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EXACT SOLUTIONS FOR A DIRAC-TYPE EQUATION WITH N-FOLD DARBOUX TRANSFORMATION*

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Abstract Based on matrix spectral problems associated with the real special orthogonal Lie algebra so(3,ℝ), a Dirac-type equation is derived by virtue of the zero-curvature equation. Further, an N-fold Darboux transformation for the Dirac-type equation is constructed by means of the gauge transformation. Finally, as its application, some exact solutions and their figures are obtained via symbolic computation software (Maple).

Keywords Dirac-type equation, the Lie algebra so(3,ℝ), Darboux transformation, exact solutions.

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1. Introduction

Nonlinear evolution equations is a research focus in the domains of physics and nonlinear sciences such as quantum mechanics, fluid mechanics, fiber-optics, oceanography, astronomy, meteorology and communication. Numerous integrable equations [38] in solitary theory like the AKNS equation, the BPT equation, and the WKI equation have been researched. At present, with the rapid progress of computer softwares, all kinds of approaches to solving soliton equations have developed which include the homogenous balance method [39, 40], the tanh method [17, 35], the Jacobi elliptic function method [20], the extended-tanh-function method [8, 28], the transformed rational function method [31, 49], the exp-function method [12, 13], the tanh-coth method [42], the F-expansion method [55], the Hirota bilinear method [14], the Lie symmetry analysis [3, 4, 7] and the multiple exp-function method [30], etc. By means of these abundant methods, various exact solutions are obtained which contain traveling wave solutions, soliton solutions, rational solutions, periodic solutions, lump solutions [6, 33, 34, 41, 46], lump-kink solutions [50, 51], interaction solutions [21, 22], algebro-geometric solutions [15, 23], complexiton solutions [24, 49], the solutions of algebraic Rossby solitary waves [11, 45] and so on. Besides the methods mentioned above, the Darboux transformation [9, 10, 25, 32, 44, 53] is a powerful

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way for solving soliton equations by finding a gauge transformation between corresponding Lax pairs. Due to its effectiveness and uniqueness, it is widely used in continuous and discrete integrable systems [5, 47, 52]. As a result, one-soliton solutions, two-soliton solutions, even N-soliton solutions can be obtained.

It is well-known that the Dirac equation is a significant research subject which can be used to describe the interactions of relativistic fermions, the dynamics of elementary molecules, and the gap solitons in optics, etc. In respect of the Dirac hierarchy, a generalized Dirac soliton hierarchy with bi-Hamiltonian structure and its integrable couplings have been generated [43, 48]. In respect of the Dirac equation, it has been researched with various approaches such as the tridiagonal matrix representation approach [1], Krylov subspace methods [2], Bosonic symmetries [37], etc.

In the Ref [26], a Dirac-type hierarchy is resulted from the real special orthogonal Lie algebra so(3,$\mathbb{R}$). Its bi-Hamiltonian structure and Liouville integrability are presented. When taking n=2 in the Dirac-type hierarchy into account, we have the Dirac-type equation [26]

$$\begin{aligned} p_t &= -q_{xx} - \tfrac{1}{2}p^2 q - \tfrac{1}{2}q^3, \\ q_t &= p_{xx} + \tfrac{1}{2}p^3 + \tfrac{1}{2}pq^2, \end{aligned} \tag{1.1}$$

where $p = p\,(x,t)\,,\ q = q\,(x,t)$ are two functions with regard to two variables x and t. For the Dirac-type equation, solvability of Dirac-type equations have been studied [16]. In this paper, we aim mainly at exploring exact solutions of the Dirac-type equation by using Darboux transformation.

The paper is organized as follows. In section 2, based on the spectral problems associated with so(3,$\mathbb{R}$), the gauge transformation is introduced so as to construct an N-fold Darboux transformation. In section 3, some explicit solutions of the Dirac-type equation are obtained by applying Darboux transformation. Then, the figures are plotted with the assistance of Maple. At last, in section 4, this paper is summarized.

2. An N-fold Darboux transformation

It is well-known that the real special orthogonal Lie algebra so(3,$\mathbb{R}$) has a basis,

$$e_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.1}$$

In the section, we shall generate a Darboux transformation for Eq. (1.1). The matrix spectral problems are presented as follows,

$$\phi_x = U\phi,\ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},\ U = \lambda e_1 + pe_2 + qe_3, \tag{2.2}$$

$$\phi_t = V\phi,\ V = (-\lambda^2 + \frac{1}{2}p^2 + \frac{1}{2}q^2)e_1 + (-p\lambda - q_x)e_2 + (-q\lambda + p_x)e_3, \tag{2.3}$$

where

$$U=\begin{bmatrix}0 & q & \lambda\\ -q & 0 & -p\\ -\lambda & p & 0\end{bmatrix},\ V=\begin{bmatrix}0 & -q\lambda+p_x & -\lambda^2+\frac{1}{2}p^2+\frac{1}{2}q^2\\ q\lambda-p_x & 0 & p\lambda+q_x\\ \lambda^2-\frac{1}{2}p^2-\frac{1}{2}q^2 & -p\lambda-q_x & 0\end{bmatrix},$$

$p=p\,(x,t)\,,\ q=q\,(x,t)$ are two potentials, λ is a spectral parameter. Through the zero-curvature equation, $U_t-V_x+[U,V]=0$, the Dirac-type equation Eq. (1.1) can be proved by direct computation.

We first introduce a gauge transformation for the spectral problems Eqs. (2.2) and (2.3),

$$\hat{\phi}=T\phi,\ T=\begin{bmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix}, \tag{2.4}$$

suppose

$$\begin{aligned}
&T_{11}=\lambda^N+\sum_{i=0}^{N-1}A_{11}^{(i)}\lambda^i,\ T_{12}=\sum_{i=0}^{N-1}A_{12}^{(i)}\lambda^i,\\
&T_{13}=\sum_{i=0}^{N-1}A_{13}^{(i)}\lambda^i,\\
&T_{21}=\sum_{i=0}^{N-1}A_{21}^{(i)}\lambda^i,\ T_{22}=\lambda^N+\sum_{i=0}^{N-1}A_{22}^{(i)}\lambda^i,\\
&T_{23}=\sum_{i=0}^{N-1}A_{23}^{(i)}\lambda^i,\\
&T_{31}=\sum_{i=0}^{N-1}A_{31}^{(i)}\lambda^i,\ T_{32}=\sum_{i=0}^{N-1}A_{32}^{(i)}\lambda^i,\\
&T_{33}=\lambda^N+\sum_{i=0}^{N-1}A_{33}^{(i)}\lambda^i,
\end{aligned}$$

N is a natural number and $A_{mn}^{(i)}(m,n=1,2,3,0\le i\le N-1)$ are the functions of x and t, which are determined later. We can easily see that the determinant of the matrix T is a $3N$th-order polynomial of λ by calculation. As a result, $detT=\prod_{j=1}^{3N}(\lambda-\lambda_j)$, where $\lambda=\lambda_j(1\le j\le 3N)$ are the roots of $detT$. Let $\psi=(\psi_1,\psi_2,\psi_3)^T,\ \varphi=(\varphi_1,\varphi_2,\varphi_3)^T,\ \mathrm{X}=(\mathrm{X}_1,\mathrm{X}_2,\mathrm{X}_3)^T$ are three basic solutions of the spectral problems Eqs. (2.2) and (2.3), which are linearly dependent. With the aid of the gauge transformation Eq. (2.4), we can get $(\hat{\psi},\hat{\varphi},\hat{\mathrm{X}})=T(\psi,\varphi,X)$ are linearly dependent as $\lambda=\lambda_j(1\le j\le 3N)$. We introduce constants $\alpha_j^{(1)}$ and $\alpha_j^{(2)}$ to obtain the following linear algebraic systems,

$$\begin{aligned}
&\sum_{i=0}^{N-1}A_{11}^{(i)}+\alpha_j^{(1)}A_{12}^{(i)}+\alpha_j^{(2)}A_{13}^{(i)}=-\lambda_j^N,\\
&\sum_{i=0}^{N-1}A_{21}^{(i)}+\alpha_j^{(1)}A_{22}^{(i)}+\alpha_j^{(2)}A_{23}^{(i)}=-\lambda_j^N\alpha_j^{(1)},\\
&\sum_{i=0}^{N-1}A_{31}^{(i)}+\alpha_j^{(1)}A_{32}^{(i)}+\alpha_j^{(2)}A_{33}^{(i)}=-\lambda_j^N\alpha_j^{(2)},
\end{aligned} \tag{2.5}$$

with

$$\alpha_j^{(1)} = \frac{\psi_2 + \gamma_j^{(1)}\varphi_2 + \gamma_j^{(2)}\mathrm{X}_2}{\psi_1 + \gamma_j^{(1)}\varphi_1 + \gamma_j^{(2)}\mathrm{X}_1},\ \alpha_j^{(2)} = \frac{\psi_3 + \gamma_j^{(1)}\varphi_3 + \gamma_j^{(2)}\mathrm{X}_3}{\psi_1 + \gamma_j^{(1)}\varphi_1 + \gamma_j^{(2)}\mathrm{X}_1},\ (1 \leq j \leq 3N). \quad (2.6)$$

Under the gauge transformation Eq. (2.4), Eqs. (2.2) and (2.3) are transformed into a new spectral problem

$$\hat{\phi}_x = \hat{U}\left(\hat{p}, \hat{q}, \lambda\right)\hat{\phi},\ \hat{\phi}_t = \hat{V}\left(\hat{p}, \hat{q}, \lambda\right)\hat{\phi}, \quad (2.7)$$

where

$$\hat{U} = (T_x + TU)T^{-1},\ \hat{V} = (T_t + TV)T^{-1}. \quad (2.8)$$

The gauge transformation Eq. (2.4) is called the Darboux transformation if the new spectral problem Eq. (2.7) is the same form as Eq. (2.2).

Proposition 2.1. The matrix $\hat{U}$ defined by Eq. (2.8) has the same type as U, in which the transformation relation between old potentials and new ones is presented by

$$\hat{p} = p + A_{12}^{(N-1)},\ \hat{q} = q + A_{23}^{(N-1)}. \quad (2.9)$$

Proof. Assume $T^{-1} = \frac{T^*}{detT}$ and

$$(T_x + TU)T^* = \begin{pmatrix} B_{11}(\lambda) & B_{12}(\lambda) & B_{13}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) & B_{23}(\lambda) \\ B_{31}(\lambda) & B_{32}(\lambda) & B_{33}(\lambda) \end{pmatrix}, \quad (2.10)$$

obviously, $B_{sl}(\lambda_j)(1 \leq s, l \leq 3, 1 \leq j \leq 3N) = 0$. In addition, $B_{13}(\lambda)$ and $B_{31}(\lambda)$ are $(3N+1)$th-order polynomials of λ. $B_{12}(\lambda)$, $B_{21}(\lambda)$, $B_{23}(\lambda)$ and $B_{32}(\lambda)$ are $(3N)$th-order polynomials of λ. $B_{11}(\lambda)$, $B_{22}(\lambda)$, $B_{33}(\lambda)$ are $(3N-1)$th-order polynomials of λ. Thus through calculation, we can prove that $B_{11} = B_{22} = B_{33} = 0$ and Eq. (2.10) is rewritten in the following form

$$(T_x + TU)T^* = (detT)C(\lambda), \quad (2.11)$$

with

$$\begin{pmatrix} C_{11}^{(0)} & C_{12}^{(0)} & C_{13}^{(1)}\lambda + C_{13}^{(0)} \\ C_{21}^{(0)} & C_{22}^{(0)} & C_{23}^{(0)} \\ C_{31}^{(1)}\lambda + C_{31}^{(0)} & C_{32}^{(0)} & C_{33}^{(0)} \end{pmatrix},$$

where $C_{sl}^{(k)}(s, l = 1, 2, 3, k = 0, 1)$ are independent of λ. By comparing the coefficients of λ in Eq. (2.11), we have

$$\begin{aligned} &C_{11}^{(0)} = 0,\ C_{12}^{(0)} = q - A_{32}^{(N-1)} = \hat{q},\ C_{13}^{(1)} = 1,\ C_{13}^{(0)} = 0, \\ &C_{21}^{(0)} = -q - A_{23}^{(N-1)} = -\hat{q},\ C_{22}^{(0)} = 0,\ C_{23}^{(0)} = A_{21}^{(N-1)} - p = -\hat{p}, \\ &C_{31}^{(1)} = -1,\ C_{31}^{(0)} = 0,\ C_{32}^{(0)} = p + A_{12}^{(N-1)} = \hat{p},\ C_{33}^{(0)} = 0. \end{aligned}$$

It is easy to see $\hat{U} = C(\lambda)$, which means $\hat{U}$ has the same type with U. The proof is completed. □

Proposition 2.2. The matrix $\hat{V}$ defined by Eq. (2.8) has the same type as V by means of the transformation Eq. (2.9).
Proof. Let

$$(T_t + TV)T^* = \begin{pmatrix} G_{11}(\lambda) & G_{12}(\lambda) & G_{13}(\lambda) \\ G_{21}(\lambda) & G_{22}(\lambda) & G_{23}(\lambda) \\ G_{31}(\lambda) & G_{32}(\lambda) & G_{33}(\lambda) \end{pmatrix}, \tag{2.12}$$

obviously, $G_{sl}(\lambda_j)(1 \le s, l \le 3, 1 \le j \le 3N) = 0$. In addition, $G_{13}(\lambda)$ and $G_{31}(\lambda)$ are $(3N+2)$th-order polynomials of λ. $G_{12}(\lambda)$, $G_{21}(\lambda)$, $G_{23}(\lambda)$ and $G_{32}(\lambda)$ are $(3N+1)$th-order polynomials of λ. $G_{11}(\lambda)$, $G_{22}(\lambda)$, $G_{33}(\lambda)$ are $(3N-1)$th-order polynomials of λ. Thus through calculation, we can prove that $G_{11} = G_{22} = G_{33} = 0$ and Eq. (2.12) is rewritten in the following form

$$(T_t + TV)T^* = (detT)D(\lambda), \tag{2.13}$$

with

$$\begin{pmatrix} D_{11}^{(0)} & D_{12}^{(1)}\lambda + D_{12}^{(0)} & D_{13}^{(2)}\lambda^2 + D_{13}^{(1)}\lambda + D_{13}^{(0)} \\ D_{21}^{(1)}\lambda + D_{21}^{(0)} & D_{22}^{(0)} & D_{23}^{(1)}\lambda + D_{23}^{(0)} \\ D_{31}^{(2)}\lambda^2 + D_{31}^{(1)}\lambda + D_{31}^{(0)} & D_{32}^{(1)}\lambda + D_{32}^{(0)} & D_{33}^{(0)} \end{pmatrix},$$

where $D_{sl}^{(k)}(s, l = 1, 2, 3, k = 0, 1, 2)$ are independent of λ. By comparing the coefficients of λ in Eq. (2.13), we arrive at

$$\begin{aligned}
&D_{11}^{(0)} = 0,\ D_{12}^{(1)} = -q + A_{32}^{(N-1)} = -\hat{q},\ D_{12}^{(0)} = \hat{p}_x,\ D_{13}^{(2)} = -1,\ D_{13}^{(1)} = 0, \\
&D_{13}^{(0)} = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\hat{q}^2,\ D_{21}^{(1)} = q + A_{23}^{(N-1)} = \hat{q},\ D_{21}^{(0)} = -\hat{p}_x,\ D_{22}^{(0)} = 0, \\
&D_{23}^{(1)} = -A_{21}^{(N-1)} + p = \hat{p},\ D_{23}^{(0)} = \hat{q}_x, D_{31}^{(2)} = 1,\ D_{31}^{(1)} = -\tfrac{1}{2}\hat{p}^2 - \tfrac{1}{2}\hat{q}^2,\ D_{31}^{(0)} = 0, \\
&D_{32}^{(1)} = -A_{12}^{(N-1)} - p = \hat{p},\ D_{32}^{(0)} = -\hat{q}_x,\ D_{33}^{(0)} = 0,
\end{aligned}$$

which completes the proof. □
Proposition 2.3. Under the Darboux transformation Eqs. (2.4) and (2.9), every solution (p, q) can be turned into a new solution $(\hat{p}, \hat{q})$, where T is uniquely determined by the linear algebraic system Eq. (2.5).

3. Exact solutions of the Dirac-type equation

In the section, we shall utilize the Darboux transformation Eqs. (2.4) and (2.9) to gain new solutions of Eq. (1.1). In order to calculate conveniently, we choose N=1 in Eqs. (2.4), (2.5) and (2.6) which are shown as follows,

$$\hat{\phi} = T\phi,\ T = \begin{bmatrix} \lambda + A_{11} & A_{12} & A_{13} \\ A_{21} & \lambda + A_{22} & A_{23} \\ A_{31} & A_{32} & \lambda + A_{33} \end{bmatrix}, \tag{3.1}$$

and

$$\begin{aligned} A_{11}+\alpha_j^{(1)}A_{12}+\alpha_j^{(2)}A_{13} &= -\lambda_j, \\ A_{21}+\alpha_j^{(1)}A_{22}+\alpha_j^{(2)}A_{23} &= -\lambda_j\alpha_j^{(1)}, \\ A_{31}+\alpha_j^{(1)}A_{32}+\alpha_j^{(2)}A_{33} &= -\lambda_j\alpha_j^{(2)}, \end{aligned} \tag{3.2}$$

with

$$\alpha_j^{(1)}=\frac{\psi_2+\gamma_j^{(1)}\varphi_2+\gamma_j^{(2)}\mathrm{X}_2}{\psi_1+\gamma_j^{(1)}\varphi_1+\gamma_j^{(2)}\mathrm{X}_1},\ \alpha_j^{(2)}=\frac{\psi_3+\gamma_j^{(1)}\varphi_3+\gamma_j^{(2)}\mathrm{X}_3}{\psi_1+\gamma_j^{(1)}\varphi_1+\gamma_j^{(2)}\mathrm{X}_1}\ (1\le j\le 3). \tag{3.3}$$

From Eq. (3.2), we can make use of Cramer law to obtain

$$A_{12}=\frac{\Delta_1}{\Delta},\ A_{23}=\frac{\Delta_2}{\Delta}, \tag{3.4}$$

with

$$\Delta=\begin{vmatrix} 1 & \alpha_1^{(1)} & \alpha_1^{(2)} \\ 1 & \alpha_2^{(1)} & \alpha_2^{(2)} \\ 1 & \alpha_3^{(1)} & \alpha_3^{(2)} \end{vmatrix},\ \Delta_1=\begin{vmatrix} 1 & -\lambda_1 & \alpha_1^{(2)} \\ 1 & -\lambda_2 & \alpha_2^{(2)} \\ 1 & -\lambda_3 & \alpha_3^{(2)} \end{vmatrix},\ \Delta_2=\begin{vmatrix} 1 & \alpha_1^{(1)} & -\lambda_1\alpha_1^{(1)} \\ 1 & \alpha_2^{(1)} & -\lambda_2\alpha_2^{(1)} \\ 1 & \alpha_3^{(1)} & -\lambda_3\alpha_3^{(1)} \end{vmatrix}. \tag{3.5}$$

Hence, Eq.(2.9) is rewritten as

$$\hat{p}=p+\frac{\Delta_1}{\Delta},\ \hat{q}=q+\frac{\Delta_2}{\Delta}. \tag{3.6}$$

It should be noted that we should make sure $\Delta\neq 0$ with selecting suitable constants $\lambda_j,\gamma_j^{(l)}(j=1,2,3,l=1,2)$.

Above all, we select the seed solutions $p=0,\ q=0$. Then the Lax pairs Eqs. (2.2) and (2.3) can be simplified down to

$$\begin{aligned} &\phi_{1x}=\lambda\phi_3,\ \phi_{3x}=-\lambda\phi_1, \\ &\phi_{1t}=-\lambda^2\phi_3,\ \phi_{3t}=-\lambda^2\phi_1. \end{aligned} \tag{3.7}$$

Hence we can figure out three basic solutions, that is,

$$\psi(\lambda)=\begin{bmatrix} \sin(\lambda^2t-\lambda x) \\ 0 \\ -\cos(\lambda^2t-\lambda x) \end{bmatrix},\ \varphi(\lambda)=\begin{bmatrix} \cos(\lambda^2t-\lambda x) \\ 0 \\ \sin(\lambda^2t-\lambda x) \end{bmatrix},\ X(\lambda)=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{3.8}$$

From Eqs. (3.3) and (3.8), we have

$$\begin{aligned} \alpha_j^{(1)} &= \frac{\gamma_j^{(2)}}{\sin(\lambda^2t-\lambda x)+\gamma_j^{(1)}\cos(\lambda^2t-\lambda x)}, \\ \alpha_j^{(2)} &= \frac{-\cos(\lambda^2t-\lambda x)+\gamma_j^{(1)}\sin(\lambda^2t-\lambda x)}{\sin(\lambda^2t-\lambda x)+\gamma_j^{(1)}\cos(\lambda^2t-\lambda x)}. \end{aligned} \tag{3.9}$$

For example, we substitute $\lambda_1=\lambda_2=1,\ \lambda_3=-1,\ \gamma_1^{(2)}=0,\ \gamma_1^{(1)}=\gamma_3^{(2)}=1,\ \gamma_2^{(1)}=\gamma_2^{(2)}=\gamma_3^{(1)}=-1$, into Eqs. (3.5) and (3.9), then we arrive at an exact real solution by using Eq. (3.6). The plots of $\hat{p}$ and $\hat{q}$ of Eq. (1.1) are presented respectively in Figs. 1 and 2.

In addition, we can get another three basic solutions according to Eq. (3.7),

$$\psi(\lambda)=\begin{bmatrix} e^{\mathrm{i}(\lambda^2 t-\lambda x)} \\ 0 \\ -\mathrm{i}e^{\mathrm{i}(\lambda^2 t-\lambda x)} \end{bmatrix},\ \varphi(\lambda)=\begin{bmatrix} \mathrm{i}e^{\mathrm{i}(\lambda x-\lambda^2 t)} \\ 0 \\ -e^{\mathrm{i}(\lambda x-\lambda^2 t)} \end{bmatrix},\ \mathrm{X}(\lambda)=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{3.10}$$

Choosing $\lambda_1=\mathrm{i}$, $\lambda_2=-\mathrm{i}$, $\lambda_3=0$, $\gamma_1^{(1)}=\gamma_3^{(1)}=-\mathrm{i}$, $\gamma_2^{(1)}=\mathrm{i}$, $\gamma_1^{(2)}=\gamma_2^{(2)}=\gamma_3^{(2)}=1$ for obtaining an another explicit complex solution.

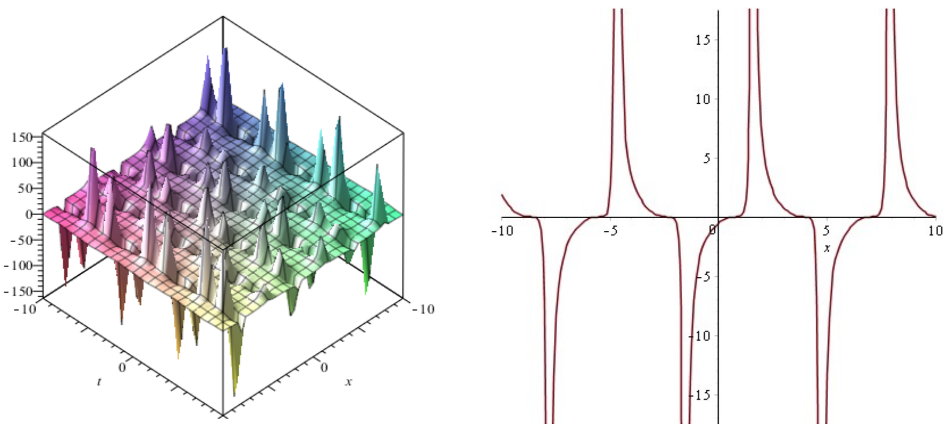

Figure 1. Plots of the intensity distribution $\hat{p}$ and the solution $\hat{p}$ at $t=0$ of Eq. (1.1) with $\lambda_1=\lambda_2=1$, $\lambda_3=-1$, $\gamma_1^{(2)}=0$, $\gamma_1^{(1)}=\gamma_3^{(2)}=1$, $\gamma_2^{(1)}=\gamma_2^{(2)}=\gamma_3^{(1)}=-1$.

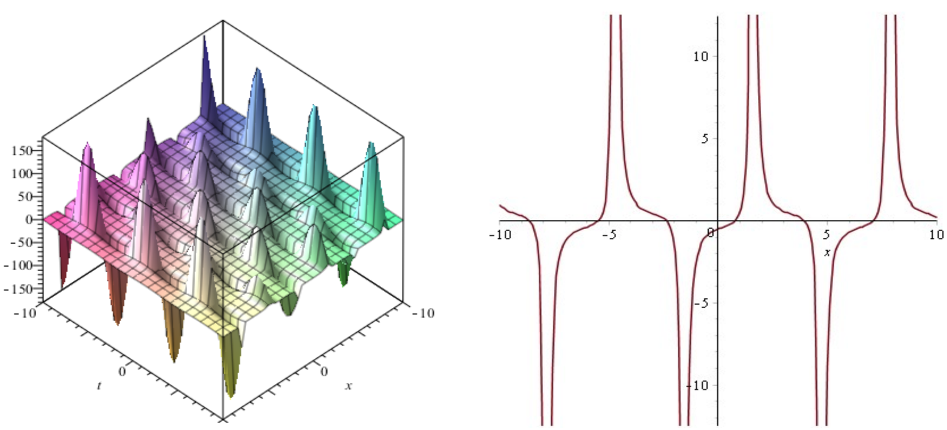

Figure 2. Plots of the intensity distribution $\hat{q}$ and the solution $\hat{q}$ at $t=0$ of Eq. (1.1) with $\lambda_1=\lambda_2=1$, $\lambda_3=-1$, $\gamma_1^{(2)}=0$, $\gamma_1^{(1)}=\gamma_3^{(2)}=1$, $\gamma_2^{(1)}=\gamma_2^{(2)}=\gamma_3^{(1)}=-1$.

4. Concluding Remarks

In this paper, we have researched the Darboux transformation of the Dirac-type equation Eq. (1.1) based on matrix spectral problems associated with so(3,$\mathbb{R}$). With the help of Maple, some explicit solutions, for example, real solutions and complex solutions have been obtained which are in accord with physical phenomena. It is noteworthy that many integrable soliton hierarchies can be derived from the the real special orthogonal Lie algebra so(3,$\mathbb{R}$) such as AKNS type, KN type and WKI type [36]. Further consideration, super integrable hierarchies with super-Hamiltonian structures [29] can be thoroughly studied in order to apply the Darboux transformation to generated soliton equations. According to the above analysis, the Darboux transformation will leave us wide research space. In addition, the binary nonlinearization [18, 19, 27, 54] of the Dirac-type equation is worth studying.

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