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BOUNDARY LAYERS FOR THE 3D PRIMITIVE EQUATIONS IN A CUBE: THE ZERO-MODE

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Dedicated to Claude-Michel Brauner on the occasion of his seventieth birthday.

Abstract We establish the vanishing viscosity limit of the zero-mode of the linearized Primitive Equations in a cube. Our method is based on the explicit construction and estimates of the boundary layers. This result, together with that in [12,15], allows us to conclude the vanishing viscosity limit of the linearized Primitive Equations in a cube.

Keywords Boundary layer, primitive equations, zero-mode, vanishing viscosity limit.

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1. Introduction.

In this article, we are interested in the study of the mode-zero case of the Linearized Primitive Equations (LPEs) at small viscosity in a rectangle $\Omega = \{(x, y) : 0 \leq x \leq L_1, 0 \leq y \leq L_2\}$. The viscous modal equations are given as

$$\begin{cases} u^\varepsilon_{0t} + \bar{U}_0 u^\varepsilon_{0x} - f v^\varepsilon_0 + \phi^\varepsilon_{0x} - \varepsilon\Delta u^\varepsilon_0 = F_{0u}, \\ v^\varepsilon_{0t} + \bar{U}_0 v^\varepsilon_{0x} + f u^\varepsilon_0 + \phi^\varepsilon_{0y} - \varepsilon\Delta v^\varepsilon_0 = F_{0v}, \\ u^\varepsilon_{0x} + v^\varepsilon_{0y} = 0, \end{cases} \tag{1.1}$$

where $(u^\varepsilon_0, v^\varepsilon_0)$ is the velocity vector field, ϕ^ε_0 is the pressure, $f = f_0(1 + \beta y)$ is the Coriolis parameter with constants $f_0, \beta > 0$, and ε is the viscosity. The subscript *0* indicates the variables of the zero mode which will be explained below. Throughout, $\bar{U}_0$ is assumed to be a positive constant. We supplement the system (1.1) with the

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initial and boundary conditions

$$(u_0^\varepsilon, v_0^\varepsilon) = 0 \text{ on } \partial\Omega, \tag{1.2}$$
$$(u_0^\varepsilon, v_0^\varepsilon) = (\tilde{u}_0, \tilde{v}_0)(x, y) \text{ at } t = 0. \tag{1.3}$$

Here it is natural to assume the following compatibility condition

$$(\tilde{u}_0, \tilde{v}_0) = 0 \text{ on } \partial\Omega. \tag{1.4}$$

The incompatible case can be treated similarly as in [9].

The viscosity ε is typically small. Formally taking the limit $\varepsilon \to 0$ in the system (1.1), one derives the inviscid problem

$$\begin{cases} u^0_{0t} + \bar{U}_0 u^0_{0x} - f v^0_0 + \phi^0_{0x} = F_{0u}, \\ v^0_{0t} + \bar{U}_0 v^0_{0x} + f u^0_0 + \phi^0_{0y} = F_{0v}, \\ u^0_{0x} + v^0_{0y} = 0. \end{cases} \tag{1.5}$$

Following [5, 13, 14], we prescribe the following boundary and initial conditions for the inviscid system (1.5):

$$u^0_0|_{x=0} = u^0_0|_{x=L_1} = 0, \tag{1.6}$$
$$v^0_0|_{x=0} = v^0_0|_{y=0} = v^0_0|_{y=L_2} = 0, \tag{1.7}$$
$$(u^0_0, v^0_0) = (\tilde{u}_0, \tilde{v}_0)(x, y) \text{ at } t = 0. \tag{1.8}$$

Note that the inviscid problem and the viscous problem have the same initial condition (1.8). Furthermore, it follows from the divergence-free condition and the first boundary condition (1.6) that

$$\int_0^{L_1} v^0_{0y}(x, y, t)dx = -\int_0^{L_1} u^0_{0x}(x, y, t)dx = 0. \tag{1.9}$$

Hence, the second boundary condition (1.7) implies that

$$\int_0^{L_1} v^0_0(x, y, t)dx = 0. \tag{1.10}$$

We will work with regular solutions to the system (1.5). It is natural to assume the following compatibility conditions at the right two corners of the domain

$$u^0_0 = 0, \quad v^0_0 = 0 \ \text{ at } (L_1, 0) \text{ and } (L_1, L_2). \tag{1.11}$$

Since $v^0_0 = 0$ at $x = 0$, one has that $v^0_{0y} = 0$ at $x = 0$. Then the divergence-free condition implies that

$$u^0_{0x} = 0 \text{ at } x = 0. \tag{1.12}$$

The well-posedness of the inviscid system (1.5) equipped with the initial-boundary conditions (1.6)–(1.8) is established in [5]. Throughout, we assume the following regularity holds

$$(u^0_0, v^0_0) \in C^1(0, T; H^3(\Omega)). \tag{1.13}$$

Our aim is to establish the rigorous vanishing viscosity limit, i.e., the convergence in L^2 of the viscous solution $(u_0^\varepsilon, v_0^\varepsilon, \phi_0^\varepsilon)$ to the inviscid one (u_0^0, v_0^0, ϕ_0^0) as $\varepsilon \to 0$. We take the classical Prandtl correctors approach. That is, one first determines the different boundary layers generated, at small viscosity, by the viscous LPEs; an approximate solution can then be defined in terms of the solution to the inviscid LPEs plus various boundary layers; finally one can try to obtain the desired convergences by the energy method. This approach allows us to establish the convergence with explicit convergence rates. The main result of this article is summarized in the following theorem

Theorem 1.1. *Let $(u_0^\varepsilon, v_0^\varepsilon)$ be the solution to the viscous problem* (1.1)*, and (u_0^0, v_0^0) be the solution to the inviscid problem* (1.5) *emanating from the same initial condition and forcing terms. Assume that $(u_0^0, v_0^0) \in C^1(0, T; H^3(\Omega))$. Then the following vanishing viscosity limit holds*

$$||(u_0^\varepsilon - u_0^0, v_0^\varepsilon - v_0^0)||_{L^\infty(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{4}}, \quad ||\phi_0^\varepsilon - \phi_0^0||_{L^2(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{4}}. \tag{1.14}$$

where C is a constant independent of ε.

Our study of the vanishing viscosity limit of the zero mode LPEs stems from the effort of resolving the issue of boundary conditions for the inviscid linearized Primitive Equations which is an important model for short term regional weather predictions. It is well-known [21, 25, 29] that the inviscid LPEs are not well-posed for any set of local boundary conditions. For the inviscid LPEs in a cube, a classical approach is to impose boundary conditions for the PEs mode by mode, after the normal mode expansion of the solution in the vertical direction. A natural way of justifying these choices of local boundary conditions for the inviscid LPEs is to prove the vanishing viscosity limit, ideally with explicit convergence rates. Below we briefly recall the procedure.

Recall that the linearized primitive equations (around the flow $\bar{U}_0$ in the direction Ox) in $\mathcal{M} = \Omega \times (-L_3, 0)$ take the following form (see e.g. [17], [18] and [26])

$$\begin{cases} u_t^\varepsilon + \bar{U}_0 u_x^\varepsilon - f v^\varepsilon + \phi_x^\varepsilon - \varepsilon \Delta_3 u^\varepsilon = F_u, \\ v_t^\varepsilon + \bar{U}_0 v_x^\varepsilon + f u^\varepsilon + \phi_y^\varepsilon - \varepsilon \Delta_3 v^\varepsilon = F_v, \\ \psi_t^\varepsilon + \bar{U}_0 \psi_x^\varepsilon + N^2 w^\varepsilon - \varepsilon \Delta_3 \psi^\varepsilon = F_\psi, \\ \phi_z^\varepsilon - \psi^\varepsilon = 0, \\ u_x^\varepsilon + v_y^\varepsilon + w_z^\varepsilon = 0, \end{cases} \tag{1.15}$$

where $\Delta_3 = \Delta + \partial^2/\partial z^2$, $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ is the 3D velocity vector field, ϕ^ε is the pressure, ψ^ε is the temperature field, N is a positive constant with N^2 representing the buoyancy frequency [22]. The boundary and initial conditions are prescribed as follows are chosen as follows:

$$(u^\varepsilon, v^\varepsilon, \psi^\varepsilon) = 0 \text{ on } \partial\Omega \times (-L_3, 0), \tag{1.16}$$

$$(\frac{\partial u^\varepsilon}{\partial z}, \frac{\partial v^\varepsilon}{\partial z}, \psi^\varepsilon, w^\varepsilon) = 0 \text{ at } z = 0, -L_3, \tag{1.17}$$

$$(u^\varepsilon, v^\varepsilon, \psi^\varepsilon) = (\tilde{u}, \tilde{v}, \tilde{\psi})(x, y, z) \text{ at } t = 0. \tag{1.18}$$

The inviscid system corresponds to the system (1.15) with $\varepsilon = 0$

$$\begin{cases} u^0_t + \bar{U}_0 u^0_x - f v^0 + \phi^0_x = F_u, \\ v^0_t + \bar{U}_0 v^0_x + f u^0 + \phi^0_y = F_v, \\ \psi^0_t + \bar{U}_0 \psi^0_x + N^2 w^0 = F_\psi, \\ \phi^0_z - \psi^0 = 0, \\ u^0_x + v^0_y + w^0_z = 0. \end{cases} \tag{1.19}$$

To prescribe appropriate local boundary conditions for the inviscid LPEs, following [15] we expand the viscous solutions by the following normal modal decompositions:

$$(u^\varepsilon, v^\varepsilon, \phi^\varepsilon) = \sum_{n\geq 0} \mathcal{U}_n(z)(u^\varepsilon_n, v^\varepsilon_n, \phi^\varepsilon_n)(x,y,t), \quad (w^\varepsilon, \psi^\varepsilon) = \sum_{n\geq 1} \mathcal{W}_n(z)(w^\varepsilon_n, \psi^\varepsilon_n)(x,y,t).$$

Here $\mathcal{U}_0 = \frac{1}{\sqrt{L_3}}$, $\mathcal{U}_n = \sqrt{\frac{2}{L_3}}\cos(\lambda_n z)$, $\mathcal{W}_n = \sqrt{\frac{2}{L_3}}\sin(\lambda_n z)$, and the frequency λ_n is given by $\lambda_n = \frac{n\pi}{L_3}$ with integer n. The corresponding normal mode decompositions are also performed to the forcing terms F_u, F_v, F_ψ and to the initial conditions. The modal equations derived are presented as follows: for $n = 0$,$(u^0_0, v^0_0) \in C^1(0,T; H^3(\Omega))$

$$\begin{cases} u^\varepsilon_{0t} + \bar{U}_0 u^\varepsilon_{0x} - f v^\varepsilon_0 + \phi^\varepsilon_{0x} - \varepsilon \Delta u^\varepsilon_0 = F_{u0}, \\ v^\varepsilon_{0t} + \bar{U}_0 v^\varepsilon_{0x} + f u^\varepsilon_0 + \phi^\varepsilon_{0y} - \varepsilon \Delta v^\varepsilon_0 = F_{v0}, \\ u^\varepsilon_{0x} + v^\varepsilon_{0y} = 0, \end{cases} \tag{1.21}$$

and for $n \geq 1$,

$$\begin{cases} u^\varepsilon_{nt} + \bar{U}_0 u^\varepsilon_{nx} - f v^\varepsilon_n + \phi^\varepsilon_{nx} + \varepsilon\lambda_n^2 u^\varepsilon_n - \varepsilon \Delta u^\varepsilon_n = F_{un}, \\ v^\varepsilon_{nt} + \bar{U}_0 v^\varepsilon_{nx} + f u^\varepsilon_n + \phi^\varepsilon_{ny} + \varepsilon\lambda_n^2 v^\varepsilon_n - \varepsilon \Delta v^\varepsilon_n = F_{vn}, \\ \psi^\varepsilon_{nt} + \bar{U}_0 \psi^\varepsilon_{nx} + N^2 w^\varepsilon_n + \varepsilon\lambda_n^2 \psi^\varepsilon_n - \varepsilon \Delta \psi^\varepsilon_n = F_{\psi n}, \\ \phi^\varepsilon_n = -\lambda_n^{-1}\psi^\varepsilon_n, \\ w^\varepsilon_n = -\lambda_n^{-1}(u^\varepsilon_{nx} + v^\varepsilon_{ny}). \end{cases} \tag{1.22}$$

Note that the case of $n = 0$ is the mode-zero case (compare to (1.1)) which is the focus of this study.

The corresponding inviscid model equations are derived in the same way and take the form of Eqs. (1.1) and (1.22) with $\epsilon = 0$. Here we do not write down the inviscid modal equations explicitly, but point out that the mode-zero case is a parabolic-elliptic system (similar to the linearized Euler equations) and the modes $n \geq 1$ are hyperbolic systems. As is explained in [25], two types of modes for $n \geq 1$ have to be further distinguished, depending on the flow of characteristics (hence different imposition of boundary conditions and resulting different PDEs). Let n_c be such that $\frac{n_c\pi}{L_3} = \lambda_{n_c} < \frac{N}{\bar{U}_0} < \lambda_{n_c+1} = \frac{(n_c+1)\pi}{L_3}$. The modes $1 \leq n \leq n_c$ are called *subcritical*, and the modes $n > n_c$ are called *supercritical*. We do not consider the non-generic case where $L_3 N/\pi\bar{U}_0$ is an integer. The result of vanishing viscosity limit for the supercritical modes and subcritical modes are established in [15] and [12], respectively, by the careful study of various boundary layers. Combining the convergence result proved in Theorem 1.1 with those established in [12, 15], we obtain the following vanishing viscosity limit for the linearized PEs system (1.15) in a cube.

Corollary 1.1. *Let* $(u^\varepsilon, v^\varepsilon, w^\varepsilon, \phi^\varepsilon, \varphi^\varepsilon)$ *be the solution to the viscous problem* (1.15) *equipped with initial boundary conditions* (1.16)–(1.18)*, and* $(u^0, v^0, w^0, \phi^0, \varphi^0)$ *be the solution to the inviscid problem* (1.19) *with initial boundary conditions prescribed mode by mode. Assume that the inviscid solution is smooth such that* $(u^0, v^0, w^0, \phi^0, \varphi^0) \in C^1(0, T; H^3(\mathcal{M}))$*. Then the following vanishing viscosity limit holds*

$$||(u^\varepsilon - u^0, v^\varepsilon - v^0, w^\varepsilon - w^0, \varphi^\varepsilon - \varphi^0)||_{L^\infty(0,T;L^2(\mathcal{M}))} \leq C\varepsilon^{\frac{1}{4}}, \tag{1.23}$$
$$||\phi^\varepsilon - \phi^0||_{L^2(0,T;L^2(\mathcal{M}))} \leq C\varepsilon^{\frac{1}{4}},$$

where C *is a constant independent of* ε*.*

The problem of vanishing viscosity limit in fluid dynamics is a difficult problem because it is a singular perturbation problem that incurs boundary layers, cf. the review articles [6,10,19] and references therein. In the case of domains with corners, as is in the current study, one also needs to address corner layers in the boundary layer analysis. The study of corner layers is very challenging, even when the equations are linear [23, 27, 28], see also [7, 16, 20] for results in non-linear settings. As we remarked earlier, the proof of Corollary 1.1 is accomplished in a series of three articles, due to the distinctive nature of the underlying inviscid problems, hence different construction of the boundary layers. The case of the supercritical modes is studied in [15] which involves the construction of parabolic boundary layers, ordinary boundary layers, elliptic boundary layers, and corner boundary layers. The supercriticality is crucial in the estimate of the corner layers for the supercritical modes. The article [12] treats the subcritical modes. The lack of damping effect in the subcritical modal equations leads to a different construction of the corner layers. The convergence involving the boundary layers in the subcritical modes is weaker than that in the supercritical modes. This work undertakes the analysis for the mode-zero case that is similar to the linearized Euler equation, see Eqs. (1.5). We avoid the construction of corner boundary layers entirely by taking advantage of the explicit construction and estimates of the parabolic boundary layers and ordinary boundary layers. This approach allows us to establish the vanishing viscosity limit. We point out that the construction of corner boundary layers is still needed if one wishes to obtain optimal convergence in the leading order expansions. The optimal convergence and convergence in the H^1 norm will be considered in a forthcoming work.

This work dealing with a subject dear to Claude-Michel Brauner (singular perturbations, see e.g. [1–4]) is dedicated to him with appreciation and friendship on the occasion of his seventieth birthday.

2. Boundary layer analysis

Define $(\vartheta_u, \vartheta_v, \vartheta_\phi) = (u_0^\varepsilon - u_0^0, v_0^\varepsilon - v_0^0, \phi_0^\varepsilon - \phi_0^0)$ as the difference between the viscous solution and the inviscid one. Subtracting (1.5) from (1.1), we find that

$$\begin{cases} \vartheta_{ut} + \bar{U}_0 \vartheta_{ux} - f\vartheta_v + \vartheta_{\phi x} - \varepsilon\Delta\vartheta_u = \varepsilon\Delta u_0^0, \\ \vartheta_{vt} + \bar{U}_0 \vartheta_{vx} + f\vartheta_u + \vartheta_{\phi y} - \varepsilon\Delta\vartheta_v = \varepsilon\Delta v_0^0, \\ \vartheta_{ux} + \vartheta_{vy} = 0, \end{cases} \tag{2.1}$$

with zero initial conditions

$$(\vartheta_u, \vartheta_v, \vartheta_\phi) = 0 \text{ at } t = 0, \tag{2.2}$$

and the boundary conditions,

$$\vartheta_u = 0 \text{ at } x = 0, L_1, \;\; \vartheta_u = -u_0^0 \text{ at } y = 0, L_2, \tag{2.3}$$

$$\vartheta_v = 0 \text{ at } x = 0, y = 0, L_2, \;\; \vartheta_v = -v_0^0 \text{ at } x = L_1. \tag{2.4}$$

Formally, the source terms in the system (2.1) are of $\mathcal{O}(\varepsilon)$. The only $\mathcal{O}(1)$ discrepancies are present on the boundaries, i.e., Eqs. (2.3) and (2.4). Following the Prandtl approach, we correct these $\mathcal{O}(1)$ discrepancies by leading order boundary layer correctors.

2.1. Parabolic Boundary Layers (PBL) at $y = 0, L_2$

We first construct the so-called parabolic boundary layers which resolve the discrepancies between the viscous solution u_0^ε and the inviscid solution u_0^0 at the boundaries $y = 0, L_2$, i.e., $\vartheta_u = -u_0^0$ at $y = 0, L_2$ as in (2.3). Since the construction at $y = 0$ and at $y = L_2$ is the same, we only present the case at $y = 0$ which is distinguished by the superscript 1 in the notation of the variables.

As in the classical boundary layer theory for Navier-Stokes equations at small viscosity, we make the following ansatz (to the leading order approximation)

$$\big(\vartheta_u, \vartheta_v, \vartheta_\phi\big) = \big(\bar{\varphi}_u^1(t, x, \bar{y}), \bar{\varphi}_v^1(t, x, \bar{y}), \gamma_\phi^1(t, x, \bar{y})\big), \tag{2.5}$$

where $\bar{y} = y/\sqrt{\varepsilon}$ is the stretched variable. Substituting the ansatz (2.5) into the system (2.1) and identifying the leading order terms, one arrives at the following equations

$$\bar{\varphi}_{ut}^1 + \bar{U}_0 \bar{\varphi}_{ux}^1 - \varepsilon\bar{\varphi}_{uyy}^1 = 0, \tag{2.6}$$

$$f\bar{\varphi}_u^1 + \gamma_{\phi y}^1 = 0, \tag{2.7}$$

$$\bar{\varphi}_{ux}^1 + \bar{\varphi}_{vy}^1 = 0. \tag{2.8}$$

Note that γ_ϕ^1 and $\bar{\varphi}_v^1$ are slave variables of $\bar{\varphi}_u^1$ via Eqs. (2.7) and (2.8), respectively.

Taking into account matching conditions, i.e., the boundary layers are (exponentially) small away from the boundary $y = 0$, one finds that the boundary layer

$\bar{\varphi}_u^1$ satisfies, for $0 < x < L_1$, $t > 0$,

$$\begin{cases} \bar{\varphi}_{ut}^1 + \bar{U}_0 \bar{\varphi}_{ux}^1 - \varepsilon \bar{\varphi}_{uyy}^1 = 0, \\ \bar{\varphi}_u^1 = -u_0^0(x,0,t) \text{ at } y = 0, \\ \bar{\varphi}_u^1 \to 0 \text{ as } y/\sqrt{\varepsilon} \to \infty, \\ \bar{\varphi}_u^1 = 0 \text{ at } x = 0, \\ \bar{\varphi}_u^1 = 0 \text{ at } t = 0, \end{cases} \tag{2.9}$$

and that the boundary layer $\bar{\varphi}_v^1$ satisfies

$$\begin{cases} \bar{\varphi}_{ux}^1 + \bar{\varphi}_{vy}^1 = 0, \\ \bar{\varphi}_v^1 = 0 \text{ at } y = 0. \end{cases} \tag{2.10}$$

The corrector γ_ϕ^1 is determined uniquely up to a constant by Eq. (2.7), once $\bar{\varphi}_u^1$ is solved from Eq. (2.9). The systems (2.9) and (2.10) imply the following conditions

$$\bar{\varphi}_u^1 = \bar{\varphi}_{ux}^1 = \bar{\varphi}_v^1 = 0, \text{ at } x = 0. \tag{2.11}$$

We note that the parabolic boundary layer systems (2.9) and (2.10) are the same as those in [12, 15]. The following lemma summarizes the estimates on $\bar{\varphi}_u^1, \bar{\varphi}_u^2$.

Lemma 2.1. *Assume that $i, l, m \geq 0$, $0 \leq 2\max\{i, l-1\} + m \leq 2$, and $0 \leq t \leq T$. Denote by γu_0^0 the trace of u_0^0 at $y = 0$. For every $j \in N^+$, there exist constants κ_j depending on j, i, l, m but independent of ε such that*

$$\left| y^j \frac{\partial^{i+l+m} \bar{\varphi}_u^1}{\partial t^i \partial x^l \partial y^m} \right|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq \kappa_j \, K_{i,l,m}^2(u_0^0) \varepsilon^{-\frac{m}{2}} \varepsilon^{\frac{j}{2}+\frac{1}{4}}, \tag{2.12}$$

where

$$\begin{aligned} K_{i,l,m}^2(u_0^0) &= \|\gamma u_0^0\|_{W^{i,\infty}(0,T;H_x^l(0,L_1))} \text{ if } m = 0, \\ K_{i,l,m}^2(u_0^0) &= \sum_{q+r=1, q,r \geq 0} \|\gamma u_0^0\|_{W^{q,\infty}(0,T;H_x^{l+r}(0,L_1))} \text{ if } m = 1, 2. \end{aligned} \tag{2.13}$$

In particular, the following pointwise estimates hold

$$\left| \frac{\partial^i \bar{\varphi}_u^1}{\partial t^i}(\cdot, y, t) \right|_{H_x^l(0,L_1)} \leq \kappa \sup_{t \in [0,T]} \left| \frac{\partial^i (\gamma u_0^0)}{\partial t^i}(\cdot, t) \right|_{H_x^l(0,L_1)} \exp\left(-c \frac{y}{\sqrt{\varepsilon}} \right), \tag{2.14}$$

$$\varepsilon \left| \frac{\partial^2 \bar{\varphi}_u^1}{\partial y^2}(\cdot, y, t) \right|_{H_x^l(0,L_1)} \leq \kappa K_{i,l,2}^2(u_0^0) \exp\left(-c \frac{y}{\sqrt{\varepsilon}} \right), \tag{2.15}$$

$$\left| \frac{\partial \bar{\varphi}_u^1}{\partial y}(\cdot, y, t) \right|_{H_x^l(0,L_1)} \leq \kappa K_{i,l,1}^2(u^0) \varepsilon^{-\frac{1}{2}} \exp\left(-c \frac{y}{\sqrt{\varepsilon}} \right). \tag{2.16}$$

Proof. The proof of Lemma 2.1 is essentially contained in [12]. We give the details here for the sake of completeness.

The systems (2.9) can be solved explicitly by Fourier transform. Introducing a function $g_u(x,t) = -\gamma_0(u_0^0)(x,t)$, $0 < x < L_1$, we obtain the following compatibility conditions from Eqs. (2.11)

$$g_u(0,t) = g_{ux}(0,t) = g_u(x,0) = 0. \tag{2.17}$$

Note that $g_u(\cdot,t) \in C^1([0,L_1])$ by the trace theorem and Sobolev embedding. We first extend g_u by zero for $x < 0$, and then smoothly for $x > L_1$ such that the extension, denoted by $\tilde{g}_u$, has compact support in $\mathbb{R}$. The compatibility condition (2.17) and the regularity assumption (1.13) imply that $\partial^i \tilde{g}_u(x,t)/\partial t^i \in H^2_x(\mathbb{R})$ for $i = 0, 1$ and

$$\left\| \frac{\partial^i \tilde{g}_u}{\partial t^i}(x,t) \right\|_{H^l_x(\mathbb{R})} \leq \kappa \left\| \frac{\partial^i g_u}{\partial t^i}(x,t) \right\|_{H^l_x(0,L_1)}, \; l = 0,1,2. \tag{2.18}$$

Likewise, we extend $\bar{\varphi}^1_u$ by zero to $x < 0$ such that,

$$\tilde{\varphi}(x,y,t) = \begin{cases} \bar{\varphi}^1_u(x,y,t), \text{ for } x > 0, \\ 0, \text{ for } x < 0, \end{cases} \tag{2.19}$$

The extended profile $\tilde{\varphi}$ is required to satisfy

$$\begin{cases} \tilde{\varphi}_t + \bar{U}_0 \tilde{\varphi}_x - \varepsilon \tilde{\varphi}_{yy} = 0, \quad x \in \mathbb{R}, y > 0 \\ \tilde{\varphi} = \tilde{g}_u(x,t) \text{ at } y = 0, \\ \tilde{\varphi} \to 0 \text{ as } y/\sqrt{\varepsilon} \to \infty, \\ \tilde{\varphi} = 0 \text{ at } t = 0. \end{cases} \tag{2.20}$$

The consistency in the construction follows from the compatibility conditions (2.11) and the uniqueness of the solution.

One now takes the Fourier transform of the Eq. (2.20) in the x variable. Solving the heat equation on a quarter plane ($\bar{y} > 0, t > 0$) gives

$$\hat{\varphi} = \sqrt{\frac{2}{\pi}} \int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty} \exp\left(-\frac{s^2}{2} - \bar{U}_0 i\omega \frac{y^2}{2\varepsilon s^2}\right) \hat{g}_u\left(\omega, t - \frac{y^2}{2\varepsilon s^2}\right) ds, \tag{2.21}$$

where the hat notation denotes the Fourier transform in x. The solution $\bar{\varphi}^1_u$ of (2.9) is determined as

$$\bar{\varphi}^1_u = \tilde{\varphi} \chi_{\{0<x<L_1\}}, \tag{2.22}$$

where $\tilde{\varphi}$ is the inverse Fourier transform of $\hat{\varphi}$ defined in (2.21).

We now derive the estimates outlined in the Lemma. First we note that $\hat{g}_u(\omega,0) = 0$, thanks to the compatibility conditions (2.17). By Parseval's identity, we derive that, for $i = 0, 1$,

$$\begin{aligned} \left| \frac{\partial^i \bar{\varphi}^1_u}{\partial t^i}(\cdot,y,t) \right|_{H^l_x(0,L_1)} &\leq \kappa \left| \frac{\partial^i \tilde{\varphi}}{\partial t^i}(\cdot,y,t) \right|_{H^l_x(\mathbb{R})} = \sum_{k=0}^{l} \kappa \left| (i\omega)^k \frac{\partial^i \hat{\varphi}}{\partial t^i}(\cdot,y,t) \right|_{L^2_\omega(\mathbb{R})} \\ &\leq \sum_{k=0}^{l} \kappa \sup_{t\in[0,T]} \left| (i\omega)^k \frac{\partial^i \hat{g}_u}{\partial t^i}(\omega,t) \right|_{L^2_\omega(\mathbb{R})} \int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty} \exp\left(-\frac{s^2}{2}\right) ds \\ &\leq \sum_{k=0}^{l} \kappa \sup_{t\in[0,T]} \left| (i\omega)^k \frac{\partial^i \hat{g}_u}{\partial t^i}(\omega,t) \right|_{L^2_\omega(\mathbb{R})} \exp\left(-c\frac{y}{\sqrt{\varepsilon}}\right) \\ &\leq \kappa \sup_{t\in[0,T]} \left| \frac{\partial^i (\gamma u^0_0)}{\partial t^i}(\cdot,t) \right|_{H^l_x(0,L_1)} \exp\left(-c\frac{y}{\sqrt{\varepsilon}}\right). \end{aligned} \tag{2.23}$$

Hence (2.12) is valid for $i = 0, 1$, $l = 0, 1, 2$, $m = 0$, in light of the estimate (2.18).

For $i = 0, m = 2$, it follows from Eq. $(2.9)_1$ that $\varepsilon|\bar{\varphi}^1_{uyy}|_{H^l(0,L_1)} \le \kappa|\bar{\varphi}^1_u|_{H^{l+1}(0,L_1)} + \kappa|\bar{\varphi}^1_{ut}|_{H^l(0,L_1)}$, $l = 0, 1$. One deduces from the estimate (2.23) that

$$\varepsilon\left|\frac{\partial^2\bar{\varphi}^1_u}{\partial y^2}(\cdot, y, t)\right|_{H^l_x(0,L_1)} \le \kappa K^2_{i,l,2}(u^0_0)\exp\left(-c\frac{y}{\sqrt{\varepsilon}}\right). \tag{2.24}$$

For $i = 0$, $m = 1$, since $\hat{g}_u(\omega, 0) = 0$, differentiating (2.21) in y gives

$$\frac{\partial\hat{\varphi}}{\partial y} = \sqrt{\frac{2}{\pi}}\int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty}\exp\left(-\frac{s^2}{2} - \bar{U}_0 i\omega\frac{y^2}{2\varepsilon s^2}\right)\cdot\left\{-\bar{U}_0 i\omega - \frac{\partial}{\partial t}\right\}\hat{g}_u\left(\omega, t - \frac{y^2}{2\varepsilon s^2}\right)\frac{y}{\varepsilon s^2}ds. \tag{2.25}$$

Then

$$\begin{aligned}
&\left|\frac{\partial\bar{\varphi}^1_u}{\partial y}(\cdot, y, t)\right|_{H^l_x(0,L_1)} \le \kappa\left|\frac{\partial\tilde{\varphi}}{\partial y}(\cdot, y, t)\right|_{H^l_x(\mathbb{R})} = \sum_{k=0}^{l}\kappa\left|(i\omega)^k\frac{\partial\hat{\varphi}}{\partial y}(\cdot, y, t)\right|_{L^2_\omega(\mathbb{R})}\\
&\quad\le \sum_{k=0}^{l}\kappa\sup_{t\in[0,T]}\left|(i\omega)^k\left\{-\bar{U}_0 i\omega - \frac{\partial}{\partial t}\right\}\hat{g}_u(\omega, t)\right|_{L^2_\omega(\mathbb{R})}\times\varepsilon^{-1}\int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty}\exp\left(-\frac{s^2}{2}\right)\frac{y}{s^2}ds\\
&\quad\le \kappa K^2_{i,l,1}(u^0_0)\varepsilon^{-1}\int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty}\exp\left(-\frac{s^2}{2}\right)\frac{y}{s^2}ds\\
&\quad\le \kappa K^2_{i,l,1}(u^0_0)\varepsilon^{-\frac{1}{2}}\exp\left(-c\frac{y}{\sqrt{\varepsilon}}\right),
\end{aligned} \tag{2.26}$$

which is valid for $l = 0, 1$. In deriving the last step of inequality (2.26) we have used

$$\begin{aligned}
\int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty}\exp\left(-\frac{s^2}{2}\right)\frac{y}{s^2}ds &= \sqrt{2\varepsilon t}\exp\left(-\frac{y^2}{4\varepsilon t}\right) - y\int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty}\exp\left(-\frac{s^2}{2}\right)ds\\
&\le \sqrt{2\varepsilon t}\exp\left(-\frac{y^2}{4\varepsilon t}\right) \le \kappa\sqrt{\varepsilon}\exp(-c\frac{y}{\sqrt{\varepsilon}}).
\end{aligned} \tag{2.27}$$

Inequality (2.12) now follows from $|y^j\exp(-\alpha\frac{y}{\varepsilon})|_{L^p(0,L_2)} \le \kappa(\frac{\varepsilon}{\alpha})^{j+\frac{1}{p}}$ for $\alpha > 0$, $1 \le p \le \infty$. The estimates (2.14)–(2.16) are clear from the process of the argument above. Lemma 2.1 is thus proved. □

The corrector $\bar{\varphi}^2_u$ at $y = L_2$ satisfies similar estimates. It follows from Eq. (2.10) that $\bar{\varphi}^1_v(t, x, y) = -\int_0^y\bar{\varphi}^1_{ux}(t, x, \tau)d\tau$. The estimates on $\bar{\varphi}^1_v$ can be readily derived from (2.12)–(2.16). For instance, one has

$$|\bar{\varphi}^1_v(t, x, y)| \le C\sqrt{\epsilon}\left[1 - \exp(-cy/\sqrt{\varepsilon})\right]. \tag{2.28}$$

It is seen from (2.14) that the trace of $\bar{\varphi}^1_u$ at $y = L_2$ is exponentially small in the L^2 norm, but may not be zero. To construct a global corrector that satisfies the correct boundary conditions, we introduce a smooth cut-off function $\sigma = \sigma(r)$ such that for $L = \min\{L_1, L_2\}$,

$$\sigma = \begin{cases} 1 \text{ for } 0 \le r \le \frac{L}{2},\\ 0 \text{ for } r \ge \frac{2L}{3}.\end{cases} \tag{2.29}$$

We then define

$$\varphi_u^1 = \bar{\varphi}_u^1 \sigma(y), \quad \varphi_u^2 = \bar{\varphi}_u^2 \sigma(L_2 - y), \quad \varphi_u = \varphi_u^1 + \varphi_u^2, \tag{2.30}$$

and similarly

$$\varphi_v^1 = \bar{\varphi}_v^1 \sigma(y), \quad \varphi_v^2 = \bar{\varphi}_v^2 \sigma(L_2 - y), \quad \varphi_v = \varphi_v^1 + \varphi_v^2. \tag{2.31}$$

The truncated profile satisfies

$$\begin{cases} \varphi_{ut} + \bar{U}_0 \varphi_{ux} - \varepsilon \varphi_{uyy} = f_\varphi^e, \\ \frac{\partial \varphi_u}{\partial x} + \frac{\partial \varphi_v}{\partial y} = h_\varphi^e, \\ \varphi_u = -u_0^0 \text{ at } y = 0 \text{ and } y = L_2, \\ \varphi_u = 0 \text{ at } x = 0, \\ \varphi_u = 0 \text{ at } t = 0, \end{cases} \tag{2.32}$$

where the error terms are defined as

$$f_\varphi^e = \varepsilon[-2\bar{\varphi}_{uy}^1 \sigma' + \bar{\varphi}_u^1 \sigma'' + 2\bar{\varphi}_{uy}^2 \sigma' + \bar{\varphi}_u^2 \sigma''], \tag{2.33}$$

$$h_\varphi^e = \bar{\varphi}_v^1 \sigma' - \bar{\varphi}_v^2 \sigma'. \tag{2.34}$$

Owing to the estimates (2.12)-(2.28), one readily derives that

$$||f_\varphi^e||_{L^\infty(0,T;L^2(\Omega))} \leq C\varepsilon^{\frac{3}{4}}, \quad ||h_\varphi^e||_{L^\infty(0,T;L^2(\Omega))} \leq C\varepsilon^{\frac{1}{2}}. \tag{2.35}$$

We now define a global pressure $\gamma_\phi(t,x,y) = \int_y^{+\infty} f\varphi_u d\tau$ (compare to Eq. (2.7)). In view of the estimate (2.14), γ_ϕ is well defined and there holds

$$|\gamma_\phi, \gamma_{\phi x}| \leq C\sqrt{\epsilon}\exp(-cy/\sqrt{\varepsilon}). \tag{2.36}$$

We note that $(\bar{\varphi}_u^1, \bar{\varphi}_v^1) = 0$ at $x = 0$ thanks to Eqs.(2.11), but may not vanish at $x = L_1$ in general. To resolve these discrepancies, we introduce the following correctors

$$\rho_u = -\varphi_u^1(t, L_1, y)\sigma(L_1 - x)\sigma(y) - \varphi_u^2(t, L_1, y)\sigma(L_1 - x)\sigma(L_2 - y), \tag{2.37}$$

$$\rho_v = -\varphi_v^1(t, L_1, y)\sigma(L_1 - x)\sigma(y) - \varphi_v^2(t, L_1, y)\sigma(L_1 - x)\sigma(L_2 - y). \tag{2.38}$$

It follows from the compatibility condition (1.11) that $\rho_u|_{y=0} = \rho_u|_{y=L_2} = 0$. By virtue of Lemma 2.1, the following estimate can be readily derived

$$|(\rho_u, \rho_v)|_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{1/4}. \tag{2.39}$$

Moreover, one can verify that the corrector (ρ_u, ρ_v) satisfies

$$\begin{cases} \rho_{ut} + \bar{U}_0 \rho_{ux} - f\rho_v - \varepsilon\Delta\rho_u = f_\rho^e, \\ \rho_{vt} + \bar{U}_0 \rho_{vx} + f\rho_u - \varepsilon\Delta\rho_v = g_\rho^e, \\ \rho_{ux} + \rho_{vy} = h_\rho^e, \\ \rho_u|_{x=0} = \rho_u|_{y=0} = \rho_u|_{y=L_2} = 0, \rho_u|_{x=L_1} = -\varphi_u|_{x=L_1}, \\ \rho_v|_{x=0} = \rho_v|_{y=0} = \rho_v|_{y=L_2} = 0, \rho_v|_{x=L_1} = -\varphi_v|_{x=L_1}, \\ (\rho_u, \rho_v)|_{t=0} = (0,0). \end{cases} \tag{2.40}$$

One can find the error forcing terms $f_\rho^e, g_\rho^e, h_\rho^e$ explicitly. We omit their definition here for simplicity, but note the following estimates which follows directly from Lemma 2.1:

$$||f_\rho^e, g_\rho^e, h_\rho^e||_{L^\infty(0,T;L^2(\Omega))} \leq C\varepsilon^{\frac{1}{4}}. \tag{2.41}$$

2.2. Ordinary boundary layers (OBL) at $x = L_1$

We now correct the discrepancy between the viscous solution v^ε and the inviscid solution v^0 at the boundary $x = L_1$, i.e., $\vartheta_v = -v^0$ at $x = L_1$ as in (2.4). Introducing the stretched variable $\bar{x} = (L_1 - x)/\varepsilon$, we find that the corrector at $x = L_1$ needs to satisfy:

$$\begin{cases} \bar{U}_0 \bar{\theta}_{v\bar{x}} - \bar{\theta}_{v\bar{x}\bar{x}} = 0, \\ \bar{\theta}_v = -v_0^0 \text{ at } x = L_1, \\ \bar{\theta}_v \to 0 \text{ as } \bar{x} \to \infty. \end{cases} \tag{2.42}$$

Notice that Eq. (2.42) is an ordinary differential equation, hence $\bar{\theta}_v$ is called the ordinary boundary layer. We can find the explicit form of $\bar{\theta}_v$:

$$\bar{\theta}_v = -v_0^0(t, L_1, y) e^{-\frac{\bar{U}_0}{\varepsilon}(L_1 - x)}. \tag{2.43}$$

Then we define $\theta_v = \bar{\theta}_v \sigma(L_1 - x)$. Here we do not introduce the divergence-free pair of $\bar{\theta}_v$. Owing to the compatibility condition (1.11), one sees that $\bar{\theta}_v = 0$ at $y = 0, L_2$. Hence the ordinary boundary layer $\bar{\theta}_v$ does not introduce discrepancy of boundary conditions at $y = 0$ and $y = L_2$.

Thanks to the explicit solution formula in (2.43), one can establish the following estimates.

Lemma 2.2. *Let $\bar{\theta}_v$ be the corrector as in (2.43). Then there exist constants $\kappa > 0$, independent of ε, such that, for $i, l, m, k \geq 0$, $0 \leq t \leq T$, $s \geq 0$,*

$$\left| (L_1 - x)^k \frac{\partial^{i+l+m} \bar{\theta}_v}{\partial t^i \partial x^l \partial y^m} \right|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa K_{i,m}^3 \varepsilon^{k-l+\frac{1}{2}}, \tag{2.44}$$

where

$$K_{i,m}^3(v_0^0) = \|\tilde{\gamma} v_0^0\|_{W^{i,\infty}(0,T;H_y^m(0,L_2))}, \tag{2.45}$$

and $\tilde{\gamma}$ is the trace operator at $x = L_1$.

The truncated profile θ_v satisfies

$$\begin{cases} \bar{U}_0 \theta_{vx} - \epsilon \theta_{vxx} = g_\theta^e, \\ \theta_v = -v_0^0 \text{ at } x = L_1, \\ \theta_v = 0 \text{ at } x = 0, \end{cases} \tag{2.46}$$

with

$$g_\theta^e = -\bar{U}_0 \bar{\theta}_v \sigma' + \epsilon[2\bar{\theta}_{vx}\sigma' - \bar{\theta}_v \sigma'']. \tag{2.47}$$

Notice that $supp(\sigma') \subset [L_1 - \frac{2L}{3}, L_1 - \frac{L}{2}]$ from the definition of the cut-off function in (2.29). It follows from the estimate (2.44) that

$$||g_\theta^e||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{k+\frac{1}{2}}, \tag{2.48}$$

for any positive integer k.

3. Convergence analysis

Collecting the inviscid solution and all the correctors, one defines an approximate solution

$$\begin{aligned} u^a &= u_0^0 + \varphi_u + \rho_u, \\ v^a &= v_0^0 + \varphi_v + \theta_v + \rho_v, \\ \phi^a &= \phi_0^0 + \gamma_\phi + \rho_\phi. \end{aligned} \tag{3.1}$$

It follows from the inviscid system (1.5), the parabolic boundary layer system (2.32), the ordinary boundary layer equation (2.46) and the corner boundary layer system (2.40) that the approximate solution (u^a, v^a, ϕ^a) satisfies

$$\begin{cases} u_t^a + \bar{U}_0 u_x^a - f v^a + \phi_x^a - \varepsilon \Delta u^a = F_u + f^e, \\ v_t^a + \bar{U}_0 v_x^a + f u^a + \phi_y^a - \varepsilon \Delta v^a = F_v + g^e, \\ u_x^a + v_y^a = h^e, \\ (u^a, v^a) = 0 \text{ on } \partial\Omega, \\ (u^a, v^a) = (\tilde{u}_0, \tilde{v}_0) \text{ at } t = 0, \end{cases} \tag{3.2}$$

where the error forcing terms are defined as

$$f^e = -\varepsilon \Delta u_0^0 - f\varphi_v - \varepsilon \varphi_{uxx} + \gamma_{\phi x} + f_\varphi^e + f_\rho^e, \tag{3.3}$$

$$g^e = -\varepsilon \Delta v_0^0 + \varphi_{vt} + \bar{U}_0 \varphi_{vx} - \varepsilon \Delta \varphi_v + \theta_{vt} - \varepsilon \theta_{vyy} + g_\theta^e + g_\rho^e, \tag{3.4}$$

$$h^e = h_\varphi^e + \theta_{vy} + h_\rho^e. \tag{3.5}$$

One may recall the definition of the error terms $f_\varphi^e, h_\varphi^e, g_\theta^e$ in (2.33), (2.34) and (2.47), respectively. Notice also that the approximate solution satisfies the same initial boundary conditions as the viscous solution, thanks to the construction of the correctors, cf. (1.2) and (1.3). In view of Lemma 2.1, Lemma 2.2, and estimates (2.35), (2.36), (2.48), the following estimate of the error forcing terms holds

$$||f^e, g^e, h^e||_{L^\infty(0,T;L^2(\Omega))} \leq C\varepsilon^{\frac{1}{4}}. \tag{3.6}$$

Introducing the error functions

$$u^e = u_0^\varepsilon - u^a, \quad v^e = v_0^\varepsilon - v^a, \quad \phi^e = \phi_0^\varepsilon - \phi^a, \tag{3.7}$$

one readily derives that

$$\begin{cases} u^e_t + \bar{U}_0 u^e_x - f v^e + \phi^e_x - \varepsilon \Delta u^e = -f^e, \\ v^e_t + \bar{U}_0 v^e_x + f u^e + \phi^e_y - \varepsilon \Delta v^e = -g^e, \\ u^e_x + v^e_y = -h^e, \\ (u^e, v^e) = 0 \text{ on } \partial\Omega, \\ (u^e, v^e) = 0 \text{ at } t = 0. \end{cases} \tag{3.8}$$

The following error estimate holds.

Theorem 3.1. *Assume the inviscid solution* $(u^0,v^0) \in C^1(0,T;H^3(\Omega))$. *Let* (u^e,v^e,ϕ^e) *be defined as in* (3.7). *The following estimate holds*

$$||u^e, v^e||_{L^\infty(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{4}}, \quad ||\phi^e||_{L^2(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{4}}. \tag{3.9}$$

Proof. The argument of the proof is standard. We provide the full details here for completeness.

We first consider an auxiliary problem

$$\begin{cases} \bar{U}_0 \tilde{u}^e_x + \tilde{\phi}^e_x - \varepsilon \Delta \tilde{u}^e + \tilde{u}^e = -f^e, \\ \bar{U}_0 \tilde{v}^e_x + \tilde{\phi}^e_y - \varepsilon \Delta \tilde{v}^e + \tilde{v}^e = -g^e, \\ \tilde{u}^e_x + \tilde{v}^e_y = -h^e, \\ (\tilde{u}^e, \tilde{v}^e) = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.10}$$

Note that the terms $(\tilde{u}^e, \tilde{v}^e)$ are added to the equations which will help to derive the L^2 error estimate below. Note also that the divergence-free condition and the homogeneous boundary conditions in (3.2) imply $\int_\Omega h^e \, dx = 0$. The well-posedness of the auxiliary problem (3.10) follows from the Lax-Milgram theorem, similar to the case of the Stokes equations with inhomogeneous divergence condition cf. [8,11,24]. Specifically, there exists a pair $(u,v) \in \mathbf{H}^1_0(\Omega)$ ([8]) such that

$$u_x + v_y = -h^e, \quad ||(u,v)||_{H^1(\Omega)} \le C||h^e||_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{4}}. \tag{3.11}$$

Now one tests the equations (3.10) by $(\tilde{u} - u, \tilde{v} - v)$. It follows that

$$\begin{aligned} &\varepsilon||\nabla(\tilde{u}^e, \tilde{v}^e)||^2_{L^2(\Omega)} + ||(\tilde{u}^e, \tilde{v}^e)||^2_{L^2(\Omega)} \\ \le& C(||(\tilde{u}^e, \tilde{v}^e)||_{L^2(\Omega)} + \varepsilon||\nabla(\tilde{u}^e, \tilde{v}^e)||_{L^2(\Omega)} + ||f^e, g^e||_{L^2(\Omega)})||(u,v)||_{H^1(\Omega)}, \end{aligned}$$

which implies

$$\varepsilon||\nabla(\tilde{u}^e, \tilde{v}^e)||^2_{L^2(\Omega)} + ||(\tilde{u}^e, \tilde{v}^e)||^2_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}. \tag{3.12}$$

The estimate of the pressure can be derived similarly. We impose that $\int_\Omega \phi^e dx = 0$ as the pressure is uniquely determined only up to constants. There exists $(u_p, v_p) \in \mathbf{H}^1_0(\Omega)$ which satisfies

$$u_{px} + v_{py} = -\tilde{\phi}^e, \quad ||(u_p, v_p)||_{H^1(\Omega)} \le C||\tilde{\phi}^e||_{L^2(\Omega)}. \tag{3.13}$$

Testing Eqs. (3.10) by (u_p, v_p) and performing integration by parts, one derives

$$||\tilde{\phi}^e||^2_{L^2(\Omega)} \leq C(||(\tilde{u}^e, \tilde{v}^e)||_{L^2(\Omega)} + \varepsilon||\nabla(\tilde{u}^e, \tilde{v}^e)||_{L^2(\Omega)})||(u_p, v_p)||_{H^1(\Omega)} \quad (3.14)$$

$$\leq C(||(\tilde{u}^e, \tilde{v}^e)||_{L^2(\Omega)} + \varepsilon||\nabla(\tilde{u}^e, \tilde{v}^e)||_{L^2(\Omega)})||\tilde{\phi}^e||_{L^2(\Omega)}. \quad (3.15)$$

It follows from the estimate (3.12) that

$$||\tilde{\phi}^e||_{L^2(\Omega)} \leq C\varepsilon^{\frac{1}{4}}. \quad (3.16)$$

Note that the variables in the system (3.10) all depend on time. By differentiating the system (3.10) in time, and repeating the same argument as above, one can further derive

$$||\partial_t(\tilde{u}^e, \tilde{v}^e)||^2_{L^2(\Omega)} \leq ||\partial_t(f^e, g^e, h^e)||^2_{L^2(\Omega)} \leq C\varepsilon^{\frac{1}{2}}. \quad (3.17)$$

Now the difference $W_u = u^e - \tilde{u}^e, W_v = v^e - \tilde{v}^e, W_\phi = \phi^e - \tilde{\phi}^e$ satisfies

$$\begin{cases} W_{ut} + \bar{U}_0 W_{ux} - fW_v + W_{\phi x} - \varepsilon\Delta W_u = -\tilde{u}^e_t + f\tilde{v}^e + \tilde{u}^e, \\ W_{vt} + \bar{U}_0 W_{vx} + fW_u + W_{\phi y} - \varepsilon\Delta W_v = -\tilde{v}^e_t - f\tilde{u}^e + \tilde{v}^e, \\ W_{ux} + W_{vy} = 0, \\ (W_u, W_v) = 0 \text{ on } \partial\Omega, \\ (W_u, W_v) = 0 \text{ at } t = 0. \end{cases} \quad (3.18)$$

Using the estimates (3.12) and (3.17), and applying the standard energy method to (3.18), one gets

$$\varepsilon^{\frac{1}{2}}||\nabla(W_u, W_v)||_{L^2(0,T;L^2(\Omega))} + ||(W_u, W_v)||_{L^\infty(0,T;L^2(\Omega))} \leq C\varepsilon^{\frac{1}{4}}. \quad (3.19)$$

We recall the vectorial space $\mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1_0(\Omega), \nabla \cdot \mathbf{u} = 0\}$ and its dual space $\mathbf{V}'$. It then follows from Eqs. (3.18) and the estimate (3.19) that

$$||(W_{ut}, W_{vt})||_{L^2(0,T;\mathbf{V}')} \leq C\varepsilon^{\frac{1}{4}}. \quad (3.20)$$

To derive the estimate on the pressure, we rewrite the system (3.18) as the classical Stokes problem

$$\begin{cases} W_{\phi x} - \varepsilon\Delta W_u = \tilde{u}^e_t + f\tilde{v}^e + \tilde{u}^e - W_{ut} - \bar{U}_0 W_{ux} + fW_v, \\ W_{\phi y} - \varepsilon\Delta W_v = \tilde{v}^e_t - f\tilde{u}^e + \tilde{v}^e - W_{vt} - \bar{U}_0 W_{vx} - fW_u, \\ W_{ux} + W_{vy} = 0, \\ (W_u, W_v) = 0 \text{ on } \partial\Omega, \\ (W_u, W_v) = 0 \text{ at } t = 0. \end{cases} \quad (3.21)$$

Note that the forcing terms on the right-hand side are bounded by $C\varepsilon^{\frac{1}{4}}$ in the $L^2(0,T;\mathbf{V}')$ norm by the estimate (3.20). A direct energy estimate gives that $||\nabla(W_u, W_v)||_{L^2(0,T;L^2(\Omega))} \leq C\varepsilon^{-\frac{3}{4}}$, which yields that $||\Delta(W_u, W_v)||_{L^2(0,T;H^{-1}(\Omega))} \leq$

$C\varepsilon^{-\frac{3}{4}}$. Here $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. It then follows from Eqs. (3.21) that $||\nabla W_\phi||_{L^2(0,T;H^{-1}(\Omega))} \le C\varepsilon^{\frac{1}{4}}$, hence that $||W_\phi||_{L^2(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{4}}$.

The desired estimate (3.9) follows immediately by an application of the triangle inequality, the estimates (3.12) and (3.16). □

Recall that $||(\varphi_u, \varphi_v)||_{L^\infty(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{4}}$ from Lemma 2.1, $||\gamma_\phi||_{L^\infty(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{3}{4}}$ from the estimate (2.36), $||(\rho_u, \rho_v)||_{L^\infty(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{4}}$ from (2.39), and $||\theta_v||_{L^\infty(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{2}}$. Theorem 3.1 implies the following result of the vanishing viscosity limit.

Corollary 3.1. *Let* $(u_0^\varepsilon, v_0^\varepsilon)$ *be the solution to the viscous problem* (1.1)*, and* (u_0^0, v_0^0) *be the solution to the inviscid problem* (1.5) *emanating from the same initial condition and forcing terms. Assume that* $(u_0^0, v_0^0) \in C^1(0, T; H^3(\Omega))$*. Then the following vanishing viscosity limit holds*

$$||(u_0^\varepsilon - u_0^0, v_0^\varepsilon - v_0^0)||_{L^\infty(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{4}}, \quad ||\phi_0^\varepsilon - \phi_0^0||_{L^2(0,T;L^2(\Omega))} \le C\varepsilon^{\frac{1}{4}}. \quad (3.22)$$

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References

[1] C. Bandle and C. M. Brauner, *Singular perturbation method in a parabolic problem with free boundary*, in *BAIL IV (Novosibirsk, 1986)*, 8 of *Boole Press Conf. Ser.*, Boole, Dún Laoghaire, 1986, 7–14.

[2] C. M. Brauner, *Perturbations singulières dans des systèmes non linéaires et applications à la biochimie*, U.E.R. Mathématique Université Paris XI Orsay, 1975, 148, 75–41.

[3] C. M. Brauner, W. Eckhaus, M. Garbey and A. Van Harten, *A nonlinear, singular perturbation problem with some unusual features*, in *Nonlinear systems of partial differential equations in applied mathematics, Part 2 (Santa Fe, N.M., 1984)*, 23 of Lectures in Appl. Math., 1984, 275–310.

[4] C. M. Brauner and B. Nicolaenko, *Transition layers, angular limiting solutions and internal layers in singularly perturbed nonlinear eigenvalue problems*, Springer Vienna, 1983, 15(5), 269–308.

[5] Q. Chen, M. C. Shiue and R. Temam, *The barotropic mode for the primitive equations*, J. Sci. Comput, 2010, 45, 167–199.

[6] W. E, *Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation*, Acta Math. Sin. (Engl. Ser.), 2000, 16(2), 207–218.

[7] M. Fei, D. Han and X. Wang, *Initial-boundary layer associated with the nonlinear Darcy-Brinkman-Oberbeck-Boussinesq system*, Phys. D, 2017, 338, 42–56.

[8] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, Springer Monographs in Mathematics, 2nd Edn, Springer, New York, 2011.

[9] G. M. Gie, *Asymptotic expansion of the Stokes solutions at small viscosity: the case of non-compatible initial data*, Commun. Math. Sci., 2014, 12(2), 383–400.

[10] G. M. Gie, C. Y. Jung and R. Temam, *Recent progresses in boundary layer theory*, Discrete Contin. Dyn. Syst., 2016, 36(5), 2521–2583.

[11] V. Girault and P. A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer-Verlag, 1986, 87(24), 167–204.

[12] M. Hamouda, D. Han, C. Y. Jung and R. Temam, *Boundary layers for the subcritical modes of the 3d primitive equations in a cube*, submitted.

[13] M. Hamouda, C. Y. Jung and R. Temam, *Boundary layers for the 2D linearized primitive equations*, Commun. Pure Appl. Anal., 2017, 8(1), 335–359.

[14] M. Hamouda, C. Y. Jung and R. Temam, *Asymptotic analysis for the 3D primitive equations in a channel*, Discrete and Continuous Dynamical Systems-Series S, 2013, 6(2), 401–422.

[15] M. Hamouda, C. Y. Jung and R. Temam, *Boundary layers for the 3D primitive equations in a cube: the supercritical modes*, Nonlinear Anal., 2016, 132, 288–317.

[16] D. Han and X. Wang, *Initial-boundary layer associated with the nonlinear Darcy-Brinkman system*, J. Differential Equations, 2014, 256(2), 609–639.

[17] J. L. Lions, R. Temam and S. H. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity, 1992, 5(2), 237–288.

[18] J. L. Lions, R. Temam and S. H. Wang, *On the equations of the large-scale ocean*, Nonlinearity, 1992, 5(5), 1007–1053.

[19] Y. Maekawa and A. Mazzucato, *The inviscid limit and boundary layers for navier-stokes flows*, in *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*, Springer, 2017, 1–48.

[20] T. T. Nguyen and F. Sueur, *Boundary-layer interactions in the plane-parallel incompressible flows*, Nonlinearity, 2012, 25(12), 3327–3342.

[21] J. Oliger and A. Sundström, *Theoretical and practical aspects of some initial boundary value problems in fluid dynamics*, SIAM J. Appl. Math., 1978, 35(3), 419–446.

[22] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer Study Edition, Springer-Verlag, New York, 1979.

[23] S. D. Shih and R. B. Kellogg, *Asymptotic analysis of a singular perturbation problem*, SIAM J. Math. Anal., 1987, 18(5), 1467–1511.

[24] R. Temam, *Navier-Stokes equations, Theory and Numerical analysis*, North-Holland Publishing Co., 1984, 45(2), 1588–1596.

[25] R. Temam and J. Tribbia, *Open boundary conditions for the primitive and Boussinesq equations*, J. Atmospheric Sci., 2003, 60(21), 2647–2660.

[26] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handbook of mathematical fluid dynamics. Vol. III, 2004, 535–657.

[27] S. Tordeux, G. Vial and M. Dauge, *Matching and multiscale expansions for a model singular perturbation problem*, C. R. Math. Acad. Sci. Paris, 2006, 343(10), 637–642.

[28] G. Vial, *Efficiency of approximate boundary conditions for corner domains coated with thin layers*, C. R. Math. Acad. Sci. Paris, 2005, 340(3), 215–220.

[29] T. Warner, R. Peterson and R. Treadon, *A tutorial on lateral boundary conditions as a basic and potentially serious limitation to regional numerical weather prediction*, Bulletin of the American Meteorological Society, 1997, 78(11), 2599–2617.