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ASYMPTOTIC BEHAVIOR IN CHEMICAL REACTION-DIFFUSION SYSTEMS WITH BOUNDARY EQUILIBRIA*

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Abstract We consider the asymptotic behavior for large time of solutions to reaction-diffusion systems modeling reversible chemical reactions. We focus on the case where multiple equilibria exist. In this case, due to the existence of so-called "boundary equilibria", the analysis of the asymptotic behavior is not obvious. The solution is understood in a weak sense as a limit of adequate approximate solutions. We prove that this solution converges in L^1 toward an equilibrium as time goes to infinity and that the convergence is exponential if the limit is strictly positive.

Keywords Reaction diffusion systems, asymptotic behavior of solution, convergence to equilibrium.

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1. Introduction and main results

The purpose of our work is to analyze the asymptotic behavior of the global solutions for reaction-diffusion systems arising in reversible chemical kinetics and with multiple equilibria. We consider the following reversible reaction process for a set of chemical species A_i $(i = 1, ..., n)$

$$\alpha_1 A_1 + ... + \alpha_n A_n \rightleftharpoons \beta_1 A_1 + ... + \beta_n A_n.$$

We assume that this takes place in a bounded regular domain $\Omega \subset \mathbb{R}^N$ with spatial diffusion phenomena. According to the mass action law for the reactive terms and to Fick's law for the diffusion (see e.g. [1]), the concentrations $u_i = u_i(t, x)$ will be assumed to satisfy the following reaction diffusion system for $i = 1, ..., n$ and for all

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$T \in (0, \infty)$:

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = (\beta_i - \alpha_i)(k_f \prod_{j=1}^n u_j^{\alpha_j} - k_r \prod_{j=1}^n u_j^{\beta_j}) & \text{in } Q_T := \Omega \times (0,T), \\ \partial_\nu u_i(x,t) = 0 & \text{on } \Gamma_T := \partial\Omega \times (0,T), \\ u_i(x,0) = u_{i0}(x) \geq 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where ∂_ν denotes the exterior normal derivative to $\partial\Omega$, $k_f, k_r \in (0,\infty)$ and α_i, β_i are nonnegative integers. Actually we will more generally assume that $\alpha_i, \beta_i \in \{0,1\} \cup [2,\infty)$ (so that the nonlinear reactive function is still of class C^2). We denote

$$I := \{i \in \{1,...,n\}; \alpha_i - \beta_i > 0\},\ J := \{j \in \{1,...,n\}; \alpha_j - \beta_j < 0\}, \tag{1.2}$$

and we naturally assume $I \neq \emptyset, J \neq \emptyset, I \cup J = \{1,...,n\}$.

We are interested here in the asymptotic behavior as $t \to +\infty$ of the *global solutions* to this system. However, the question of existence of global solutions is delicate and we need to recall some facts. First, let us introduce the standard approximate system where $1 \leq i \leq n$ and $\epsilon \in (0,1)$

$$\begin{cases} \partial_t u_i^\epsilon - d_i \Delta u_i^\epsilon = f_i^\epsilon(u^\epsilon) & \text{in } Q_T, \\ \partial_\nu u_i^\epsilon = 0 & \text{on } \Gamma_T, \\ u_i^\epsilon(\cdot,0) = u_{i0}^\epsilon \geq 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

$$\begin{cases} f_i^\epsilon(u) = \frac{f_i(u)}{1+\epsilon\sum_{j=1}^n |f_j(u)|},\ u_{i0}^\epsilon = \inf\{u_{i0}, \epsilon^{-1}\},\ f_i(u) := (\beta_i - \alpha_i)F(u), \\ F(u) := k_f \prod_{j=1}^n u_j^{\alpha_j} - k_r \prod_{j=1}^n u_j^{\beta_j}. \end{cases} \tag{1.4}$$

Notice that $f^\epsilon = (f_1^\epsilon, ..., f_n^\epsilon)$ is locally Lipschitz continuous, quasi-positive and uniformly bounded by $1/\epsilon$. By standard arguments, existence and uniqueness of a classical nonnegative solution u^ϵ to (1.3) holds for all $T > 0$.

It is proved in [13] that u^ϵ converges as $\epsilon \to 0$ (up to a subsequence) to a so-called *renormalized solution* to (1.1). Let us recall the main facts proved in [13] and that we will use in this paper. For this, let us introduce:

$$\begin{cases} L_i(s) := s(\log s - 1 + \mu_i) + e^{-\mu_i} \geq 0 \textit{ for all } s \in [0,\infty), \\ \mu_i := [\log k_f - \log k_r]/[n(\alpha_i - \beta_i)],\ i = 1,...,n, \end{cases} \tag{1.5}$$

and for all $r \in (0,\infty)$, let $T_r \in C^2([0,\infty);[0,\infty))$ with

$$\begin{cases} 0 \leq T_r'(s) \leq 1,\ T_r''(s) \leq 0 \text{ for all } s \in [0,\infty), \\ T_r(s) = s \text{ for } s \in [0,r],\ T_r'(s) = 0 \ \text{ for } s \in [2r,\infty). \end{cases} \tag{1.6}$$

Proposition 1.1. *Assume that $u_0 = (u_{01}, ..., u_{0n})$ belongs to $L^1(\Omega)^n$ and satisfies for all $i = 1, ..., n$,*

$$u_{i0}(x) \geq 0 \ a.e. x \in \Omega,\ \textit{and} \ \int_\Omega u_{i0}|\log u_{i0}| < \infty.$$

Then, along a subsequence as $\epsilon \downarrow 0$ *and for all* $i = 1, ..., n, T \in (0, \infty)$

$$u_i^\epsilon \to u_i \text{ a.e. in } Q_\infty \text{ and in } L^1(Q_T),\ \nabla\sqrt{u_i^\epsilon} \to \nabla\sqrt{u_i} \text{ weakly in } L^2(Q_T). \tag{1.7}$$

Moreover, the limit $u = (u_1, ..., u_n)$ *satisfies the following properties:*
(I) (entropy inequality)

$$\begin{cases} u_i, u_i \log u_i \in L^\infty\left(0,\infty; L^1(\Omega)\right),\ \sqrt{u_i} \in L^2\left(0,\infty; H^1(\Omega)\right),\ \text{and a.e.t}, \\ \int_\Omega \sum_{i=1}^n L_i(u_i(t)) + \int_0^t \int_\Omega \{\sum_{i=1}^n d_i \frac{|\nabla u_i|^2}{u_i} - \sum_{i=1}^n f_i(u)[\mu_i + \log(u_i)]\} \\ \le \int_\Omega \sum_{i=1}^n L_i(u_{i0}). \end{cases} \tag{1.8}$$

(II) (a renormalized property) For $v_i := u_i + \eta \sum_{j\neq i} u_j, \eta \in (0, 1]$,

$$\begin{cases} \partial_t T_r(v_i) - d_i \Delta T_r(v_i) = G_1 + G_2 + \nabla \cdot G_3, \\ \partial_\nu T_r(v_i) = 0 \text{ on } \partial\Omega,\ \ T_r(v_i)(0) = T_r(v_{i0}),\ v_{i0} := u_{i0} + \eta \sum_{j\neq i} u_{j_0}, \end{cases} \tag{1.9}$$

where

$$\begin{cases} G_1 := T_r'(v_i)[f_i(u) + \eta \sum_{j\neq i} f_j(u)] \in L^\infty(Q_\infty), \\ G_2 := -\eta \sum_{j\neq i} (d_j - d_i) T_r''(v_i) \nabla v_i \nabla u_j - T_r''(v_i) |\nabla v_i|^2 \in L^1\left(0,\infty; L^1(\Omega)\right), \\ G_3 := \eta \sum_{j\neq i} (d_j - d_i) T_r'(v_i) \nabla u_j \in L^2\left(0,\infty; L^2(\Omega)\right). \end{cases} \tag{1.10}$$

We will come back to this proposition with more comments on its meaning and on its proof (see the beginning of Section 2). But let us continue with the main purpose of this paper. Let us define the equilibrium set associated with $u_0 \in L^1(\Omega)^{+n}$ and which contains the expected asymptotic limits. Note that they are constant functions.

$$\begin{cases} \mathbb{E}_{u_0} := \{e = (e_1, ..., e_n) \in [0,\infty)^n; e \text{ satisfies } (E1) \text{ and } (E2)\}, \\ (E1) \quad k_f \prod_i e_i^{\alpha_i} - k_r \prod_i e_i^{\beta_i} = 0, \\ (E2) \quad \frac{e_i}{\alpha_i - \beta_i} + \frac{e_j}{\beta_j - \alpha_j} = A_i + B_j,\ \ \forall (i,j) \in I \times J, \\ A_i := \frac{\overline{u}_{i0}}{\alpha_i - \beta_i}, B_j := \frac{\overline{u}_{j0}}{\beta_j - \alpha_j},\ \forall (i,j) \in I \times J, \end{cases} \tag{1.11}$$

where for $v \in L^1(\Omega)$, we denote $\overline{v} := ⨍_\Omega v = |\Omega|^{-1} \int_\Omega v$.

The main result of this paper is the following.

Theorem 1.1. *Under the assumptions of Proposition 1.1, there exists* $u^\infty \in \mathbb{E}_{u_0}$ *such that the solution* $u(t) = (u_1(t), ..., u_n(t))$ *satisfies*

$$u(t) \to u^\infty \text{ in } L^1(\Omega)^n \text{ as } t \to +\infty. \tag{1.12}$$

If moreover $u^\infty \in (0,\infty)^n$*, then the convergence is exponential: there exist* $C, \lambda \in (0,\infty)$ *such that*

$$\|u(t) - u^\infty\|_{L^1(\Omega)^n} \le Ce^{-\lambda t}.$$

Remark 1.1. The main interest of this result is that it applies to systems *with boundary equilibria*, that is when $\mathbb{E}_{u_0}$ is not reduced to a single point in $(0,\infty)^n$ and contains equilibria e whose components e_i do vanish for some i. It says moreover that, if for a given u_0 the limit $u^\infty \in (0,\infty)^n$, then the convergence is exponential even if $\mathbb{E}_{u_0}$ contains a boundary equilibrium. The convergence together with the exponential rate is well-known for the associated ODE system (see e.g. [14]), but it is quite more delicate for the full PDE system.

It is known for the systems considered here and more generally for so-called *complex balanced systems*, that there exists a unique element of $\mathbb{E}_{u_0}$ belonging to $(0,\infty)^n$ for all $u_0 \in L^1(\Omega)^{+n}$ with positive initial mass (i.e. with $\min_{i\in I} A_i + \min_{j\in J} B_j > 0$, see [16]). For a subclass of these systems, $\mathbb{E}_{u_0}$ may even be reduced to this only positive equilibrium *for all such u_0 with positive initial mass.* Exponential convergence has been proved for such kinds of systems, first for some particular nonlinearities, then for general ones, see [7, 8, 10–12, 21].

Among the systems (1.1) considered here, $\mathbb{E}_{u_0}$ is reduced to its positive equilibrium for positive initial masses when chemical species are "separated"

$$\alpha_1 A_1 + \ldots \alpha_m A_m \rightleftharpoons \beta_{m+1} A_{m+1} + \ldots + \beta_n A_n. \tag{1.13}$$

This means there exists $m \in \{1, \ldots, n\}$ such that $\alpha_j = 0, \beta_i = 0, j = m+1, \ldots, n,\ i = 1, \ldots, m$. Exponential convergence towards the positive equilibrium then holds (see [10, 12, 21]).

It may now happen that $\mathbb{E}_{u_0}$ is reduced to its only positive equilibrium *for some u_0 with initial mass, but not all.* It is then interesting to find conditions on u_0 so that this property holds. This is the purpose of Proposition 1.2 below. It may also happen that $u^\infty \in (0,\infty)^n$ while $\mathbb{E}_{u_0}$ contains also a boundary equilibrium (see Remark 1.2). Theorem 1.1 then states that the convergence is again exponential. We also refer to [10, 12] for results on the asymptotic behavior of some specific systems with boundary equilibria, including models for more than two chemical reactions.

The analysis of the asymptotic behavior of global-in-time solutions for system (1.1) is mainly studied by the entropy method introduced and widely exploited in [7, 8] and then extended in the references [10–12, 21]. Here, the proof does also exploit the entropy estimates, but is different and consists in the following steps.

1) We prove that the trajectories $t \to u(t)$ are relatively compact in $L^1(\Omega)^n$. This part is strongly based on the study of the compactness of the trajectories of $t \to w_{i,r}(t) := T_r\left(u_i(t) + \eta \sum_{j\neq i} u_j(t)\right)$ for $\eta \in (0,1)$ small and where $T_r(\cdot) : [0,\infty) \to [0,\infty)$ are usual regularizations of the truncations $s \in [0,\infty) \to \min\{s, r\}$. Letting $r \to +\infty$ and $\eta \to 0$ carries the compactness of these truncated trajectories, valid for all r, η, over to $u(t)$ itself. Similar techniques were also used in [20] to study the asymptotic behavior in the case of nonhomogeneous boundary conditions. Here we use also some of the renormalized properties of u proved in [13].

This first step requires only part of the structure of System (1.1) and could be extended to quite more general systems.

2) Next we prove the convergence in $C\left([0,T]; L^1(\Omega)\right)$ as $t_m \to +\infty$ of the translated functions $\tau \in [0,T] \to w_{i,r}(t_m + \tau)$ where $w^{i,r}(t_m)$ converges in $L^1(\Omega)$. Again, this property, valid for all r, η, carries over to $\tau \to u^m(t_m + \tau)$ as well. Together with the estimates coming from the decrease of entropy, we deduce that

all $L^1(\Omega)^n$–limit points are constant functions and that the limit points are unique. Whence the asymptotic convergence in $L^1(\Omega)^n$.

3) Finally, coupling with previous approaches, we recover that, when the limit u^∞ in (1.12) is positive, then the asymptotic convergence is exponential, this even if $\mathbb{E}_{u_0}$ contains boundary equilibria. This is essentially a consequence of Lemma 2.6.

When the solution is uniformly bounded (i.e. $\sup_t \|u(t)\|_{L^\infty} < +\infty$), one can prove that the asymptotic limit is positive as soon as $\overline{u}_{i0} > 0$ for all $i = 1, ...n$, and the convergence is therefore exponential. This is probably true in general but this does not seem easily extendable to very weak solutions with so poor regularity as those of Proposition 1.1 (see Section 3 and Remark 3.1.)

We can however state a sufficient condition on the data u_0 so that $\mathbb{E}_{u_0}$ be reduced to its unique positive equilibrium. Let us define for all $i = 1, ..., n, \sigma_i := \min\{\alpha_i, \beta_i\}$ and $K := \{k \in \{1, ..., n\}; \sigma_k > 0\}$. Then, the function F may be rewritten

$$F(X) = (\Pi_{k\in K} X_k^{\sigma_k})\, H(X), \quad H(X) := k_f \Pi_{i\in I} X_i^{\alpha_i - \beta_i} - k_r \Pi_{j\in J} X_j^{\beta_j - \alpha_j}, \tag{1.14}$$

where I, J were defined in (1.2). Let us also denote for $u_0 \in L^1(\Omega)^{+n}$

$$A_i := \frac{\overline{u}_{i0}}{\alpha_i - \beta_i}, \ \forall\, i \in I, \quad B_j := \frac{\overline{u}_{j0}}{\beta_j - \alpha_j}, \ \forall\, j \in J.$$

If $I \cap K = \emptyset$ (resp. $J \cap K = \emptyset$), we set $\min_{i\in I\cap K} A_i := +\infty$ (resp. $\min_{j\in J\cap K} B_j := +\infty$). Note that, in the separate case (1.13), $K = \emptyset$ so that $\min_{i\in I\cap K} A_i = +\infty = \min_{j\in J\cap K} B_j$.

Proposition 1.2. *Let $u_0 \in L^1(\Omega)^{+n}$. In addition to the assumptions of Theorem 1.1, suppose that*

$$\min_{i\in I} A_i < \min_{i\in I\cap K} A_i, \quad \min_{j\in J} B_j < \min_{j\in J\cap K} B_j, \quad \min_{i\in I} A_i + \min_{j\in J} B_j > 0. \tag{1.15}$$

Then, $\mathbb{E}_{u_0}$ has no boundary equilibrium, i.e. $\mathbb{E}_{u_0} = \{Z\}, Z \in (0,\infty)^n, H(Z) = 0$ and $u(t)$ converges exponentially to Z in $L^1(\Omega)^n$.

Remark 1.2. As already noticed, the assumption (1.15) holds in the separate case (1.13). But it holds in many more situations like the following elementary one (given as an example)

$$\alpha_1 A_1 + A_2 \rightleftharpoons \beta_1 A_1 + A_3, \quad \alpha_1 > \beta_1,$$

when $0 < (\alpha_1 - \beta_1)\overline{u}_{20} < \overline{u}_{10}$. On the other hand, when $0 < \overline{u}_{10} \le (\alpha_1 - \beta_1)\overline{u}_{20}$, then $\mathbb{E}_{u_0}$ contains 2 elements:

$$\mathbb{E}_{u_0} = \left\{\left(0\,, \overline{u}_{20} - (\alpha_1 - \beta_1)^{-1}\overline{u}_{10}, (\alpha_1 - \beta_1)^{-1}\overline{u}_{10} + \overline{u}_{30}\right),\ Z\right\} \ \textit{where}\ Z \in (0,\infty)^3.$$

2. Proof of Theorem 1.1

Preliminary remarks. Let us first make some comments on Proposition 1.1 which is proved in [13]. Note that the entropy inequality (1.8) can be directly proved for the solution u^ϵ of the approximate problem (1.3). For this we use

$$\partial_t \int_\Omega L_i(u_i^\epsilon(t)) = \int_\Omega (\log u_i^\epsilon + \mu_i)\partial_t u_i^\epsilon = \int_\Omega (\log u_i^\epsilon + \mu_i)[d_i \Delta u_i^\epsilon + f_i^\epsilon(u^\epsilon)]. \tag{2.1}$$

Then, after an integration by parts for the term with Δu_i^ϵ, we sum over i to get the estimate (1.8) with u^ϵ in place of u. Then, as proved in [13], this is preserved for the limit u of u^ϵ as $\epsilon \to 0$ along an adequate subsequence. The point is that (recall the definitions of F, μ_i in (1.4), (1.5))

$$\sum_i [\log u_i^\epsilon(t) + \mu_i] f_i^\epsilon(u^\epsilon(t)) = -\frac{[\log k_f \Pi_i (u_i^\epsilon)^{\alpha_i} - \log k_r \Pi_i (u_i^\epsilon)^{\beta_i}]F}{1 + \epsilon \sum_j |f_j(u^\epsilon)|} \leq 0. \tag{2.2}$$

This implies the following estimates

$$\begin{cases} \sup_t \int_\Omega u_i^\epsilon(t) + u_i^\epsilon(t)|\log u_i^\epsilon(t)|, \int_{Q_\infty} |\nabla \sqrt{u_i^\epsilon}|^2 \leq C, \\ \int_{Q_\infty} \left[\log k_f \Pi_i (u_i^\epsilon)^{\alpha_i} - \log k_r \Pi_i (u_i^\epsilon)^{\beta_i}\right] F \leq C, \end{cases} \tag{2.3}$$

with $C \in (0, +\infty)$ independent of ϵ. This is strongly used in [13] to prove the convergence of u^ϵ a.e. and in $L^1_{loc}([0, \infty); L^1(\Omega))$. Actually, by using known a priori L^2-estimates on u_i^ϵ, we could show as in [19] that the convergence also holds in $L^2_{loc}\left([0, \infty); L^2(\Omega)\right)$. The estimates (2.3) are preserved at the limit for u by using Fatou's lemma for the first and the third and by weak L^2-convergence of ∇u_i^ϵ for the second one. Thus

$$\begin{cases} \operatorname{ess\,sup}_t \int_\Omega u_i(t) + u_i(t)|\log u_i(t)|, \int_{Q_\infty} |\nabla \sqrt{u_i}|^2 < +\infty, \\ \int_{Q_\infty} \left[\log k_f \Pi_i (u_i)^{\alpha_i} - \log k_r \Pi_i (u_i)^{\beta_i}\right] F < +\infty. \end{cases} \tag{2.4}$$

We deduce that $\nabla u_i \chi_{[u_i \leq 2r]}$ is bounded in $L^2\left(0, \infty; L^2(\Omega)\right)$ for all $r \in (0, \infty)$. Indeed,

$$+\infty > C \geq \int_0^{+\infty} \int_\Omega \frac{|\nabla u_i|^2}{u_i} \geq \int_0^{+\infty} \int_{[u_i \leq 2r]} \frac{[\nabla u_i|^2}{2r}. \tag{2.5}$$

As a consequence, since

$$v_i = u_i + \eta \sum_{j \neq i} u_j, \nabla T_r(v_i) = T_r'(u_i + \eta \sum_{j \neq i} u_j) \left(\nabla u_i + \eta \sum_{j \neq i} \nabla u_j\right),$$

we obtain

$$\begin{cases} \int_{Q_\infty} |\nabla T_r(v_i)|^2 \leq \int_0^\infty \int_{[u_i \leq 2r]} 2|\nabla u_i|^2 + \sum_{j \neq i} \int_0^\infty \int_{[u_j \leq 2r/\eta]} 2n\eta^2 |\nabla u_j|^2 \\ \leq C(r, \eta) < +\infty. \end{cases} \tag{2.6}$$

The L^1-estimate on G_2 and the L^2-estimate on G_3 in (1.10) of Proposition 1.1 follow. The L^∞-estimate on G_1 is obvious by the definition of T_r and the local boundedness of the f_i.

Next the equation (1.9) has to be understood in a variational sense: for all $\psi \in C^\infty\left([0, \infty) \times \overline{\Omega}\right)$ and all $T \in (0, \infty)$

$$\begin{cases} \int_\Omega T_r(v_i)(T)\psi(T) - T_r(v_i)(0)\psi(0) + \int_{Q_T} -T_r(v_i)\partial_t \psi + d_i \nabla T_r(v_i) \nabla \psi \\ = \int_{Q_T} [G_1 + G_2 + \nabla \cdot G_3]\psi. \end{cases} \tag{2.7}$$

We will exploit quite a lot this identity which is a consequence of the fact that the limit u is a so-called *renormalized solution* of (1.1) as defined in [13], Definition 1.

In order to obtain it, we choose $\xi(u) := T_r(u_i + \eta \sum_{j\neq i} u_j), b \equiv 0, g \equiv 0, A_i = d_i I$ in this definition.

The identity (2.7) can also be written in terms of the heat semigroup $(S_{d_i}(t))_{t\geq 0}$ with homogeneous Neumann boundary conditions on $L^1(\Omega)$ (see e.g. the appendix in [2] for the equivalence of definitions). We can write,

$$\begin{cases} T_r(v_i(t)) = S_{d_i}(t)T_r(v_{i0}) + \int_0^t S_{d_i}(t-s)[G_1(s)+G_2(s)]ds + w_3(t), \\ v_{i0} := u_{i0} + \eta \sum_{j\neq i} u_{j0}, \end{cases} \tag{2.8}$$

and w_3 is the variational solution of (see e.g. [6], Chapter XVIII for more details)

$$\begin{cases} w_3 \in C\left([0,\infty); L^2(\Omega)\right) \cap L^2\left(0,\infty; H^1(\Omega)\right), \\ \partial_t w_3 - d_i \Delta w_3 = \nabla \cdot G_3 \in L^2\left(0,\infty; H^{-1}(\Omega)\right), \\ \partial_\nu w_3 = 0 \text{ on } \Sigma_\infty,\ w_3(0) = 0. \end{cases} \tag{2.9}$$

Let us recall some useful properties of this semigroup that we will use later (see e.g. Lemma 1.3 in [22]):

$$\begin{cases} \|S_{d_i}(t)w\|_{L^p(\Omega)} \leq \|w\|_{L^p(\Omega)},\ \forall\, p \in [1,\infty], t \geq 0, w \in L^p(\Omega), \\ \|S_{d_i}(t)w - \fint_\Omega w\|_{L^1(\Omega)} \leq Ce^{-\lambda t}\|w\|_{L^1(\Omega)} \text{ for some } C, \lambda \in (0,\infty), \\ \|S_{d_i}(t)[w - \fint_\Omega w]\|_{C^\alpha(\Omega)} \leq C(1+t^{-\beta})e^{-\lambda t}\|w - \fint_\Omega w\|_{L^\infty(\Omega)}, \\ \text{for some } \alpha, \beta \in (0,1), C, \lambda \in (0,\infty). \end{cases} \tag{2.10}$$

Note also that $[t \to T_r(v_i(t))] \in C\left([0,\infty); L^1(\Omega)\right)$ (according for instance to (2.8), (2.10)). One can actually deduce that $u \in C\left([0,\infty); L^1(\Omega)^n\right)$: this is checked below in Lemma 2.3.

Note finally that if $\sup_{\epsilon\in(0,1)} \|F(u^\epsilon)\|_{L^1(Q_T)} < +\infty$ for all $T > 0$, or even if $F(u) \in L^1(Q_T)$ for all $T > 0$, then u is a *weak solution* of (1.1), that is $u_i(t) = S_{d_i}(t)u_{i0} + \int_0^t S_{d_i}(t-s)f_i(u(s))ds$ for all $t \in [0,\infty), i = 1, ..., n$, see [13,17,18]. This L^1-estimate does hold if F is at most quadratic (see e.g. [9]). In some cases, classical global solutions may even be obtained (see [3, 4, 15]), but it is an open problem for the general system (1.1).
End of preliminary remarks.

Let us prepare the proof of Theorem 1.1 by several lemmas.

Lemma 2.1. *For all $r \in (0,\infty), \eta \in (0,1)$, the trajectory $\{T_r\big(v_i(t)\big), t \geq 0\}$ is relatively compact in $L^1(\Omega)$.*

Remark 2.1. A subset $\mathcal{F} \subset L^1(\Omega)$ is said to be *relatively compact*, if its closure is compact. It is well-known that it is equivalent to saying that, from any sequence in $\mathcal{F}$, one can extract a subsequence which converges in $L^1(\Omega)$. It is also equivalent to saying that $\mathcal{F}$ is *precompact*, which means that, for all $\epsilon > 0$, there exists a finite number of functions $f_i \in L^1(\Omega), i = 1, ..., N_\epsilon$ such that $\mathcal{F} \subset \cup_{i=1}^{N_\epsilon} B(f_i, \epsilon)$, where $B(f_i, \epsilon)$ is the open ball centered at f_i and of radius ϵ in $L^1(\Omega)$.

Proof of Lemma 2.1. Let us introduce $\tau \mapsto w_j^T(\tau), j = 1,2,3$ the variational solutions on $[0,\infty)$ of

$$\begin{cases} \partial_\tau w_j^T - d_i \Delta w_j^T = G_1(T+\cdot), G_2(T+\cdot), \text{respectively } \nabla\cdot G_3(T+\cdot), \\ \partial_\nu w_j^T = 0 \text{ on } \Sigma_\infty,\ w_j^T(0) = 0. \end{cases} \tag{2.11}$$

When $T = 0$, we will write $w_j^0 = w_j, j = 1,2,3$ (which is compatible with (2.9)). We then have

$$w_j(t) = S_{d_i}(t-T)w_j(T) + w_j^T(t-T) \text{ for all } t \geq T \geq 0. \tag{2.12}$$

$$T_r\big(v_i(t)\big) = S_{d_i}(t)T_r(v_{i0}) + w_1(t) + w_2(t) + w_3(t) \text{ for all } t \in [0,\infty). \tag{2.13}$$

See (2.8), (2.11) for this last formula.

Goal 2.1. We will show that, as $t \to +\infty$,

- $w_2(t)$ has a limit in $L^1(\Omega)$.
- $w_3(t)$ has a limit in $L^2(\Omega)$ and therefore in $L^1(\Omega)$.
- $\{w_1(t); t \geq 0\}$ is relatively compact in $L^\infty(\Omega)$ and therefore in $L^1(\Omega)$.

Since $S_{d_i}(t)T_r(v_{i,0})$ converges to $⨍_\Omega T_r(v_{i,0})$ in $L^1(\Omega)$ as $t \to +\infty$ (see (2.10)), and according to (2.13), the compactness property announced in the Lemma 2.1 will follow.

Study of w_2. From the definition of w_2^T, we have for all $\tau \in [0,\infty)$

$$\begin{cases} \|w_2^T(\tau)\|_{L^1(\Omega)} \leq \int_0^\tau \|G_2(T+s)\|_{L^1(\Omega)} ds = \int_T^{T+\tau} \|G_2(s)\|_{L^1(\Omega)} ds \\ \qquad\qquad\qquad\quad \leq \int_T^\infty \|G_2(s)\|_{L^1(\Omega)} ds. \end{cases} \tag{2.14}$$

Thus, using (2.12), we deduce for $0 \leq T \leq t \leq t+h$

$$\begin{cases} \|w_2(t+h) - w_2(t)\|_{L^1(\Omega)} \leq \|S_{d_i}(t+h-T)w_2(T) - S_{d_i}(t-T)w_2(T)\|_{L^1(\Omega)} \\ \qquad\qquad\qquad\qquad\qquad\qquad + 2\int_T^\infty \|G_2(s)\|_{L^1(\Omega)} ds. \end{cases}$$

But, we know that $S_{d_i}(s)w_2(T)$ converges as $s \to +\infty$ to $⨍_\Omega w_2(T)$ in $L^1(\Omega)$ (see (2.10)). Thus, the previous inequality implies

$$\limsup_{t,t+h\to+\infty} \|w_2(t+h) - w_2(t)\|_{L^1(\Omega)} \leq 2\int_T^\infty \|G_2(s)\|_{L^1(\Omega)} ds, \text{ for all } T > 0.$$

As a consequence, since $G_2 \in L^1\left(0,\infty; L^1(\Omega)\right)$, $w_2(t)$ has a limit in $L^1(\Omega)$ as $t \to +\infty$.

Study of w_3. Note that $\nabla\cdot G_3 \in L^2\left(0,\infty; H^{-1}(\Omega)\right)$ and $G_3\cdot\nu = 0$ on $(0,\infty)\times\partial\Omega$ (in a variational sense). Multiplying the equation (2.11) in w_3^T by w_3^T and integrating on $[0,\tau]$ lead to

$$\begin{cases} \frac{1}{2}\int_\Omega w_3^T(\tau)^2 + d_i\int_0^\tau\int_\Omega |\nabla w_3^T|^2 = \int_0^\tau\int_\Omega w_3^T \nabla\cdot G_3(T+\cdot) \\ = -\int_0^\tau\int_\Omega G_3(T+\cdot)\cdot\nabla w_3^T \leq \int_0^\tau\int_\Omega \frac{d_i}{2}|\nabla w_3^T|^2 + C(d_i)\int_0^\tau\int_\Omega G_3(T+\cdot)^2 \\ \leq \int_0^\tau\int_\Omega \frac{d_i}{2}|\nabla w_3^T|^2 + C(d_i)\int_T^\infty\int_\Omega G_3(\cdot)^2. \end{cases}$$

It implies that, for all fixed $T > 0$

$$\|w_3^T(\tau)\|^2_{L^2(\Omega)} \leq 2C(d_i) \int_T^\infty \|G_3(s)\|^2_{L^2(\Omega)} ds \ \text{ for all } T > 0. \tag{2.15}$$

Thus, using (2.12), we deduce for $0 \leq T \leq t \leq t+h$,

$$\begin{cases} \|w_3(t+h) - w_3(t)\|_{L^2(\Omega)} \leq \|S_{d_i}(t+h-T)w_3(T) - S_{d_i}(t-T)w_3(T)\|_{L^2(\Omega)} \\ \qquad\qquad\qquad\qquad +2\left[2C(d_i)\int_T^\infty \|G_3(s)\|^2_{L^2(\Omega)} ds\right]^{1/2}. \end{cases}$$

But, we know that $S_{d_i}(s)w_3(T)$ converges as $s \to +\infty$ to $\fint_\Omega w_3(T)$ in $L^2(\Omega)$ (see (2.10)). Thus, the previous inequality implies that, for all $T > 0$,

$$\limsup_{t,t+h\to+\infty} \|w_3(t+h) - w_3(t)\|_{L^2(\Omega)} \leq 2\left[2C(d_i)\int_T^\infty \|G_3(s)\|^2_{L^2(\Omega)} ds\right]^{1/2}.$$

As a consequence, since $G_3 \in L^2\left(0,\infty; L^2(\Omega)\right)$, $w_3(t)$ has a limit in $L^2(\Omega)$ as $t \to +\infty$.

Study of w_1. Recall that w_1 is solution on $(0,\infty)$ of

$$\begin{cases} \partial_t w_1 - d_i \Delta w_1 = G_1, \\ \partial_\nu w_1 = 0, \ w_1(0) = 0. \end{cases} \tag{2.16}$$

First we have, $\partial_t \int_\Omega w_1(t) = \int_\Omega G_1(t)$ and therefore $\fint_\Omega w_1(t) = \int_0^t \fint_\Omega G_1(s)ds$. It follows that

$$w_1(t) - \fint_\Omega w_1(t) = \int_0^t S_{d_i}(t-s)\left[G_1(s) - \fint_\Omega G_1(s)\right] ds.$$

Remember the regularizing effect (see (2.10))

$$\|S_{d_i}(\tau)[u_0 - \fint_\Omega u_0]\|_{C^\alpha(\Omega)} \leq C(1+\tau^{-\beta})e^{-\lambda\tau}\|u_0 - \fint_\Omega u_0\|_{L^\infty(\Omega)},$$

for some $\alpha, \beta \in (0,1)$ and $C, \lambda \in (0,\infty)$. This implies

$$\|w_1(t) - \fint_\Omega w_1(t)\|_{C^\alpha(\Omega)} \leq \int_0^t C[1+(t-s)^{-\beta}]e^{-\lambda(t-s)}\|G_1(s) - \fint G_1(s)\|_{L^\infty(\Omega)} ds.$$

Since $G_1 \in L^\infty(Q_\infty)$, we deduce that

$$\sup_t \|w_1(t) - \fint_\Omega w_1(t)\|_{C^\alpha(\Omega)} \leq 2C\|G_1\|_{L^\infty(Q_\infty)} \sup_t \int_0^t [1+(t-s)^{-\beta}]e^{-\lambda(t-s)} ds < +\infty$$

so that $\{w_1(t) - \fint_\Omega w_1(t), t \geq 0\})$ is relatively compact in $L^\infty(\Omega)$.

But, by (2.13), $w_1 = T_r(v_i) - S_{d_i}(t)T_r(v_{i0}) - w_2 - w_3$ where each of these four functions is in $L^\infty\left(0,\infty; L^1(\Omega)\right)$. Thus so is w_1. As a consequence, $\fint_\Omega w_1(t)$ lies in a compact set of $\mathbb{R}$. It follows that $w_1(t) = \left[w_1(t) - \fint_\Omega w_1(t)\right] + \fint_\Omega w_1(t)$ is relatively compact in $L^\infty(\Omega)$. This ends the proof of **Goal 2.1** and therefore of Lemma 2.1. □

Lemma 2.2. *Any $L^1(\Omega)$-limit point of $T_r(v_i(t))$ as $t \to +\infty$ is a constant function.*

Proof. Let V^∞ be an $L^1(\Omega)$-limit point of $T_r(v_i(t))$ as $t \to +\infty$. Let $(t_m)_m$ be a sequence of times with

$$\lim_{m\to+\infty} t_m = +\infty, \;\; T_r(v_i(t_m)) \to \; V^\infty \text{ in } L^1(\Omega).$$

Let $T > 0$ and $V^m(\tau) := T_r(v_i(t_m + \tau))$ for $\tau \in [0, T]$. We will prove that, at least up to a subsequence,

$$V^m \to V \textit{ in } C\left([0,T]; L^1(\Omega)\right) \text{ as } m \to +\infty. \tag{2.17}$$

Then V will in particular satisfy $V(0) = V^\infty$. Moreover, we know from (2.6) that

$$\int_0^T \int_\Omega |\nabla V^m|^2 = \int_{t_m}^{t_m+T} \int_\Omega |\nabla T_r(v_i)|^2 \le \int_{t_m}^{\infty} \int_\Omega |\nabla T_r(v_i)|^2 \to 0 \text{ as } m \to +\infty.$$

By lower-semicontinuity of the norm for the weak-L^2-convergence of ∇V^m to ∇V, $\int_0^T \int_\Omega |\nabla V|^2 = 0$, that is $\nabla V(t) = 0$ a.e. $t \in [0, T]$. This implies that $V(t, \cdot)$ is a constant function for a.e. t and therefore for all $t \in [0,T]$ since $V \in C\left([0,T]; L^1(\Omega)\right)$. It is in particular the case for $V(0) = V^\infty$, whence the statement of the lemma.

Thus let us prove (2.17). Let us introduce the same decomposition as in the previous proof, using the equation (1.9) and the notation (2.11), namely

$$V^m(\tau) = T_r(v_i(t_m+\tau)) = S_{d_i}(\tau)(T_r(v_i(t_m)) + w_1^{t_m}(\tau) + w_2^{t_m}(\tau) + w_3^{t_m}(\tau). \tag{2.18}$$

From the estimates in the previous proof (see (2.14), (2.15)) we have,

$$\lim_{m\to\infty} \left\{ \sup_{\tau\in[t_m,\infty)} \|w_2^{t_m}(\tau)\|_{L^1(\Omega)} \right\} = 0 = \lim_{m\to\infty} \left\{ \sup_{\tau\in[t_m,\infty)} \|w_3^{t_m}(\tau)\|_{L^2(\Omega)} \right\}.$$

Let $T > 0$. The function $w_1^{t_m}$ is solution on $[0, T]$ of

$$\partial_\tau w_1^{t_m} - d_i \Delta w_1^{t_m} = G_1(t_m + \cdot), \;\; \partial_\nu w_1^{t_m} = 0, \;\; w_1^{t_m}(0) = 0.$$

Since G_1 is bounded in $L^\infty((t_m, t_m+T) \times \Omega)$, it follows that $w_1^{t_m}$ is relatively compact in $L^\infty((0,T) \times \Omega)$ and therefore in $C([0,T]; L^1(\Omega))$. On the other hand, by the property of L^1-contraction of the semigroup, $S_{d_i}(\cdot)(T_r(v_i(t_m))$ converges in $C\left([0,T]; L^1(\Omega)\right)$ to $S_{d_i}(\cdot)V^\infty$. Going back to (2.18), together with the convergence (resp. compactness) in $C\left([0,T]; L^1(\Omega)\right)$ of $w_2^{t_m}, w_3^{t_m}$ (resp. $w_1^{t_m}$), we deduce that, up to a subsequence, V^m converges in $C\left([0,T]; L^1(\Omega)\right)$ to some function V. This proves (2.17) and ends the proof of Lemma 2.2. □

Lemma 2.3. *The solution u defined in Proposition 1.1 is in $C\left([0,\infty); L^1(\Omega)^n\right)$ with $u(0) = u_0$. The trajectory $\{u(t); t \in [0,\infty)\}$ is relatively compact in $L^1(\Omega)^n$ and any limit point as $t \to +\infty$ is a constant function.*

Proof. Throughout this proof, we simply write $\|\cdot\|$ for $\|\cdot\|_{L^1(\Omega)}$. To prove the continuity of $t \to u(t) \in L^1(\Omega)$, we write for a.e. $t, s \ge 0$

$$\begin{cases} \|u_i(t) - u_i(s)\| \le h_i(t) + h_{i,r}(t) + \|T_r(v_i(t)) - T_r(v_i(s))\| + h_{i,r}(s) + h_i(s), \\ h_i(t) := \|u_i(t) - v_i(t)\|, \; h_{i,r}(t) := \|v_i(t) - T_r(v_i(t))\|, \\ h_{i,r}(t) \le \int_\Omega (u_i(t) + \eta \sum_{j\ne i} u_j(t) - r)^+. \end{cases} \tag{2.19}$$

Recall that, for all i (see (2.4)),

$$\operatorname*{ess\,sup}_{t\geq 0}\{\|u_i(t)\|+\|u_i(t)\log u_i(t)\|\}<+\infty.$$

Thus, for some various $C\in(0,\infty)$ independent of i,t,η,r where $r\geq 1$

$$a.e.t,\ h_i(t)\leq \eta\sum_{j=1}^n\|u_j(t)\|\leq \eta\, C, \tag{2.20}$$

$$\begin{cases} a.e.t,\ h_{i,r}(t)\leq \int_\Omega (u_i(t)-r)^+ +\eta\sum_{j=1}^n\|u_j(t)\| \\ \qquad\qquad \leq \frac{1}{\log r}\int_\Omega u_i(t)|\log u_i(t)|+\eta\, C\leq (\frac{1}{\log r}+\eta)\, C. \end{cases} \tag{2.21}$$

We deduce from (2.19), (2.20), (2.21)

$$a.e.t,\ \ \|u_i(t)-u_i(s)\|\leq 2\eta C+2[(\log r)^{-1}+\eta]C+\|T_r(v_i(t))-T_r(v_i(s))\|.$$

By continuity of $s\to T_r(v_i(s))$ at $s=t$, it follows that

$$a.e.t,\ \ \operatorname*{ess\,lim\,sup}_{s\to t}\|u_i(t)-u_i(s)\|\leq 2\eta C+2[(\log r)^{-1}+\eta]C.$$

Whence the expected continuity of a representative of u by letting $\eta\to 0, r\to+\infty$. We obtain in a similar way that

$$\|u_i(t)-u_{i0}\|\leq 2\eta C+2[(\log r)^{-1}+\eta]C+\|T_r(v_i(t))-T_r(v_{i0})\|.$$

Then using that $T_r(v_i(t))$ tends to $T_r(v_{i0})$ in $L^1(\Omega)$ as $t\to 0$, we deduce that $u_i(0)=u_{i0}$.

Let us now prove the compactness property. By Lemma 2.1, we know that $\{T_r(v_i(t); t\in[0,\infty)\}$ is relatively compact (or precompact, see Remark 2.1) in $L^1(\Omega)$ for all $i,r\in(0,\infty),\eta\in(0,1]$, where $v_i=u_i+\eta\sum_{j\neq i}u_j$. For any $f\in L^1(\Omega)$, we may write

$$\|u_i(t)-f\|\leq h_i(t)+h_{i,r}(t)+\|T_r(v_i(t))-f\|, \tag{2.22}$$

where $h_i, h_{i,r}$ are defined in (2.19). We deduce from (2.20), (2.21)

$$\|u_i(t)-f\|\leq[(\log r)^{-1}+2\eta]C+\|T_r(v_i(t))-f\|. \tag{2.23}$$

Let $\epsilon\in(0,1)$. Let us choose (and fix) r large enough and η small enough so that $(\log r)^{-1}+2\eta\leq\epsilon/4C$. By precompactness of $\{T_r(v_i(t)), t\in[0,\infty), i=1,...,n\}$, there exists a finite number $f_k\in L^1(\Omega), k=1,...,K_\epsilon$ such that

$$T_r(v_i(t))\in\cup_{k=1}^{K_\epsilon}B\left(f_k,\epsilon/2\right),\ \text{for all } t\in[0,\infty) \text{ and all } i.$$

Together with the estimate (2.23) and the choice of r,η, this implies

$$u_i(t)\in\cup_{k=1}^{K_\epsilon}B\left(f_k,\epsilon\right),\ \text{for all } t\in[0,\infty) \text{ and all } i.$$

Whence the precompactness announced in Lemma 2.3.

Now since any limit point of $T_r(v_i(t))$ as $t\to+\infty$ is a constant function for all r,η,i, the same property follows for all limit points of $u_i(t)$ itself. $\square$

Lemma 2.4. *There exists $u^\infty \in \mathbb{E}_{u_0}$, as defined in (1.11), such that $u(t)$ converges to u^∞ in $L^1(\Omega)^n$ as $t \to +\infty$, where u is the solution defined in Proposition 1.1.*

Proof. By Lemma 2.3, we know that the trajectory $\{u(t), t \geq 0\}$ is relatively compact in $L^1(\Omega)^n$. We will prove the uniqueness of the limit points as $t \to +\infty$. It will follow that $u(t)$ converges toward this unique limit point as $t \to +\infty$.

Let $u^\infty \in L^1(\Omega)^n$ be a limit point. Let (t_m) be a sequence of times such that $t_m \to +\infty, u(t_m) \to u^\infty$ in $L^1(\Omega)^n$ as $m \to +\infty$. Let us consider again the functions $\tau \in [0,T] \to u^m(\tau) := u(t_m + \tau)$. As in the proof of the previous lemma, we write again more simply $\|\cdot\| := \|\cdot\|_{L^1(\Omega)}$ and we recall the notation $V^m(\tau) = T_r(v_i(t_m + \tau)))$. We may write

$$\|u_i^m(\tau) - u_i^p(\tau)| \leq h_i(t_m+\tau) + h_{i,r}(t_m+\tau) + \|V^m(\tau) - V^p(\tau)\| + h_i(t_p+\tau) + h_{i,r}(t_p+\tau),$$

where the functions $h_i, h_{i,r}$ are defined in (2.19). We proved in Lemma 2.2 that V^m is converging in $C\left([0,T]; L^1(\Omega)\right)$. Using the estimates (2.20), (2.21), we deduce

$$\limsup_{m,p\to+\infty} \{ \sup_{\tau\in[0,T]} \|u_i^m(\tau) - u_i^p(\tau)\| \} \leq 2\eta C + 2[(\log r)^{-1} + \eta]C.$$

Letting $\eta \to 0, r \to +\infty$, we deduce that u_i^m converges in $C\left([0,T]; L^1(\Omega)\right)$ to some U_i. We saw in Lemma 2.2 that the limit V of V^m does not depend on the x-variable. This being true for all i, η, r, it implies the same property for the limit $U = (U_1, ..., U_n)$ of u^m, i.e. $U(\tau) \in [0,\infty)^n$ for all τ. Note also that $U(0) = u^\infty$.

Now we use the estimate (2.4) which implies

$$\lim_{m\to+\infty} \int_0^T \int_\Omega \left[\log k_f \Pi_i (u_i^m)^{\alpha_i} - \log k_r \Pi_i (u_i^m)^{\beta_i}\right] \left[k_f \Pi_i (u_i^m)^{\alpha_i} - k_r \Pi_i (u_i^m)^{\beta_i}\right] = 0.$$

Up to a subsequence, we may assume that u^m converges a.e. (τ, x) to U. Using Fatou's lemma and the nonnegativity of the integrand, we deduce

$$F(U(\tau)) = k_f \Pi_i\, U_i(\tau)^{\alpha_i} - k_r \Pi_i\, U_i(\tau)^{\beta_i} = 0 \; a.e. \tau \in [0,T], \tag{2.24}$$

and this even holds for all $\tau \in [0,T]$ by continuity of U. It is the case in particular for $U(0) = u^\infty$, namely

$$k_f \Pi_i (u_i^\infty)^{\alpha_i} - k_r \Pi_i (u_i^\infty)^{\beta_i} = 0. \tag{2.25}$$

Now we use the invariant quantities (1.11). The invariance holds for the approximate solution u^ϵ of Problem (1.3) since

$$\partial_t \int_\Omega (\beta_j - \alpha_j) u_i^\epsilon(t) + (\alpha_i - \beta_i) u_j^\epsilon(t) = \int_\Omega (\beta_j - \alpha_j) f_i^\epsilon(u^\epsilon) + (\alpha_i - \beta_i) f_j^\epsilon(u^\epsilon) = 0,$$

so that

$$\frac{\overline{u}_i^\epsilon(t)}{\alpha_i - \beta_i} + \frac{\overline{u}_j^\epsilon(t)}{\beta_j - \alpha_j} = A_i + B_j \text{ for all } i \in I,\ j \in J,\ t \geq 0.$$

By convergence of a subsequence as $\epsilon \to 0$ of $u^\epsilon(t)$ for a.e. t in $L^1(\Omega)$, the identities are valid at the limit for $u(t)$, at least a.e. t, but even for all $t \geq 0$ by continuity of u. By translation, they hold for $u^m(\tau)$, and by convergence in $L^1(\Omega)^n$ as $m \to +\infty$, they also hold for $U(\tau)$ and in particular for $U(0) = u^\infty$, namely

$$\frac{u_i^\infty}{\alpha_i - \beta_i} + \frac{u_j^\infty}{\beta_j - \alpha_j} = A_i + B_j \text{ for all } i \in I,\ j \in J. \tag{2.26}$$

Let us prove that these relations, together with (2.25), are satisfied by only a finite number of points in $[0,\infty)^n$. Let $X \in [0,\infty)^n$ satisfy

$$\begin{cases} F(X) = k_f \Pi_i X_i^{\alpha_i} - k_r \Pi_i X_i^{\beta_i} = 0, \\ \frac{X_i}{\alpha_i - \beta_i} + \frac{X_j}{\beta_j - \alpha_j} = A_i + B_j, \forall (i,j) \in I \times J. \end{cases} \tag{2.27}$$

As already noticed in (1.14), the function F may be rewritten

$$F(X) = (\Pi_{k\in K} X_k^{\sigma_k})\, H(X) := (\Pi_{k\in K} X_k^{\sigma_k}) \left[k_f \Pi_{i\in I} X_i^{\alpha_i - \beta_i} - k_r \Pi_{j\in J} X_j^{\beta_j - \alpha_j} \right],$$

where $\sigma_i := \min\{\alpha_i, \beta_i, i = 1, ..., n\}, K := \{k \in \{1, ..., n\}; \sigma_k > 0\}$. In the following, we assume, without loss of generality, that $1 \in I$. Then, the identities in (2.27) allow to solve all X_i, X_j in terms of X_1, namely

$$\begin{cases} X_j = (\beta_j - \alpha_j) \left[A_1 + B_j - (\alpha_1 - \beta_1)^{-1} X_1 \right], \ \forall j \in J, \\ X_i = (\alpha_i - \beta_i) \left[(\alpha_1 - \beta_1)^{-1} X_1 + A_i - A_1 \right], \ \forall i \in I. \end{cases} \tag{2.28}$$

Now $F(X) = 0$ implies one of the two situations:
Case 1: $\Pi_{k\in K} X_k^{\sigma_k} = 0$.
Case 2: $H(X) = k_f \Pi_{i\in I} X_i^{\alpha_i - \beta_i} - k_r \Pi_{j\in J} X_j^{\beta_j - \alpha_j} = 0$.
Case 1: assume for instance, without loss of generality, that $X_1 = 0$. All $X_i, i \in I, X_j, j \in J$ are then uniquely determined by the relations (2.28).
Case 2: let us denote $h(X_1) := H(X) = k_f \Pi_{i\in I} X_i^{\alpha_i - \beta_i} - k_r \Pi_{j\in J} X_j^{\beta_j - \alpha_j}$ where each of the X_i, X_j are given in terms of X_1 as in (2.28). Since all X_i are increasing functions of X_1 and all X_j are decreasing function of X_1, h is an increasing function of $X_1 \in [X_1^-, X_1^+]$ where

$$X_1^- := (\alpha_1 - \beta_1)(A_1 - \min_{i\in I} A_i)^+ \le X_1^+ := (\alpha_1 - \beta_1)(A_1 + \min_{j\in J} B_j).$$

Moreover $h(X_1^-) \le 0, h(X_1^+) \ge 0$. Thus there exists a unique $X_1 \in [X_1^-, X_1^+]$ such that $h(X_1) = 0$ and therefore a unique $X \in [0,\infty)^n$ solution of (2.27) in this Case 2.

A main consequence is that the set $\mathcal{X}^\infty \subset [0,\infty)^n$ of solutions X of (2.27) is finite : the above analysis proves that $\mathcal{X}^\infty$ has at most $|K| + 1$ elements where $|K|$ is the number of elements of K. According to (2.25), (2.26), $u^\infty \in \mathcal{X}^\infty$. Since u^∞ is arbitrary as a limit point, we deduce that

$$\omega(u_0) = \{u^\infty \in L^1(\Omega)^n; \exists\, t_m \to +\infty, u(t_m) \to u^\infty \,\mathrm{in}\, L^1(\Omega)^n \,\mathrm{as}\, m \to +\infty\} \subset \mathcal{X}^\infty.$$

Since $u \in C\left([0,\infty); L^1(\Omega)^n\right)$, it is known that $\omega(u_0)$ is connected. Since $\mathcal{X}^\infty$ has a finite number of points, it follows that $\omega(u_0)$ is reduced to one point. Moreover this point belongs to $\mathbb{E}_{u_0}$ since it satisfies (2.27).

This ends the proof of Lemma 2.4. □

Remark 2.2. In the Case 2 of the previous proof, it is easy to check that

$$\min_{i\in I} A_i + \min_{j\in J} B_j > 0 \ \Rightarrow \ h(X_1^-) < 0, \ h(X_1^+) > 0. \tag{2.29}$$

Thus, in this case, there exists a unique $X_1 \in (X_1^-, X_1^+)$ satisfying $h(X_1) = 0$. Moreover, we check that all X_i, X_j given by (2.28) are then strictly positive. In other words, there exists a unique $Z \in (0,\infty)^n$ such that

$$H(Z) = 0,\ \frac{Z_i}{\alpha_i - \beta_i} + \frac{Z_j}{\beta_j - \alpha_j} = A_i + B_j \text{ for all } i \in I,\ j \in J. \tag{2.30}$$

$\square$

The convergence part of Theorem 1.1 is a consequence of Lemma 2.4. It remains to show that the convergence is exponential when the limit as $t \to +\infty$ of $u(t)$ in $L^1(\Omega)^n$ belongs to $(0,\infty)^n$. This will be a consequence of the two following lemmas.

For $w \in L^1(\Omega)^{+n}$ with $w_i \log w_i \in L^1(\Omega)$ for all $i = 1, ..., n$, we denote (see (1.5))

$$\mathbf{E}(w) := \fint_\Omega \sum_{i=1}^n L_i(w_i),\ \ L_i(s) = s(\log s - 1 + \mu_i) + e^{-\mu_i},\ \forall\, s \in [0,\infty).$$

Lemma 2.5. *For the solution u defined in Proposition 1.1, we have*

$$\frac{d}{dt}\mathbf{E}(u(t)) \leq -D(u(t)), \tag{2.31}$$

in the sense of distributions on $(0,\infty)$, where

$$\begin{cases} D(u) := 4\sum_{i=1}^n \fint_\Omega d_i |\nabla\sqrt{u_i}|^2 \\ \qquad + \fint_\Omega [\log k_f \Pi_{i=1}^n u_i^{\alpha_i} - \log k_r \Pi_{i=1}^n u_i^{\beta_i}][k_f \Pi_{i=1}^n u_i^{\alpha_i} - k_r \Pi_{i=1}^n u_i^{\beta_i}]. \end{cases} \tag{2.32}$$

Recall the writing

$$F(X) = (\Pi_{k\in K} X_k^{\sigma_k})\, H(X) = (\Pi_{k\in K} X_k^{\sigma_k}) \left[k_f \Pi_{i\in I} X_i^{\alpha_i - \beta_i} - k_r \Pi_{j\in J} X_j^{\beta_j - \alpha_j}\right],$$

as introduced in (1.14) for $X \in [0,\infty)^n$. We also denote $\sqrt{X} := (\sqrt{X_i})_{1\leq i\leq n} \in [0,\infty)^n$.

Lemma 2.6. *Let $u \in L^1(\Omega)^{+n}$ with $u_i \log u_i \in L^1(\Omega), \sqrt{u_i} \in H^1(\Omega)$ for all $i = 1, ..., n$. Let us denote*

$$A_i := \frac{\overline{u}_i}{\alpha_i - \beta_i} \text{ for all } i \in I,\ \ B_j := \frac{\overline{u}_j}{\beta_j - \alpha_j} \text{ for all } j \in J. \tag{2.33}$$

Assume that $V := \min_{i\in I} A_i + \min_{j\in J} B_j > 0$. Let $Z \in (0,\infty)^n$ be defined by (2.30). Then,

$$D(u) \geq C\,(\Pi_{k\in K} \overline{u}_k^{\sigma_k})\,[\mathbf{E}(u) - \mathbf{E}(Z)], \tag{2.34}$$

where $C \in (0,\infty)$ depends only V, on $U := \max_{i=1}^n \overline{u}_i$ and on the data $|\Omega|, \alpha_i, \beta_i, \mu_i$, $i = 1, ..., n$.

We postpone the proof of these two lemmas and show how they imply Theorem 1.1.

Proof of Theorem 1.1. The convergence part is a direct consequence of Lemma 2.4.

For the exponential rate, let us assume that $u^\infty := \lim_{t\to+\infty} u(t) \in (0,\infty)^n$. Then (see (1.14)) $F(u^\infty) = 0$ implies in fact $H(u^\infty) = 0$. Let us recall the invariance property

$$\begin{cases} \frac{\overline{u}_i(t)}{\alpha_i-\beta_i} + \frac{\overline{u}_j(t)}{\beta_j-\alpha_j} = A_i + B_j \text{ for all } (i,j) \in I\times J, \\ A_i := \frac{\overline{u}_{i0}}{\alpha_i-\beta_i},\;\; B_j := \frac{\overline{u}_{j0}}{\beta_j-\alpha_j} \text{ for all } (i,j)\in I\times J. \end{cases} \tag{2.35}$$

Since $\lim_{t\to+\infty} \overline{u}_i(t) = u_i^\infty > 0$ for all $i \in I$ and $\lim_{t\to+\infty} \overline{u}_j(t) = u_j^\infty > 0$ for all $j \in J$, it follows that V:=$\min_{i\in I} A_i + \min_{j\in J} B_j > 0$. In particular $u^\infty = Z$ as defined in (2.30). Then by (2.31), (2.34), we deduce

$$\frac{d}{dt}\left[\mathbf{E}(u(t)) - \mathbf{E}(Z)\right] \le -C\left(\Pi_{k\in K}\overline{u}_k(t)^{\sigma_k}\right)\left[\mathbf{E}(u(t)) - \mathbf{E}(Z)\right].$$

By assumption, there exists $T \in (0,\infty)$ such that $\Pi_{k\in K}\overline{u}_k(t)^{\sigma_k} \ge \Pi_{k\in K}\overline{e}_k^{\sigma_k}/2 =: \Lambda > 0$ for all $t \ge T$. This implies

$$\mathbf{E}(u(t)) - \mathbf{E}(Z) \le e^{-C\Lambda(t-T)}[\mathbf{E}(u(t)) - \mathbf{E}(Z)]. \tag{2.36}$$

Let us finally prove that, for some other constant $C \in (0,\infty)$ (depending only on V, U and the data $|\Omega|, \alpha_i, \beta_i, \mu_i$ as in Lemma 2.6, and like all constants C used in this proof)

$$\|u(t) - Z\|_{L^1(\Omega)^n} \le C\left[\mathbf{E}(u(t)) - \mathbf{E}(Z)\right]. \tag{2.37}$$

This will end the proof of Theorem 1.1 since by (2.36)

$$\mathbf{E}(u(t)) - \mathbf{E}(Z) \le \hat{C}e^{-\lambda t},\ \lambda := C\Lambda,\ \hat{C} := \sup_t \mathbf{E}(u(t)).$$

The proof of (2.37) is made as usual by using the Cziszár-Kullback-Pinsker estimate (see Theorem 31 in [5]):

$$\left[⨍_\Omega |u_i(t) - \overline{u}_i(t)|\right]^2 \le 4\overline{u}_i(t)⨍_\Omega u_i(t)\log\frac{u_i(t)}{\overline{u}_i(t)} \le C⨍_\Omega u_i(t)\log\frac{u_i(t)}{\overline{u}_i(t)}. \tag{2.38}$$

On the other hand, we use the structure of the entropy $\mathbf{E}$. We check that

$$⨍_\Omega L_i(u_i) - L_i(Z_i) = ⨍_\Omega u_i \log\frac{u_i}{\overline{u}_i} + (\overline{u}_i - Z_i)(\mu_i + \log Z_i) + \overline{u}_i(\log\frac{\overline{u}_i}{Z_i} - 1) + Z_i,$$

so that

$$\begin{cases} \mathbf{E}(u(t)) - \mathbf{E}(Z) = \sum_i ⨍_\Omega u_i \log\frac{u_i}{\overline{u}_i} + (\overline{u}_i - Z_i)(\mu_i + \log Z_i) \\ \qquad\qquad\qquad\qquad + \overline{u}_i(\log\frac{\overline{u}_i}{Z_i} - 1) + Z_i. \end{cases} \tag{2.39}$$

Using the estimate $s(\log s - 1) + 1 \ge C(M)(s-1)^2$ for $s \in [0, M]$, we also have

$$\begin{cases} |\overline{u}_i - Z_i|^2 = Z_i^2\left|\frac{\overline{u}_i}{Z_i} - 1\right|^2 \le CZ_i^2\left[\frac{\overline{u}_i}{Z_i}\left(\log\frac{\overline{u}_i}{Z_i} - 1\right) + 1\right] \\ \qquad\qquad \le C\left[\overline{u}_i(\log\frac{\overline{u}_i}{Z_i} - 1) + Z_i\right]. \end{cases} \tag{2.40}$$

Using (2.38), (2.40), then (2.39), and summing over $i = 1, ..., n$ lead to

$$\begin{cases} \|u(t) - Z\|^2_{L^1(\Omega)^n} \le C\sum_i \|u_i(t) - \overline{u}_i(t)\|^2_{L^1(\Omega)} + \|\overline{u}_i(t) - Z_i\|^2_{L^1(\Omega)} \\ \qquad\qquad \le C|\Omega|^2\left[\mathbf{E}(u(t)) - \mathbf{E}(Z) - R\right], \end{cases}$$

$$\text{with} \quad R = \sum_i (\overline{u}_i - Z_i)(\mu_i + \log Z_i).$$

It turns out that $R = 0$ (whence (2.37)). Indeed, by (2.30), (2.35), we have $(\overline{u}_i - Z_i) = (\alpha_i - \beta_i)A_1$ for all $i = 1, ..., n$ so that (recall the definition of μ_i in (1.5))

$$R = A_1 \sum_{i=1}^{n} (\alpha_i - \beta_i)(\mu_i + \log Z_i) = A_1 \log\left(k_f \Pi_{i=1}^n Z_i^{\alpha_i} / k_r \Pi_{i=1}^n Z_i^{\beta_i}\right) = 0.$$

□

Remark 2.3. For a future use, we notice the identity deduced from (2.39) and from $R = 0$: for all $u \in L^1(\Omega)^{+n}$ with $\sqrt{u_i} \in H^1(\Omega), u_i \log u_i \in L^1(\Omega), i = 1, ..., n,$

$$\mathbf{E}(u) - \mathbf{E}(Z) = \fint_\Omega \sum_i u_i \log \frac{u_i}{\overline{u}_i} + \overline{u}_i \left(\log \frac{\overline{u}_i}{Z_i} - 1\right) + Z_i, \tag{2.41}$$

where Z is defined in (2.30) and

$$A_i := \frac{\overline{u}_i}{\alpha_i - \beta_i} \text{ for all } i \in I, \quad B_j := \frac{\overline{u}_j}{\beta_j - \alpha_j} \text{ for all } j \in J.$$

Remark 2.4. It is clear from the previous proof that, to obtain the exponential rate, it is sufficient to assume that $u_k^\infty > 0$ for all $k \in K$. See also the comments given in Remark 3.1.

Proof of Lemma 2.5. This inequality is essentially proved in [21]. Let us just recall here the main ingredients of the proof.

For the approximate and regular solution u^ϵ of (1.3), we know that

$$\frac{d}{dt}\mathbf{E}(u^\epsilon) + D_\epsilon(u^\epsilon) = 0, \tag{2.42}$$

$$\text{with} \quad D_\epsilon(u) = 4\fint_\Omega \sum_{i=1}^n d_i |\nabla\sqrt{u_i}|^2 + \fint_\Omega \frac{F(u)}{1 + \epsilon \sum_i |f_j(u)|} \log \frac{k_f \prod_{i=1}^n u_i^{\alpha_i}}{k_r \prod_{i=1}^n u_i^{\beta_i}} \geq 0. \tag{2.43}$$

The equality (2.42) means that for all $\phi \in C_0^\infty\left([0, \infty)\right)^+$

$$\phi(0)\mathbf{E}(u_0^\epsilon) + \int_0^\infty \phi'(t)\mathbf{E}(u^\epsilon(t))\ dt = \int_0^\infty \phi(t) D_\epsilon(u^\epsilon(t))\ dt. \tag{2.44}$$

By Proposition 1.1, we know that u is obtained as a limit of u^ϵ along a subsequence $\epsilon^p \to 0$. We easily obtain that

$$\int_0^\infty \phi(t) D(u(t)) \leq \liminf_{\epsilon_p \to 0} \int_0^\infty \phi(t) D_{\epsilon_p}(u^{\epsilon_p}(t))\, dt.$$

This is a consequence of the weak convergence of $\nabla\sqrt{u_i^{\epsilon_p}}$ to $\nabla\sqrt{u_i}$ in $L^2(Q_T)$ for all $T \in (0, \infty)$ and of Fatou's Lemma applied to the other integral.

The next (more difficult) point is to prove that, at least up to a subsequence

$$\mathbf{E}(u^{\epsilon_p}(t)) \to \mathbf{E}(u(t)) \ \text{ a.e.} t \in (0, \infty) \text{ as } \epsilon_p \to 0, \tag{2.45}$$

Since $\mathbf{E}(u^\epsilon(t)) \leq \mathbf{E}(u_0^\epsilon) \leq C < +\infty$, (2.45) allows to pass to the limit in (2.44) to obtain

$$\phi(0)\mathbf{E}(u_0) + \int_0^\infty \phi'(t)\mathbf{E}(u(t))\ dt \geq \int_0^\infty \phi(t)D(u(t))\ dt,$$

which is the claim of Lemma 2.5. The main point in the proof of (2.45) is the fact that u^ϵ is bounded in $L^2\left((\tau,T)\times\Omega\right)$ for all $0<\tau<T<+\infty$ as proved in Lemma 4 of [19]. Together with the a.e convergence of u^ϵ, this implies the convergence of subsequences of $u_i^{\epsilon_p}\log u_i^{\epsilon_p}$ in $L^1\left((\tau,T)\times\Omega\right)$ and therefore in $L^1(\Omega)$ for a.e. $t\in(0,\infty)$. ☐

Proof of Lemma 2.6. In this proof, the meaning of the constants C will vary from one line to the other, but it depends only on the quantities U, V, defined in the Lemma 2.6 and on the various data. By (2.41), we have

$$\mathbf{E}(u) - \mathbf{E}(Z) = \fint_\Omega \sum_i u_i \log\frac{u_i}{\overline{u}_i} + \overline{u}_i\left(\log\frac{\overline{u}_i}{Z_i} - 1\right) + Z_i.$$

By the logarithmic Sobolev inequality (see e.g. Theorem 17 in [5])

$$\fint_\Omega \sum_i u_i \log\frac{u_i}{\overline{u}_i} \leq C \fint_\Omega \sum_i |\nabla\sqrt{u_i}|^2.$$

Using the definition of $D(u)$ in (2.32) and the fact that $\Pi_{k\in K}\overline{u}^{\sigma_k} \leq C$, we deduce

$$\fint_\Omega \sum_i u_i \log\frac{u_i}{\overline{u}_i} \leq C\, D(u) \leq C\, D(u)/\Pi_{k\in K}\overline{u}_k^{\sigma_k}.$$

It remains to prove that, similarly

$$L(\overline{u}, Z) := \sum_{i=1}^n \overline{u}_i\left(\log\frac{\overline{u}_i}{Z_i} - 1\right) + Z_i \leq C\, D(u)/\Pi_{k\in K}\overline{u}_k^{\sigma_k}. \tag{2.46}$$

Let us enumerate the steps to be proved to reach this inequality.

$$\text{(Step 1)}\ \ L(\overline{u}, Z) \leq C\sum_{i=1}^n |\sqrt{\overline{u}_i} - \sqrt{Z_i}|^2. \tag{2.47}$$

$$\text{(Step 2)}\ \ \sum_{i=1}^n |\sqrt{\overline{u}_i} - \sqrt{Z_i}|^2 \leq C\, H_m\big(\sqrt{\overline{u}}\big)^2, \tag{2.48}$$

where $H_m(X) = \sqrt{k_f}\,\Pi_{i\in I}X_i^{\alpha_i-\beta_i} - \sqrt{k_r}\,\Pi_{j\in J}X_j^{\beta_j-\alpha_j}$.

$$\text{(Step 3)}\ \ F_m\big(\sqrt{\overline{u}}\big)^2 \leq C\fint_\Omega F_m\big(\sqrt{u}\big)^2 + \sum_{i=1}^n |\nabla\sqrt{u_i}|^2, \tag{2.49}$$

where $F_m(X) := (\Pi_{k\in K}X_k^{\sigma_k})\, H_m(X)$.

$$\text{(Step 4)}\ \fint_\Omega F_m\big(\sqrt{u}\big)^2 \leq \fint_\Omega F(u)\left[\log k_f\Pi_{i=1}u_i^{\alpha_i} - \log k_r\Pi_{i=1}^n u_i^{\beta_i}\right]. \tag{2.50}$$

Using these four steps yields (2.46), and therefore Lemma 2.6. Indeed, combining them leads to

$$\begin{cases} L(\overline{u}, Z) \leq C\, H_m\big(\sqrt{\overline{u}}\big)^2 = \frac{C\, F_m\left(\sqrt{\overline{u}}\right)^2}{\Pi_{k\in K}\overline{u}_k^{\sigma_k}} \\ \qquad \leq \frac{C}{\Pi_{k\in K}\overline{u}_k^{\sigma_k}} \fint_\Omega F(u)\left[\log k_f \Pi_{i=1} u_i^{\alpha_i} - \log k_r \Pi_{i=1}^n u_i^{\beta_i}\right] + \sum_{i=1}^n |\nabla\sqrt{u}_i|^2 \\ \qquad = \frac{C\, D(u)}{\Pi_{k\in K}\overline{u}_k^{\sigma_k}}. \end{cases}$$

Let us now prove successively these four steps.

Proof of Step 1. We use the inequality $s(\log s - 1) + 1 \leq C(M)|\sqrt{s}-1|^2$ for $s \in [0, M]$ to obtain:

$$\overline{u}_i(\log\frac{\overline{u}_i}{Z_i} - 1) + Z_i \leq C\, Z_i|\sqrt{\frac{\overline{u}_i}{Z_i}} - 1)|^2 \leq C\,|\sqrt{\overline{u}_i} - \sqrt{Z_i}|^2.$$

Then we sum over $i = 1, ..., n$.

Proof of Step 2. It is sufficient to prove

$$\sum_{i=1}^n |\overline{u}_i - Z_i|^2 \leq C\, H(\overline{u})^2. \tag{2.51}$$

Indeed, assuming this estimate (2.51), we deduce (again with different constants C as explained above)

$$\begin{cases} \sum_{i=1}^n |\sqrt{\overline{u}_i} - \sqrt{Z_i}|^2 = \sum_i \frac{|\overline{u}_i - Z_i|^2}{(\sqrt{\overline{u}_i}+\sqrt{Z_i})^2} \leq C\, H(\overline{u})^2 \\ \qquad = C\, H_m\big(\sqrt{\overline{u}}\big)^2\left[\sqrt{k_f\Pi_{i\in I}\overline{u}_i^{\alpha_i-\beta_i}} + \sqrt{k_r\Pi_{j\in J}\overline{u}_j^{\beta_j-\alpha_j}}\right] \\ \qquad \leq C H_m\big(\sqrt{\overline{u}}\big)^2. \end{cases}$$

To obtain (2.51), we write the variation of H as follows, with $\xi(s) := (1-s)Z + s\overline{u}$,

$$\begin{cases} H(\overline{u}) = H(\overline{u}) - H(Z) = \int_0^1 \nabla H(\xi(s))\cdot(\overline{u} - Z)ds \\ \qquad = A_1\int_0^1 \nabla H(\xi(s))\cdot\gamma ds =: A_1\theta, \end{cases} \tag{2.52}$$

where

$$\overline{u} - Z = A_1\gamma, \quad \gamma := (\alpha_i - \beta_i)_{1\leq i\leq n}, \quad A_1 := (\overline{u}_1 - Z_1)/(\alpha_1 - \beta_1),$$

due to the identity (see (2.30), (2.33)),

$$\frac{\overline{u}_i - Z_i}{\alpha_i - \beta_i} = \frac{\overline{u}_1 - Z_1}{\alpha_1 - \beta_1} \text{ for all } i \in I, \quad \frac{\overline{u}_j - Z_j}{\beta_j - \alpha_j} = -\frac{\overline{u}_1 - Z_1}{\alpha_1 - \beta_1} \text{ for all } j \in J.$$

To obtain (2.51), it is sufficient to prove that $\theta \geq C > 0$. Indeed

$$\sum_{i=1}^n |\overline{u}_i - Z_i|^2 = \sum_i (\alpha_i - \beta_i)^2 A_1^2 \leq A_1^2 \max_{1\leq i\leq n}(\alpha_i - \beta_i)^2 = C\, A_1^2,$$

so that, by (2.52), $H(\overline{u})^2 \geq C\theta^2 \sum_{i=1}^n |\overline{u}_i - Z_i|^2$. But $\theta = \int_0^1 \nabla H(\xi(s)) \cdot \gamma ds$ where

$$\begin{cases} \nabla H(\xi(s)) \cdot \gamma = \sum_{i\in I} k_f(\alpha_i - \beta_i)^2 \xi_i(s)^{-1} \Pi_{l\in I} \xi_l(s)^{\alpha_l - \beta_l} \\ \qquad\qquad\qquad + \sum_{j\in J} k_r(\beta_j - \alpha_j)^2 \xi_j(s)^{-1} \Pi_{l\in J} \xi_l(s)^{\beta_l - \alpha_l}. \end{cases}$$

This quantity is positive and bounded from below by a positive number depending on $\min_{1\leq i\leq n} Z_i$ and of $V = \min\{\min_{i\in I} A_i + \min_{j\in J} B_j\}$ which is assumed to be positive in Lemma 2.6.

Proof of Step 3. This inequality is actually valid for any C^2 function G in place of F_m. This is proved in Lemma 2.7 below. The condition $\alpha_i, \beta_i \in \{0,1\} \cup [2,+\infty)$ implies that F_m is C^2 on $[0,\infty)^n$. Actually, it would be sufficient for our purpose to assume $\alpha_i, \beta_i \in \{0\} \cup [1,\infty)$. In this case, F_m would be C^1 on $[0,\infty)^n$ and C^2 only on $(0,\infty)^n$. But, since Lemma 2.6 is used only in the case when the limit of $\overline{u}(t)$ is in $(0,\infty)^n$, we can avoid the difficulty near 0.

Proof of Step 4. Note that

$$\begin{cases} F_m(\sqrt{u})^2 = (\Pi_{k\in K} u_k^{\sigma_k}) \left[\sqrt{k_f \Pi_{i\in I} u_i^{\alpha_i - \beta_i}} - \sqrt{\Pi_{i\in J} u_j^{\beta_j - \alpha_j}} \right]^2 \\ \qquad\qquad = \left[\sqrt{k_f \Pi_{i=1}^n u_i^{\alpha_i}} - \sqrt{k_r \Pi_{i=1}^n u_i^{\beta_i}} \right]^2. \end{cases}$$

Thus Step 4 is a consequence of the scalar inequality

$$(\sqrt{X} - \sqrt{Y})^2 \leq (X - Y) \log \frac{X}{Y} \text{ for all } X, Y \in (0,\infty).$$

We apply it to $X := k_f \Pi_{i=1}^n u_i^{\alpha_i}, Y := k_r \Pi_{i=1}^n u_i^{\beta_i}$ and the estimate (2.50) follows. □

Lemma 2.7. *Let $G \in C^2([0,\infty)^n; \mathbb{R})$. Then, there exists a constant C depending only on*

$$U := \max_{1\leq i\leq n} \sqrt{\overline{u}_i}, \text{ and } \mathcal{G} := \max_{r\in[0,U+1)^n} \{|G(r)|, \|\nabla G(r)\|, \|D^2 G(r,r)\|\},$$

such that

$$G\big(\sqrt{\overline{u}}\big)^2 \leq C \fint_\Omega G(\sqrt{u})^2 + \sum_{i=1}^n |\nabla \sqrt{u_i}|^2 \;\; \textit{for all } u \in L^1(\Omega)^{+n}, \tag{2.53}$$

where $\sqrt{u} = (\sqrt{u_i}\,)_{1\leq i\leq n}$.

Proof. This proof follows closely the steps of the proof of Lemma 13 in [21]. All constants C may differ from each other, but will depend only on the two values $U, \mathcal{G}$ defined in the lemma. Let us introduce $\sigma = \sigma(x) \in \mathbb{R}^n$ for $x \in \Omega$ by $\sqrt{u} = \sqrt{\overline{u}} + \sigma$. First, we have

$$G(\sqrt{u})^2 = G(\sqrt{\overline{u}} + \sigma)^2 = \Big(G(\sqrt{\overline{u}}) + \nabla G(\sqrt{\overline{u}}) \cdot \sigma + M\Big)^2,$$

where $M = \int_0^1 (1-s) D^2 G(\sqrt{\overline{u}} + s\sigma)[\sigma,\sigma]\ ds$. Using $(\nabla G(\sqrt{\overline{u}})\cdot\sigma + M)^2 \geq 0$, this implies

$$G(\sqrt{u})^2 \geq G(\sqrt{\overline{u}})^2 + 2G(\sqrt{\overline{u}})\nabla G(\sqrt{\overline{u}})\cdot\sigma + 2G(\sqrt{\overline{u}})M.$$

By Young's inequality and the estimate $|\nabla G(\sqrt{\overline{u}}\,)\cdot\sigma| \leq C\|\sigma\|$, we have

$$2G(\sqrt{\overline{u}})\nabla G(\sqrt{\overline{u}})\cdot\sigma \geq -\frac{1}{2}G(\sqrt{\overline{u}})^2 - 2(\nabla G(\sqrt{\overline{u}})\cdot\sigma)^2 \geq -\frac{1}{2}G(\sqrt{\overline{u}})^2 - C\|\sigma\|^2.$$

It follows from the two previous inequalities and $|G(\sqrt{\overline{u}}\,)| \leq C$ that

$$G(\sqrt{u})^2 \geq \frac{1}{2}G(\sqrt{\overline{u}})^2 - C(\|\sigma\|^2 + |M|). \tag{2.54}$$

Next, since $\sqrt{u} \geq 0$ implies $\sigma \geq -\sqrt{\overline{u}}$ in $\mathbb{R}^n$, we have the partition $\Omega = \Omega_1 \cup \Omega_2$ where

$$\Omega_1 = \{x\in\Omega \mid -\sqrt{\overline{u}_i} \leq \sigma_i(x) \leq 1,\ \forall\ 1\leq i\leq n\},$$
$$\Omega_2 = \cup_{1\leq i\leq n}\{x\in\Omega \mid \sigma_i(x) > 1\}.$$

For $x \in \Omega_1, s\in[0,1]$, one has: $0 \leq \sqrt{\overline{u}_i} + s\sigma_i \leq 1 + \sqrt{\overline{u}}_i$, so that

$$|M| \leq \int_0^1 (1-s)\|D^2G(\sqrt{\overline{u}} + s\sigma)\|\ ds\cdot\|\sigma\|^2 \leq C\|\sigma\|^2,\ x\in\Omega_1.$$

Together with (2.54), we deduce

$$\int_{\Omega_1} G(\sqrt{u})^2\ dx \geq \int_{\Omega_1}\left[\frac{1}{2}G(\sqrt{\overline{u}})^2 - C\|\sigma\|^2\right]dx. \tag{2.55}$$

We also have

$$\int_{\Omega_2} G(\sqrt{\overline{u}})^2\ dx = |\Omega_2| G(\sqrt{\overline{u}})^2 \leq G(\sqrt{\overline{u}})^2 \sum_{i=1}^n \left|[\sigma_i^2 > 1]\right|$$

with

$$\left|[\sigma_i^2 > 1]\right| = \int_{[\sigma_i^2>1]} dx \leq \int_{[\sigma_i^2>1]} \sigma_i^2\ dx \leq \int_\Omega \sigma_i^2\ dx,$$

which implies

$$\int_{\Omega_2} G(\sqrt{\overline{u}})^2\ dx \leq G(\sqrt{\overline{u}})^2 \int_\Omega \|\sigma\|^2\ dx \leq C\int_\Omega \|\sigma\|^2\ dx. \tag{2.56}$$

By (2.55)-(2.56), we obtain

$$G(\sqrt{\overline{u}})^2 = \fint_\Omega G(\sqrt{\overline{u}})^2\ dx \leq C\fint_\Omega [G(\sqrt{u})^2 + \|\sigma\|^2]\ dx. \tag{2.57}$$

Then, using in particular the Schwarz inequality "$\sqrt{\overline{u}_i} \geq \fint_\Omega \sqrt{u_i}$", we have

$$\fint_\Omega \sigma_i^2 = \fint_\Omega u_i - 2\sqrt{\overline{u}_i}\sqrt{u_i} + \overline{u}_i \leq 2\left\{\fint u_i - \left(\fint_\Omega \sqrt{u_i}\right)^2\right\} = 2\fint_\Omega\left(\sqrt{u_i} - \fint_\Omega\sqrt{u_i}\right)^2.$$

Using Poincaré-Wirtinger's inequality implies that

$$\fint_\Omega \sigma_i^2 = 2\fint_\Omega\left(\sqrt{u_i} - \fint_\Omega\sqrt{u_i}\right)^2 \leq C\fint_\Omega |\nabla\sqrt{u_i}|^2.$$

Whence (2.53) by plugging the sum of these inequalities for $i = 1,...,n$ into (2.57).
$\square$

3. Proof of Proposition 1.2 and some more comments

Proof of Proposition 1.2. Recall that the set $\mathbb{E}_{u_0}$ is defined by the relations

$$\begin{cases} e \in [0,\infty)^n, \;\; F(e)=0, \;\; \frac{e_i}{\alpha_i-\beta_i}+\frac{e_j}{\beta_j-\alpha_j}=A_i+B_j, \; \forall\, i\in I, \forall j \in J, \\ A_i := \frac{\overline{u}_{i0}}{\alpha_i-\beta_i}, \; \forall\, i \in I, \;\; B_j := \frac{\overline{u}_{j0}}{\beta_j-\alpha_j} \; \forall\, j\in J. \end{cases} \tag{3.1}$$

And (see (1.14)),

$$F(e) = (\Pi_{k\in K} e_k^{\sigma_k})\, H(e), \; H(e) = k_f \Pi_{i\in I} e_i^{\alpha_i-\beta_i} - k_r \Pi_{j\in J} e_j^{\beta_j-\alpha_j}.$$

By assumption, $\min_{i\in I} A_i + \min_{j\in J} B_j > 0$. Thus, for $e \in \mathbb{E}_{u_0}$, if $e_{i_*}=0$ for some $i_* \in I$, then $e_j > 0$ for all $j\in J$. This implies $H(e) = -k_r \Pi_{j\in J} e_j^{\beta_j-\alpha_j} \neq 0$. Since $F(e)=0$, necessarily, $i_* \in K$.

On the other hand, since

$$\frac{e_i}{\alpha_i-\beta_i} - \frac{e_{i_*}}{\alpha_{i_*}-\beta_{i_*}} = A_i - A_{i_*} \;\; \forall\, i \in I,$$

$e_{i_*}=0$ implies also that $A_i \geq A_{i_*}$ for all $i\in I$ so that $\min_{i\in I} A_i = 0 = \min_{i\in I\cap K} A_i$. This is a contradiction with the assumption of Proposition 1.2. Therefore $e_{i_*} = 0, i_* \in I$ is impossible for $e\in\mathbb{E}_{u_0}$. This implies that $e_i > 0$ for all $i \in I$. We prove similarly that $e_j > 0$ for all $j \in J$. Therefore $e \in (0,\infty)^n$ and, since $H(e)=0$, necessarily $\mathbb{E}_{u_0} = \{Z\}$ where Z is defined in (2.30).

By Theorem 1.1, the convergence toward $u^\infty = Z$ is then exponential. □

Remark 3.1. When $\min_{i\in I} A_i + \min_{j\in J} B_j > 0$, it follows from the expression of $F(e) = (\Pi_{k\in K} e_k^{\sigma_k})\, H(e)$ that (see (2.30))

$$e \in \mathbb{E}_{u_0}, \; e_k > 0 \,\forall\, k\in K, \;\; \Rightarrow \;\; e = Z \in (0,\infty)^n.$$

Thus positivity on K is enough to deduce $e\in(0,\infty)^n$ and $e=Z$. And in this case, convergence is exponential.

Actually here are some possible situations.

1. If $K=\emptyset$ and $\min_{i\in I} A_i + \min_{j\in J} B_j > 0$, then $\mathbb{E}_{u_0} = \{Z\}$ and $u(t)$ converges to Z exponentially.
2. If $K=\emptyset$ and there exist $i_*\in I, j_*\in J$ with $\overline{u}_{i_*} = 0 = \overline{u}_{j_*}$, then $u_{i_*}(t) = 0 = u_{j_*}(t)$ for all $t\geq 0$. This implies, $F(u(t)) = 0$ so that, for all i, u_i is the solution of the linear heat equation with initial data u_{i0} and $u_i(t)$ converges exponentially to $\overline{u}_{i0}$ as $t\to+\infty$ (see (2.10)).
3. If $K\neq\emptyset$ and there exists $k\in K$ such that $\overline{u}_{k0}=0$, then $u_k(t)=0$ for all $t\geq 0$ and again $F(u(t))=0$ and u_i is again solution of the linear equation for all i.
4. If $K\neq\emptyset$ and there exist $i_*\in I, j_*\in J$ with $\overline{u}_{i_*}=0=\overline{u}_{j_*}$, then the situation is 'linear' as in 2 and 3.
5. If $K\neq\emptyset$, then the condition $\min_{i\in I} A_i + \min_{j\in J} B_j > 0$ is not sufficient to claim that $u^\infty \in (0,\infty)^n$. Indeed, we may choose $\overline{u}_{k0}=0$ for some $k\in K$, and $\overline{u}_{i0} > 0$ for all $i\neq k$. Then, by the point 3 above, $u_k^\infty = 0$ while $\min_{i\in I} A_i + \min_{j\in J} B_j > 0$.

6. It is natural to think that if $\overline{u}_{i0} > 0$ for all $i = 1, ..., n$, then $u^\infty \in (0,\infty)^n$. This is the case if $\|u(t)\|_{L^\infty(\Omega)}$ is bounded as $t \to +\infty$. Indeed, in this case, we get compactness of the trajectory in $L^\infty(\Omega)^n$ thanks to the C^α-estimate recalled in (2.10) and we can argue as follows.
Assume by contradiction that, for instance, $1 \in I$ and $u_1^\infty = 0$. It implies that $1 \in K$ and $u_j^\infty > 0$ for all $j \in J$. We deduce that, as $t \to +\infty$,

$$H(u(t)) = k_f \Pi_{i\in I} u_i(t)^{\alpha_i - \beta_i} - k_r \Pi_{j\in J} u_j(t)^{\beta_j - \alpha_j} \to -k_r \Pi_{j\in J} (u_j^\infty)^{\beta_j - \alpha_j} < 0,$$

and this convergence is uniform. Thus $-H(u(t,x)) \geq \eta > 0$ for t large enough and for all $x \in \Omega$. Consequently, for t large enough (recall that $1 \in I$ so that $\beta_1 - \alpha_1 < 0$)

$$\partial_t \int_\Omega u_1(t) = \int_\Omega (\beta_1 - \alpha_1) \Pi_{k\in K} u_k(t)^{\sigma_k} H(u(t)) \geq 0.$$

Then $\lim_{t\to+\infty} \int_\Omega u_1(t) = 0$ is possible only if there exists $T < +\infty$ such that $u_1(T) \equiv 0$. But this is not possible either since

$$\begin{cases} \partial_t \int_\Omega u_1(t) \geq -C \int_\Omega u_1(t), \\ C := (\alpha_1 - \beta_1) \sup_t \left\| u_1(t)^{\sigma_1 - 1} \Pi_{k\in K, k\neq 1} u_k(t)^{\sigma_k} H(u(t)) \right\|_{L^\infty(\Omega)}. \end{cases}$$

Therefore $\int_\Omega u_1(t) \geq e^{-Ct} \int_\Omega u_{10} > 0$ for all $t > 0$. $\square$
Unfortunately, it is not clear how to extend such a proof to the weak solutions as defined in Proposition 1.1.

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References

[1] D. Bothe, *On the Maxwell-Stefan approach to multicomponent diffusion*, In Progress in Nonlinear Differential Equations and their Applications, Springer, Basel, 2011, 81–93.

[2] D. Bothe and M. Pierre, *Quasi-steady-state approximation for a reaction-diffusion with fast intermediate*, J. Math. Ana. Appl., 2010, 368, 120–132.

[3] J. A. Cañizo, L. Desvillettes and K. Fellner, *Improved duality estimates and applications to reaction-diffusion equations*, Comm. Partial Differential Equations, 2014, 39, 1185–204.

[4] M. C. Caputo, T. Goudon and A. Vasseur, *Solutions of the 4-species quadratic reaction-diffusion system are bounded and C^∞-smooth, in any space dimension*, 2017, preprint.

[5] J. A. Carrillo, A. Jüngel, P.A. Markovitch, G. Toscani and A. Unterreiter, *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities*, Monatsch. Math., 2001, 133, 1–82.

[6] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technnology*, Springer, Berlin, 1988.

[7] L. Desvillettes and K. Fellner, *Entropy methods for reaction-diffusion equations: slowly growing a-priori bounds*, Rev. Mat. Iberoamericana, 2008, 24, 407–431.

[8] L. Desvillettes and K. Fellner, *Exponential Decay toward Equilibrium via Entropy Methods for Reaction-Diffusion Equations*, JMAA, 2006, 319(1), 157–176.

[9] L. Desvillettes, K. Fellner, M. Pierre and J. Vovelle, *About global existence for quadratic systems of reaction-diffusion*, J. Adv. Nonlinear Stud., 2007, 7, 491–511.

[10] L. Desvillettes, K. Fellner and B. Q. Tang, *Trend to equilibrium for reaction-diffusion systems arising from complex balanced chemical reaction networks*, to appear is SIAM J. Math. Ana..

[11] K. Fellner and E. Laamri, *Exponential decay towards equilibrium and global classical solutions for nonlinear reaction-diffusion systems*, Journal of Evolution Equations, 2016, 16(3), 681–704.

[12] K. Fellner and B. Q. Tang, *Explicit exponential convergence to equilibrium for mass action reaction-diffusion systems with detailed balance condition*, Nonlinear Analysis TMA, 2017, 159, 145–180.

[13] J. Fischer, *Global existence of renormalized solutions to entropy-dissipating reaction-diffusion systems*, Arch. Ration. Mech. Anal., 2015, 218(1), 553–587.

[14] I. Gentil and B. Zegarlinski, *Asymptotic Behavior of a general reversible chemical reaction-diffusion equation*, Kinet. Relat. Models, 2010, 3, 427–444.

[15] T. Goudon and A. Vasseur, *Regularity analysis for systems of reaction-diffusion equations*, Ann. Sci. Ec. Norm. Super., 2010, 43(4), 117–141.

[16] F. J. M. Horn, *Necessary and suffcient conditions for complex balancing in chemical kinetics*, Arch. Rational Mech. Anal., 1972, 49, 172–186.

[17] M. Pierre, *Weak solutions and supersolutions in L^1 for reaction-diffusion systems*, J. Evol. Equ., 2003, 3, 153–168.

[18] M. Pierre, *Global existence in reaction-diffusion systems with control of mass: a survey*, Milan. J. Math., 2010, 78, 417–455.

[19] M. Pierre and G. Rolland, *Global existence for a class of quadratic reactiondiffusion systems with nonlinear diffusions and L^1 initial data*, Nonlinear Analysis TMA, 2016, 138, 369–387.

[20] M. Pierre, T. Suzuki and H. Umakoshi, *Global-in-time behavior of weak solution to reaction diffusion system with inhomogeneous dirichlet boundary condition*, Nonlinear Analysis, 2017, 159, 393–407.

[21] M. Pierre, T. Suzuki and R. Zou, *Asymptotic behavior of solutions to chemical reaction-diffusion systems*, J. Math. Anal. Appl., 2017, 450, 152–168.

[22] M. Winkler, *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model*, J. Diff. Equ., 2010, 248(12), 2889–2905.