{w} + \begin{pmatrix} R_{uu}(u_0,v_0) & R_{uv}(u_0,v_0) \\ R_{vu}(u_0,v_0) & R_{vv}(u_0,v_0) \end{pmatrix} \mathbf{w} + \nabla R\big(\mathbf{w}\big)\Big](x,t)\end{aligned}$$

with initial condition $\mathbf{w}(x,0) = (0,0)$. Take $L_2 \times L_2$-inner product with $\mathbf{w}$, it gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\mathcal{N}^{\frac{1}{2}}\mathbf{w}(\cdot,t)\|^2 + \|\mathcal{N}^{\frac{1}{2}}\mathbf{w}_x(\cdot,t)\|^2) \\ =& -\Big\langle \partial_x\Big(\mathcal{N}\mathbf{v}_0 + \nabla R(\mathbf{v}_0)\Big), \mathbf{w}\Big\rangle - \Big\langle \partial_x\Big[\mathcal{N}\mathbf{w} + \begin{pmatrix} R_{uu}(u_0,v_0) & R_{uv}(u_0,v_0) \\ R_{vu}(u_0,v_0) & R_{vv}(u_0,v_0) \end{pmatrix} \mathbf{w}\Big], \mathbf{w}\Big\rangle \\ &- \langle \partial_x \nabla R(\mathbf{w}), \mathbf{w}\rangle.\end{aligned}$$

Since

$$\langle \partial_x \mathcal{N}\mathbf{w}, \mathbf{w}\rangle = \langle \partial_x \nabla R(\mathbf{w}), \mathbf{w}\rangle = 0,$$

and notice that $\|\nabla R(\mathbf{v}_0)\|$ is bounded by a number times $\|\mathbf{v}_0\|_s^2$ due to that H^s is continuously embedded in L_4. It follows immediately

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathcal{N}^{\frac{1}{2}}\mathbf{w}(\cdot,t)\|_1^2\\ \le&\|\mathcal{N}\mathbf{v}_0+\nabla R(\mathbf{v}_0)\|\|\mathbf{w}_x\|+2\max\{|\alpha|,|\beta|,|\gamma|,|\delta|\}(\|u_0\|+\|v_0\|)|\mathbf{w}|_\infty\|\mathbf{w}_x\|\\ \le&\Big(\|\mathcal{N}\mathbf{v}_0\|+\|\nabla R(\mathbf{v}_0)\|\Big)\|\mathbf{w}_x\|+2\max\{|\alpha|,|\beta|,|\gamma|,|\delta|\}(\|u_0\|+\|v_0\|)|\mathbf{w}|_1^2\\ \le&c_s\big(\|\mathcal{N}^{\frac{1}{2}}\mathbf{w}(\cdot,t)\|_1+\|\mathcal{N}^{\frac{1}{2}}\mathbf{w}(\cdot,t)\|_1^2\big),\end{aligned}$$

where $c_s=c(\|\mathbf{v}_0\|_s)$ is a constant only dependent on $\|\mathbf{v}_0\|_s$. Solving the last inequality yields

$$\|\mathcal{N}^{\frac{1}{2}}\mathbf{w}(\cdot,t)\|_1\le e^{c_st}-1. \tag{4.27}$$

In consequence,

$$\|\mathbf{v}(\cdot,t)\|_s\le\|\mathbf{w}(\cdot,t)\|_1+\|\mathbf{v}_0\|_s\le\lambda^*\|\mathcal{N}^{\frac{1}{2}}\mathbf{w}(\cdot,t)\|_1+\|\mathbf{v}_0\|_s,\le c_{s,1}e^{c_st}. \tag{4.28}$$

where λ^* is given in (4.12). The first part of (4.23) is established. It remains to show the second estimate, namely $0\le s\le\frac{1}{4}$.

Denote $\mathbf{v}_1=\mathcal{K}\Big(\mathbf{v}_0+\mathbf{G}(\mathbf{v}_0)\Big)$. By Proposition 3.1, it lies in $H^{\frac{1}{2}+s-\epsilon}\times H^{\frac{1}{2}+s-\epsilon}\cap C_b\times C_b$ for any $\epsilon>0$ with

$$\begin{aligned}&\|\mathbf{v}_1\|_{\frac{1}{2}+s-\epsilon}\le\|\mathbf{v_0}\|+\tilde{c}\left(\int(1+\xi^2)^{s-\frac{3}{2}-\epsilon}|\xi|^{2-2s}\,d\xi\right)^{\frac{1}{2}}\|\mathbf{v}\|_s^2,\\ &|\mathbf{v}_1|_\infty\le\|\mathbf{v_0}\|+\frac{1}{3}\tilde{c}\,\|\mathbf{v}_0\|^2,\end{aligned} \tag{4.29}$$

where the constant $\tilde{c}=6\max\{|A|,|B|,\cdots,|F|\}$.

Write $\mathbf{z}$ defined in (3.14) as

$$\mathbf{z}=\mathbf{v}-\mathbf{v}_0-tK*\Big(\mathbf{v}_0+\mathbf{G}(\mathbf{v}_0)\Big)=\mathbf{w}-t\mathbf{v_1}.$$

System (3.16) can be rewritten in terms of variable vector $\mathbf{z}$ as

$$\begin{aligned}\mathbf{z}(x,t)=&\int_0^tK*\Big[\mathbf{z}+\mathbf{tv_1}+\begin{pmatrix}P_u(u_0,v_0)&P_v(u_0,v_0)\\Q_u(u_0,v_0)&Q_v(u_0,v_0)\end{pmatrix}(\mathbf{z}+\mathbf{tv_1})+\mathbf{G}\big(\mathbf{z}+\mathbf{tv_1}\big)\Big](x,t)\,dt\\ =&\int_0^tK*\Big[\mathbf{z}+\begin{pmatrix}P_u(\mathbf{v}_0+t\mathbf{v}_1)&P_v(\mathbf{v}_0+t\mathbf{v}_1)\\Q_u(\mathbf{v}_0+t\mathbf{v}_1)&Q_v(\mathbf{v}_0+t\mathbf{v}_1)\end{pmatrix}\mathbf{z}+\mathbf{G}(\mathbf{z})\Big](x,t)\,dt\\ &+\frac{1}{2}t^2K*\Big\{\mathbf{v}_1+\begin{pmatrix}P_u(\mathbf{v}_0)&P_v(\mathbf{v}_0)\\Q_u(\mathbf{v}_0)&Q_v(\mathbf{v}_0)\end{pmatrix}\mathbf{v_1}\Big\}(x)+\frac{1}{3}t^3K*\mathbf{G}(\mathbf{v}_1)(x).\end{aligned}$$

It is sufficient to estimate a prior bound $\|\mathbf{z}(\cdot,t)\|_1$.

Notice that its equivalent differential equations are

$$\begin{cases} \mathbf{z}_t + \mathbf{z}_x - \mathbf{z}_{xxt} + \partial_x \mathbf{G}(\mathbf{z}) + \partial_x \begin{pmatrix} P_u(\mathbf{v}_0 + t\mathbf{v}_1) & P_v(\mathbf{v}_0 + t\mathbf{v}_1) \\ Q_u(\mathbf{v}_0 + t\mathbf{v}_1) & Q_v(\mathbf{v}_0 + t\mathbf{v}_1) \end{pmatrix} \mathbf{z} \\ \quad + t\partial_x \Big\{ \mathbf{v_1} + \begin{pmatrix} P_u(\mathbf{v}_0) & P_v(\mathbf{v}_0) \\ Q_u(\mathbf{v}_0) & Q_v(\mathbf{u}_0) \end{pmatrix} \mathbf{v_1} \Big\} + \mathbf{t^2} \partial_\mathbf{x} \mathbf{G}(\mathbf{v_1}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \mathbf{z}(x,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases} \tag{4.30}$$

Multiply the system by the matrix $\mathcal{N}$, and use the relation (4.6) between $\mathcal{N}$ and $\mathbf{G}$, it follows

$$(1-\partial_{xx})\mathcal{N}\mathbf{z}_t + \partial_x \Big[\mathcal{N}\mathbf{z} + \begin{pmatrix} R_{uu}(\mathbf{v}_0 + t\mathbf{v}_1) & R_{uv}(\mathbf{v}_0 + t\mathbf{v}_1) \\ R_{vu}(\mathbf{v}_0 + t\mathbf{v}_1) & R_{vv}(\mathbf{v}_0 + t\mathbf{v}_1) \end{pmatrix} \mathbf{z} + \nabla R(\mathbf{z}) \Big]$$

$$+ t\partial_x \Big\{ \mathcal{N}\mathbf{v}_1 + \begin{pmatrix} R_{uu}(u_0, v_0) & R_{uv}(u_0, v_0) \\ R_{vu}(u_0, v_0) & R_{vv}(u_0, v_0) \end{pmatrix} \mathbf{v_1} \Big\} + t^2 \partial_x \nabla R(\mathbf{v}_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Take the $L_2 \times L_2$ inner product with $\mathbf{z}$, use the property that $< \partial_x \mathcal{N}\mathbf{z}, \mathbf{z} >= 0$ and $< \partial_x \nabla R(\mathbf{z}), \mathbf{z} >= 0$, then after integrations by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{N}^{\frac{1}{2}}\mathbf{z}\|_1^2 \le \tilde{c}_1 \|\mathbf{v}_0 + t\mathbf{v}_1\| \|\mathbf{z}\|_1^2 + \tilde{c}_2 (\|\mathbf{v_1}\| + \|\mathbf{v}_0\| |\mathbf{v}_1|_\infty) t \|\mathbf{z}\|_1 + \tilde{c}_3 (\|\mathbf{v}_1\| |\mathbf{v}_1|_\infty)) t^2 \|\mathbf{z}\|_1,$$

where $\tilde{c}_1, \tilde{c}_2$ and $\tilde{c}_3$ are constants only dependent on a, b, c and $\alpha, \cdots, \delta$. The estimate (4.29) with positive definiteness of the matrix $\mathcal{N}$ implies that

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{N}^{\frac{1}{2}}\mathbf{z}\|_1^2 \le c_1(1+t)\|\mathcal{N}^{\frac{1}{2}}\mathbf{z}\|_1^2 + c_2 t \|\mathcal{N}^{\frac{1}{2}}\mathbf{z}\|_1 + c_3 t^2 \|\mathcal{N}^{\frac{1}{2}}\mathbf{z}\|_1,$$

where c_1, c_2 and c_3 are constants only depend on $\|\mathbf{v}_0\|$. Solving the differential inequality, it follows that

$$\|\mathcal{N}^{\frac{1}{2}}\mathbf{z}\|_1 \le e^{c_1(t+\frac{1}{2}t^2)} \int_0^t (c_2\tau + c_3\tau^2) e^{-c_1(\tau+\frac{1}{2}\tau^2)}\, d\tau.$$

Let $c_4 = \int_0^\infty (c_2\tau + c_3\tau^2) e^{-c_1(\tau+\frac{1}{2}\tau^2)}\, d\tau$, then

$$\|\mathcal{N}^{\frac{1}{2}}\mathbf{z}\|_1 \le c_4 e^{c_1(t+\frac{1}{2}t^2)} \le c_4 e^{\frac{1}{2}c_1} e^{c_1 t^2}.$$

Since $\mathcal{N}^{\frac{1}{2}}$ is a positive definite matrix, $\|\mathcal{N}^{\frac{1}{2}}\mathbf{z}\|_1$ is equivalent to $\|\mathbf{z}\|_1$. Thus

$$\|\mathbf{v}(\cdot,t)\|_s \le \|\mathbf{z}(\cdot,t)\|_1 + \|\mathbf{v}_0\|_s + t\|\mathbf{v}_1\|_s \le c_{s,1} e^{c_s t^2}$$

for some constants $c_{s,1}$ and c_s which only depend on $\|\mathbf{v}_0\|_s$. The second part of (4.23) is demonstrated, and the theorem is complete. □

4.3. Global well-posedness under $4ac - b^2 = 0$

Now we turn our attention to the case $4ac - b^2 = 0$, the quadratic form $\Omega(u, v)$ is lack of strict positivity. The method in the last section fails. However, the system (1.3) can be reduced to a single BBM equation and a perturbed BBM equation.

Apparently a, c cannot be both zero, otherwise $b = 0$ and $(a, b, c) = (0, 0, 0)$ would be a trivial solution of (1.6). Without loss of generality, we may assume $a = 1$ and introduce a new variable $\tilde{u}(x, t) = u(x, t) + (b/2)v(x, t)$, then (1.3) is rewritten as follows

$$\begin{cases} \tilde{u}_t + \tilde{u}_x - \tilde{u}_{xxt} + (\tilde{A}\tilde{u}^2)_x = 0, \\ v_t + v_x - v_{xxt} + (D\tilde{u}^2 + \tilde{E}\tilde{u}v + \tilde{F}v^2)_x = 0, \\ \tilde{u}(x, 0) = \tilde{u}_0(x), \quad v(x, 0) = v_0(x), \end{cases} \tag{4.31}$$

where $\tilde{A} = 2A + bD, \tilde{E} = E - bD, \tilde{F} = \frac{b^2}{4}D - \frac{b}{2}E + F$ and $\tilde{u}_0 = u_0 + \frac{b}{2}v_0$. Drop tildes, it obtains

$$\begin{cases} u_t + u_x - u_{xxt} + (Au^2)_x = 0, \\ v_t + v_x - v_{xxt} + (Du^2 + Euv + Fv^2)_x = 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x). \end{cases} \tag{4.32}$$

It is a special case of system (1.3) where $P(u, v) = Au^2$, namely, one way coupled BBM equation. Here is our results of global well-posedness for (4.32).

Theorem 4.3. *System* (4.32) *is well posed globally in time in* $H^s \times H^s$ *for any* $s \geq 1$. *i.e the distributional solution* $\mathbf{v} = (u, v)$ *lies in* $C(0, \infty; H^s \times H^s)$. *Moreover,*

$$\|u(\cdot, t)\|_s \leq c(1 + t)^{\frac{2}{3}(s-1)+\frac{1}{3}(s-\lfloor s \rfloor)}, \tag{4.33}$$

$$\|v(\cdot, t)\|_s \leq c_{s,1}e^{c_{s,2}t}. \tag{4.34}$$

Proof. It is a known fact that the first equation, i.e. u-equation, is globally well-posed in H^s for any $s \geq 1$, and $\|u(\cdot, t)\|_1 = \|u_0\|_1$ is a constant, the estimate (4.33) can be derived from Theorem 4.1, or see Chen [9].

To estimate (4.34), we start with calculating $\|v(\cdot, t)\|_1$. In (4.32), multiply the second equation by v and integrate over $\mathbb{R}$ with respect to x, upon integration by parts and simplifications, it follows that

$$\frac{1}{2}\frac{d}{dt}\|v(\cdot, t)\|_1^2 = \int (Du^2 v_x + Euvv_x)\, dx \leq |D|\|u\|_1^2\|v\|_1 + |E|\|u\|_1\|v\|_1^2,$$

or what is the same,

$$\frac{d}{dt}\|v\|_1 \leq c_{1,2}(1 + \|v\|_1),$$

where $c_{1,2}$ is a constant only dependent on D, E and $\|u_0\|_1$. Grownwall inequality yields the following

$$\|v(\cdot, t)\|_1 \leq (1 + \|v_0\|_1)e^{c_{1,2}t} - 1 < (1 + \|v_0\|_1)e^{c_{1,2}t}. \tag{4.35}$$

Estimate (4.34) is true for $s = 1$.

Our attention turns to estimate a priori bound of $\|v(\cdot,t)\|_s$ for $s>1$. We use mathematical induction. First consider that s is a positive integer and assume that the estimate (4.34) is true for

$$\|v(\cdot,t)\|_j \le c_{j,1}e^{c_{j,2}t} \quad for \quad j=2,\cdots,s-1. \tag{4.36}$$

Rewrite the v-equation of (4.32) as follows,

$$v_{xxt} = v_t + v_x + (Du^2 + Euv + Fv^2)_x.$$

Integrate with respect to t over interva $[0,t]$, it follows

$$\partial_{xx}(v(x,t)-v_0(x)) = v(x,t)-v_0(x)+\int_0^t \partial_x\big(v+Du^2+Euv+Fv^2\big)(x,\tau)\,d\tau. \tag{4.37}$$

Taking the L_2 norm yields

$$\|\partial_{xx}(v(\cdot,t)-v_0)\| \le \|v(\cdot,t)-v_0\|+\int_0^t (\|v\|_1+|D|\|u\|_1^2+|E|\|u\|_1\|v\|_1+|F|\|v\|_1^2)(\cdot,\tau)\,d\tau,$$

applying $\|u(\cdot,t)\|_1 = \|u_0\|\|_1$, it transpires that,

$$\|\partial_{xx}(v(\cdot,t)-v_0)\| \le c_{1,1}e^{c_{1,2}t}$$

where $c_{1,1}$ and $c_{1,2}$ are constants dependent only on $\|u_0\|_1$ and $\|v_0\|_1$. In (4.34), take derivative with respect to x $s-2$ times, it yields

$$\begin{aligned}
&\partial_x^s(v(x,t)-v_0(x))\\
=&\partial_x^{s-2}(v(x,t)-v_0(x)) + \int_0^t \partial_x^{s-1}\Big(v+Du^2+Euv+Fv^2)\Big)(x,\tau)\,d\tau\\
=&\partial_x^{s-2}(v(x,t)-v_0(x)) + \int_0^t \partial_x^{s-1}v(x,\tau)\,d\tau\\
&+\sum_{j=0}^{s-1}\binom{s-1}{j}\int_0^t \Big(D\partial_x^j u\partial_x^{s-1-j}u + E\partial_x^j u\partial_x^{s-1-j}v + F\partial_x^j v\partial_x^{s-1-j}v\Big)(x,\tau)\,d\tau.
\end{aligned} \tag{4.38}$$

Take $L_2(\mathbb{R})$ norm in the above expression,

$$\begin{aligned}
&\|\partial_x^s(v(\cdot,t)-v_0)\|\\
\le&\|\partial_x^{s-2}(v(\cdot,t)-v_0)\| + \int_0^t \|\partial_x^{s-1}v(\cdot,\tau)\|\,d\tau\\
&+\sum_{j=0}^{s-1}\binom{s-1}{j}\int_0^t \big\|D\partial_x^j u\partial_x^{s-j}u + E\partial_x^j u\partial_x^{s-1-j}v + F\partial_x^j v\partial_x^{s-1-j}v\big\|(\cdot,\tau)\,d\tau\\
\le&\|v(\cdot,t)-v_0\|_{s-2} + \int_0^t \|v(\cdot,\tau)\|_{s-1}\,d\tau\\
&+\kappa\int_0^t \Big(\|u(\cdot,\tau)\|_{s-1}^2 + \|u(\cdot,\tau)\|_{s-1}\|v(\cdot,\tau)\|_{s-1} + \|v(\cdot,\tau)\|_{s-1}^2\Big)\,d\tau.
\end{aligned} \tag{4.39}$$

where κ is a constant only dependent on D, E and F. As the integrants increasing no faster than exponential, see (4.33) and inductive assumption (4.36), it is seen that there are constants $c_{s,1}, c_{s,2}$ dependent on only $\|u_0\|_s$ an $\|v_0\|_s$ such that

$$\|v(\cdot,t)\|_s \le c_{s,1}e^{c_{s,2}t}. \tag{4.40}$$

It indicates that (4.34) holds true for s to be a positive integer. What remains is to show the estimates is true for $s > 1$ to be non-integer. Denote $\lfloor s \rfloor = n$, so $v \in C(0,\infty; H^n)$ and $\|v(\cdot,t)\|_n$ is bounded by an exponential function, $c_{n,1}e^{c_{n,2}t}$, say, with constants $c_{n,1}$ and $c_{n,2}$ dependent only on $\|u_0\|_n$ and $\|v_0\|_n$.

Observe (4.34) again to see that $v(\cdot,t)-v_0$ lies in $C(0,\infty; H^{n+1})$. Take derivative with respect to x $n-1$ times in (4.34), it yields

$$\begin{aligned}\partial_x^{n+1}(v(x,t)-v_0(x)) =& \partial_x^{n-1}(v(x,t)-v_0(x)) \\ &+ \int_0^t \partial_x^n\Big(v + Du^2 + Euv + Fv^2\Big)\Big)(x,\tau)\,d\tau \\ =& \partial_x^{s-1}(v(x,t)-v_0(x)) + \int_0^t \|\partial_x^n v(\cdot,\tau)\|\,d\tau \\ &+ \sum_{j=0}^n \binom{n}{j}\int_0^t\Big(D\partial_x^j u\partial_x^{n-j}u + E\partial_x^j u\partial_x^{n-j}v + F\partial_x^j v\partial_x^{n-j}v\Big)(x,\tau)d\tau.\end{aligned} \tag{4.41}$$

Take $L_2(\mathbb{R})$ norm in the above expression,

$$\begin{aligned}\|\partial_x^{n+1}(v(\cdot,t)-v_0)\| \le& \|\partial_x^{n-1}(v(\cdot,t)-v_0)\| + \int_0^t \|\partial_x^n v(x,\tau)\|\,d\tau \\ &+ \sum_{j=0}^n \binom{n}{j}\int_0^t \Big\|D\partial_x^j u\partial_x^{s-j}u + E\partial_x^j u\partial_x^{n-j}v + F\partial_x^j v\partial_x^{n-j}v\Big\|(\cdot,\tau)\,d\tau.\end{aligned} \tag{4.42}$$

The same argument as that in the case where $s > 1$ is integer concludes

$$\|v(\cdot,t)-v_0\|_{n+1} \le c_{n,1}e^{c_{n,2}t}.$$

It follows readily that

$$\|v(\cdot,t)\|_s \le \|v_0\|_s + \|v(\cdot,t)-v_0\|_{n+1} \le c_{s,1}e^{c_{s,2}t}.$$

The estimate in (4.34) is shown, and the theorem is established. □

Theorem 4.4. *The problem* (4.32) *is well posed globally in time in $H^s \times H^s$ for any $0 \le s < 1$. Furthermore, the growth bounds of the H^s-norm in time is*

$$\begin{aligned}\|u(\cdot,t)\|_s &\le c_{s,1}e^{|A|\|u_0\|t} \\ \|v(\cdot,t)\|_s &\le c_{s,1}e^{e^{c_{s,2}t}}\end{aligned} \quad \textit{for} \quad \frac{1}{4} < s < 1, \tag{4.43}$$

$$\begin{aligned}\|(u(\cdot,t)\|_s &\le c_{s,1}e^{c_{s,2}t^2} \\ \|v(\cdot,t)\|_s &\le c_{s,1}e^{e^{c_{s,2}t^2}}\end{aligned} \quad \textit{for} \quad 0 \le s \le \frac{1}{4}, \tag{4.44}$$

where $c_{s,1}, c_{s,2}$ are constants dependent only on the corresponding Sobolev norms of the initial data $\|u_0\|_s, \|v_0\|_s$.

Proof. Since the general system (1.3) is well posed locally in time in $H^s \times H^s$, to show that time interval can be extended to infinity, it is sufficient to estimate the upper bound of $\|v(\cdot,t)\|_s$ so that it does not blow up in any finite time.

We begin with $\frac{1}{4} < s < 1$, so $\mathbf{w} = \mathbf{v} - \mathbf{v}_0 \in C(0,T; H^1 \times H^1)$ satisfies (3.16), or its alternative form as follows,

$$(1-\partial_{xx})\mathbf{w}_t(x,t) = -\partial_x\Big(\mathbf{v}_0(x) + \mathbf{G}(\mathbf{v}_0(x)\Big) \\ -\partial_x\Big[\mathbf{w} + \begin{pmatrix} P_u(u_0,v_0) & P_v(u_0,v_0) \\ Q_u(u_0,v_0) & Q_v(u_0,v_0)\end{pmatrix}\mathbf{w} + \begin{pmatrix} P(\mathbf{w}) \\ Q(\mathbf{w})\end{pmatrix}\Big](x,t),$$

where $\mathbf{G} = \begin{pmatrix} P \\ Q \end{pmatrix}$ and $P(u,v) = Au^2$, $Q(u,v) = Du^2 + Euv + Fv^2$. Let two components of the vector w be w_1 and w_2, then the last system can be written component-wise as follows,

$$\begin{cases} w_{1t} + w_{1x} - w_{1xxt} + (Aw_1^2)_x + 2A(u_0w_1)_x + (u_0 + Au_0^2)_x = 0, \\ w_{2t} + w_{2x} - w_{2xxt} + F(w_2^2)_x + \Big((Q_v(u_0,v_0) + Ew_1)w_2\Big)_x \\ \quad + \Big(Q_u(u_0,v_0)w_1 + Dw_1^2 + v_0 + Q(u_0,v_0)\Big)_x = 0, \\ w_1(x,0) = 0, \quad w_2(x,0) = 0. \end{cases} \tag{4.45}$$

Take the L_2 inner product of w_1 equation with w_1, after integrations by parts and simplification, it yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_1(\cdot,t)\|_1^2 = & \int \Big(2Au_0w_1w_{1x} + (u_0 + Au_0^2)w_{1x}\Big)\,dx \\ \le & |A|\|u_0\|\|w_1(\cdot,t)\|_1^2 + \|u_0 + Au_0^2\|\|w_{1x}(\cdot,t)\|_1 \\ \le & |A|\|u_0\|\|w_1(\cdot,t)\|_1^2 + (\|u_0\| + \kappa|A|\|u_0\|_s^2)\|w_{1x}(\cdot,t)\|_1, \end{aligned} \tag{4.46}$$

where κ is an embedding constant of $H^s \subset L_4$. Solve this inequality, it obtains,

$$\|u(\cdot,t) - u_0\|_1 = \|w_1(\cdot,t)\|_1 \le c_{s,1}e^{|A|\|u_0\|t} \tag{4.47}$$

where $c_{s,1}$ is a constant dependent only on $\|u_0\|_s$. It transpires that

$$\|u(\cdot,t)\|_s \le \|u(\cdot,t) - u_0\|_1 + \|u_0\|_s.$$

The first estiamte in (4.43) follows.

Take the L_2 inner product of w_2 equation with w_2, similar calculations as above leads to

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|w_2(\cdot,t)\|_1^2 \\ \le & \Big\|Q_v(u_0,v_0) + Ew_1\Big\|\|w_2(\cdot,t)\|_1^2 + \Big\|Q_u(u_0,v_0)w_1 + Dw_1^2 + v_0 + Q(u_0,v_0)\Big\|\|w_2(\cdot,t)\|_1 \\ \le & c\Big(\|\mathbf{v_0}\|_s^2 + |E|c_{s,1}e^{|A|\|u_0\|t}\Big)\|w_2(\cdot,t)\|_1^2 + c\Big(\|\mathbf{v}_0\|_s^2 + \|\mathbf{v}_0\|_s + e^{2|A|\|u_0\|t}\Big)\|w_2(\cdot,t)\|_1 \end{aligned}$$

where c is a constant only dependent on $\|u_0\|_s$. What is the same,

$$\begin{aligned}&\frac{d}{dt}\|w_2(\cdot,t)\|_1\\ &\le c\Big(\|\mathbf{v_0}\|_s^2+|E|c_{s,1}e^{|A|\|u_0\|t}\Big)\|w_2(\cdot,t)\|_1+c\Big(\|\mathbf{v_0}\|_s^2+\|\mathbf{v_0}\|_s+e^{2|A|\|u_0\|t}\Big).\end{aligned}$$

It follows readily that

$$\|v(\cdot,t)-v_0)\|_1=\|w_2(\cdot,t)\|_1\le c_{s,1}e^{e^{c_{s,2}t}}. \tag{4.48}$$

This finishes off (4.43).

The attention is now turned to $0\le s\le \frac{1}{4}$. By Theorem 3.1, there is $T>0$ dependent on $\|u_0\|$ and $\|v_0\|$ such that

$$\mathbf{z}=\mathbf{v}-\mathbf{v}_0-tK*\Big(\mathbf{v}_0+\mathbf{G}(\mathbf{v}_0)\Big)$$

lies in $C(0,T;H^1\times H^1)$ and satisfy equations (4.30), i.e.

$$\begin{cases}\mathbf{z}_t+\mathbf{z}_x-\mathbf{z}_{xxt}+\partial_x\mathbf{G}(\mathbf{z})+\partial_x\begin{pmatrix}P_u(\mathbf{v}_0+t\mathbf{v}_1) & P_v(\mathbf{v}_0+t\mathbf{v}_1)\\ Q_u(\mathbf{v}_0+t\mathbf{v}_1) & Q_v(\mathbf{u}_0+t\mathbf{v}_1)\end{pmatrix}\mathbf{z}\\ \quad+t\partial_x\Big\{\mathbf{v}_1+\begin{pmatrix}P_u(\mathbf{v}_0) & P_v(\mathbf{v}_0)\\ Q_u(\mathbf{u}_0) & Q_v(\mathbf{v}_0)\end{pmatrix}\mathbf{v_1}\Big\}+t^2\partial_x\mathbf{G}(\mathbf{v}_1)=\mathbf{0},\\ \mathbf{z}(x,0)=\mathbf{0},\end{cases} \tag{4.49}$$

where the vector $\mathbf{v}_1=\begin{pmatrix}u_1\\ v_1\end{pmatrix}$ is given as

$$\mathbf{v}_1=K*\Big(\mathbf{v}_0+\mathbf{G}(\mathbf{v}_0)\Big)=K*\begin{pmatrix}u_0+P(u_0,v_0)\\ v_0+Q(u_0,v_0)\end{pmatrix}=\begin{pmatrix}K*(u_0+Au_0^2)\\ K*(v_0+Du_0^2+Eu_0v_0+Fv_0^2)\end{pmatrix}.$$

Let ξ and η be two components of vector $\mathbf{z}$ i.e.

$$\begin{aligned}\xi&=u-u_0-tK*(u_0+Au_0^2),\\ \eta&=v-v_0-tK*(v_0+Du_0^2+Eu_0v_0+Fv_0^2),\end{aligned} \tag{4.50}$$

then they both lie in $C(0,T;H^1)$ and satisfy the following

$$\begin{cases}\xi_t+\xi_x-\xi_{2xxt}+A(\xi^2)_x+2A((u_0+tu_1)\xi)_x+\big(t(u_1+2Au_0u_1)+At^2u_1^2\big)_x=0,\\ \eta_t+\eta_x-\eta_{xxt}+F(\eta^2)_x+\Big((Q_v(u_0+tu_1,v_0+tv_1)+E\xi)\eta\Big)_x\\ \quad+\Big(Q_u(u_0+tu_1,v_0+tv_1)\xi+D\xi^2\Big)_x\\ \quad+t\Big\{v_1+Q_u(u_0,v_0)u_1+Q_v(u_0,v_0)v_1\Big\}_x+t^2Q(u_1,v_1)_x=0,\\ \xi(x,0)=\eta(x,0)=0.\end{cases} \tag{4.51}$$

The same tedious calculations as those for w_1 and w_2 provide the following estimates:

$$\frac{1}{2}\frac{d}{dt}\|\xi(\cdot,t)\|_1^2 = \|2A((u_0+tu_1)\|\|\xi(\cdot,t)\|_1^2 + \left\|t(u_1+2Au_0u_1)+At^2u_1^2\right\|\|\xi(\cdot,t)\|_1.$$

Since $v_1 \in L_2 \cap C_b$, see (4.29), we have

$$\frac{d}{dt}\|\xi(\cdot,t)\|_1 \le c_1(1+t)\|\xi(\cdot,t)\|_1 + c_2t(1+t),$$

where constants c_1 and c_2 depend only on $\|u_0\|$. Solving the inequality, we have

$$\begin{aligned}\|\xi(\cdot,t)\|_1 &\le e^{c_1(t+\frac{1}{2}t^2)} + c_2e^{c_1(t+\frac{1}{2}t^2)}\int_0^t e^{-c_1(\tau+\frac{1}{2}\tau^2)}\tau(1+\tau)d\tau\\ &\le e^{\frac{1}{2}c_1}e^{c_1t^2} + c_2e^{\frac{1}{2}c_1}\int_0^\infty e^{-c_1(\tau+\frac{1}{2}\tau^2)}\tau(1+\tau)d\tau\, e^{c_1t^2}\\ &= \kappa_1 e^{c_1t^2}\end{aligned}$$

in which $\kappa_1 = e^{\frac{1}{2}c_1} + c_2e^{\frac{1}{2}c_1}\int_0^\infty e^{-c_1(\tau+\frac{1}{2}\tau^2)}\tau(1+\tau)d\tau$ is a constant. And

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\eta(\cdot,t)\|_1^2 \le &\left\|Q_v(u_0+tu_1,v_0+tv_1)+E\xi)\right\|\|\eta(\cdot,t)\|_1^2\\ &+\left\|Q_u(u_0+tu_1,v_0+tv_1)\xi + D\xi^2\right\|\|\eta(\cdot,t)\|_1\\ &+t\left\|v_1+Q_u(u_0,v_0)u_1+Q_v(u_0,v_0)v_1\right\|\|\eta(\cdot,t)\|_1\\ &+t^2\|Q(u_1,v_1)\|\|\eta(\cdot,t)\|.\end{aligned}$$

Upon simplification and making use of the last property of $\|\xi(\cdot,t)\|_1$,

$$\begin{aligned}\frac{d}{dt}\|\eta(\cdot,t)\|_1 \le &\Big(\|Q_v(u_0+tu_1,v_0+tv_1)\| + |E|\kappa_1e^{c_1t^2}\Big)\|\eta(\cdot,t)\|_1\\ &+\|Q_u(u_0+tu_1,v_0+tv_1)\|\kappa_1e^{c_1t^2} + |D|\kappa_1e^{2c_1t^2}\\ &+t\left\|v_1+Q_u(u_0,v_0)u_1+Q_v(u_0,v_0)v_1\right\|\\ &+t^2\|Q(u_1,v_1)\|.\end{aligned}$$

Solving it, we get

$$\|\eta(\cdot,t)\|_1 \le \kappa_2 e^{e^{c_st^2}}. \tag{4.52}$$

From definition (4.50),

$$\|u(\cdot,t)\|_s \le \|\xi(\cdot,t)\|_s + \|u_0+tu_1\|_s \le \|\xi(\cdot,t)\|_1 + \|u_0+tu_1\|_s,$$

and

$$\|v(\cdot,t)\|_s \le \|\eta(\cdot,t)\|_s + \|v_0+tv_1\|_s \le \|\eta(\cdot,t)\|_1 + \|v_0+tv_1\|_s.$$

They imply the estimate (4.44).

The theorem is complete. □

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