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A LINEAR OPERATOR ASSOCIATED WITH THE MITTAG-LEFFLER FUNCTION AND RELATED CONFORMAL MAPPINGS

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Abstract In the present paper, we introduce a linear operator associated with the Mittag-Leffler function. Some convolution properties of meromorphic functions involving this operator are given.

Keywords Meromorphic functions, conformal mapping, Mittag-Leffler function, second-order differential subordination, Hadamard product (or convolution), convex functions, univalent functions.

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1. Introduction

The familiar Mittag-Leffler function $E_\alpha(z)$ introduced by Mittag-Leffler [5] and its generalization $E_{\alpha,\beta}(z)$ introduced by Wiman [12] are defined by the following series:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (z, \alpha \in \mathbb{C};\ \Re(\alpha) > 0) \tag{1.1}$$

and

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (z, \alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0), \tag{1.2}$$

respectively. These functions are natural extensions of the exponential, hyperbolic and trigonometric functions, since

$$E_1(z) = E_{1,1}(z) = e^z, \quad E_2(z^2) = E_{2,1}(z^2) = \cosh z \quad \text{and}$$
$$E_2(-z^2) = E_{2,1}(-z^2) = \cos z.$$

The above-defined functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, as well as their various further generalizations, arise naturally in the solution of fractional differential equations and fractional integro-differential equations which are associated with (for example) the kinetic equation, random walks, Lévy flights, super-diffusive transport problems

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and in the study of complex systems. In particular, the Mittag-Leffler function is an explicit formula for the resolvent of Riemann-Liouville fractional integrals by Hille and Tamarkin. Several properties of the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, together with their generalizations, can be found in a number of recent works (see [1–3] and [7–11]).

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1.3}$$

which are analytic in the *punctured* open unit disk

$$\mathbb{U}_0 = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\}.$$

The class $\Sigma(p)$ is closed under the Hadamard product (or convolution):

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,1} a_{n-p,2} z^{n-p} = (f_2 * f_1)(z),$$

where

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p} \in \Sigma(p) \quad (j = 1, 2).$$

For $f \in \Sigma(p)$, we consider the following operator $T_{\alpha,\beta} : \Sigma(p) \to \Sigma(p)$ associated with the Mittag-Leffler function:

$$\begin{aligned} T_{\alpha,\beta} f(z) &= \left(\Gamma(\beta) z^{-p} E_{\alpha,\beta}(z)\right) * f(z) \\ &= z^{-p} + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)} a_{n-p} z^{n-p}, \end{aligned} \tag{1.4}$$

where $z, \alpha, \beta \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Let $\mathcal{P}$ be the class of functions h with $h(0) = 1$, which are analytic and convex univalent in the open unit disk $\mathbb{U} = \mathbb{U}_0 \cup \{0\}$.

For functions f and g analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$, if g is univalent in $\mathbb{U}$, $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Now we introduce the following new subclass of $\Sigma(p)$.

Definition 1.1. A function $f \in \Sigma(p)$ is said to be in the class $\mathcal{M}_{\alpha,\beta}(\lambda; h)$ if it satisfies the second order differential subordination:

$$\frac{\lambda - 1}{p} z^{p+1} \left(T_{\alpha,\beta} f(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(T_{\alpha,\beta} f(z)\right)'' \prec h(z), \tag{1.5}$$

where $\lambda, \alpha, \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ and $h \in \mathcal{P}$.

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.6}$$

which are analytic in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\gamma)$ if

$$\Re\left(\frac{z f'(z)}{f(z)}\right) > \gamma \quad (z \in \mathbb{U}) \tag{1.7}$$

for some γ $(\gamma < 1)$. When $0 \leqq \gamma < 1$, $\mathcal{S}^*(\gamma)$ is the class of starlike functions of order γ in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be prestarlike of order γ in $\mathbb{U}$ if

$$\frac{z}{(1-z)^{2(1-\gamma)}} * f(z) \in \mathcal{S}^*(\gamma) \quad (\gamma < 1). \tag{1.8}$$

We denote this class by $\mathcal{R}(\gamma)$ (see [6]). It is obvious that a function $f \in \mathcal{A}$ is in the class $\mathcal{R}(0)$ if and only if f is convex univalent in $\mathbb{U}$ and $\mathcal{R}\left(\frac{1}{2}\right) = \mathcal{S}^*\left(\frac{1}{2}\right)$.

The study of the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ is a recent interesting topic in geometric function theory. In the present paper we shall make a further contribution to the subject by showing some convolution properties for meromorphic functions involving the Mittag-Leffler functions.

The following lemmas will be used in our investigation.

Lemma 1.1 ([6]). *Let $\gamma < 1$, $f \in \mathcal{S}^*(\gamma)$ and $g \in \mathcal{R}(\gamma)$. Then, for analytic function F in $\mathbb{U}$,*

$$\frac{g * (fF)}{g * f}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U})),$$

where $\overline{co}(F(\mathbb{U}))$ denotes the closed convex hull of $F(\mathbb{U})$.

Lemma 1.2 ([4]). *Let $g(z) = 1 + \sum_{n=m}^{\infty} b_n z^n$ $(m \in \mathbb{N})$ be analytic in $\mathbb{U}$. If $\Re(g(z)) > 0$ $(z \in \mathbb{U})$, then*

$$\Re\left(g(z)\right) \geqq \frac{1-|z|^m}{1+|z|^m} \quad (z \in \mathbb{U}).$$

2. Hadamard product properties

In this section we shall derive several Hadamard product properties for functions in the class $\mathcal{M}_{\alpha,\beta}(\lambda; h)$.

Theorem 2.1. *Let $f \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$, $g \in \Sigma(p)$ and $\Re\left(z^p g(z)\right) > \frac{1}{2}$ $(z \in \mathbb{U})$. Then $f * g \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$.*

Proof. For $f \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$ and $g \in \Sigma(p)$, we have

$$\begin{aligned}
&\frac{\lambda-1}{p} z^{p+1}\left(T_{\alpha,\beta}(f*g)(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2}\left(T_{\alpha,\beta}(f*g)(z)\right)'' \\
=&\frac{\lambda-1}{p}\left(z^p g(z)\right) * \left(z^{p+1}(T_{\alpha,\beta}f(z))'\right) + \frac{\lambda}{p(p+1)}\left(z^p g(z)\right) * \left(z^{p+2}(T_{\alpha,\beta}f(z))''\right) \\
=&\left(z^p g(z)\right) * \psi(z),
\end{aligned} \tag{2.1}$$

where

$$\psi(z) = \frac{\lambda-1}{p} z^{p+1}\left(T_{\alpha,\beta}f(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2}\left(T_{\alpha,\beta}f(z)\right)'' \prec h(z). \tag{2.2}$$

In view of the conditions of Theorem 2.1, the function $z^p g(z)$ has the Herglotz representation:

$$z^p g(z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in \mathbb{U}), \tag{2.3}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$. Since the function h is convex univalent in $\mathbb{U}$, it follows from (2.1) to (2.3) that

$$\begin{aligned}&\frac{\lambda-1}{p} z^{p+1} \left(T_{\alpha,\beta}(f*g)(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(T_{\alpha,\beta}(f*g)(z)\right)'' \\ &= \int_{|x|=1} \psi(xz) d\mu(x) \prec h(z).\end{aligned}$$

This shows that $f * g \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$. The proof of Theorem 2.1 is completed. □

Theorem 2.2. *Let $f \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$, $g \in \Sigma(p)$ and $z^{p+1}g(z) \in \mathcal{R}(\gamma)$ $(\gamma < 1)$. Then $f * g \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$.*

Proof. From (2.1) we can write

$$\begin{aligned}&\frac{\lambda-1}{p} z^{p+1} \left(T_{\alpha,\beta}(f*g)(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(T_{\alpha,\beta}(f*g)(z)\right)'' \\ &= \frac{\left(z^{p+1}g(z)\right) * \left(z\psi(z)\right)}{\left(z^{p+1}g(z)\right) * z},\end{aligned} \tag{2.4}$$

where the function ψ is defined as in (2.2).

Since the function h is convex univalent in $\mathbb{U}$,

$$\psi(z) \prec h(z), \quad z^{p+1}g(z) \in \mathcal{R}(\gamma) \quad \text{and} \quad z \in \mathcal{S}^*(\gamma)\ (\gamma < 1),$$

from (2.4) and Lemma 1.1, we obtain the desired result. The proof of Theorem 2.2 is completed. □

Taking $\gamma = 0$ and $\gamma = \frac{1}{2}$ in Theorem 2.2, we have the following consequence.

Corollary 2.1. *Let $f \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$. Also let $g \in \Sigma(p)$ satisfy either of the following conditions*:

(i) *$z^{p+1}g(z)$ is convex univalent in $\mathbb{U}$*

or

(ii) *$z^{p+1}g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$.*

Then $f * g \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$.

Theorem 2.3. *Let $\lambda \leqq 0$ and*

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p} \in \mathcal{M}_{\alpha,\beta}(\lambda; h_j) \quad (j = 1, 2), \tag{2.5}$$

where

$$h_j(z) = \frac{1 + A_j z}{1 + B_j z} \quad \text{and} \quad -1 \leqq B_j < A_j \leqq 1. \tag{2.6}$$

If $f \in \Sigma(p)$ is defined by

$$\left(T_{\alpha,\beta} f(z)\right)' = -\frac{1}{p}\left(\left(T_{\alpha,\beta} f_1(z)\right)' * \left(T_{\alpha,\beta} f_2(z)\right)'\right), \tag{2.7}$$

then $f \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$, where

$$h(z) = \gamma + (1-\gamma)\frac{1+z}{1-z} \tag{2.8}$$

and γ is given by

$$\gamma = \begin{cases} 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \left(1 + \frac{p+1}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u} du\right) & (\lambda < 0) \\ 1 - \frac{2(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} & (\lambda = 0). \end{cases} \tag{2.9}$$

The bound γ is sharp when $B_1 = B_2 = -1$.

Proof. We consider the case when $\lambda < 0$. By setting

$$H_j(z) = \frac{\lambda - 1}{p} z^{p+1} \left(T_{\alpha,\beta} f_j(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(T_{\alpha,\beta} f_j(z)\right)'' \quad (j = 1, 2)$$

for f_j $(j = 1, 2)$ given by (2.5), we find that

$$H_j(z) = 1 + \sum_{n=1}^{\infty} b_{n,j} z^n \prec \frac{1 + A_j z}{1 + B_j z} \quad (j = 1, 2) \tag{2.10}$$

and

$$\left(T_{\alpha,\beta} f_j(z)\right)' = \frac{p(p+1)}{\lambda} z^{\frac{(1-\lambda)(p+1)}{\lambda}} \int_0^z t^{-\frac{p+1}{\lambda}-1} H_j(t) dt \quad (j = 1, 2). \tag{2.11}$$

Now, if $f \in \Sigma(p)$ is defined by (2.7), we find from (2.11) that

$$\begin{aligned} \left(T_{\alpha,\beta} f(z)\right)' &= -\frac{1}{p} \left(\left(T_{\alpha,\beta} f_1(z)\right)' * \left(T_{\alpha,\beta} f_2(z)\right)' \right) \\ &= -\frac{1}{p} \left(\frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{\lambda}-1} H_1(uz) du \right) \\ &\quad * \left(\frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{\lambda}-1} H_2(uz) du \right) \\ &= \frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{\lambda}-1} H(uz) du, \end{aligned} \tag{2.12}$$

where

$$H(z) = -\frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda}-1} (H_1 * H_2)(uz) du. \tag{2.13}$$

Also, by using (2.10) and the Herglotz theorem, we see that

$$\Re\left\{ \left(\frac{H_1(z) - \gamma_1}{1 - \gamma_1} \right) * \left(\frac{1}{2} + \frac{H_2(z) - \gamma_2}{2(1 - \gamma_2)} \right) \right\} > 0 \quad (z \in \mathbb{U}),$$

which leads to

$$\Re\{(H_1 * H_2)(z)\} > \gamma_0 = 1 - 2(1 - \gamma_1)(1 - \gamma_2) \quad (z \in \mathbb{U}),$$

where

$$0 \leqq \gamma_j = \frac{1 - A_j}{1 - B_j} < 1 \quad (j = 1, 2).$$

According to Lemma 1.2, we have

$$\Re\{(H_1 * H_2)(z)\} \geqq \gamma_0 + (1 - \gamma_0) \frac{1 - |z|}{1 + |z|} \quad (z \in \mathbb{U}). \tag{2.14}$$

Now it follows from (2.12) to (2.14) that

$$\begin{aligned}
&\Re\left\{\frac{\lambda-1}{p}z^{p+1}\left(T_{\alpha,\beta}f(z)\right)' + \frac{\lambda}{p(p+1)}z^{p+2}\left(T_{\alpha,\beta}f(z)\right)''\right\} = \Re\{H(z)\}\\
&= -\frac{p+1}{\lambda}\int_0^1 u^{-\frac{p+1}{\lambda}-1}\Re\{(H_1 * H_2)(uz)\}du\\
&\geqq -\frac{p+1}{\lambda}\int_0^1 u^{-\frac{p+1}{\lambda}-1}\left(\gamma_0 + (1-\gamma_0)\frac{1-u|z|}{1+u|z|}\right)du\\
&> \gamma_0 - \frac{(p+1)(1-\gamma_0)}{\lambda}\int_0^1 u^{-\frac{p+1}{\lambda}-1}\frac{1-u}{1+u}du\\
&= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1+\frac{p+1}{\lambda}\int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u}du\right)\\
&= \gamma,
\end{aligned}$$

which proves that $f \in \mathcal{M}_{\alpha,\beta}(\lambda; h)$ for the function h given by (2.8).

When $B_1 = B_2 = -1$, we consider the functions f_j $(j = 1, 2)$ defined by

$$\left(T_{\alpha,\beta}f_j(z)\right)' = \frac{p(p+1)}{\lambda}z^{\frac{(1-\lambda)(p+1)}{\lambda}}\int_0^z t^{-\frac{p+1}{\lambda}-1}\frac{1+A_j t}{1-t}dt \quad (j = 1, 2), \tag{2.15}$$

for which we have

$$\begin{aligned}
H_j(z) &= \frac{\lambda-1}{p}z^{p+1}\left(T_{\alpha,\beta}f_j(z)\right)' + \frac{\lambda}{p(p+1)}z^{p+2}\left(T_{\alpha,\beta}f_j(z)\right)''\\
&= \frac{1+A_j z}{1-z} \quad (j = 1, 2)
\end{aligned}$$

and

$$\begin{aligned}
(H_1 * H_2)(z) &= \frac{1+A_1 z}{1-z} * \frac{1+A_2 z}{1-z}\\
&= 1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-z}.
\end{aligned}$$

Hence, for the function f given by (2.7), we have

$$\begin{aligned}
&\frac{\lambda-1}{p}z^{p+1}\left(T_{\alpha,\beta}f(z)\right)' + \frac{\lambda}{p(p+1)}z^{p+2}\left(T_{\alpha,\beta}f(z)\right)''\\
&= -\frac{p+1}{\lambda}\int_0^1 u^{-\frac{p+1}{\lambda}-1}\left(1-(1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz}\right)du\\
&\to 1-(1+A_1)(1+A_2)\left(1+\frac{p+1}{\lambda}\int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u}du\right)
\end{aligned}$$

as $z \to -1$.

Finally, for the case when $\lambda = 0$, the proof of Theorem 2.3 is simple, and we choose to omit the details involved. Now the proof of Theorem 2.3 is completed. $\square$

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