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EXISTENCE OF THREE POSITIVE SOLUTIONS FOR A CLASS OF RIEMANN-LIOUVILLE FRACTIONAL *Q*-DIFFERENCE EQUATION

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Abstract In this paper, we confirm the existence of three positive solutions for a class of Riemann-Liouville fractional q-difference equation which satisfies the boundary conditions. We gain several sufficient conditions for the existence of three positive solutions of this boundary value problem by applying the Leggett-Williams fixed point theorem.

Keywords Fractional q-integral of the Riemann-Liouville, fractional q-derivative of the Riemann-Liouville, fixed point theorem, fractional q-difference equation, positive solutions.

MSC(2010) 39A13, 34A08, 34B18.

1. Introduction

We consider the following fractional q-difference equation with boundary conditions

$$(D_q^\alpha x)(t) + f(t, x(t)) = 0,\ 0 < q < 1,\ 0 < t < 1, \tag{1.1}$$

$$x(0) = D_q x(0) = D_q x(1) = 0, \tag{1.2}$$

whereD_q^α denotes the Riemann-Liouville fractional q-derivative of order α, $2 < \alpha < 3$, $f \in C([0,1] \times [0,+\infty)) \to [0,+\infty)$.

Since the q-difference calculus or quantum calculus was founded by Jackson [6,7]. People show great interests in fractional q-difference equations. After the fractional q-difference calculus was developed by Al-Salam and Agarwal [1,2], a lot of papers on the fractional q-difference equation have been emerged in recent years(consult [4], [8–16], and the other references). The authors employed the existence of solutions or positive solutions for fractional q-difference equations by applying the monotone

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iterative method, the upper and lower solutions method and some distinguished fixed-point theories such as Krasnosel'skiĭ and Schauder fixed point theorems, respectively. For example, Ferreira [4] employed the existence of positive solutions for the following nonlinear q-fractional boundary value problem

$$(D_q^\alpha y)(x) = -f(x, y(x)),\ 0 < x < 1,$$
$$y(0) = D_q y(0) = 0, D_q y(1) = \beta \geqslant 0$$

by applying a fixed point theorem in cones, where $D_q^\alpha,\ q, \alpha$ is similar as (1.1)-(1.2).

Li, et al. [10] gained the conditions of the existence of positive solutions for the following nonlinear fractional q-difference equation with parameter

$$(D_q^\alpha y)(x) + \lambda f(y(x)),\ 0 < x < 1,$$
$$y(0) = D_q y(0) = D_q y(1) = 0$$

depending on the range of parameter λ by using the known Guo-Krasnosel'skii fixed point theorem in cones, where $D_q^\alpha,\ q, \alpha$ is similar as (1.1)-(1.2).

By using the monotone iterative method and upper and lower solutions method, Zhai and Ren [15] obtained the existence of positive or negative solutions for problem (1.1)-(1.2).

As we all know, few people can solve the existence of three positive solutions for a Riemann-Liouville fractional q-difference equation boundary value problem (1.1)-(1.2), by using Leggett-Williams fixed point theorem to study. Inspired by papers [17, 19], we confirm the existence of three positive solutions for problem (1.1)-(1.2) by using properties of the Green's function and the Leggett-Williams fixed-point theorem in this paper.

In the followings, we will provide basic definitions and some lemmas in order to prove our main results in Section 2. In Section 3, we establish some results for the existence of three positive solutions to the problem (1.1)-(1.2). In Section 4, we will give some examples to corroborate our results.

2. Preliminaries

In order to demonstrate our main results, we show some basic definitions and some lemmas in this piece.

Set $0 < q < 1$, we define

$$[s]_q = \frac{1 - q^s}{1 - q}, s \in \mathbb{R}.$$

The q-analogue of the power function $(s - t)^n$ with $n \in \mathbb{N}$ is

$$(s - t)^0 = 1, (s - t)^n = \prod_{k=0}^{n-1} (s - tq^k), n \in \mathbb{N}, s, t \in \mathbb{R}.$$

If $\alpha \in \mathbb{R}$, then

$$(s - t)^\alpha = s^\alpha \prod_{n=0}^{\infty} \frac{s - tq^n}{s - tq^{n+\alpha}}.$$

We define the q-gamma function as following

$$\Gamma_q(s) = (1-q)^{(s-1)}(1-q)^{1-s}, s \in \mathbb{R}\backslash\{0, -1, -2, \cdots\}$$

and satisfies $\Gamma_q(s+1) = [s]_q\Gamma_q(s)$.

For $0 < q < 1$, the q-derivative of a function f is defined by

$$D_q f(s) = \frac{\mathrm{d}_q}{\mathrm{d}_q s} f(s) = \frac{f(s) - f(qs)}{(1-q)s},\ (D_q f)(0) = \lim_{s\to 0}(D_q f)(s), s \neq 0.$$

The higher order q-derivatives are defined by

$$D_q^0 f(s) = f(s),\ (D_q^n f)(s) = D_q(D_q^{n-1} f)(s), n \in \mathbb{N}.$$

The q-integral of a function f defined on the interval $[0, b]$ is given by

$$I_q f(s) = \int_0^s f(t)\mathrm{d}_q t = s(1-q)\sum_{n=0}^{\infty} f(sq^n)q^n, s \in [0, b]$$

provided that the series converges.

If f is defined on the interval $[0, b]$ and $a \in [0, b]$, its q-integral from a to b is defined by

$$\int_a^b f(s)\mathrm{d}_q s = \int_0^b f(s)\mathrm{d}_q s - \int_0^a f(s)\mathrm{d}_q s.$$

The higher order q-integrals are defined by

$$I_q^0 f(s) = f(s), (I_q^n f)(s) = I_q(I_q^{n-1} f)(s), n \in \mathbb{N}.$$

We note that $D_q I_q f(s) = f(s)$ and if f is continuous at $s = 0$, we get $I_q D_q f(s) = f(s) - f(0)$.

To gain more details of the basic material on q-calculus, the readers can refer to [1,2,6,7].

Definition 2.1. The fractional q-integral of the Riemann-Liouville of order $\alpha > 0$ for a continuous function $f : [0, 1] \to \mathbb{R}$ is given by

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t - qs)^{(\alpha-1)} f(s)\mathrm{d}_q s,\ t \in [0, 1],$$

where $\Gamma_q(\alpha) = (1-q)^{(\alpha-1)}(1-q)^{1-\alpha},\ 0 < q < 1$, and satisfies $\Gamma_q(\alpha+1) = [\alpha]_q\Gamma_q(\alpha)$.

Definition 2.2. The fractional q-derivative of the Riemann-Liouville of order $\alpha > 0$ is given by $D_q^0 f(t) = f(t)$ and

$$D_q^\alpha f(t) = D_q^n I_q^{n-\alpha} f(t),$$

where n is the smallest integer greater than or equal to α.

Lemma 2.1 (Theorem 2.0.5, [4])**.** *Let $\alpha > 0$ and n be the smallest integer greater than or equal to α. Then for $t \in [0, 1]$, the following equality holds*

$$(I_q^\alpha\ D_q^\alpha f)(t) = f(t) + \sum_{k=0}^{n-1} \frac{t^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} D_q^k f(0).$$

Lemma 2.2 (Lemma 2.5, [10]). *Given $g(t) \in C[0,1]$, the unique solution of the following problem*

$$(D_q^\alpha x)(t) + g(t) = 0,\ 0 < t < 1, 2 < \alpha < 3, \tag{2.1}$$

$$x(0) = D_q x(0) = D_q x(1) = 0 \tag{2.2}$$

is

$$x(t) = \int_0^1 G(t, qs) g(s) d_q s,$$

where

$$G(t, qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-qs)^{(\alpha-2)} t^{\alpha-1} - (t-qs)^{(\alpha-1)}\ , & 0 \leqslant qs \leqslant t \leqslant 1, \\ (1-qs)^{(\alpha-2)} t^{\alpha-1}, & 0 \leqslant t \leqslant qs \leqslant 1. \end{cases} \tag{2.3}$$

Here $G(t, qs)$ is called the Green's function of boundary value problem (2.1)-(2.2).

Lemma 2.3 (Lemma 2.6, [10]). *Suppose $2 < \alpha < 3$, then the function $G(t, qs)$ defined by (2.3) satisfies the following inequalities:*

(i) $G(t, qs) \geqslant 0,\ t, s \in [0,1]$.

(ii) $t^{\alpha-1} G(1, qs) \leqslant G(t, qs) \leqslant G(1, qs),\ t, s \in [0,1]$.

Definition 2.3. If P is a cone of the real Banach space E, a mapping $\theta : P \to [0, \infty)$ is continuous and with

$$\theta(tx + (1-t)y) \geqslant t\theta(x) + (1-t)\theta(y),\ \ x, y \in P,\ t \in [0,1],$$

is called a nonnegative concave continuous functional θ on P.

We will employ the following notations on positive constants r, r_1, r_2 for the later content.

$$P_r = \{x \in P : \|x\| < r\},$$
$$\overline{P}_r = \{x \in P : \|x\| \leqslant r\},$$

and

$$P(\theta, r_1, r_2) = \{x \in P : \theta(x) \geqslant r_1, \|x\| \leqslant r_2\}.$$

For the convenience, we show the Leggett-Williams fixed point theorem as follows.

Lemma 2.4 (P347 Theorem 5.4, [5]). *Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ be a cone of E, and $r_3 > 0$ be a constant. Suppose there exists a concave nonnegative continuous functional θ on P with $\theta(x) \leqslant \|x\|$ for $x \in \overline{P}_{r_3}$. Let $T : \overline{P}_{r_3} \to \overline{P}_{r_3}$ be a completely continuous operator. Assume there are numbers r, r_1 and r_2 with $0 < r < r_1 < r_2 \leq r_3$ such that*

(i) The set $\{x \in P(\theta, r_1, r_2) : \theta(x) > r_1\}$ is nonempty and $\theta(Tx) > r_1$ for all $x \in P(\theta, r_1, r_2)$;

(ii) $\|Tx\| < r$ for $x \in \overline{P}_r$;

(iii) $\theta(Tx) > r_1$ for all $x \in P(\theta, r_1, r_3)$ with $\|Tx\| > r_2$.

Then T has at least three fixed points x_1, x_2 and $x_3 \in \overline{P}_{r_3}$. Furthermore we have

$$\max_{t\in[0,1]} x_1(t) < r, r_1 < \min_{t\in[0,1]} x_2(t) < \max_{t\in[0,1]} x_2(t) < r_3 \ \ \textit{and}\ \ r < \max_{t\in[0,1]} x_3(t) \leq r_3, \min_{t\in[0,1]} x_3(t) < r_1.$$

3. Existence of three positive solutions

This section, we will obtain the main results of this paper by making use of the lemmas in Section 2. Let $C[0,1]$ be the space of all continuous real functions defined on $[0,1]$ with the maximum norm $\|x\| = \max\limits_{t\in[0,1]}|x(t)|$. It is obviously a Banach space. Define the cone $P \subset C[0,1]$ as following:

$$P = \{x \in C[0,1] : x(t) \geqslant 0,\ t \in [0,1]\}.$$

From Lemma 2.2, we know that $x(t)$ is a solution of boundary value problem (1.1)-(1.2) if and only if it satisfy

$$x(t) = \int_0^1 G(t, qs) f(s, x(s)) \mathrm{d}_q s,\ t \in [0,1]. \tag{3.1}$$

Then, the positive solutions $x(t)$ of problem (1.1)-(1.2) are the fixed points of T in $C[0,1]$ that defined by

$$(Tx)(t) = \int_0^1 G(t, qs) f(s, x(s)) \mathrm{d}_q s,\ t \in [0,1], \tag{3.2}$$

In line with Lemma 3.1 of [10], $T : P \to P$ is completely continuous.

Next, by using Lemma 2.4, we obtain sufficient conditions for the existence of three positive solutions to the problem (1.1)-(1.2). To acquire our results, we take a positive number $\nu \in (0,1)$, set the nonnegative concave continuous function θ on P be defined by

$$\theta(x) = \min_{t\in[\nu,1]} x(t). \tag{3.3}$$

Denote

$$f^0 = \limsup_{x\to 0} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \quad f^\infty = \limsup_{x\to +\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x};$$
$$\rho^{-1} = \int_0^1 G(1, qs) \mathrm{d}_q s, \quad \omega^{-1} = \int_\nu^1 G(1, qs) \mathrm{d}_q s.$$

And suppose that the function $f(t,x)$ satisfies the following condition:

(H) $f(t,x)$ is a nonnegative continuous function on $[0,1]\times[0,+\infty)$ and there exists $t_n \to 0$ such that $f(t_n, x(t_n)) > 0, n = 1, 2, \cdots$.

Theorem 3.1. *Suppose the condition (H) holds and there exist constants* $0 < r < r_1$ *such that*

(H1) $f(t,x) < \rho\, r$ *for* $(t,x) \in [0,1] \times [0,r]$;

(H2) $f(t,x) \geqslant \dfrac{\omega}{\nu^{\alpha-1}} r_1$ *for* $(t,x) \in [\nu,1] \times [r_1, r_3]$, *where* $r_3 > \dfrac{r_1}{\nu^{\alpha-1}}$;

(H3) $f(t,x) \leqslant \kappa x + \beta$ *for* $(t,x) \in [0,1] \times [0,+\infty)$, *where* κ, β *are positive numbers.*

Then the boundary value problem (1.1)-(1.2) has at least three positive solutions $x_1,\ x_2$ *and* x_3.

Proof. Set $r_3 > \max\{\dfrac{\beta}{\rho-\kappa}, \dfrac{r_1}{\nu^{\alpha-1}}\}$, then for $x \in \overline{P}_{r_3}$, we have from (3.2) and (H3)

$$\begin{aligned}\|Tx\| =& \max_{t\in[0,1]} \int_0^1 G(t,qs)f(s,x(s))\mathrm{d}_q s\\ \leqslant& \int_0^1 G(1,qs)(\kappa x(s)+\beta)\mathrm{d}_q s\\ \leqslant&(\kappa\|x\|+\beta)\int_0^1 G(1,qs)\mathrm{d}_q s\\ =&\frac{(\kappa\|x\|+\beta)}{\rho}\\ <&r_3,\end{aligned}$$

i.e, $Tx \in P_{r_3}$. Therefore $T : \overline{P}_{r_3} \to \overline{P}_{r_3}$ be a completely continuous operator. By (H1), we can get

$$\begin{aligned}\|Tx\| \leqslant& \int_0^1 G(1,qs)f(s,x(s))\mathrm{d}_q s\\ <&\rho r\int_0^1 G(1,qs)\mathrm{d}_q s\\ =&r.\end{aligned}$$

Hence assumption (ii) of Lemma 2.4 is satisfied.

We choose $x_0 = \frac{(\nu^{\alpha-1}+1)r_1}{2\nu^{\alpha-1}}$ for $t \in [\nu,1]$, then $x_0 \in \{x \in P(\theta, r_1, \dfrac{1}{\nu^{\alpha-1}}r_1) : \theta(x) > r_1\}$, which implies $\{x \in P(\theta, r_1, \frac{1}{\nu^{\alpha-1}}r_1) : \theta(x) > r_1\}$ is nonempty set. If $x \in P(\theta, r_1, \frac{1}{\nu^{\alpha-1}}r_1)$, then $r_1 \leqslant x(t) \leqslant \frac{1}{\nu^{\alpha-1}}r_1$ for $\nu \leqslant t \leqslant 1$. Thus

$$\begin{aligned}\theta(Tx) =& \min_{\nu\leqslant t\leqslant 1} \int_0^1 G(t,qs)f(s,x(s))\mathrm{d}_q s\\ >& \int_\nu^1 \min_{\nu\leqslant t\leqslant 1} G(t,qs)f(s,x(s))\mathrm{d}_q s\\ >& \int_\nu^1 \nu^{\alpha-1}G(1,qs)f(s,x(s))\mathrm{d}_q s\\ \geqslant&\frac{\omega}{\nu^{\alpha-1}}r_1\int_\nu^1 \nu^{\alpha-1}G(1,qs)\mathrm{d}_q s\\ =& \ r_1.\end{aligned}$$

From the above inequality, we see that $\theta(Tx) > r_1$ for all $x \in P(\theta, r_1, \frac{1}{\nu^{\alpha-1}}r_1)$. This shows that condition (i) of Lemma 2.4 is satisfied.

Finally, for $x \in P(\theta, r_1, r_3)$ with $\|Tx\| > \frac{1}{\nu^{\alpha-1}}r_1$, we get

$$\begin{aligned}\theta(Tx) =& \min_{\nu\leqslant t\leqslant 1} \int_0^1 G(t,qs)f(s,x(s))\mathrm{d}_q s\\ \geqslant& \int_0^1 \min_{\nu\leqslant t\leqslant 1} t^{\alpha-1}G(1,qs)f(s,x(s))\mathrm{d}_q s\end{aligned}$$

$$
\begin{aligned}
&\geqslant \nu^{\alpha-1}\int_0^1 G(1,qs)f(s,x(s))\mathrm{d}_q s\\
&\geqslant \nu^{\alpha-1}\int_0^1 \max_{0\leqslant t\leqslant 1} G(t,qs)f(s,x(s))\mathrm{d}_q s\\
&> r_1.
\end{aligned}
$$

This confirms that condition (iii) of Lemma 2.4 is fulfilled. By virtue of Lemma 2.4, the boundary value problem (1.1)-(1.2) has at least three solutions x_1, x_2 and x_3. Take into account that condition (H) holds, we have $x_i(t) > 0,\ 0 < t < 1,\ i = 1,2,3$. The proof is completed. □

Theorem 3.2. *Assume the condition (H) holds. There exist constants* $0 < r < r_1 < r_3$ *(*$r_3 > \frac{r_1}{\nu^{\alpha-1}}$*) such that (H1),(H2) and (H4) are satisfied, where*

(H4) $f(t,x) \leqslant \rho r_3$ *for* $(t,x) \in [0,1]\times[0,r_3]$.

Then the boundary value problem (1.1)-(1.2) has at least three positive solutions x_1, x_2 *and* x_3 *such that*

$$
\begin{aligned}
&\max_{t\in[0,1]} x_1(t) < r,\ r_1 < \min_{t\in[\nu,1]} x_2(t) < \max_{t\in[0,1]} x_2(t) < r_3\\
\textit{and}\quad & r < \max_{t\in[0,1]} x_3(t) \leqslant r_3,\ \min_{t\in[\nu,1]} x_3(t) < r_1.
\end{aligned}
$$

Proof. From (H4), we get

$$
\begin{aligned}
\|Tx\| &= \max_{t\in[0,1]}\int_0^1 G(t,qs)f(s,x(s))\mathrm{d}_q s\\
&\leqslant \int_0^1 G(1,qs)f(s,x(s))\mathrm{d}_q s\\
&< \rho r_3\int_0^1 G(1,qs)\mathrm{d}_q s\\
&= r_3.
\end{aligned}
$$

Therefore $T:\overline{P}_{r_3}\to\overline{P}_{r_3}$. The remainder of proof is similar to the proof of Theorem 3.1 and is therefore omitted. By Lemma 2.4, the boundary value problem (1.1)-(1.2) has at least three positive solutions x_1, x_2 and x_3 satisfying

$$
\begin{aligned}
&\max_{t\in[0,1]} x_1(t) < r,\ r_1 < \min_{t\in[\nu,1]} x_2(t) < \max_{t\in[0,1]} x_2(t) < r_3\\
\text{and}\quad & r < \max_{t\in[0,1]} x_3(t) \leqslant r_3,\ \min_{t\in[\nu,1]} x_3(t) < r_1.
\end{aligned}
$$

The proof is complete. □

Theorem 3.3. *Assume the condition (H) holds. There exist constants* $0 < r < r_1$ *such that (H1), (H2) are satisfied, and function* $f(t,x)$ *satisfies*

(H5) $f^\infty < \rho$.

Then the boundary vale problem (1.1)-(1.2) has at least three positive solutions.

Proof. From the assumption (H5), there exist $0 < \sigma < \rho$ and $R > 0$, when $x \geqslant R$, we have

$$f(t,x) \leqslant \sigma x.$$

Set $M = \max\limits_{(t,x)\in[0,1]\times[0,R]} f(t,x)$, consequently

$$0 \leqslant f(t,x) \leqslant \sigma x + M,\ \ 0 \leqslant x < +\infty.$$

This shows that condition (H3) of Theorem 3.1 is satisfied. By Theorem 3.1, the boundary value problems (1.1)-(1.2) has at least three positive solutions. The proof is completed. □

Theorem 3.4. *Assume there exist two positive constants r_1, r_3 ($r_3 > \frac{1}{\nu^{\alpha-1}} r_1$) such that the condition (H), (H2)and (H4) hold. And function $f(t,x)$ satisfies*

(H6) $f^0 < \rho$.

Then the boundary vale problems (1.1)-(1.2) has at least three positive solutions.

Proof. In line with (H6), it is easy to see that there exists a positive constant $r < r_1$ such that for $\|x\| < r$, we have

$$f(t,x(t)) < \rho x.$$

Namely,

$$f(t,x(t)) < \rho r, \|x\| < r.$$

This implies that conditions of Theorem 3.2 is satisfied. By Theorem 3.2, the boundary vale problem (1.1) has at least three positive solutions. The proof is completed. □

In the light of the proof of Theorem 3.3 and Theorem 3.4, we obtain one theorem and four corollaries as follows.

Theorem 3.5. *Assume the function $f(t,x)$ satisfies conditions (H), (H2), (H5) and (H6). Then the boundary vale problem (1.1)-(1.2) has at least three positive solutions.*

Corollary 3.1. *Assume the conditions (H),(H2) and (H3) hold. The function $f(t,x)$ satisfies $f^0 = 0$. Then the boundary vale problem (1.1)-(1.2) has at least three positive solutions.*

Corollary 3.2. *Assume the conditions (H),(H1) and (H2) hold. The function $f(t,x)$ satisfies $f^\infty = 0$. Then the boundary vale problem (1.1)-(1.2) has at least three positive solutions.*

Corollary 3.3. *Assume the conditions (H),(H2) and (H4) hold. The function $f(t,x)$ satisfies $f^\infty = 0$. Then the boundary vale problem (1.1)-(1.2) has at least three positive solutions.*

Corollary 3.4. *Assume the conditions (H) and (H2) hold. The function $f(t,x)$ satisfies $f^0 = 0$ and $f^\infty = 0$. Then the boundary vale problem (1.1)-(1.2) has at least three positive solutions.*

4. Examples

This section, we present three examples to illustrate our results. We take $\alpha = \frac{5}{2},\ q = \frac{1}{2},\ \nu = \frac{1}{4}$, by estimating, we then have $\rho > 7.8,\ \omega < 3.0015$.

Example 4.1. We take

$$f(t,x) = \begin{cases} 0.26t + 25x^2, & (t,x) \in [0,1] \times [0,1], \\ 0.26t + 24 + x, & (t,x) \in [0,1] \times (1,+\infty). \end{cases}$$

There exist constants $r = 0.04$ and $r_1 = 1.03$ such that

$f(t,x) = 0.26t + 25x^2 < 7.8 \times 0.04 < \rho r$ for $(t,x) \in [0,1] \times [0,r]$.

$f(t,x) = 0.26t+24+x > 3.0015 \cdot \frac{1}{\nu^{\alpha-1}} r > \frac{\omega}{\nu^{\alpha-1}} r_1$ for $(t,x) \in [\nu,1] \times [r_1, \frac{r_1}{\nu^{\alpha-1}} + 1]$.

$f(t,x) \leqslant 0.26 + 25x$ for $(t,x) \in [0,1] \times (0,+\infty)$.

All the conditions of Theorem 3.1 hold. Thus, this moment, by virtue of Theorem 3.1 we know that the boundary value problem (1.1)-(1.2) has three positive solutions.

Example 4.2. We take

$$f(t,x) = \begin{cases} 52.5\ x^3 \cdot 2^{-t}, & (t,x) \in [0,1] \times [0,1], \\ 2^{-t}[45x^{\frac{1}{2}} + 7.5x], & (t,x) \in [0,1] \times (1,+\infty). \end{cases}$$

There exist constants $r = 0.36$ and $r_1 = 1.15$ such that

$f^{\infty} = 7.5 < \rho$ for $(t,x) \in [0,1] \times [0,+\infty)$.

$f(t,x) \geqslant 3.0015 \cdot \frac{r_1}{\nu^{\alpha-1}} > \frac{\omega}{\nu^{\alpha-1}} r_1$ for $(t,x) \in [\nu,1] \times [r_1, \frac{r_1}{\nu^{\alpha-1}} + 1]$.

$f(t,x) \leqslant 52.5\ r^3 < \rho\ r$ for $(t,x) \in [0,1] \times [0,r]$.

All the conditions of Theorem 3.3 hold. Hence, under these circumstances we know that the boundary value problem (1.1)-(1.2) has three positive solutions, by using Theorem 3.3.

Example 4.3. We seek

$$f(t,x) = \begin{cases} 6x + 36x^4, & (t,x) \in [0,1] \times [0,1], \\ 35x^{\frac{1}{2}} + 7x, & (t,x) \in [0,1] \times (1,+\infty). \end{cases}$$

There exist constant $r_1 = 3$ such that

$f^{\infty} = 7 < 7.8 < \rho$.

$f^{0} = 6 < 7.8 < \rho$.

$f(t,x) = 35x^{\frac{1}{2}} + 7x > \frac{\omega}{\nu^{\alpha-1}} r_1$ for $(t,x) \in [\nu,1] \times [r_1, \frac{r_1}{\nu^{\alpha-1}} + 1]$.

All the conditions of Theorem 3.5 hold. Thus, in this case, by using Theorem 3.5 we know that the boundary value problem (1.1)-(1.2) has three positive solutions.

Competing interests

The authors declare that they have no competing interests.

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