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OSCILLATORY PROPERTIES OF CERTAIN NONLINEAR FRACTIONAL NABLA DIFFERENCE EQUATIONS

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Abstract In this paper, we investigate the oscillation of a class of nonlinear fractional nabla difference equations. Some oscillation criteria are established.

Keywords Oscillation, nonlinear fractional nabla difference equation, Riemann-Liouville fractional nabla difference operator.

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1. Introduction

As we know, the theory of fractional differential equations and their applications have been investigated extensively in the past few years. For example, see the literatures [6,14–17,23–28] and the references therein. In 1980s, Miller and Ross [21] and Gray and Zhang [11] firstly introduced the definitions of non–integer order differences and sums, which were the origination of the theory of discrete fractional calculus. After then, several authors had a strong interest in studying the theory of fractional difference equations. A lot of excellent results have been established. For example, we refer the readers to [1–5,7–10,12,13,18–20,22] and the references therein.

The oscillation theory is a very important part of the qualitative theory of fractional difference equations. However, to the best of authors' knowledge, up to now, very little is known regarding the oscillatory behavior of fractional difference equations [7,8,18–20,22].

In this paper, we investigate the oscillation of fractional nabla difference equations of the form

$$\begin{cases} \nabla(\nabla_a^\alpha x(t)) + q(t)f(x(t)) = g(t), & t \in \mathbb{N}_a, \\ \nabla_a^{-(1-\alpha)} x(t)\big|_{t=a} = c, \end{cases} \tag{1.1}$$

where $\nabla f(t) = f(t) - f(t-1)$, c and α are constants, $0 < \alpha < 1$, $\nabla_a^\alpha x$ is the Riemann–Liouville fractional nabla difference operator of order α of x, $a \geq 0$ is a real number, and $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$.

Throughout this paper, we always assume that

(A) $f : \mathbb{R} \to \mathbb{R}$, and $xf(x) > 0$ for $x \neq 0$, $g : \mathbb{N}_a \to \mathbb{R}$, and $q(t) \geq 0$, $t \in \mathbb{N}_a$.

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A solution $x(t)$ of the Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

The paper is organized as follows. In Section 2, we present some basic definitions and lemmas in order to prove our main results. In Section 3, we establish some results for the oscillation of the Eq. (1.1). In Section 4, we construct some examples to show that the assumptions of our main results can not be dropped.

2. Preliminaries

In this section, we collect some basic definitions and lemmas that will be important to us in what follows. These and other related results and their proofs can be found in [1,3,13].

Definition 2.1. Let $\nu > 0$. The ν-th fractional sum f is defined by

$$\nabla_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t} (t-s+1)^{\overline{\nu-1}} f(s), \tag{2.1}$$

for $t \in \mathbb{N}_a$, where Γ is the gamma function, and

$$t^{\overline{\nu}} = \frac{\Gamma(t+\nu)}{\Gamma(t)}. \tag{2.2}$$

Definition 2.2. Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer. Set $\nu = m - \mu$. The μ-th fractional nabla difference is defined as

$$\nabla_a^{\mu} f(t) = \nabla_a^{m-\nu} f(t) = \nabla_a^{m} \nabla_a^{-\nu} f(t). \tag{2.3}$$

Lemma 2.1. *Let f be a real–valued function defined on $\mathbb{N}_a$, and let $\mu, \nu > 0$. Then*

$$\nabla_a^{-\nu}[\nabla_a^{-\mu} f(t)] = \nabla_a^{-(\mu+\nu)} f(t) = \nabla_a^{-\mu}[\nabla_a^{-\nu} f(t)], \tag{2.4}$$

and

$$\nabla_{a+1}^{-\nu} \nabla f(t) = \nabla \nabla_a^{-\nu} f(t) - \frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)} f(a). \tag{2.5}$$

Lemma 2.2. *For every $t \in \mathbb{N}_a$,*

$$\nabla_a^{-\nu} (t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a+1)^{\overline{\nu+\mu}}. \tag{2.6}$$

Lemma 2.3. *Let*

$$E(t) = \sum_{s=a}^{t} (t-s+1)^{\overline{-\alpha}} x(s), \ \ t \in \mathbb{N}_a. \tag{2.7}$$

Then

$$\nabla E(t) = \Gamma(1-\alpha) \nabla_a^{\alpha} x(t). \tag{2.8}$$

Proof. Using Definition 2.1, it follows from (2.7) that

$$\begin{aligned} E(t) &= \sum_{s=a}^{t} (t-s+1)^{\overline{-\alpha}} x(s) = \sum_{s=a}^{t} (t-s+1)^{\overline{(1-\alpha)-1}} x(s) \\ &= \Gamma(1-\alpha) \nabla_a^{-(1-\alpha)} x(t). \end{aligned} \tag{2.9}$$

Using Definition 2.2, it follows from (2.9) that

$$\nabla E(t) = \Gamma(1-\alpha)\nabla\nabla_a^{-(1-\alpha)}x(t) = \Gamma(1-\alpha)\nabla_a^{\alpha}x(t).$$

The proof of Lemma 2.3 is complete. □

3. Main results

In this section, we establish the oscillation results of Eq. (1.1). First, we give two lemmas.

Lemma 3.1. *If $x(t) > 0$ is a solution of the Eq. (1.1), then $x(t)$ satisfies the difference inequality*

$$\nabla(\nabla_a^{\alpha}x(t)) \leq g(t),\ t \in \mathbb{N}_a. \tag{3.1}$$

Proof. Noting the assumption (A), from the Eq. (1.1), we have

$$\nabla(\nabla_a^{\alpha}x(t)) = -q(t)f(x(t)) + g(t) \leq g(t),$$

which shows $x(t) > 0$ is a solution of the inequality (3.1). The proof is complete. □

Similarly we have the following lemma.

Lemma 3.2. *If $x(t) < 0$ is a solution of the Eq. (1.1), then $x(t)$ satisfies the difference inequality*

$$\nabla(\nabla_a^{\alpha}x(t)) \geq g(t),\ t \in \mathbb{N}_a. \tag{3.2}$$

Next, we introduce our main results. By Lemma 3.1 and Lemma 3.2, we immediately obtain the following conclusion.

Theorem 3.1. *If the inequality (3.1) has no eventually positive solutions and the inequality (3.2) has no eventually negative solutions, then every solution $x(t)$ of the Eq. (1.1) is oscillatory.*

Theorem 3.2. *Assume that $x(t)$ is a solution of Eq. (1.1) and there exists $t_0 \in \mathbb{N}_a$ such that $\nabla_a^{\alpha}x(t)|_{t=t_0} = C$ exists. If*

$$\liminf_{t\to\infty}\left\{(t-a)^{1-\alpha}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\alpha-1}}\left[C + \sum_{\xi=t_0+1}^{s}g(\xi)\right]\right\} = -\infty, \tag{3.3}$$

and

$$\limsup_{t\to\infty}\left\{(t-a)^{1-\alpha}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\alpha-1}}\left[C + \sum_{\xi=t_0+1}^{s}g(\xi)\right]\right\} = +\infty, \tag{3.4}$$

then the solution $x(t)$ of the Eq. (1.1) is oscillatory.

Proof. Suppose to the contrary that the solution $x(t)$ is a non-oscillatory solution of Eq. (1.1). Then $x(t)$ is eventually positive or eventually negative.

If $x(t) > 0,\ t \geq t_0$, by Lemma 3.1, we obtain

$$\nabla(\nabla_a^{\alpha}x(t)) \leq g(t),\ t \in \mathbb{N}_{t_0}. \tag{3.5}$$

Summing both sides of (3.5) from t_0+1 to t, we obtain

$$\nabla_a^\alpha x(t) \le \nabla_a^\alpha x(t_0) + \sum_{s=t_0+1}^{t} g(s) = C + \sum_{s=t_0+1}^{t} g(s). \tag{3.6}$$

Applying the $\nabla_{a+1}^{-\alpha}$ operator to the above inequality (3.6), we have

$$\nabla_{a+1}^{-\alpha}\nabla_a^\alpha x(t) \le \nabla_{a+1}^{-\alpha}\Big[C + \sum_{s=t_0+1}^{t} g(s)\Big]. \tag{3.7}$$

Using Definition 2.2, Lemma 2.1 in the left–hand side of (3.7) and noting the initial condition of Eq. (1.1), we obtain

$$\begin{aligned}\nabla_{a+1}^{-\alpha}\nabla_a^\alpha x(t) &= \nabla_{a+1}^{-\alpha}\nabla\nabla_a^{-(1-\alpha)}x(t)\\ &= \nabla\nabla_a^{-\alpha}\nabla_a^{-(1-\alpha)}x(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\nabla_a^{-(1-\alpha)}x(a)\\ &= x(t) - \frac{c}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}}.\end{aligned} \tag{3.8}$$

Using Definition 2.1, it follows from the right–hand side of (3.7) that

$$\begin{aligned}&\nabla_{a+1}^{-\alpha}\Big[C + \sum_{s=t_0+1}^{t} g(s)\Big]\\ &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\alpha-1}}\Big[C + \sum_{\xi=t_0+1}^{s} g(\xi)\Big].\end{aligned} \tag{3.9}$$

Combining (3.7)–(3.9), we have

$$\begin{aligned}x(t) \le{}& \frac{c}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}}\\ &+\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\alpha-1}}\Big[C + \sum_{\xi=t_0+1}^{s} g(\xi)\Big].\end{aligned} \tag{3.10}$$

It follows from (3.10) that

$$\begin{aligned}&\Gamma(\alpha)(t-a)^{1-\alpha}x(t)\\ &\le c(t-a+1)^{\overline{\alpha-1}}(t-a)^{1-\alpha}\\ &\quad+(t-a)^{1-\alpha}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\alpha-1}}\Big[C + \sum_{\xi=t_0+1}^{s} g(\xi)\Big].\end{aligned} \tag{3.11}$$

By using the Stirling's formula [7]

$$\lim_{t\to\infty}\frac{\Gamma(t)t^\varepsilon}{\Gamma(t+\varepsilon)} = 1,\ \varepsilon > 0,$$

we obtain

$$\begin{aligned}
&\lim_{t\to\infty}(t-a)^{1-\alpha}(t-a+1)^{\overline{\alpha-1}}\\
&=\lim_{t\to\infty}(t-a)^{1-\alpha}\frac{\Gamma(t-a+1+\alpha-1)}{\Gamma(t-a+1)}\\
&=\lim_{t\to\infty}(t-a)^{1-\alpha}\frac{\Gamma(t-a+\alpha)}{(t-a)\Gamma(t-a)}\\
&=\lim_{t\to\infty}\frac{\Gamma(t-a+\alpha)}{(t-a)^{\alpha}\Gamma(t-a)}\\
&=1.
\end{aligned} \tag{3.12}$$

Noting (3.12) and taking $t\to\infty$ in (3.11), we have

$$\liminf_{t\to\infty}\Big\{(t-a)^{1-\alpha}x(t)\Big\}\le-\infty,$$

which contradicts with $x(t)>0$.

If $x(t)<0,\ t\ge t_0$, by Lemma 3.2, using the above mentioned method, we easily obtain a contradiction. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Assume that $x(t)$ is a solution of Eq. (1.1) and there exists $t_0\in\mathbb{N}_a$ such that $\nabla_a^{\alpha}x(t)|_{t=t_0}=C$ exists. If*

$$\liminf_{t\to\infty}\Big\{\sum_{s=t_0+1}^{t}\Big(1-\frac{s-1}{t}\Big)g(s)\Big\}=-\infty, \tag{3.13}$$

and

$$\limsup_{t\to\infty}\Big\{\sum_{s=t_0+1}^{t}\Big(1-\frac{s-1}{t}\Big)g(s)\Big\}=+\infty, \tag{3.14}$$

then the solution $x(t)$ of the Eq. (1.1) is oscillatory.

Proof. Suppose to the contrary that there is a non-oscillatory solution $x(t)$. Then $x(t)$ is eventually positive or eventually negative.

If $x(t)>0,\ t\ge t_0$. As in the proof of Theorem 3.2, we obtain (3.6). Using Lemma 2.3, it follows from (3.6) that

$$\nabla E(t)\le\Gamma(1-\alpha)\Big[C+\sum_{s=t_0+1}^{t}g(s)\Big]. \tag{3.15}$$

Summing both sides of (3.15) from t_0+1 to t, we have

$$\begin{aligned}
E(t)&\le E(t_0)+\Gamma(1-\alpha)\sum_{s=t_0+1}^{t}\Big[C+\sum_{\xi=t_0+1}^{s}g(\xi)\Big]\\
&=E(t_0)+C\Gamma(1-\alpha)(t-t_0)+\Gamma(1-\alpha)\sum_{s=t_0+1}^{t}(t-s+1)g(s).
\end{aligned} \tag{3.16}$$

Therefore,

$$\frac{E(t)}{t}\le\frac{E(t_0)}{t}+C\Gamma(1-\alpha)\Big(1-\frac{t_0}{t}\Big)+\Gamma(1-\alpha)\sum_{s=t_0+1}^{t}\Big(1-\frac{s-1}{t}\Big)g(s). \tag{3.17}$$

Letting $t \to \infty$ in (3.17) and noting the assumption (3.13), we obtain

$$\liminf_{t\to\infty} \frac{E(t)}{t} = -\infty,$$

which contradicts with $E(t) > 0$.

If $x(t) < 0,\ t \geq t_0$, Noting the condition (3.14) and using the above mentioned method, we easily obtain a contradiction. The proof of Theorem 3.3 is complete. $\square$

4. Examples

In this section, we introduce some examples to illustrate our main results.

Example 4.1. Consider the following fractional nabla difference equation

$$\begin{cases} \nabla(\nabla_1^{\frac{1}{2}} x(t)) + \dfrac{\Gamma(\frac{1}{3})\Gamma(t)}{\Gamma(t+\frac{1}{2})} x(t) = \Gamma(\dfrac{1}{3}),\ t \in \mathbb{N}_1, \\ \nabla_1^{-\frac{1}{2}} x(t)\Big|_{t=1} = \dfrac{\sqrt{\pi}}{2}. \end{cases} \tag{4.1}$$

Here $\alpha = \frac{1}{2}, q(t) = \frac{\Gamma(\frac{1}{3})\Gamma(t)}{\Gamma(t+\frac{1}{2})},\ f(x(t)) = x(t),\ g(t) = \Gamma(\frac{1}{3})$. By careful calculation, we find that $x(t) = t^{\overline{\frac{1}{2}}} > 0$ is a non-oscillatory solution of Eq. (4.1).

In fact, using Lemma 2.2, we have

$$\begin{aligned} \nabla_1^{-\frac{1}{2}} x(t) &= \nabla_1^{-\frac{1}{2}} t^{\overline{\frac{1}{2}}} = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{2}+\frac{1}{2}+1)} t^{\overline{\frac{1}{2}+\frac{1}{2}}} \\ &= \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{\Gamma(2)} t^{\overline{1}} = \frac{\sqrt{\pi}}{2} t. \end{aligned} \tag{4.2}$$

By Definition 2.2, we obtain

$$\begin{aligned} \nabla_1^{\frac{1}{2}} x(t) &= \nabla_1^{\frac{1}{2}} t^{\overline{\frac{1}{2}}} = \nabla_1 \nabla_1^{-\frac{1}{2}} t^{\overline{\frac{1}{2}}} \\ &= \nabla_1 \left(\frac{1}{2}\sqrt{\pi} t \right) = \frac{\sqrt{\pi}}{2}. \end{aligned} \tag{4.3}$$

Therefore,

$$\nabla(\nabla_1^{\frac{1}{2}} x(t)) = \nabla(\nabla_1^{\frac{1}{2}} t^{\overline{\frac{1}{2}}}) = 0. \tag{4.4}$$

Using the relation (2.2) of Definition 2.1, we obtain

$$x(t) = t^{\overline{\frac{1}{2}}} = \frac{\Gamma(t+\frac{1}{2})}{\Gamma(t)}. \tag{4.5}$$

Combining (4.2)–(4.5), we conclude that $x(t) = t^{\overline{\frac{1}{2}}}$ is a solution of Eq. (4.1).

For the solution $x(t) = t^{\overline{\frac{1}{2}}}$ of Eq. (4.1), it is easy to see that there exists $t_0 \in \mathbb{N}_1$ such that $\nabla_1^{\frac{1}{2}} x(t)|_{t=t_0} = C = \frac{\sqrt{\pi}}{2}$ exists, and

$$\begin{aligned} &(t-1)^{\frac{1}{2}} \sum_{s=2}^{t} (t-s+1)^{\overline{-\frac{1}{2}}} \left[C + \sum_{\xi=t_0+1}^{s} g(\xi) \right] \\ &= (t-1)^{\frac{1}{2}} \sum_{s=2}^{t} (t-s+1)^{\overline{-\frac{1}{2}}} \left[\frac{\sqrt{\pi}}{2} + \sum_{\xi=t_0+1}^{s} \Gamma(\frac{1}{3}) \right] \\ &> 0,\ t \in \mathbb{N}_1,\ t \geq 2, \end{aligned} \tag{4.6}$$

which shows that the condition (3.3) of Theorem 3.2 does not hold.

Example 4.2. Consider the following fractional nabla difference equation

$$\begin{cases} \nabla(\nabla_1^{\frac{1}{3}}x(t)) + \dfrac{2\Gamma(\frac{1}{4})\Gamma(t)}{\Gamma(t+\frac{1}{3})}x(t) = 2\Gamma(\dfrac{1}{4}),\ t \in \mathbb{N}_1, \\ \nabla_1^{-\frac{2}{3}}x(t)\Big|_{t=1} = \dfrac{1}{3}\Gamma(\dfrac{1}{3}). \end{cases} \tag{4.7}$$

Here $\alpha = \frac{1}{3}$, $q(t) = \frac{2\Gamma(\frac{1}{4})\Gamma(t)}{\Gamma(t+\frac{1}{3})}$, $f(x(t)) = x(t)$, $g(t) = 2\Gamma(\frac{1}{4})$. Obviously, there exists $t_0 \in \mathbb{N}_1$ such that

$$\sum_{s=t_0+1}^{t}\Big(1-\frac{s-1}{t}\Big)g(s) = 2\Gamma(\frac{1}{4})\sum_{s=t_0+1}^{t}\Big(1-\frac{s-1}{t}\Big) > 0. \tag{4.8}$$

Thus, the condition (3.13) of Theorem 3.3 does not hold. In fact, using a similar way in Example 4.1, we can verify that $x(t) = t^{\overline{\frac{1}{3}}} > 0$ is a non-oscillatory solution of Eq. (4.7).

Example 4.3. Consider the following fractional nabla difference equation

$$\begin{cases} \nabla(\nabla_1^{\frac{1}{2}}x(t)) + \dfrac{2t}{\Gamma(t+\frac{1}{2})}x(t) = (-1)^t e^t - (-1)^{t-1}e^{t-1},\ t \in \mathbb{N}_1, \\ \nabla_1^{-\frac{1}{2}}x(t)\Big|_{t=1} = c_1,\ (c_1 \text{ is a constant}). \end{cases} \tag{4.9}$$

Assume that $x(t)$ is a solution of Eq. (4.9) and there exists $t_0 \in \mathbb{N}_1$ such that $\nabla_1^{\frac{1}{2}}x(t)|_{t=t_0} = C$ exists.

Here $\alpha = \frac{1}{2}$, $q(t) = \frac{2t}{\Gamma(t+\frac{1}{2})}$ $f(x(t)) = x(t)$, $g(t) = (-1)^t e^t - (-1)^{t-1}e^{t-1}$. Obviously,

$$\sum_{\xi=t_0+1}^{s} g(\xi) = (-1)^s e^s - (-1)^{t_0}e^{t_0}. \tag{4.10}$$

It is easy to see that $(t-s+1)^{\overline{\alpha-1}} = (t-s+1)^{\overline{-\frac{1}{2}}}$ is increased for s, and

$$(t-s+1)^{\overline{-\frac{1}{2}}} \le \sqrt{\pi} \quad \text{for } s = 2,3,\cdots,t. \tag{4.11}$$

Let

$$M(t) = \sum_{s=2}^{t}(t-s+1)^{\overline{-\frac{1}{2}}}\Big[C + \sum_{\xi=t_0+1}^{s} g(\xi)\Big].$$

It follows from (4.10) and (4.11) that $M(t)$ is oscillatory and $|M(t)|$ is a strictly monotone increasing function. Therefore, we obtain

$$\begin{aligned} &\liminf_{t\to\infty}\Big\{(t-a)^{1-\alpha}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\alpha-1}}\Big[C + \sum_{\xi=t_0+1}^{s} g(\xi)\Big]\Big\} \\ &= \liminf_{t\to\infty}\Big\{(t-1)^{\frac{1}{2}}\sum_{s=2}^{t}(t-s+1)^{\overline{-\frac{1}{2}}}\Big[C + (-1)^s e^s - (-1)^{t_0}e^{t_0}\Big]\Big\} \\ &= -\infty, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\left\{(t-a)^{1-\alpha}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\alpha-1}}\Big[C+\sum_{\xi=t_0+1}^{s}g(\xi)\Big]\right\}\\&=\limsup_{t\to\infty}\left\{(t-1)^{\frac{1}{2}}\sum_{s=2}^{t}(t-s+1)^{\overline{-\frac{1}{2}}}\Big[C+(-1)^s e^s-(-1)^{t_0}e^{t_0}\Big]\right\}\\&=+\infty,\end{aligned}\tag{4.13}$$

which show that the conditions in Theorem 3.2 are satisfied. By Theorem 3.2, the solution $x(t)$ of the Eq. (4.9) is oscillatory.

Remark 4.1. Example 4.1 and Example 4.2 show that the assumptions of Theorem 3.2 and Theorem 3.3 can not be dropped, respectively.

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