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# ESTIMATE FOR EVOLUTIONARY SURFACES OF PRESCRIBED MEAN CURVATURE AND THE CONVERGENCE*

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**Abstract** In the paper, we will discuss the gradient estimate for the evolutionary surfaces of prescribed mean curvature with Neumann boundary value under the condition $f_\tau \geq -\kappa$, which is the same as the one in the interior estimate by K. Ecker and generalizes the condition $f_\tau \geq 0$ studied by Gerhardt etc. Also, based on the elliptic result obtained recently, we will show the longtime behavior of surfaces moving by the velocity being equal to the mean curvature.

**Keywords** Mean curvature flow, gradient estimate, convergence.

**MSC(2010)** 35B45, 35J92.

## 1. Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ and $u$ be a smooth function defined on it. As it is known to us, the mean curvature of the graph of $u$ is

$$H = \mathrm{div}(\frac{Du}{\sqrt{1+|Du|^2}}). \tag{1.1}$$

This is no doubt one of the most important geometrical quantities of submanifolds and lots of references have appeared to study it and the quasilinear equations concerned, for instance, one can refer to [1,3,6,14,15,17,22–24,28–35] and the references therein.

Parallel with the elliptic case, the parabolic case which was historically named as the "mean curvature flow" is also an important and interesting subject in geometric analysis and partial differential equations. It usually includes two equations, one is

$$\frac{\partial u}{\partial t} = \mathrm{div}(\frac{Du}{\sqrt{1+|Du|^2}}) \tag{1.2}$$

and the other is

$$\frac{\partial u}{\partial t} = \sqrt{1+|Du|^2}\mathrm{div}(\frac{Du}{\sqrt{1+|Du|^2}}). \tag{1.3}$$

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For the first one, it describes that a set of graphs of $u(\cdot,\, t)$, denoted by $M_t$, move at the speed of the mean curvature along the $x^{n+1}$ direction. While for the second case, a series of graphs move at the speed of its mean curvature vector. Both of the flows attract the interest of many mathematicians and lots of interesting results have been deduced, one can refer to the references of this paper.

In this paper, we focus on a class of equations concerned with the first flow (1.2) which can be expressed as follows

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) - f(x,\, u), \tag{1.4}$$

where $f(x,\ \tau)$ is defined on $\Omega \times\ \mathbb{R}$. It is important both on the fundamental theory of parabolic differential equations and on the geometrical applications.

For this known equation, Gerhardt ( [5]) proved the $C^0$ estimates of $u$, $|\frac{\partial u}{\partial t}|$ and an interior gradient bound under the condition

$$\frac{\partial f}{\partial \tau} \geq 0, \tag{1.5}$$

which is an analogue to the one for capillary surfaces given by Concus and Finn ( [3]).

In 1982, Ecker ( [4]) derived the interior estimate of (1.4) under the weaker condition

$$\frac{\partial f}{\partial \tau} \geq -\kappa, \tag{1.6}$$

where $\kappa$ is a nonnegative constant. What he has proved is the following generalization of Gerhardt ( [5]).

**Lemma 1.1** (Lemma 1, [5])**.** *Assume $\Omega$ is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $u(x,\ t)$ be the solution to the parabolic equation*

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) - f(x,\ u) \ \ in \ \ \Omega \times (0,\ T), \\ u(x,0) = u_0(x) \qquad\qquad\qquad\qquad on \ \ \Omega, \end{cases} \tag{1.7}$$

*where $f(x,\ \tau)$ satisfies that $\frac{\partial f}{\partial \tau} \geq -\kappa$ with $\kappa \geq 0$. Then for $x_0 \in \Omega$ and $t \in [0,\ T)$ we have the estimate*

$$|Du(x_0,\ t)| \leq C, \tag{1.8}$$

*where $C = C(n, T, \operatorname{dist}(x_0, \partial\Omega),\, \kappa,\, |f|_{C^0(\Omega\times\mathbb{R}},\, |D_x f|_{C^0(\Omega)})$.*

Besides the interior gradient bound, Ecker in the same paper also derived the global gradient estimate for solutions of the Dirichlet problem to (1.7) as follows with $\frac{\partial f}{\partial \tau} \geq -\kappa$

$$u = \varphi \qquad \text{on} \qquad \partial\Omega \times [0,\, T], \tag{1.9}$$

and also some Hölder continuity of the solutions under several kinds of boundary conditions was derived.

However, in many researches including Guan ( [8,9]) and Xu ( [36]), concerned with the second mean curvature flow (1.3) with the following form and Neumann boundary data or prescribed contact angle boundary value,

$$\begin{cases} \frac{\partial u}{\partial t} = \sqrt{1+|Du|^2}\operatorname{div}(\frac{Du}{\sqrt{1+|Du|^2}}) - f(x,\ u) \text{ in } & \Omega \times (0,\ T), \\ u(x,0) = u_0(x) & \text{on } \ \Omega, \end{cases} \tag{1.10}$$

the condition $\frac{\partial f}{\partial \tau} \geq 0$ is crucial during the proof of the long time existence and the convergence. Remark that in the capillary surfaces problems

$$\operatorname{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = f(x,\ u), \tag{1.11}$$

the sign of $\frac{\partial f}{\partial \tau}$ describes one of the characters of the gravitational field. It then naturally arises the question whether we can get the long time existence, namely the gradient estimate of (1.7) with some kind of boundary data under the same condition as Ecker's in the interior estimate.

In this paper, we come to estimate the gradient of the solution of (1.4) under the conditions as Ecker's with Neumann boundary value conditions. Finally, we will reach the following results.

**Theorem 1.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $\partial\Omega \in C^3$, $n \geq 2$. $\nu$ is the inner unit normal. Suppose $f, \varphi$ are functions defined on $\Omega \times \mathbb{R}$ and $\overline{\Omega}$ respectively. Let $u(x,\ t)$ be the solution to the parabolic equation*

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}(\frac{Du}{\sqrt{1+|Du|^2}}) - f(x,\ u) \text{ in } & \Omega \times [0,\ T], \\ u(x,0) = u_0(x) & \text{on } \ \Omega, \\ \frac{\partial u}{\partial \nu} = \varphi(x) & \text{on } \ \partial\Omega \times [0,\ T], \end{cases} \tag{1.12}$$

*where $f(x,\ \tau)$ satisfies that $\frac{\partial f}{\partial \tau} \geq -\kappa$ with $\kappa \geq 0$ and $|\varphi|_{C^3(\overline{\Omega})} \leq L$. Then for $t \in [0,\ T]$ we have the estimate*

$$|D_x u(\ ,\ t)| \leq C \tag{1.13}$$

*in $\overline{\Omega}$, where $C = C(n, T, L, \kappa, |f|_{C^0(\Omega\times\mathbb{R})}, |D_x f|_{C^0(\Omega)})$.*

Joint with the bounds of $u$ and $\frac{\partial u}{\partial t}$ we will derive in Section 3, one can easily get the long time existence of the equation with Neumann boundary value.

**Corollary 1.1.** *Under the same conditions as described in Theorem 1.1, the parabolic equation*

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}(\frac{Du}{\sqrt{1+|Du|^2}}) - f(x,\ u) \text{ in } & \Omega \times [0,\ +\infty), \\ u(x,0) = u_0(x) & \text{on } \ \Omega, \\ \frac{\partial u}{\partial \nu} = \varphi(x) & \text{on } \ \partial\Omega \times [0,\ +\infty) \end{cases} \tag{1.14}$$

*has a smooth solution $u = u(x,\ t)$.*

In [23], for the second mean curvature flow (1.3), they got the long time existence and the convergence which showed that the solution will converge to a translating solution with constant speed. In the same paper, they also derived a compatible result for the mean curvature equation as follows.

**Lemma 1.2** (Lemma 2, [23]). *Let $\Omega$ be a strictly convex bounded domain in $\mathbb{R}^n$ with smooth boundary. For any $\varphi \in C^\infty(\overline{\Omega})$, there exists a unique $\lambda_0 \in \mathbb{R}$ and a function $\omega \in C^\infty(\overline{\Omega})$ solving*

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \lambda_0 & \text{in } \Omega\,, \\ u_\nu = \varphi\,(x) & \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

*where $\nu$ is an inward unit normal vector to $\partial\Omega$. Moreover, the solution $\omega$ is unique up to a constant.*

Based on this lemma and Theorem 1.1, we can describe the convergence result of the mean curvature flow (1.2) with Neumann boundary data.

**Theorem 1.2.** *Assume $\Omega$ is strictly convex bounded domain in $\mathbb{R}^n$, $n \geq 2$. Let $u(x,\ t)$ be the solution to the mean curvature flow*

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}(\frac{Du}{\sqrt{1+|Du|^2}}) & \text{in} \quad \Omega \times [0,\ +\infty), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = \varphi(x) & \text{on} \quad \partial\Omega \times [0,\ +\infty), \end{cases} \tag{1.16}$$

*where $(u_0)_\nu = \varphi(x)$ and $|\varphi|_{C^3(\overline{\Omega})}$ is bounded. Then $u(x,t)$ will converge to a translating solution as $\lambda_0 t + \omega$, where $\omega$ is a suitable solution to equation* (1.15)*.*

For the arrangement we proceed as below. In section 2, we list the notations and the preliminaries used during the process of the proof. In section 3, we will give the bound of $u_t$ and it then follows the $C^0$ estimate of the solution. In section 4, the gradient estimate near boundary of the solution will be deduced and then we can conclude the longtime existence of the parabolic equation by combining with the interior estimate already derived by Ecker. In the last section, we will draw out the convergence result of the evolutionary surfaces moving with velocity being equals to its mean curvature.

## 2. Notations and preliminaries

As an important preparation, we list here some properties of the distance function to the boundary of the domain.

Let $d(x)$ be the distance from the point $x$ to $\partial\Omega$ and $\nu$ be the inner unit normal along $\partial\Omega$. Denoted by

$$\Omega_\mu = \{x \in \Omega|\, d(x) < \mu\}.$$

We can know from [24] that there exists a $\mu_1 > 0$ such that $d(x) \in C^3(\Omega_{\mu_1})$ and in this annular domain we can take $Dd$ as the extension of $\nu$, denoted by $\nu$ as before. It follows that in $\Omega_{\mu_1}$

$$\begin{aligned} &|D\nu| + |D^2\nu| \le C(n, \Omega); \\ &|\nu| = 1,\ D\nu \perp \nu,\ D_\nu \nu = 0. \end{aligned} \tag{2.1}$$

We also note the following facts. When $\Omega$ is a strictly convex smooth domain, there exists a smooth defining function $h$ for $\Omega$ such that $h < 0$ in $\Omega$ and $h = 0$ on $\partial\Omega$, $\{h_{ij}\} \ge k_0\{\delta_{ij}\}$ for a constant $k_0 > 0$ and $\sup_\Omega |Dh| \le 1$, $h_\nu = -1$ and $|Dh| = 1$ on $\partial\Omega$. Because of the strict convexity of the domain, we may assume that the curvature matrix of $\partial\Omega$ satisfies $\{\kappa_{ij}\}_{1\le i,j\le n-1} \ge k_1\{\delta_{ij}\}_{1\le i,j\le n-1}$, where $k_1 > 0$ is the minimum principal curvature of the boundary.

Remark that we will use the notation $O(z)$ to indicate that $|O(z)| \le C|z|$ as $z$ is large enough, $C$ is a universal constant dependent upon some prescribed factors. This type of notations will be adopted frequently during the whole paper.

# 3. Estimate of $u_t$ and the $C^0$ estimate of $u$

For convenience we change the parabolic equation (1.12) into the following form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^n A_{ij}(Du)u_{ij} - f(x,\ u), \tag{3.1}$$

here we denote $A_{ij}$ to be $\frac{1}{v}(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2})$ and $v = \sqrt{1+|Du|^2}$ for convenience.

Derivative with parameter $t$ on both sides we then get

$$\frac{\partial u_t}{\partial t} = \sum_{i,j=1}^n A_{ij}(Du)(u_t)_{ij} + \sum_{i,j,k=1}^n A_{ij,k}(Du)u_{ij}(u_t)_k - f_\tau(x,\ u)u_t. \tag{3.2}$$

Therefore,

$$\begin{aligned} \frac{\partial(\mathbf{e}^{-\kappa t}u_t)}{\partial t} = &\sum_{i,j=1}^n A_{ij}(Du)(\mathbf{e}^{-\kappa t}u_t)_{ij} + \sum_{i,j,k=1}^n A_{ij,k}(Du)u_{ij}(\mathbf{e}^{-\kappa t}u_t)_k \\ &- (f_\tau(x,\ u) + \kappa)\mathbf{e}^{-\kappa t}u_t. \end{aligned} \tag{3.3}$$

So, according to maximum principle, if the nonnegative maximal value achieves at the point $(x_0,\ t_0)$, then only one of the following three cases would possibly occur:

(1) $t_0 = 0$;

(2) $t_0 > 0$ and $\mathbf{e}^{-\kappa t}u_t$ is a constant(thus must be $u_t(x, 0)$) on $\Omega \times [0,\ t_0]$;

(3) $t_0 > 0$ and $x_0 \in \partial\Omega$.

For the third case, thanks to Hopf lemma we have $u_{t\nu} < 0$. But on the other hand, by the boundary value, $u_{t\nu} = \varphi_t = 0$, this is a contradiction.

For the two cases remained, we can deduce that

$$u_t(x,\ t) \le \max\{\sup_{x\in\Omega} \mathbf{e}^{\kappa t}u_t(x,\ 0),\ 0\}. \tag{3.4}$$

Similarly, we can get the lower bound

$$u_t(x,\ t) \ge \min\{\inf_{x\in\Omega} \mathbf{e}^{\kappa t}u_t(x,\ 0),\ 0\}. \tag{3.5}$$

Based on the discussion about $u_t$, we then immediately get the $C^0$ estimate of $u$. In fact, by the mean value theorem, we have

$$|u(x,\, t) - u_0(x)| \le Ct\mathrm{e}^{\kappa t}, \tag{3.6}$$

where $C = \sup_{x\in\Omega} |u_t(x,\, 0)|$ is determined by $u_0,\, f$.

**Remark 3.1.** For the mean curvature flow (1.2), it is obvious that $\kappa = 0$ and we can get the estimate of $u_t$ as follows which is independent upon $t$.

$$\inf_{x\in\Omega} u_t(x,\, 0) \le u_t(x,\, t) \le \sup_{x\in\Omega} u_t(x,\, 0). \tag{3.7}$$

## 4. $C^1$ estimate of $u$

In this section, we set out to prove Theorem 1.1. The technique and the idea are mainly from [22, 23], but at the same time the main difference is the choice of the auxiliary function.

**Proof.** Based on Lemma 1.1, we only need to bound the gradient of $u$ near the boundary $\partial\Omega$. Let $w = u - \varphi(x)d$ and set

$$\Phi = \log(\mathrm{e}^{-\lambda t}|Dw|^2) + \alpha d, \qquad x \in \overline{\Omega}_{\mu_0}, \tag{4.1}$$

where $\mu_0$ is a small positive constant less than $\mu_1$ and only depends on $n,\, \Omega,\, f,\, \varphi$. Both of $\alpha,\ \lambda$ are positive constants determined later.

Set $A = |Dw|^2$ for convenience. We assume that $\Phi$ reaches its maximum at a point $(x_0,\, t_0)$ where $x_0 \in \overline{\Omega}_{\mu_0}$. In the following, we split the whole proof into three cases.

**Case 1:** $x_0 \in \partial\Omega_{\mu_0} \cap \Omega$.

If this case occurs, we then immediately get the gradient estimate by Lemma 1.1.

**Case 2:** $x_0 \in \partial\Omega$. By an observation we easily get that $\frac{\partial w}{\partial \nu} = 0$ which equivalently states that $Dw|_{\partial\Omega}$ is a tangent vector field along $\partial\Omega$. Let $D'$ be the connection of $\partial\Omega$ induced by $D$.

In this case, we can deduce that

$$\begin{aligned} 0 \ge \frac{\partial\Phi(x_0)}{\partial\nu} =& \frac{\sum_{i=1}^{n} D_i(|Dw|^2)\nu^i}{A} + \alpha \\ =& \frac{2\sum_{i,k=1}^{n} w_k w_{ik}\nu^i}{A} + \alpha \\ =& \frac{2}{A}\left( < D(\frac{\partial w}{\partial\nu}),\ Dw > - \sum_{i,k=1}^{n} w_i D_k(\nu^i) w_k \right) + \alpha \\ =& \frac{2}{A}\left( < D'(\frac{\partial w}{\partial\nu}),\ Dw > - \sum_{i,k=1}^{n} w_i D_k(\nu^i) w_k \right) + \alpha \\ =& \alpha - \frac{2\sum_{i,k=1}^{n} w_i D_k(\nu^i) w_k}{A}, \end{aligned} \tag{4.2}$$

which will cause a contradiction if we choose $\alpha$ to be a large constant determined by the geometry of $\partial\Omega$, for instance, $\alpha = 2\bar{k}+1$, where $\bar{k}$ is the maximum of the principal curvature. It indicates that this case will not occur at all.

**Case 3:** $x_0 \in \Omega_{\mu_0}$.

Under this assumption, if $t_0 = 0$, the proof is done. So in the following, we assume without loss of generality that $t_0 > 0$. Remark that all the calculations will be done at the fixed point $(x_0,\ t_0)$, and at this point a special coordinate is chosen such that $u_1 = |Du|,\ u_i = 0(i = 2, \cdots, n)$ and $u_{ij}(2 \le i, j \le n)$ is diagonal. Obviously, if we change the equation into the following form

$$\sum_{i,j=1}^{n} a_{ij}u_{ij} - v^3\frac{\partial u}{\partial t} = f(x,\ u)v^3, \qquad v = \sqrt{1+|Du|^2},$$

where $a_{ij} = (1+|Du|^2)\delta_{ij} - u_iu_j$. At this point we have

$$a_{11} = 1, a_{ii} = 1+u_1^2,\ i = 2, \cdots, n\,.$$

In the following, we come to control $|Du|$ at this point and thus obtain the whole gradient bound near the boundary. Also, we remark that if $|Du|$ is large enough, then $|Du|,\ v,\ w_1,\ |Dw|$ are equivalent with each other.

We obviously have

$$0 = \Phi_i = \frac{|Dw|^2{}_i}{A} + \alpha d_i. \tag{4.3}$$

Therefore,

$$\sum_{l=1}^{n} w_l\left(u_{li} - G_{li}\right) = -\frac{\alpha A d_i}{2}, \tag{4.4}$$

where we denoted by $G(x) = \varphi(x)d(x)$ for simplicity.

Thus, for $i = 1$,

$$u_{11} = -\frac{\alpha A d_1}{2w_1} - \sum_{l=2}^{n}\frac{w_l}{w_1}u_{1l} + O(1), \tag{4.5}$$

and for $i > 1$,

$$u_{1i} = -\frac{\alpha A d_i}{2w_1} - \frac{w_i}{w_1}u_{ii} + O(1). \tag{4.6}$$

Now, plugging (4.6) into (4.5) we then have

$$\begin{aligned} u_{11} =& -\frac{\alpha A d_1}{2w_1} - \sum_{l=2}^{n}\frac{w_l}{w_1}\left(-\frac{\alpha A d_l}{2w_1} - \frac{w_l}{w_1}u_{ll} + O(1)\right) + O(1) \\ =& -\frac{\alpha A d_1}{2w_1} + \sum_{l=2}^{n}\frac{\alpha A w_l d_l}{2{w_1}^2} + \sum_{l=2}^{n}\left(\frac{w_l}{w_1}\right)^2 u_{ll} + O(1). \end{aligned} \tag{4.7}$$

On the other hand, according to the classical differential geometry we have

$$\sum_{i,j=1}^{n} a_{ij}u_{ij} = Hv^3, \tag{4.8}$$

therefore,

$$\Delta u = Hv + \frac{{u_1}^2}{v^2}u_{11}, \qquad Hv = \frac{u_{11}}{v^2} + \sum_{l=2}^{n} u_{ll}. \tag{4.9}$$

It is a direct calculation that

$$
\begin{aligned}
v^3\Phi_t =&\frac{v^3|Dw|^2{}_t}{A} - \lambda v^3\\
=&\frac{2v^3\sum\limits_{i=1}^{n} w_i u_{it}}{A} - \lambda v^3\\
=&\frac{2v^3\sum\limits_{i=1}^{n} w_i (H - f(x,u))_i}{A} - \lambda v^3\\
=&\frac{1}{A}\left(2v^3 < Dw, DH > - 2v^3 < Dw, D_x f > - 2v^3 f_\tau < Dw, Du > - \lambda v^3 |Dw|^2\right).
\end{aligned} \tag{4.10}
$$

According to (4.3), we can deduce that

$$
\Phi_{ij} = \frac{|Dw|^2{}_{ij}}{A} - \alpha^2 d_i d_j + \alpha d_{ij}. \tag{4.11}
$$

Setting $\lambda = 2(\kappa + 1)$, then we have at the point concerned

$$
\begin{aligned}
0 \geq& \sum_{i,j=1}^{n} a_{ij}\Phi_{ij} - v^3\frac{\partial \Phi}{\partial t}\\
=&\frac{\sum\limits_{i,j=1}^{n} a_{ij}|Dw|^2{}_{ij}}{A} - \alpha^2\sum_{i,j=1}^{n} a_{ij} d_i d_j + \alpha\sum_{i,j=1}^{n} a_{ij} d_{ij}\\
&- \frac{1}{A}\left(2v^3 < Dw, DH > - 2v^3 < Dw, D_x f > - 2v^3 f_\tau < Dw, Du > - \lambda v^3 |Dw|^2\right)\\
\geq&\frac{1}{A}\left(\sum_{i,j=1}^{n} a_{ij}|Dw|^2{}_{ij} - 2v^3 < Dw, DH >\right) + v^3\\
&+ \left(\frac{2v^3 < Dw, D_x f >}{A} - \alpha^2\sum_{i,j=1}^{n} a_{ij} d_i d_j + \alpha\sum_{i,j=1}^{n} a_{ij} d_{ij}\right)\\
=& I + II + III.
\end{aligned} \tag{4.12}
$$

It is an observation that

$$
III = O(v^2). \tag{4.13}
$$

In the following, we come to settle the remaining term $I$.

$$
\begin{aligned}
I =&\frac{1}{A}\left(\sum_{i,j=1}^{n} a_{ij}|Dw|^2{}_{ij} - 2v^3 < Dw, DH >\right)\\
=&\frac{1}{A}\left(2\sum_{i,j,k=1}^{n} w_k a_{ij}(u_{ijk} - G_{ijk}) + 2\sum_{i,j,k=1}^{n} a_{ij} w_{ik} w_{jk} - 2v^3 < Dw, DH >\right)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{A}\left(2\left(\sum_{i,j,k=1}^{n} w_k a_{ij} u_{ijk} - v^3 < Dw, DH >\right) + 2\sum_{i,j,k=1}^{n} a_{ij} w_{ik} w_{jk}\right) + O(v)\\
=&\frac{I_1 + I_2}{A} + O(v).
\end{aligned}
\tag{4.14}
$$

For the first term $I_1$, joining with (4.8) and (4.9), we have

$$
\begin{aligned}
I_1 =&2\left(\sum_{i,j,k=1}^{n} w_k a_{ij} u_{ijk} - v^3 < Dw, DH >\right)\\
=&2\left(\sum_{k=1}^{n} w_k (\sum_{i,j=1}^{n} a_{ij} u_{ij})_k - \sum_{i,j,k=1}^{n} w_k a_{ij,k} u_{ij} - v^3 < Dw, DH >\right)\\
=&2 < Dw, D(Hv^3) > -2\sum_{i,j,k,l=1}^{n} w_k (2u_l u_{kl} \delta_{ij} - 2u_{ik} u_j) u_{ij} - 2v^3 < Dw, DH >\\
=&u_1(6Hv - 4\Delta u)\sum_{k=1}^{n} w_k u_{1k} + 4u_1 \sum_{i,k=1}^{n} w_k u_{ik} u_{1i}\\
=&u_1(2Hv - \frac{4{u_1}^2}{v^2} u_{11})\sum_{k=1}^{n} w_k u_{1k} + 4u_1 \sum_{i,k=1}^{n} w_k u_{ik} u_{1i}\\
=&I_{11} + I_{12}.
\end{aligned}
\tag{4.15}
$$

For $I_{11}$, by (4.4) and (4.7),

$$
\begin{aligned}
I_{11} =&u_1\left(2\left(\frac{u_{11}}{v^2} + \sum_{l=2}^{n} u_{ll}\right) - \frac{4{u_1}^2}{v^2} u_{11}\right)\sum_{k=1}^{n} w_k u_{1k}\\
=&u_1\left(2\sum_{l=2}^{n} u_{ll} + \frac{2 - 4{u_1}^2}{v^2} u_{11}\right)\sum_{k=1}^{n} w_k u_{1k}\\
=&u_1\left(2\sum_{l=2}^{n} u_{ll} + \frac{2 - 4{u_1}^2}{v^2}\left(-\frac{\alpha A d_1}{2w_1} + \sum_{l=2}^{n} \frac{\alpha A w_l d_l}{2{w_1}^2} + \sum_{l=2}^{n}\left(\frac{w_l}{w_1}\right)^2 u_{ll} + O(1)\right)\right)\\
&\times\left(-\frac{\alpha A d_1}{2} + O(v)\right)\\
=&O(v^4) + \sum_{l=2}^{n} O(v^3) u_{ll}.
\end{aligned}
\tag{4.16}
$$

For $I_{12}$, by (4.4), (4.6) and (4.7),

$$
\begin{aligned}
I_{12} =&4u_1 \sum_{i,k=1}^{n} w_k u_{ik} u_{1i}\\
=&4u_1\left(-\frac{\alpha A d_1}{2} + O(v)\right)\left(-\frac{\alpha A d_1}{2w_1} + \sum_{l=2}^{n} \frac{\alpha A w_l d_l}{2{w_1}^2} + \sum_{l=2}^{n}\left(\frac{w_l}{w_1}\right)^2 u_{ll} + O(1)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+4u_1\sum_{l=2}^{n}\left(-\frac{\alpha Ad_l}{2}+O(v)\right)\left(-\frac{\alpha Ad_l}{2w_1}-\frac{w_l}{w_1}u_{ll}+O(1)\right)\\
=&O(v^4)+\sum_{l=2}^{n}O(v^2)u_{ll}.
\end{aligned}
\tag{4.17}
$$

Therefore, combining (4.15), (4.16) and (4.17) we then have

$$
I_1=O(v^4)+\sum_{l=2}^{n}O(v^3)u_{ll}. \tag{4.18}
$$

For the term $I_2$,

$$
\begin{aligned}
I_2=&2a_{ij}w_{ik}w_{jk}=2a_{ij}u_{ik}u_{jk}-4a_{ij}u_{ik}G_{jk}+2a_{ij}G_{ik}G_{jk}\\
=&I_{21}+I_{22}+I_{23},
\end{aligned}
\tag{4.19}
$$

and we deal with these terms one by one.

Direct calculation shows that

$$
\begin{aligned}
I_{21}=&2a_{ij}u_{ik}u_{jk}=2\sum_{k=1}^{n}{u_{1k}}^2+2v^2\sum_{l=2}^{n}{u_{1l}}^2+2v^2\sum_{l=2}^{n}{u_{ll}}^2\\
=&2{u_{11}}^2+2(1+v^2)\sum_{l=2}^{n}{u_{1l}}^2+2v^2\sum_{l=2}^{n}{u_{ll}}^2.
\end{aligned}
\tag{4.20}
$$

For the second term,

$$
\begin{aligned}
I_{22}=&-4a_{ij}u_{ik}G_{jk}=-4u_{k1}G_{k1}-4v^2\sum_{l=2}^{n}u_{l1}G_{l1}-4v^2\sum_{l=2}^{n}u_{ll}G_{ll}\\
=&-4u_{11}G_{11}-4(1+v^2)\sum_{l=2}^{n}u_{l1}G_{l1}-4v^2\sum_{l=2}^{n}u_{ll}G_{ll}.
\end{aligned}
\tag{4.21}
$$

So,

$$
\begin{aligned}
I_{21}+I_{22}=&v^2\sum_{l=2}^{n}{u_{ll}}^2+2(u_{11}-G_{11})^2+2(1+v^2)\sum_{l=2}^{n}(u_{1l}-G_{1l})^2-2{G_{11}}^2\\
&+v^2\sum_{l=2}^{n}(u_{ll}-2G_{ll})^2-2(1+v^2)\sum_{l=2}^{n}{G_{1l}}^2-4v^2\sum_{l=2}^{n}{G_{ll}}^2\\
\geq&v^2\sum_{l=2}^{n}{u_{ll}}^2+O(v^2).
\end{aligned}
\tag{4.22}
$$

For the term $I_{23}$,

$$
I_{23}=2a_{ij}G_{ik}G_{jk}=O(v^2). \tag{4.23}
$$

Combining (4.19), (4.22) and (4.23) we conclude that

$$
I_2\geq v^2\sum_{l=2}^{n}{u_{ll}}^2+O(v^2). \tag{4.24}
$$

Now we can derive by (4.14), (4.18) and (4.24) that

$$I \geq \frac{1}{A}\left(v^2\sum_{l=2}^{n}{u_{ll}}^2+\sum_{l=2}^{n}O(v^3)u_{ll}+O(v^4)\right)\geq \frac{O(v^4)}{A}=O(v^2), \tag{4.25}$$

where we have used the fact that $at^2+bt\geq -\frac{b^2}{4a}$ for $a>0$.

We finally get

$$0\geq \sum_{i,j=1}^{n}a_{ij}\Phi_{ij}-v^3\frac{\partial \Phi}{\partial t}\geq v^3+O(v^2), \tag{4.26}$$

which forces $v$ to be bounded at this point.

After the discussion of the three cases we get the gradient estimate of $u$ near the boundary and thus finish the whole proof of Theorem 1.1.

□

## 5. Convergence of evolutionary surfaces moving by mean curvature

In this section, we come to prove Theorem 1.2.

Based on the long time existence result in Corollary 1.1, we need to prove a prior uniform gradient estimate being independent of the $||u||_{C^0}$, for the solution to (1.16). This is the crucial step in establishing the infinite time convergence of solutions. In this step we will make strong use of the strict convexity of the domain.

**Theorem 5.1.** *Let $\Omega$ be a smooth strictly convex bounded domain in $\mathbb{R}^n$ and $n\geq 2$. Suppose that $u(x,t)\in C^{3,2}(\Omega\times[0,\ T))$ is a solution to (1.16). Then there exists a constant $C_0=C_0(n,\Omega,u_0,\varphi(x))>0$ such that*

$$\sup_{\Omega\times[0,\ T)}|Du|\leq C_0.$$

**Proof.** Also, the idea of the proof of this lemma mainly follows [23]. To reach the conclusion of the lemma, we only need to prove that $|Du|$ can be bounded on $\overline{\Omega}\times[0,\ T']$ uniformly in $T'\in[0,T]$.

Let

$$\Phi(x,\ t)=\log|Dw|^2+f(h),$$

where

$$w=u+\varphi(x)h,\ \ f=\alpha h,$$

and $\alpha$ can be determined later. For convenience we denote by $G=-\varphi(x)h$.

We firstly show the maximum of $\Phi(x,\ t)$ on $\overline{\Omega}\times\ [0,T']$ can not be achieved at the boundary $\partial\Omega\times\ [0,T']$.

Let $n$ denote the unit inner normal vector and $1\leq i\leq n-1$ denote the tangential derivative. $D$ denotes the derivative in $\mathbb{R}^n$. By the boundary condition, $w_n=0$ on $\partial\Omega$, which means that $Dw|_{\partial\Omega}$ is a tangent vector along $\partial\Omega$. If $\Phi(x,\ t)$ attains its maximum at $(x_0, t_0) \in \partial\Omega \times [0, T']$, then at $(x_0, t_0)$, we have

$$\begin{aligned} 0 \geq \Phi_n =& \frac{|Dw|_n^2}{|Dw|^2} - \alpha \\ =& \frac{2\sum_{k=1}^{n-1} w_k D_{kn} w}{|Dw|^2} - \alpha \\ =& \frac{2}{|Dw|^2}\left(\sum_{k=1}^{n-1} w_k w_{nk} + \sum_{i,k=1}^{n-1} w_k w_i \kappa_{ik}\right) - \alpha \\ =& \frac{2\sum_{i,k=1}^{n-1} w_k w_i \kappa_{ik}}{|Dw|^2} - \alpha \\ \geq& 2k_1 - \alpha. \end{aligned} \tag{5.1}$$

By taking $0 < \alpha < 2k_1$, the maximum of $\Phi$ can only be achieved in $\Omega \times [0, T']$.

Now, only the following two cases are left to be discussed.

**Case 1:** $\Phi$ attains its maximum $(x_0, 0) \in \Omega \times \{0\}$, then there exists a constant $C = C(u_0) > 0$ such that

$$\max_{\overline{\Omega}\times[0,\ T']} v \leq C. \tag{5.2}$$

**Case 2:** $\Phi$ attains its maximum at $(x_0, t_0) \in \Omega \times (0, T']$.

As in section 4, at this point a special coordinate is chosen such that $u_1 = |Du|,\ u_i = 0 (i = 2, \cdots, n)$ and $u_{ij} (2 \leq i, j \leq n)$ is diagonal. It is obvious that

$$a_{11} = 1, a_{ii} = 1 + u_1^2,\ i = 2, \cdots, n\,.$$

Denoted $A$ to be $|Dw|^2$, we have at $(x_0, t_0)$

$$\Phi_t = \frac{|Dw|_t^2}{A}, \tag{5.3}$$

and

$$0 = \Phi_i = \frac{(|Dw|^2)_i}{A} + \alpha h_i. \tag{5.4}$$

Therefore,

$$\begin{aligned} \Phi_{ij}(x_0, t_0) =& \frac{|Dw|_{ij}^2}{|Dw|^2} - \frac{|Dw|_i^2 |Dw|_j^2}{|Dw|^4} + \alpha h_{ij} \\ =& \frac{|Dw|_{ij}^2}{|Dw|^2} + \alpha h_{ij} - \alpha^2 h_i h_j. \end{aligned} \tag{5.5}$$

Thus at $(x_0,\ t_0)$ we have

$$\begin{aligned} 0 \geq \sum_{i,j=1}^{n} a_{ij}\Phi_{ij} - v^3\Phi_t =& \frac{\sum_{i,j=1}^{n} a_{ij}(|Dw|^2)_{ij} - v^3|Dw|_t^2}{A} \\ & - \alpha^2 \sum_{i,j=1}^{n} a_{ij} h_i h_j + \alpha \sum_{i,j=1}^{n} a_{ij} h_{ij} \\ \triangleq& I + II + III. \end{aligned} \tag{5.6}$$

From (5.4), we deduce that for $i = 1, 2, \cdots, n$,

$$\sum_{l=1}^{n} w_l u_{li} = \sum_{l=1}^{n} w_l w_{li} + \sum_{l=1}^{n} w_l G_{li} = -\frac{\alpha A}{2} h_i + O(v). \tag{5.7}$$

Also it is remarkable that as $v$ is large enough, $u_1,\ v,\ w_1$ and $|Dw|$ are equivalent with each other.

It follows that for $i > 1$,

$$w_1 u_{1i} + w_i u_{ii} = O(v) - \frac{\alpha A}{2} h_i,$$

therefore,

$$u_{1i} = O(1) - \frac{\alpha A}{2w_1} h_i - \frac{w_i}{w_1} u_{ii}, \tag{5.8}$$

and for $i = 1$,

$$w_1 u_{11} + \sum_{l=2}^{n} w_l u_{l1} = O(v) - \frac{\alpha A}{2} h_1. \tag{5.9}$$

Combining (5.8) with (5.9), we then have

$$\begin{aligned} u_{11} =& O(1) - \frac{\alpha A}{2w_1} h_1 - \sum_{l=2}^{n} \frac{w_l}{w_1} \left( O(1) - \frac{\alpha A}{2w_1} h_l - \frac{w_l}{w_1} u_{ll} \right) \\ =& O(1) - \frac{\alpha A}{2w_1} h_1 + \sum_{l=2}^{n} \left( \frac{w_l}{w_1} \right)^2 u_{ll}. \end{aligned} \tag{5.10}$$

Similar discussions as in section 4 and by (5.10), we derive

$$\Delta u = Hv + \frac{{u_1}^2}{v^2} u_{11} = Hv + O(1) - \frac{{u_1}^2 \alpha A h_1}{2v^2 w_1} + \sum_{l=2}^{n} \left( \frac{u_1 w_l}{v w_1} \right)^2 u_{ll}, \tag{5.11}$$

and

$$Hv = \frac{u_{11}}{v^2} + \sum_{l=2}^{n} u_{ll} = O(v^{-2}) - \frac{\alpha A h_1}{2v^2 w_1} + \sum_{l=2}^{n} \left[ 1 + \left( \frac{w_l}{v w_1} \right)^2 \right] u_{ll}. \tag{5.12}$$

In the following, we come to settle (5.6).

It's easy to get

$$II = -\alpha^2 \left( {h_1}^2 + (1 + {u_1}^2) \sum_{i=2}^{n} {h_i}^2 \right), \tag{5.13}$$

and

$$III = \sum_{1 \le i,j \le n} \alpha a_{ij} h_{ij} \ge \alpha k_0 \left( n + (n-1) u_1^2 \right). \tag{5.14}$$

We settle the term $I$ in the rest.

Direct calculation shows that

$$\begin{aligned}
&\sum_{i,j=1}^{n} a_{ij}(|Dw|^2)_{ij} - v^3|Dw|_t^2 \\
=&2\left(\sum_{i,j,l=1}^{n} a_{ij}u_{ijl}w_l - v^3\sum_{l=1}^{n} w_l u_{lt}\right) - 2\sum_{i,j,l=1}^{n} a_{ij}G_{ijl}w_l \\
&+ 2\sum_{i,j,l=1}^{n} a_{ij}u_{il}u_{jl} - 4\sum_{i,j,l=1}^{n} a_{ij}u_{il}G_{jl} + 2\sum_{i,j,l=1}^{n} a_{ij}G_{il}G_{jl} \\
=&I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{5.15}$$

In the following, we compute these terms one by one.

For the term $I_1$, by differentiating the equation, we have

$$\begin{aligned}
I_1 =&2\sum_{l=1}^{n} w_l\left(\left(Hv^3\right)_l - \sum_{i,j=1}^{n} a_{ij,l}u_{ij} - v^3 u_{lt}\right) \\
=&6Hv\sum_{k,l=1}^{n} u_k u_{kl} w_l - 4\Delta u\sum_{k,l=1}^{n} u_k u_{kl} w_l + 4\sum_{i,j,l=1}^{n} u_i u_{ij} u_{jl} w_l \\
=&\left(6Hv - 4\Delta u\right) u_1\sum_{l=1}^{n} u_{1l} w_l + 4u_1\sum_{j,l=1}^{n} u_{1j}u_{jl}w_l \\
=&I_{11} + I_{12}.
\end{aligned} \tag{5.16}$$

For the term $I_{11}$, jointing with (5.7), (5.11) and (5.12), we derive

$$\begin{aligned}
I_{11} =&u_1\left[6Hv - 4\left(Hv + O(1) - \frac{{u_1}^2\alpha A h_1}{2v^2w_1} + \sum_{l=2}^{n}\left(\frac{u_1w_l}{vw_1}\right)^2 u_{ll}\right)\right]\left(-\frac{\alpha A}{2}h_1 + O(v)\right) \\
=&u_1\left(2Hv + O(1) + \frac{2\alpha A {u_1}^2}{v^2w_1}h_1 - 4\sum_{l=2}^{n}\left(\frac{u_1w_l}{vw_1}\right)^2 u_{ll}\right)\left(-\frac{\alpha A}{2}h_1 + O(v)\right) \\
=&-Hvu_1\alpha A h_1 + O(v^3) - \frac{\alpha^2A^2{u_1}^3}{v^2w_1}{h_1}^2 + \sum_{l=2}^{n} O_l(v)u_{ll} \\
=&\left(O(v^{-2}) + \frac{\alpha A}{2v^2w_1}h_1 - \sum_{l=2}^{n}\left[1 + \left(\frac{w_l}{vw_1}\right)^2\right]u_{ll}\right)u_1\alpha A h_1 \\
&+ O(v^3) - \frac{\alpha^2A^2{u_1}^3}{v^2w_1}{h_1}^2 + \sum_{l=2}^{n} O_l(v)u_{ll} \\
=&O(v^3) - \frac{{u_1}^3\alpha^2A^2{h_1}^2}{v^2w_1} + \sum_{l=2}^{n}\left(O_l(v) - u_1\alpha A h_1\right)u_{ll},
\end{aligned} \tag{5.17}$$

and for the term $I_{12}$, also by (5.7), (5.8) and (5.10), we obtain

$$\begin{aligned} I_{12} =&4u_1 \sum_{j=1}^{n} u_{1j} \left( -\frac{\alpha A}{2} h_j + O(v) \right) \\ =&4u_1 \left( O(1) - \frac{\alpha A}{2w_1} h_1 + \sum_{l=2}^{n} \left( \frac{w_l}{w_1} \right)^2 u_{ll} \right) \left( -\frac{\alpha A}{2} h_1 + O(v) \right) \\ &+ 4u_1 \sum_{l=2}^{n} \left( O(1) - \frac{\alpha A}{2w_1} h_l - \frac{w_l}{w_1} u_{ll} \right) \left( -\frac{\alpha A}{2} h_l + O(v) \right) \\ =&O(v^3) + \frac{u_1 \alpha^2 A^2 {h_1}^2}{w_1} + \sum_{l=2}^{n} \frac{u_1 \alpha^2 A^2 {h_l}^2}{w_1} + \sum_{l=2}^{n} O_l(v^2) u_{ll}. \end{aligned} \tag{5.18}$$

Thus by (5.17) and (5.18), we have

$$I_1 \geq O(v^3) + \frac{u_1 \alpha^2 A^2 {h_1}^2}{v^2 w_1} + \sum_{j=2}^{n} \frac{u_1 \alpha^2 A^2 {h_j}^2}{w_1} + \sum_{l=2}^{n} \left( O(v^2) - u_1 \alpha A h_1 \right) u_{ll}. \tag{5.19}$$

It is easy to observe that

$$I_2 = O(v^3), \quad I_5 = O(v^2). \tag{5.20}$$

For the term $I_4$ we get

$$\begin{aligned} I_4 =& -4u_{11} G_{11} - 4(1+v^2) \sum_{l=2}^{n} u_{1l} G_{1l} - 4v^2 \sum_{l=2}^{n} u_{ll} G_{ll} \\ \geq& - \left( 2{u_{11}}^2 + 2{G_{11}}^2 + \frac{1+v^2}{2} \sum_{l=2}^{n} {u_{1l}}^2 + 8(1+v^2) \sum_{l=2}^{n} {G_{1l}}^2 \right) \\ &- \left( \frac{v^2}{2} \sum_{l=2}^{n} {u_{ll}}^2 + 8v^2 \sum_{l=2}^{n} {G_{ll}}^2 \right). \end{aligned} \tag{5.21}$$

Therefore we have by (5.8)

$$\begin{aligned} \sum_{i=2}^{5} I_i \geq& \frac{3}{2}(1+v^2) \sum_{l=2}^{n} {u_{1l}}^2 + \frac{3v^2}{2} \sum_{l=2}^{n} {u_{ll}}^2 + O(v^3) \\ =& \frac{3}{2}(1+v^2) \sum_{l=2}^{n} \left( O(1) - \frac{\alpha A}{2w_1} h_l - \frac{w_l}{w_1} u_{ll} \right)^2 + \frac{3v^2}{2} \sum_{l=2}^{n} {u_{ll}}^2 + O(v^3) \\ =& O(v^3) + \frac{3(1+v^2)}{8} \sum_{l=2}^{n} \frac{\alpha^2 A^2}{{w_1}^2} {h_l}^2 + \sum_{l=2}^{n} \left( \frac{3v^2}{2} + O(1) \right) {u_{ll}}^2 + \sum_{l=2}^{n} O(v^2) u_{ll}. \end{aligned} \tag{5.22}$$

Combining (5.19) and (5.22), we then have

$$\begin{aligned}\sum_{i=1}^{5} I_i \geq & O(v^3) + \frac{u_1\alpha^2 A^2 {h_1}^2}{v^2 w_1} + \sum_{j=2}^{n} \frac{u_1 \alpha^2 A^2 {h_j}^2}{w_1} + \frac{3(1+v^2)}{8} \sum_{l=2}^{n} \frac{\alpha^2 A^2}{{w_1}^2} {h_l}^2 \\ & + \sum_{l=2}^{n} \left( \frac{3v^2}{2} + O(1) \right) {u_{ll}}^2 + \sum_{l=2}^{n} \left( O(v^2) - u_1 \alpha A h_1 \right) u_{ll} \\ \geq & O(v^3) + \frac{u_1\alpha^2 A^2 {h_1}^2}{v^2 w_1} + \sum_{j=2}^{n} \frac{u_1 \alpha^2 A^2 {h_j}^2}{w_1} + \frac{3(1+v^2)}{8} \sum_{l=2}^{n} \frac{\alpha^2 A^2}{{w_1}^2} {h_l}^2 \\ & - \sum_{l=2}^{n} \frac{\left[ O(v^2) - u_1 \alpha A h_1 \right]^2}{6v^2 + O(1)},\end{aligned} \tag{5.23}$$

where in the last formula, for each term $2 \leq l \leq n$, we once again use the fact that $at^2 + bt \geq -\frac{b^2}{4a}$ for $a > 0$.

Since $v$ has been assumed to be large enough, we have

$$\frac{u_1\alpha^2 A {h_1}^2}{v^2 w_1} + \sum_{j=2}^{n} \frac{u_1 \alpha^2 A {h_j}^2}{w_1} + \frac{3(1+v^2)}{8} \sum_{l=2}^{n} \frac{\alpha^2 A}{{w_1}^2} {h_l}^2 \geq \alpha^2 v^2 \sum_{l=2}^{n} {h_l}^2, \tag{5.24}$$

and

$$-\sum_{l=2}^{n} \frac{\left[ O(v^2) - u_1 \alpha A h_1 \right]^2}{6v^2 A + AO(1)} \geq -\frac{n-1}{5} \alpha^2 v^2 h_1^2 + O(1). \tag{5.25}$$

By (5.23)–(5.25), it follows that

$$I \geq O(v) + \alpha^2 v^2 \sum_{l=2}^{n} {h_l}^2 - \frac{n-1}{5} \alpha^2 v^2 h_1^2. \tag{5.26}$$

Then by (5.6), (5.13), (5.14) and (5.26), we obtain

$$\begin{aligned}0 \geq \sum_{i,j=1}^{n} a_{ij} \Phi_{ij} - v^3 \Phi_t \geq & O(v) + \alpha^2 v^2 \sum_{i=2}^{n} {h_i}^2 - \frac{n-1}{5} \alpha^2 v^2 h_1^2 + \alpha k_0 \left[ n + (n-1){u_1}^2 \right] \\ & - \alpha^2 \left( {h_1}^2 + (1 + {u_1}^2) \sum_{i=2}^{n} {h_i}^2 \right) \\ \geq & O(v) + \alpha (n-1) k_0 v^2 - \frac{n-1}{5} \alpha^2 v^2 h_1^2.\end{aligned} \tag{5.27}$$

Taking $0 < \alpha < \min\{2k_0, 2k_1\}$, we know $|Du|$ must be bounded at this point. And by an easy argument we then reach

$$\sup_{\Omega \times [0,\ T)} |Du| \leq C_0$$

for a universal constant $C_0$ depending upon the quantities described in the lemma.

Combining all the cases above, we finish the proof of the theorem. □

The longtime existence of the solution to equation (1.16) is obvious to us according to Corollary 1.1 and now we study its asymptotic behavior on the strictly convex bounded domain in $\mathbb{R}^n$. Remark that we have already obtained uniform estimates on $\frac{\partial u}{\partial t}, |Du|$ as long as a smooth solution exists in Remark 3.1 and Theorem 5.1. Thus, the proof of the convergence results is almost the same as the corresponding part in [23], we would like to omit it and one can refer to the procedure in [23] for details. This complete the proof of Theorem 1.2.

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