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SOLITARY WAVE SOLUTIONS TO THE TZITZÉICA TYPE EQUATIONS OBTAINED BY A NEW EFFICIENT APPROACH

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Abstract The properties of Tzitzéica equations in nonlinear optics have received a great attention of many recent studies. In this work, the so-called generalized exponential rational function method (GERFM) has been applied for finding the analytical solution of two nonlinear partial differential equations type of equations, namely Tzitzéica-Dodd-Bullough and Tzitzéica equation. The proposed method provides a wide range of closed-form travelling solutions leading to a very effective and simply-applied method by means of a symbolic computation system. The method not only provides a general form of solutions with some free parameters but also shows potential application to other types of nonlinear partial differential equations.

Keywords Symbolic computations, nonlinear partial differential equations,exponential rational function method, solitary wave solutions, Tzitzéica-Dodd-Bullough equation, Tzitzéica equation.

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1. Introduction

A great number of real life phenomena rising in different fields such as high-energy physics, optics, quantum mechanics, chemical physics, biology, fluid mechanics, electricity, propagation of shallow water waves and so on, can be described by partial differential equations [1–42]. In order to understand these phenomena, it is important to seek the exact solutions of their fundamental equations. Most of these equations are nonlinear and, in general, are often very complicated to solve explicitly. Therefore, the powerful and efficient methods to find exact solutions of nonlinear equations has a lot of interest. Numerous approaches and techniques such as Exp-function method [10], extended Tanh-coth method [2], the unified method (UM) [22–25], the generalized unified method (GUM) [26–30,41], the extended Sinh-Gordon equation expansion method [5,6,42], etc. has been used. Those techniques are based on the physical properties of the traveling wave solutions which can be expressed in terms of the trigonometrical functions. In this paper, we present an analytic solution for such nonlinear equations using the well-known exponential rational function method. Then, in order to make obvious the effectiveness and usefulness of the method, we apply it to find solitary wave solutions of the Tzitzéica-

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Dodd-Bullough equation

$$u_{xt} - e^{-u} - e^{-2u} = 0, \tag{1.1}$$

and Tzitzéica equation

$$u_{tt} - u_{xx} - e^{u} + e^{-2u} = 0, \tag{1.2}$$

where $u = u(x,t)$ is the displacement of the point x at time t. The Tzitzéica equation [32, 33] and other related Tzitzéica type equations play an important role in in nonlinear optics. For a better understanding of those equations the reader can be referred to the [1, 8, 9, 15–17, 37] and references therein.

The outline of this paper is organized as follows: In Section 2, we give the description of the proposed method. Then the application of GERFM to the Tzitzéica-Dodd-Bullough equation and the Tzitzéica equation are presented in Section 3. In order to see the physical features and mechanism for some of the obtained results, we provide Section 4 and illustrated some features with suitable choices of the parameters. Finally, some conclusive remarks are included in Section 5.

2. Description of the GERFM

Following previous works on similar application in [7, 21], we introduce the key steps of GERF method as follows:

Step (i) We suppose that given nonlinear partial differential equation for $u(x,t)$ to be in the form

$$\mathcal{N}(u, u_x, u_t, u_{xx}, \ldots) = 0, \tag{2.1}$$

in order to blackuce a nonlinear partial differential equation can be blackuced to an ODE

$$\mathcal{N}(u, u', u'', u''', \ldots) = 0, \tag{2.2}$$

by the transformation $\xi = kx + lt$ is the wave variable, where k and l are constants to be determined later.

Step (ii) Consider

$$\Phi(\xi) = \frac{\alpha_1 e^{\beta_1 \xi} + \alpha_2 e^{\beta_2 \xi}}{\alpha_3 e^{\beta_3 \xi} + \alpha_4 e^{\beta_4 \xi}}, \tag{2.3}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_4$ and $\alpha_1, \alpha_2, \ldots, \alpha_4$ are real or complex numbers such that the travelling wave solution of Eq.2.1 can be expressed as follows:

$$u(\xi) = A_0 + \sum_{n=1}^{m} A_n \Phi(\xi)^n + \sum_{n=1}^{m} B_k \Phi(\xi)^{-n}. \tag{2.4}$$

Unknown coefficients $A_k (0 \le n \le N)$ and $B_k (1 \le n \le N)$ are constants to be determined, such that solution 2.4 satisfies the nonlinear ordinary differential equation 2.2. Note that, the positive integer N can be determined by applying the homogeneous balance method between the highest order derivatives and nonlinear terms appearing in Eq. 2.2.

Step (iii) Inserting (2.4) into Eq. (2.2) with known value of m obtained in Step (ii). Collecting the coefficients of the resulting polynomials in terms of $Y_i = e^{\beta_i \xi}$, for $i = 1, 2, 3, 4$, then setting the coefficients to zero, we can get a set of equations for $A_0, A_k, B_n (n = 1, 2, \ldots, N), k, l$ with the aid of symbolic computation in Maple.

Step (iv) Solving the algebraic equations in Step (iii), then substituting the solutions in (2.4).

3. Application of GERFM

In this section, we apply the method to obtain travelling wave solutions of the Tzitzéica-Dodd-Bullough equation(1.1) and Tzitzéica equation (1.2).

3.1. Tzitzéica-Dodd-Bullough equation

Taking advantage of the transformation as $v = e^{?u}$ or $u =? \ln v$, we can rewrite equation (1.1) as follows

$$-vv_{xt} - v_t v_x - v^3 - v^4 = 0. \tag{3.1}$$

let us consider the wave transformation $\xi = kx + lt$, where $k, l \neq 0$ to be determined later. The travelling wave variable $v = v(\xi)$ permits us to convert the Eq.(3.1) to the following ordinary differential equation:

$$kl\left(v'^2 - vv''\right) - v^3 - v^4 = 0. \tag{3.2}$$

According to Step ii of the method, considering the homogeneous balance method between the vv'' and v^4 in (3.2), we obtain $N = 1$. So from 2.4, the soliton wave solution of the equation will be as

$$u(\xi) = A_0 + A_1\Phi(\xi) + B_1\Phi(\xi)^{-1}. \tag{3.3}$$

Substituting (3.3) into Eq.(3.2) and following step iii of the GERFM, we derive a set of algebraic equations for k, l, A_0, A_1, B_1 and α_i, β_i for $i = 1, 2, 3, 4$. Solving the above algebraic equations obtained of (1.2), we have the following sets of coefficients for the non-trivial solutions of (1.2) as given below:

Family 1: $\alpha = [2, 1, 1, 1]$ and $\beta = [1, 0, 1, 0]$, which gives

$$\Phi\left(\xi\right) = \frac{2\,e^{\xi} + 1}{e^{\xi} + 1}. \tag{3.4}$$

Case 1.1:

$$l = -\frac{1}{k}, A_0 = 1, A_1 = 0, B_1 = -2,$$

and

$$v\left(\xi\right) = -\frac{1}{1 + 2\,e^{\xi}}$$

therefore new exact solutions of the Tzitzéica equation is obtained as

$$u_1\left(x, t\right) = \ln\left(-\frac{1}{1 + 2\,e^{k\,x - \frac{t}{k}}}\right).$$

Case 1.2:

$$l = -\frac{1}{k}, A_0 = -2, A_1 = 0, B_1 = 2,$$

and

$$v(\xi) = -\frac{2e^{\xi}}{2\,e^{\xi}+1},$$

therefore we get

$$u_2(x,t) = \ln\left(-\frac{2e^{k\,x-\frac{t}{k}}}{2\,e^{k\,x-\frac{t}{k}}+1}\right).$$

Case 1.3: $l=-\frac{1}{k}, A_0=-2, A_1=1, B_1=0,$

$$v(\xi) = -\frac{1}{1+e^{\xi}},$$

therefore we get

$$u_3(x,t) = \ln\left(-\frac{1}{1+e^{k\,x-\frac{t}{k}}}\right).$$

Case 1.4: $l=-\frac{1}{k}, A_0=1, A_1=-1, B_1=0,$

$$v(\xi) = -\frac{e^{\xi}}{1+e^{\xi}},$$

therefore we get

$$u_4(x,t) = \ln\left(-\frac{e^{k\,x-\frac{t}{k}}}{1+e^{k\,x-\frac{t}{k}}}\right).$$

Family 2: $\alpha=[1+i, 1-i, 1, 1]$ and $\beta=[i, -i, , i, -i]$, which gives

$$\Phi(\xi) = \frac{-\sin(\xi)+\cos(\xi)}{\cos(\xi)}. \tag{3.5}$$

Case 2.1: $l=\frac{1}{4k}, A_0=-\frac{1}{2}\mp\frac{i}{2}, A_1=0, B_1=\pm i$

$$v(\xi) = \frac{\pm 2\,i\sin(\xi)\cos(\xi)-2\,\cos^2(\xi)+1\pm i}{4\,\cos^2(\xi)-2},$$

therefore we get

$$u_{5,6}(x,t)=\ln\left(\frac{\pm 2i\sin\left(kx+\frac{t}{4k}\right)\cos\left(kx+\frac{t}{4k}\right)-2\cos^2\left(kx+\frac{t}{4k}\right)+1\pm i}{4\cos^2\left(kx+\frac{t}{4k}\right)-2}\right).$$

Case 2.2: $kl=-\frac{1}{4k}, A_0=-1, A_1=\frac{1}{2}, B_1=1,$

$$v(\xi) = -\frac{1}{2\cos(\xi)\left(\sin(\xi)-\cos(\xi)\right)},$$

therefore we get

$$u_7(x,t)=-\ln\left(2\cos\left(kx-\frac{t}{4k}\right)\left(\cos\left(kx-\frac{t}{4k}\right)-\sin\left(kx-\frac{t}{4k}\right)\right)\right).$$

Case 2.3: $l = \frac{1}{4k}, A_0 = -\frac{1}{2} \pm \frac{i}{2}, A_1 = \mp\frac{i}{2}, B_1 = 0$

$$v(\xi) = \frac{\pm i \sin(\xi) - \cos(\xi)}{2\cos(\xi)},$$

therefore we get

$$u_{8,9}(x,t) = \ln\left(\frac{\pm i \sin\left(k\,x + \frac{t}{4k}\right) - \cos\left(k\,x + \frac{t}{4k}\right)}{2\cos\left(k\,x + \frac{t}{4k}\right)}\right).$$

Family 3: $\alpha = [1-i, 1+i, 1, 1]$ and $\beta = [i, -i, i, -i]$, which gives

$$\Phi(\xi) = \frac{\cos(\xi) + \sin(\xi)}{\cos(\xi)}. \tag{3.6}$$

Case 3.1: $l = \frac{1}{4k}, A_0 = -\frac{1}{2} \mp \frac{i}{2}, A_1 = 0, B_1 = \pm i,$

$$v(\xi) = \frac{\mp 2\, i \sin(\xi)\cos(\xi) - 2\,\cos^2(\xi) + 1 \pm i}{4\,\cos^2(\xi) - 2},$$

therefore we get

$$u_{10,11}(x,t) = \ln\left(\frac{\mp 2i \sin\left(kx + \frac{t}{4k}\right)\cos\left(kx + \frac{t}{4k}\right) - 2\cos^2\left(kx + \frac{t}{4k}\right) + 1 \pm i}{4\cos^2\left(kx + \frac{t}{4k}\right) - 2}\right).$$

Case 3.2: $l = -\frac{1}{4k}, A_0 = -1, A_1 = \frac{1}{2}, B_1 = 1$

$$v(\xi) = \frac{1}{2\cos(\xi)\left(\cos(\xi) + \sin(\xi)\right)},$$

therefore we get

$$u_{12}(x,t) = -\ln\left(2\cos\left(kx - \frac{t}{4k}\right)\left(\cos\left(kx - \frac{t}{4k}\right) + \sin\left(kx - \frac{t}{4k}\right)\right)\right).$$

Family 4: $\alpha = [-3, 2, 1, 1]$ and $\beta = [1, 0, 1, 0]$, which gives

$$\phi(\xi) = \frac{-2 - 3\,e^{\xi}}{1 + e^{\xi}}. \tag{3.7}$$

Case 4.1: $l = -\frac{1}{k}, A_0 = 2, A_1 = 0, B_1 = 6,$

$$v(\xi) = \frac{-2}{2 + 3\,e^{\xi}},$$

therefore we get

$$u_{13}(x,t) = \ln\left(\frac{-2}{2 + 3\,e^{k\,x - \frac{t}{k}}},\right).$$

Case 4.2: $l = -\frac{1}{k}, A_0 = -3, A_1 = 0, B_1 = -6,$

$$v(\xi) = \frac{-3e^{\xi}}{2 + 3\,e^{\xi}},$$

therefore we get

$$u_{14}(x,t) = \ln\left(\frac{-3e^{k\,x - \frac{t}{k}}}{2 + 3\,e^{k\,x - \frac{t}{k}}}\right).$$

Case 4.3: $l=-\frac{1}{25k}, A_0=-1, A_1=-\frac{1}{5}, B_1=-\frac{6}{5}$,

$$v(\xi)=-\frac{e^{\xi}}{15\,e^{2\xi}+25\,e^{\xi}+10},$$

therefore we get

$$u_{15}(x,t)=\ln\left(-\frac{e^{k\,x-\frac{t}{25k}}}{15\,e^{2k\,x-\frac{2t}{25k}}+25\,e^{k\,x-\frac{t}{25k}}+10}\right).$$

Family 5: $\alpha=[-2,-1,1,1]$ and $\beta=[1,0,1,0]$, which gives

$$\phi(\xi)=\frac{-1-2\,e^{\xi}}{1+e^{\xi}}. \tag{3.8}$$

Case 5.1: $l=-\frac{1}{9k}, A_0=-1, A_1=\frac{1}{3}, B_1=\frac{2}{3}$,

$$v(\xi)=-\frac{e^{\xi}}{6\,e^{2\xi}+9\,e^{\xi}+3},$$

therefore we get

$$u_{16}(x,t)=\ln\left(-\frac{e^{k\,x-\frac{t}{9k}}}{6\,e^{2k\,x-\frac{2t}{9k}}+9\,e^{k\,x-\frac{t}{9k}}+3}\right).$$

Family 6: $\alpha=[-2-i,-2+i,1,1]$ and $\beta=[i,-i,i,-i]$, which gives

$$\Phi(\xi)=\frac{\sin(\xi)-2\,\cos(\xi)}{\cos(\xi)}. \tag{3.9}$$

Case 6.1: $l=\frac{1}{4k}, A_0=-\frac{1}{2}\pm i, A_1=0, B_1=\pm\frac{5}{2}\,i$,

$$v(\xi)=\frac{\mp 5\,i\cos(\xi)\sin(\xi)-5\,\cos^2(\xi)+1\mp 2\,i}{10\,\cos^2(\xi)-2},$$

therefore we get

$$u_{17,18}(x,t)=\ln\left(\frac{\mp 5i\cos\left(kx+\frac{t}{4k}\right)\sin\left(kx+\frac{t}{4k}\right)-5\cos^2\left(kx+\frac{t}{4k}\right)+1\mp 2i}{10\cos^2\left(kx+\frac{t}{4k}\right)-2}\right).$$

Case 6.2: $l=-\frac{1}{16k}, A_0=-1, A_1=-\frac{1}{4}, B_1=-\frac{5}{4}$,

$$v(\xi)=-\frac{1}{4\cos(\xi)\left(\sin(\xi)-2\,\cos(\xi)\right)},$$

therefore we get

$$u_{19}(x,t)=-\ln\left(4\cos\left(kx-\frac{t}{16k}\right)\left(2\cos\left(kx-\frac{t}{16k}\right)-\sin\left(kx-\frac{t}{16k}\right)\right)\right).$$

Family 7: $\alpha=[1,2,1,1]$ and $\beta=[1,0,1,0]$, which gives

$$\phi(\xi)=\frac{2+e^{\xi}}{1+e^{\xi}}. \tag{3.10}$$

Case 7.1: $l=-\frac{1}{k}, A_0=1, A_1=0, B_1=-2$,

$$v(\xi)=-\frac{e^{\xi}}{2+e^{\xi}},$$

therefore we get

$$u_{20}(x,t)=\ln\left(-\frac{e^{k\,x-\frac{t}{k}}}{2+e^{k\,x-\frac{t}{k}}}\right).$$

Case 7.2: $l=-\frac{1}{k}, A_0=-2, A_1=0, B_1=2$,

$$v(\xi)=-\frac{2}{2+e^{\xi}},$$

therefore we get

$$u_{21}(x,t)=\ln\left(-\frac{2}{2+e^{k\,x-\frac{t}{k}}}\right).$$

Case 7.3: $l=-\frac{1}{9k}, A_0=-1, A_1=\frac{1}{3}, B_1=\frac{2}{3}$,

$$v(\xi)=-\frac{e^{\xi}}{(3+3\,e^{\xi})(2+e^{\xi})},$$

therefore we get

$$u_{22}(x,t)=\ln\left(-\frac{e^{k\,x-\frac{t}{9k}}}{\left(3+3\,e^{k\,x-\frac{t}{9k}}\right)\left(2+e^{k\,x-\frac{t}{9k}}\right)}\right).$$

Family 8: $\alpha=[2,3,1,1]$ and $\beta=[1,0,1,0]$, which gives

$$\phi(\xi)=\frac{3+2\,e^{\xi}}{1+e^{\xi}}. \tag{3.11}$$

Case 8.1: $l=-\frac{1}{k}, A_0=2, A_1=0, B_1=-6$,

$$v(\xi)=-\frac{2e^{\xi}}{3+2\,e^{\xi}},$$

therefore we get

$$u_{23}(x,t)=\ln\left(-\frac{2e^{k\,x-\frac{t}{k}}}{3+2\,e^{k\,x-\frac{t}{k}}}\right).$$

Case 8.2: $l=-\frac{1}{k}, A_0=-3, A_1=0, B_1=6$,

$$v(\xi)=-\frac{3}{3+2\,e^{\xi}},$$

therefore we get

$$u_{24}(x,t)=\ln\left(-\frac{3}{3+2\,e^{k\,x-\frac{t}{k}}}\right).$$

Case 8.3: $l=-\frac{1}{25k}, A_0=-1, A_1=\frac{1}{5}, B_1=\frac{6}{5}$,

$$v(\xi)=-\frac{e^{\xi}}{10\,e^{2\xi}+25\,e^{\xi}+15},$$

therefore we get

$$u_{25}(x,t)=\ln\left(-\frac{e^{k\,x-\frac{t}{25k}}}{10\,e^{2k\,x-\frac{2t}{25k}}+25\,e^{k\,x-\frac{t}{25k}}+15}\right).$$

Family 9: $\alpha=[2-i,2+i,1,1]$ and $\beta=[i,-i,i,-i]$, which gives

$$\Phi(\xi)=\frac{\sin(\xi)+2\cos(\xi)}{\cos(\xi)}. \tag{3.12}$$

Case 9.1: $l=-\frac{1}{16k}, A_0=-1, A_1=\frac{1}{4}, B_1=\frac{5}{4}$,

$$v(\xi)=\frac{1}{4\cos(\xi)(\sin(\xi)+2\cos(\xi))},$$

therefore we get

$$u_{26}(x,t)=-\ln\left(\cos\left(kx-\frac{t}{16k}\right)\left(4\sin\left(kx-\frac{t}{16k}\right)+8\cos\left(kx-\frac{t}{16k}\right)\right)\right).$$

Family 10: $\alpha=[1-i,-1-i,-1,1]$ and $\beta=[i,-i,i,-i]$, which gives

$$\Phi(\xi)=\frac{-\sin(\xi)+\cos(\xi)}{\sin(\xi)}. \tag{3.13}$$

Case 10.1: $l=\frac{1}{4k}, A_0=-\frac{1}{2}\pm\frac{i}{2}, A_1=0, B_1=\pm i$,

$$v(\xi)=\frac{\pm 2\,i\cos(\xi)\sin(\xi)-2\cos^2(\xi)+1\pm i}{4\cos^2(\xi)-2},$$

therefore we get

$$u_{27,28}(x,t)=\ln\left(\frac{\pm 2i\cos\left(kx+\frac{t}{4k}\right)\sin\left(kx+\frac{t}{4k}\right)-2\cos^2\left(kx+\frac{t}{4k}\right)+1\pm i}{4\cos^2\left(kx+\frac{t}{4k}\right)-2}\right).$$

Case 10.2: $l=-\frac{1}{4k}, A_0=-1, A_1=-\frac{1}{2}, B_1=-1$,

$$v(\xi)=\frac{1}{2\sin(\xi)(\sin(\xi)-\cos(\xi))},$$

therefore we get

$$u_{29}(x,t)=-\ln\left(2\sin\left(kx-\frac{t}{4k}\right)\left(\sin\left(kx-\frac{t}{4k}\right)-\cos\left(kx-\frac{t}{4k}\right)\right)\right).$$

Case 10.3: $l=\frac{1}{4k}, A_0=-\frac{1}{2}\pm\frac{i}{2}, A_1=\pm\frac{i}{2}, B_1=0$,

$$v(\xi)=\frac{\pm i\cos(\xi)-\sin(\xi)}{2\sin(\xi)},$$

therefore we get

$$u_{30,31}(x,t)=\ln\left(\frac{\pm i\cos\left(k\,x+\frac{t}{4k}\right)-\sin\left(k\,x+\frac{t}{4k}\right)}{2\sin\left(k\,x+\frac{t}{4k}\right)}\right).$$

Family 11: $\alpha = [-2-i, 2-i, -1, 1]$ and $\beta = [i, -i, i, -i]$, which gives

$$\Phi(\xi) = \frac{\cos(\xi) + 2\sin(\xi)}{\sin(\xi)}. \tag{3.14}$$

Case 11.1: $l = \frac{1}{4k}, A_0 = -\frac{1}{2} \mp i, A_1 = 0, B_1 = \pm\frac{5}{2} i,$

$$v(\xi) = \frac{\pm 5\, i \cos(\xi)\sin(\xi) - 5\cos^2(\xi) + 4 \mp 2\, i}{10\cos^2(\xi) - 8},$$

therefore we get

$$u_{32,33}(x,t) = \ln\left(\frac{\pm 5i\cos\left(kx + \frac{t}{4k}\right)\sin\left(kx + \frac{t}{4k}\right) - 5\cos^2\left(kx + \frac{t}{4k}\right) + 4 \mp 2\, i}{10\cos^2\left(kx + \frac{t}{4k}\right) - 8}\right).$$

Case 11.2: $l = -\frac{1}{16k}, A_0 = -1, A_1 = \frac{1}{4}, B_1 = \frac{5}{4},$

$$v(\xi) = \frac{1}{4\sin(\xi)\left(\cos(\xi) + 2\sin(\xi)\right)},$$

therefore we get

$$u_{34}(x,t) = -\ln\left(4\sin\left(kx - \frac{t}{16k}\right)\left(\cos\left(kx - \frac{t}{16k}\right) + 2\sin\left(kx - \frac{t}{16k}\right)\right)\right).$$

Family 12: $\alpha = [-1-i, 1-i, -1, 1]$ and $\beta = [i, -i, i, -i]$, which gives

$$\Phi(\xi) = \frac{\cos(\xi) + \sin(\xi)}{\sin(\xi)}. \tag{3.15}$$

Case 12.1: $l = -\frac{1}{4k}, A_0 = -1, A_1 = \frac{1}{2}, B_1 = 1,$

$$v(\xi) = \frac{1}{2\sin(\xi)\left(\cos(\xi) + \sin(\xi)\right)},$$

therefore we get

$$u_{35}(x,t) = -\ln\left(2\sin\left(kx - \frac{t}{4k}\right)\left(\cos\left(kx - \frac{t}{4k}\right) + \sin\left(kx - \frac{t}{4k}\right)\right)\right).$$

Family 13: $\alpha = [2-i, 2-i, -1, 1]$ and $\beta = [i, -i, i, -i]$, which gives

$$\Phi(\xi) = \frac{\cos(\xi) - 2\sin(\xi)}{\sin(\xi)}. \tag{3.16}$$

Case 13.1: $l = -\frac{1}{16k}, A_0 = -1, A_1 = -\frac{1}{4}, B_1 = -\frac{5}{4},$

$$v(\xi) = -\frac{1}{4\sin(\xi)\left(\cos(\xi) - 2\sin(\xi)\right)},$$

therefore we get

$$u_{36}(x,t) = -\ln\left(\sin\left(kx - \frac{t}{16k}\right)\left(8\sin\left(kx - \frac{t}{16k}\right) - 4\cos\left(kx - \frac{t}{16k}\right)\right)\right).$$

Family 14: $\alpha = [i, -i, 1, 1]$ and $\beta = [i, -i, i, -i]$, which gives

$$\Phi(\xi) = -\frac{\sin(\xi)}{\cos(\xi)}. \tag{3.17}$$

Case 14.1: $l = \frac{1}{16k}, A_0 = -\frac{1}{2}, A_1 = \pm\frac{i}{4}, B_1 = \mp\frac{i}{4}$,

$$v(\xi) = \frac{-2\cos(\xi)\sin(\xi) \pm 2i\cos^2(\xi) \mp i}{4\cos(\xi)\sin(\xi)},$$

therefore we get

$$u_{37,38}(x,t) = \ln\left(\frac{-2\cos\left(kx+\frac{t}{16k}\right)\sin\left(kx+\frac{t}{16k}\right) \pm 2i\cos^2\left(kx+\frac{t}{16k}\right) \mp i}{4\cos\left(kx+\frac{t}{16k}\right)\sin\left(kx+\frac{t}{16k}\right)}\right).$$

Family 15: $\alpha = [1, -1, 1, 1]$ and $\beta = [1, -1, 1, -1]$, which gives

$$\Phi(\xi) = \frac{\sinh(\xi)}{\cosh(\xi)}. \tag{3.18}$$

Case 15.1: $l = -\frac{1}{4k}, A_0 = -\frac{1}{2}, A_1 = \pm\frac{1}{2}, B_1 = 0$,

$$v(\xi) = \frac{-\cosh(\xi) \pm \sinh(\xi)}{2\cosh(\xi)},$$

therefore we get

$$u_{39,40}(x,t) = \ln\left(\frac{-\cosh\left(kx - \frac{t}{4k}\right) \pm \sinh\left(kx - \frac{t}{4k}\right)}{2\cosh\left(kx - \frac{t}{4k}\right)}\right).$$

Case 15.2: $l = -\frac{1}{16k}, A_0 = -\frac{1}{2}, A_1 = \pm\frac{1}{4}, B_1 = \pm\frac{1}{4}$,

$$v(\xi) = \frac{\pm 2\cosh^2(\xi) - 2\sinh 2(\xi)\cosh(\xi) \mp 1}{4\sinh^2(\xi)\cosh^2(\xi)}$$

therefore we get

$$u_{41,42}(x,t) = \ln\left(\frac{\pm 2\cosh^2(kx-\frac{t}{16k}) - 2\sinh 2(kx-\frac{t}{16k})\cosh(kx-\frac{t}{16k}) \mp 1}{4\sinh^2(kx-\frac{t}{16k})\cosh^2(kx-\frac{t}{16k})}\right).$$

3.2. Tzitzéica equation

Now, we apply the method to obtain travelling wave solutions of the Tzitzéica equation (1.2). Under the given transformation $v = e^{??u}$ or $u =? \ln v$, it can be blackuced as follows

$$vv_{tt} - vv_{xx} - v_t^2 + v_x^2 - v^3 + 1 = 0. \tag{3.19}$$

Let us consider the wave transformation $\xi = kx + lt$, where $k, l \neq 0$ to be determined later. The travelling wave variable $v = v(\xi)$ permits us to convert the Eq.(3.19) to the following ordinary differential equation:

$$\left(l^2 - k^2\right)\left(v'^2 - vv''\right) - v^3 + 1 = 0. \tag{3.20}$$

The solution of Eq.(3.20) can be written in the form of Eq.(2.4). Balancing the vv'' and v^3 gives $m = 2$. According to method and from (2.4), the soliton wave solution of the equation will be as

$$u(\xi) = A_0 + A_1\Phi(\xi) + A_2\Phi^2(\xi) + B_1\Phi(\xi)^{-1} + B_2\Phi(\xi)^{-2}. \tag{3.21}$$

By substituting solution (3.21), into Eq. (3.20), and following the step as described in Section 2, we will derive a nonlinear algebraic system whose solution gives:

Family 10: $\alpha = [1-i, -1-i, -1, 1]$ and $\beta = [i, -i, i, -i]$, which gives

$$\Phi(\xi) = \frac{-\sin(\xi) + \cos(\xi)}{\sin(\xi)}. \tag{3.22}$$

Case 10.1:

$$l = \pm\sqrt{k^2 - \frac{3}{4}}, A_0 = -2, A_1 = 0, A_2 = 0, B_1 = -6, B_2 = -6,$$

and

$$v(\xi) = \frac{2\cos(\xi)\sin(\xi) + 2}{2\cos(\xi)\sin(\xi) - 1},$$

therefore we get

$$u_1(x,t) = \ln\left(\frac{2\cos\left(kx \pm t\sqrt{k^2 - \frac{3}{4}}\right)\sin\left(kx \pm t\sqrt{k^2 - \frac{3}{4}}\right) + 2}{2\cos\left(kx \pm t\sqrt{k^2 - \frac{3}{4}}\right)\sin\left(kx \pm t\sqrt{k^2 - \frac{3}{4}}\right) - 1}\right).$$

Case 10.2:

$$l = \pm\sqrt{k^2 \pm 3\sqrt{\frac{3}{8}}\,i + \frac{3}{8}}, A_0 = \pm i\sqrt{3} + 1, A_1 = A_2 = 0, B_1 = B_2 = 3 \pm 3\,i\sqrt{3},$$

and

$$v(\xi) = \frac{(\sin(\xi)\cos(\xi) + 1)\left(\mp i\sqrt{3} - 1\right)}{2\sin(\xi)\cos(\xi) - 1},$$

therefore we get

$$u_{2,3}(x,t) = \ln\left(\frac{\left(\sin\left(kx \pm t\sqrt{k^2 + \frac{3}{8} \pm 3\sqrt{\frac{3}{8}}\,i}\right)\cos\left(kx \pm t\sqrt{k^2 + \frac{3}{8} \pm 3\sqrt{\frac{3}{8}}\,i}\right) + 1\right)\left(\mp i\sqrt{3} - 1\right)}{2\sin\left(kx \pm t\sqrt{k^2 + \frac{3}{8} \pm 3\sqrt{\frac{3}{8}}\,i}\right)\cos\left(kx \pm t\sqrt{k^2 + \frac{3}{8} \pm 3\sqrt{\frac{3}{8}}\,i}\right) - 1}\right).$$

Case 10.3:

$$l = \pm\sqrt{k^2 - \frac{3}{4}}, A_0 = -2, A_1 = -3, A_2 = -\frac{3}{2}, B_1 = B_2 = 0,$$

and

$$v(\xi) = -\frac{2\cos^2(\xi) + 1}{2\sin^2(\xi)},$$

therefore we get

$$u_4(x,t) = \ln\left(-\frac{2\cos^2\left(kx \pm t\sqrt{k^2 - \frac{3}{4}}\right) + 1}{2\sin^2\left(kx \pm t\sqrt{k^2 - \frac{3}{4}}\right)}\right).$$

Case 10.4:

$$l=\pm\sqrt{k^2+\frac{3}{8}\mp\frac{3}{8}i\sqrt{3}}, A_0=1\mp i\sqrt{3}, A_1=\frac{3}{2}\mp\frac{3}{2}i\sqrt{3},$$
$$A_2=\frac{3}{4}\mp\frac{3}{4}i\sqrt{3}, B_1=B_2=0,$$

and

$$v\left(\xi\right)=\frac{\left(2\,\cos^2\left(\xi\right)+1\right)\left(\mp i\sqrt{3}+1\right)}{4\sin^2\left(\xi\right)},$$

therefore we get

$$u_{5,6}(x,t)=\ln\left(\frac{\left(2\cos^2\left(kx\pm t\sqrt{k^2+\frac{3}{8}\mp\frac{3}{8}i\sqrt{3}}\right)+1\right)\left(\mp i\sqrt{3}+1\right)}{4\sin^2\left(kx\pm t\sqrt{k^2+\frac{3}{8}\mp\frac{3}{8}i\sqrt{3}}\right)}\right).$$

Family 11: $\alpha=[-2-i,2-i,-1,1]$ and $\beta=[i,-i,i,-i]$, which gives

$$\Phi\left(\xi\right)=\frac{\cos\left(\xi\right)+2\,\sin\left(\xi\right)}{\sin\left(\xi\right)}. \tag{3.23}$$

Case 11.1:

$$l=\pm\sqrt{k^2-\frac{3}{4}}, A_0=-1\frac{3}{2}, A_1=0, A_2=0, B_1=30, B_2=-\frac{75}{2},$$

and

$$v\left(\xi\right)=\frac{50\,\cos^4\left(\xi\right)+60\,\cos\left(\xi\right)\sin\left(\xi\right)-35\,\cos^2\left(\xi\right)-28}{2\left(5\,\cos^2\left(\xi\right)-4\right)^2},$$

therefore we get

$$u_7\left(x,t\right)=\ln\left(\frac{50\,\cos^4\left(\xi\right)+60\,\cos\left(\xi\right)\sin\left(\xi\right)-35\,\cos^2\left(\xi\right)-28}{2\left(5\,\cos^2\left(\xi\right)-4\right)^2}\right),$$

where

$$\xi=k\,x\pm t\sqrt{k^2-\frac{3}{4}}.$$

Case 11.2:

$$l=\pm\sqrt{k^2+\frac{3}{8}\pm\frac{3\,i}{8}\sqrt{3}}, A_0=\pm\frac{13\,i}{4}\sqrt{3}+\frac{13}{4}, A_1=A_2=0,$$
$$B_1=\mp 15\,i\sqrt{3}-15, B_2=\pm\frac{75\,i}{4}\sqrt{3}+\frac{75}{4},$$

and

$$v(\xi)=\frac{\left(\pm i\sqrt{3}-1\right)\left(50\cos^4(\xi)-35\cos^2(\xi)+60\sin(\xi)\cos(\xi)-28\right)}{4\left(5\cos^2(\xi)-4\right)^2},$$

therefore we get

$$u_{8,9}(x,t)=\ln\left(\frac{\left(\pm i\sqrt{3}-1\right)\left(50\cos^4(\xi)-35\cos^2(\xi)+60\sin(\xi)\cos(\xi)-28\right)}{4\left(5\cos^2(\xi)-4\right)^2}\right).$$

where

$$\xi = k\,x \pm t\sqrt{k^2+\frac{3}{8}\pm\frac{3\,i}{8}\sqrt{3}}.$$

Family 12: $\alpha=[-1-i,1-i,1,-1]$ and $\beta=[i,-i,i,-i]$, which gives

$$\Phi\left(\xi\right)=\frac{\cos\left(\xi\right)+\sin\left(\xi\right)}{\sin\left(\xi\right)}. \tag{3.24}$$

Case 12.1:

$$l=\pm\sqrt{k^2-\frac{3}{4}}, A_0=-2, A_1=A_2=0, B_1=6, B_2=-6,$$

and

$$v\left(\xi\right)=\frac{2\,\sin\left(\xi\right)\cos\left(\xi\right)-2}{2\,\sin\left(\xi\right)\cos\left(\xi\right)+1},$$

therefore we get

$$u_{10}\left(x,t\right)=\ln\left(\frac{2\,\sin\left(k\,x\pm t\sqrt{k^2-\frac{3}{4}}\right)\cos\left(k\,x\pm t\sqrt{k^2-\frac{3}{4}}\right)-2}{2\,\sin\left(k\,x\pm t\sqrt{k^2-\frac{3}{4}}\right)\cos\left(k\,x\pm t\sqrt{k^2-\frac{3}{4}}\right)+1}\right).$$

Case 12.2:

$$l=\pm\sqrt{k^2+\frac{3}{8}\pm\frac{3}{8}\,i\sqrt{3}}, A_0=\pm i\sqrt{3}+1, A_1=A_2=0,$$
$$B_1=-3\mp 3\,i\sqrt{3}, B_2=3\pm 3\,i\sqrt{3},$$

and

$$v\left(\xi\right)=\frac{\left(\sin\left(\xi\right)\cos\left(\xi\right)-1\right)\left(-1\mp i\sqrt{3}\right)}{2\,\sin\left(\xi\right)\cos\left(\xi\right)+1},$$

therefore we get

$$u_{11,12}(x,t)=\ln\left(\frac{\left(\sin\left(k\,x\pm t\sqrt{k^2+\frac{3}{8}\pm\frac{3}{8}\,i\sqrt{3}}\right)\cos\left(k\,x\pm t\sqrt{k^2+\frac{3}{8}\pm\frac{3}{8}\,i\sqrt{3}}\right)-1\right)\left(-1\mp i\sqrt{3}\right)}{2\,\sin\left(k\,x\pm t\sqrt{k^2+\frac{3}{8}\pm\frac{3}{8}\,i\sqrt{3}}\right)\cos\left(k\,x\pm t\sqrt{k^2+\frac{3}{8}\pm\frac{3}{8}\,i\sqrt{3}}\right)+1}\right).$$

Family 13: $\alpha=[2-i,-2-i,-1,1]$ and $\beta=[i,-i,i,-i]$, which gives

$$\Phi\left(\xi\right)=\frac{\cos\left(\xi\right)-2\,\sin\left(\xi\right)}{\sin\left(\xi\right)}. \tag{3.25}$$

Case 13.1:

$$l=\pm\sqrt{k^2-\frac{3}{4}}, A_0=-\frac{13}{2}, A_1=0, A_2=0, B_1=-30, B_2=-\frac{75}{2},$$

and

$$v(\xi) = \frac{50\ \cos^4(\xi) - 60\ \sin(\xi)\cos(\xi) - 35\ \cos^2(\xi) - 28}{2\left(5\ \cos^2(\xi) - 4\right)^2},$$

therefore we get

$$u_{13}(x,t) = \ln\left(\frac{50\ \cos^4(\xi) - 60\ \sin(\xi)\cos(\xi) - 35\ \cos^2(\xi) - 28}{2\left(5\ \cos^2(\xi) - 4\right)^2}\right),$$

where

$$\xi = k\,x \pm t\sqrt{k^2 - \frac{3}{4}}.$$

Case 13.2:

$$l = \pm\sqrt{k^2 + \frac{3}{8} \pm \frac{3}{8}\,i\sqrt{3}}, A_0 = \pm\frac{13\,i}{4}\sqrt{3} + \frac{13}{4}, A_1 = A_2 = 0,$$
$$B_1 = \pm 15\,i\sqrt{3} + 15, B_2 = \pm\frac{75\,i}{4}\sqrt{3} + \frac{75}{4},$$

and

$$v(\xi) = \frac{\left(1 \pm i\sqrt{3}\right)\left(-50\cos^4(\xi) + 60\cos(\xi)\sin(\xi) + 35\cos^2(\xi) + 28\right)}{4\left(5\cos^2(\xi) - 4\right)^2},$$

therefore we get

$$u_{14,15}(x,t) = \ln\left(\frac{\left(1 \pm i\sqrt{3}\right)\left(-50\cos^4(\xi) + 60\cos(\xi)\sin(\xi) + 35\cos^2(\xi) + 28\right)}{4(5\cos^2(\xi) - 4)^2}\right).$$

where

$$\xi = k\,x \pm t\sqrt{k^2 + \frac{3}{8} \pm \frac{3}{8}\,i\sqrt{3}}.$$

Family 14: $\alpha = [i, -i, 1, 1]$ and $\beta = [i, -i, i, -i]$, which gives

$$\Phi(\xi) = -\frac{\sin(\xi)}{\cos(\xi)}. \tag{3.26}$$

Case 14.1:

$$l = \pm\sqrt{k^2 - \frac{3}{4}}, A_0 = -\frac{1}{2}, A_1 = 0, A_2 = -\frac{3}{2}, B_1 = B_2 = 0,$$

and

$$v(\xi) = \frac{2\ \cos^2(\xi) - 3}{2\cos^2(\xi)},$$

therefore we get

$$u_{16}(x,t) = \ln\left(\frac{2\ \cos^2\left(k\,x \pm t\sqrt{k^2 - \frac{3}{4}}\right) - 3}{2\cos^2\left(k\,x \pm t\sqrt{k^2 - \frac{3}{4}}\right)}\right).$$

Case 14.2:

$$l=\pm\sqrt{k^2+\frac{3}{8}\pm\frac{3}{8}\,i\sqrt{3}},A_0=\pm\frac{i}{4}\sqrt{3}+\frac{1}{4},A_2=\pm\frac{3}{4}\,i\sqrt{3}+\frac{3}{4},$$
$$A_1=B_1=B_2=0,$$

and

$$v\left(\xi\right)=\frac{\left(-1\mp i\sqrt{3}\right)\left(\cos^2\left(\xi\right)-\frac{3}{2}\right)}{2\cos^2\left(\xi\right)},$$

therefore we get

$$u_{17,18}(x,t)=\ln\left(\frac{\left(-1\mp i\sqrt{3}\right)\left(\cos^2\left(kx\pm t\sqrt{k^2+\frac{3}{8}\pm\frac{3}{8}i\sqrt{3}}\right)-\frac{3}{2}\right)}{2\cos^2\left(kx\pm t\sqrt{k^2+\frac{3}{8}\pm\frac{3}{8}i\sqrt{3}}\right)}\right).$$

Case 14.3:

$$l=\pm\sqrt{k^2-\frac{3}{16}},A_0=\frac{1}{4},A_1=0,A_2=-\frac{3}{8},B_1=0,B_2=-\frac{3}{8},$$

and

$$v\left(\xi\right)=\frac{-8\ \cos^4\left(\xi\right)+8\ \cos^2\left(\xi\right)-3}{8\cos^2\left(\xi\right)\sin^2\left(\xi\right)},$$

therefore we get

$$u_{19}(x,t)=\ln\left(\frac{-8\cos^4\left(kx\pm t\sqrt{k^2-\frac{3}{16}}\right)+8\cos^2\left(kx\pm t\sqrt{k^2-\frac{3}{16}}\right)-3}{8\cos^2\left(kx\pm t\sqrt{k^2-\frac{3}{16}}\right)\sin^2\left(kx\pm t\sqrt{k^2-\frac{3}{16}}\right)}\right).$$

Case 14.4:

$$l=\pm\sqrt{k^2+\frac{3}{32}\pm\frac{3\,i}{32}\sqrt{3}},A_0=\mp i/8\sqrt{3}-1/8,A_1=B_1=0,$$
$$A_2=B_2=\pm\frac{3}{16}\,i\sqrt{3}+\frac{3}{16},$$

and

$$v\left(\xi\right)=\frac{\left(8\cos^4\left(\xi\right)-8\cos^2\left(\xi\right)+3\right)\left(1\pm i\sqrt{3}\right)}{16\cos^2\left(\xi\right)\sin^2\left(\xi\right)},$$

therefore we get

$$u_{20,21}\left(x,t\right)=\ln\left(\frac{\left(8\cos^4\left(\xi\right)-8\cos^2\left(\xi\right)+3\right)\left(1\pm i\sqrt{3}\right)}{16\cos^2\left(\xi\right)\sin^2\left(\xi\right)}\right),$$

where

$$\xi=k\,x\pm t\sqrt{k^2+\frac{3}{32}\pm\frac{3\,i}{32}\sqrt{3}}.$$

Family 15: $\alpha=[1,-1,1,1]$ and $\beta=[1,-1,1,-1]$, which gives

$$\Phi\left(\xi\right)=\frac{\sinh\left(\xi\right)}{\cosh\left(\xi\right)}. \tag{3.27}$$

Case 15.1:

$$l = \pm\sqrt{k^2 + \frac{3}{4}}, A_0 = -\frac{1}{2}, A_1 = 0, A_2 = \frac{3}{2}, B_1 = B_2 = 0$$

and

$$v(\xi) = \frac{2\cosh^2(\xi) - 3}{2\cosh^2(\xi)},$$

therefore we get

$$u_{20,21}(x,t) = \ln\left(\frac{2\cosh^2\left(k\,x \pm t\sqrt{k^2 + \frac{3}{4}}\right) - 3}{2\cosh^2\left(k\,x \pm t\sqrt{k^2 + \frac{3}{4}}\right)}\right).$$

Case 15.2:

$$l = \pm\sqrt{k^2 - \frac{3}{8} \pm \frac{3\,i}{8}\sqrt{3}}, A_0 = \mp\frac{i}{4}\sqrt{3} + \frac{1}{4}, A_1 = 0,$$
$$A_2 = \pm\frac{3}{4}\,i\sqrt{3} - \frac{3}{4}, B_1 = B_2 = 0$$

and

$$v(\xi) = \frac{\left(2\cosh^2(\xi) - 3\right)\left(\pm i\sqrt{3} - 1\right)}{4\cosh^2(\xi)}$$

therefore we get

$$u_{22,23}(x,t) = \ln\left(\frac{\left(2\cosh^2\left(k\,x \pm t\sqrt{k^2 - \frac{3}{8} \pm \frac{3i}{8}\sqrt{3}}\right) - 3\right)\left(\pm i\sqrt{3} - 1\right)}{4\cosh^2\left(kx \pm t\sqrt{k^2 - \frac{3}{8} \pm \frac{3i}{8}\sqrt{3}}\right)}\right).$$

Case 15.3:

$$l = \pm\sqrt{k^2 + \frac{3}{4}}, A_0 = -\frac{1}{2}, A_1 = A_2 = B_1 = 0, B_2 = \frac{3}{2}$$

and

$$v(\xi) = \frac{2\cosh^2(\xi) + 1}{2\cosh^2(\xi)},$$

therefore we get

$$u_{24}(x,t) = \ln\left(\frac{2\cosh^2\left(k\,x \pm t\sqrt{k^2 + \frac{3}{4}}\right) + 1}{2\cosh^2\left(k\,x \pm t\sqrt{k^2 + \frac{3}{4}}\right)}\right).$$

Case 15.4:

$$l = \pm\sqrt{k^2 + \frac{3}{16}}, A_0 = \frac{1}{4}, A_1 = 0, A_2 = \frac{3}{8}, B_1 = 0, B_2 = \frac{3}{8}$$

and

$$v(\xi) = \frac{8\cosh^4(\xi) - 8\cosh^2(\xi) + 3}{8\sinh^2(\xi)\cosh^2(\xi)},$$

therefore we get

$$u_{25}(x,t)=\ln\left(\frac{8\cosh^4\left(kx\pm t\sqrt{k^2+\frac{3}{16}}\right)-8\cosh^2\left(k\,x\pm t\sqrt{k^2+\frac{3}{16}}\right)+3}{8\sinh^2\left(kx\pm t\sqrt{k^2+\frac{3}{16}}\right)\cosh^2\left(kx\pm t\sqrt{k^2+\frac{3}{16}}\right)}\right).$$

Family 16: $\alpha=[-1,0,1,1]$ and $\beta=[0,0,1,0]$, which gives

$$\Phi(\xi)=-\frac{1}{1+\mathrm{e}^{\xi}}. \tag{3.28}$$

Case 16.1:

$$l=\pm\sqrt{k^2+3}, A_0=1, A_1=A_2=6, B_1=B_2=0,$$

and

$$v(\xi)=\frac{\mathrm{e}^{2\,\xi}-4\,\mathrm{e}^{\xi}+1}{\left(1+\mathrm{e}^{\xi}\right)^2},$$

therefore we get

$$u_{26}(x,t)=\ln\left(\frac{\mathrm{e}^{2\,k\,x\pm t\sqrt{k^2+3}}-4\,\mathrm{e}^{k\,x\pm t\sqrt{k^2+3}}+1}{\left(1+\mathrm{e}^{k\,x\pm t\sqrt{k^2+3}}\right)^2}\right).$$

Case 16.2:

$$l=\pm\sqrt{k^2-\frac{3}{2}\pm\frac{3}{2}\,i\sqrt{3}}, A_0=-\frac{1}{2}\pm\frac{1}{2}\,i\sqrt{3},$$
$$A_1=A_2=-3\pm 3\,i\sqrt{3}, B_1=B_2=0,$$

and

$$v(\xi)=\frac{\left(-4\,\mathrm{e}^{\xi}+\mathrm{e}^{2\,\xi}+1\right)\left(\pm i\sqrt{3}-1\right)}{2\left(1+\mathrm{e}^{\xi}\right)^2},$$

therefore we get

$$u_{27}(x,t)=\ln\left(\frac{\left(-4\,\mathrm{e}^{k\,x\pm t\sqrt{k^2-\frac{3}{2}\pm\frac{3}{2}\,i\sqrt{3}}}+\mathrm{e}^{2\,k\,x\pm t\sqrt{k^2-\frac{3}{2}\pm\frac{3}{2}\,i\sqrt{3}}}+1\right)\left(\pm i\sqrt{3}-1\right)}{2\left(1+\mathrm{e}^{k\,x\pm t\sqrt{k^2-\frac{3}{2}\pm\frac{3}{2}\,i\sqrt{3}}}\right)^2}\right).$$

Remark 3.1. We have checked the entire set of the new solutions obtained in sections 3 and 4 with Maple, and found that all the obtained solutions satisfied corresponding partial differential equation.

4. Physical interpretation

In this section, we present some three dimensional figures of the modulus of some of the obtained solutions presented in the immediate section. The construction of the figures is carried out by taking suitable values of the parameters in order to see the mechanism of the original Eqs. (1) and (2). The physical features of the solutions $u_{20,21}(x,t)$, $u_{40,41}(x,t)$, $u_{39,40}(x,t)$, $u_{8,9}(x,t)$, $u_{5,6}(x,t)$, $u_{19}(x,t)$ are shown in Figures 1-6, respectively. One can see that, $u_{20,21}(x,t)$ and $u_{39,40}(x,t)$ are singular solutions, $u_{40,41}(x,t)$ is a kink type solution, $u_{8,9}(x,t)$ and $u_{19}(x,t)$ are periodic wave solutions and $u_{5,6}(x,t)$ is a periodic wave solution.

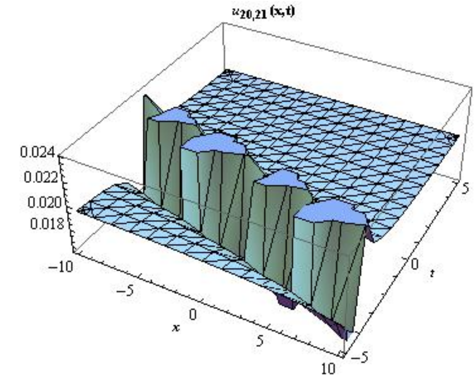

Figure 1. 3D plot of solution $u_{20,21}(x,t)$ with $k = 1.7$.

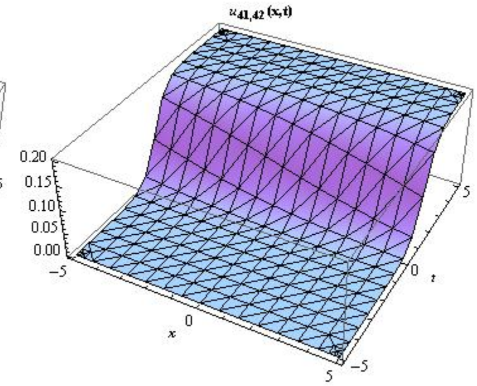

Figure 2. 3D plot of $u_{40,41}(x,t)$ with $k = 1.7$.

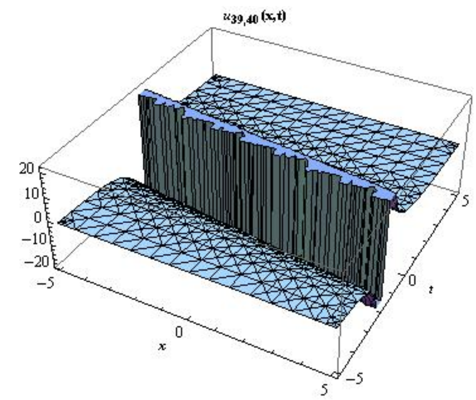

Figure 3. 3D plot of $u_{39,40}(x,t)$ with $k = 0.5$.

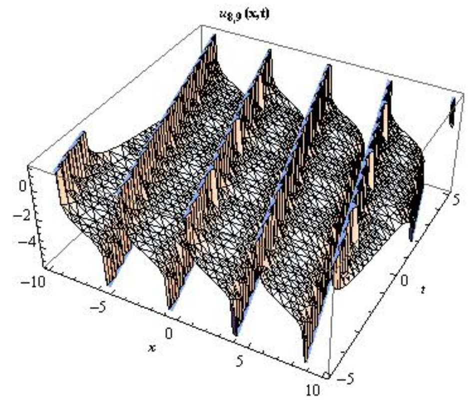

Figure 4. 3D plot of $u_{8,9}(x,t)$ with $k = 0.5$.

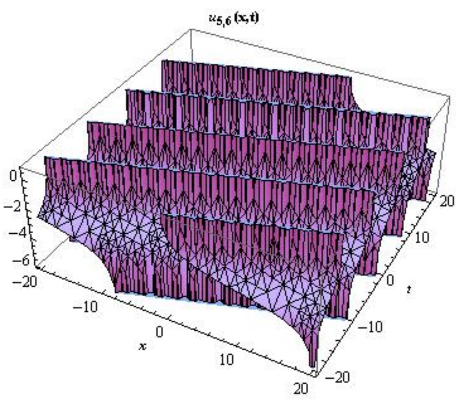

Figure 5. 3D plot of $u_{5,6}(x,t)$ with $k = 1$.

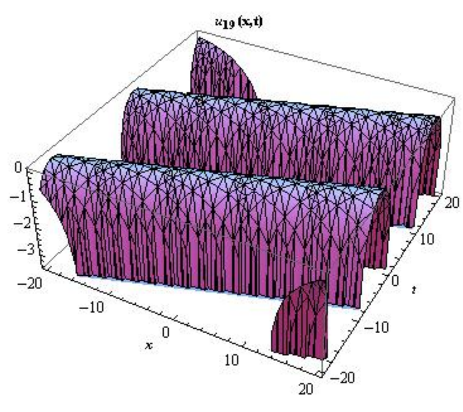

Figure 6. 3D plot of $u_{19}(x,t)$ with $k = 1$.

5. Final Remarks

In this paper, we introduced a so-called generalized exponential rational function method (GERFM) for acquiring exact solutions of the nonlinear partial differential equations. It has been observed that GERFM provided a wide range of closed-form travelling solutions of two nonlinear evolution equations namely Tzitzéica-Dodd-Bullough and Tzitzéica equation. The most important feature of the new method is that it is very effective and simple. The main merits of the proposed method is that it generates more general solutions with some free parameters and can be applied to other types of nonlinear partial differential equations. Physical features for some of the obtained results are presented in Figures 1-6.

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