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INVENTORY CONTROL WITH FIXED COST AND PRICE OPTIMIZATION IN CONTINUOUS TIME*

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Article dedicated to Claude Brauner for his 70^{th} *anniversary.*

Abstract We continue to study the problem of inventory control, with simultaneous pricing optimization in continuous time. In our previous paper [8], we considered the case without set up cost, and established the optimality of the base stock-list price (BSLP) policy. In this paper we consider the situation of fixed price. We prove that the discrete time optimal strategy (see [11]), i.e., the (s, S, p) policy can be extended to the continuous time case using the framework of quasi-variational inequalities (QVIs) involving the value function. In the process we show that an associated second order, nonlinear two-point boundary value problem for the value function has a unique solution yielding the triplet (s, S, p). For application purposes the explicit knowledge of this solution is needed to specify the optimal inventory and pricing strategy. Selecting a particular demand function we are able to formulate and implement a numerical algorithm to obtain good approximations for the optimal strategy.

Keywords (s, S, p) policy, stochastic inventory control, quasi-variational inequalities, stochastic dynamic programming.

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1. INTRODUCTION

In recent years optimization of inventory control strategies in order to reach higher profit margins has received considerable attention in the literature. The pioneering work by Arrow, Harris and Marschak [1] established the optimality of the Base-Stock policy for both the deterministic and stochastic cases, and in addition proved the optimality of the (s, S) policy when fixed cost is associated with replenishment (see also [19] where the optimality of the (s, S) policy is shown using a K-convexity type argument).

Numerous extensions have been added to the original works [1], [19], to include more complex models and different business applications (see, e.g.,the papers by Veinott [24] and Caplin [10]). Also, new methods of proving optimality have emerged (we refer the interested reader to the papers by Sawaki and Sato [18], Benoussan, Liu and Sethi [6], Presman and Sethi [17], and Benkherouf [2], and the references therein).

Models with dynamic pricing were also considered. We see from [14] that combining the two normally separate fields of pricing optimization and inventory control can have a positive effect on the profit margins. In particular, the authors show in [14] that the optimal strategy to the case of no fixed cost follows a Base Stock List Price policy, originally coined by [16], for both the finite and infinite horizon cases. In the case of fixed cost the work by Chen and Simchi-Levi [11], [12] proves that the optimal strategy follows an (s, S, p) format for both the finite and the infinite horizon cases. They also prove that this strategy is optimal when the continuous review model is applied (i.e., when the inventory is monitored continuously, see [13]).

In this paper we address the problem of inventory control, with simultaneous pricing optimization in continuous time. This paper can be compared with the major contributions of Chen and Simchi-Levi [11], however, in this work we consider a continuous time model, whereas [11] considers the discrete time case.

For convenience we briefly describe the results for both the fixed cost and no fixed cost cases. With no fixed cost, the optimal policy is the Base Stock-List Price (BSLP) policy, in which one orders when the inventory is below the threshold S (Base Stock) up to S. It is complemented by a pricing policy, which is constant (optimal price) when the inventory is below S, and depends on the inventory level when the inventory is above S. In that case, the price is a decreasing function of the inventory, hence a rebate is granted to reduce the stock to its optimal level S. When fixed cost is inserted, the base stock is replaced by an (s, S) policy, where replenishment occurs when the inventory is below s, to reach S. When price is incorporated as a control variable, then [11] shows that the (s, S) policy still holds and is complemented by a pricing policy depending on the inventory, which they write as the triplet (s, S, p). In this paper we obtain a similar result in continuous time for the infinite horizon case, however the methodology used to obtain these results differ from that of [11]. We apply quasi variational inequalities (QVIs), see [7] as a general reference on QVI's, and use similar methodology to obtain an optimal policy as in [8], where the no fixed cost case was considered in continuous time. By using the QVIs framework we construct a second order, nonlinear two point boundary value problem on a semi-infinite interval with a moving (unknown) left boundary, and in which a singularity at infinity occurs. We establish the existence of a unique solution to the boundary value problem leading to the solution triplet (s, S, p) constituting the optimal strategy for the inventory control problem. Obtaining good numerical approximations for the solution triplet (s, S, p) is essential for application purposes. To avoid numerical difficulties caused by the singularity at infinity we consider regularizations (the so called associated epsilon problems) of the boundary value problem and obtain approximations for (s, S, p) by solving the associated regularized problems for decreasing epsilon values. As an important byproduct we are able to inspect the properties of the pricing policy as a function of the inventory. In particular, we show that the rebate property still holds when the inventory is larger than S, whereas the price increases for inventory values below S.

2. GENERAL PRESENTATION OF THE MODEL

2.1. MODEL AND ASSUMPTIONS

The model is the same as in our previous paper, except, of course the structure of costs and the type of policy. We consider a probability space $(\Omega, \mathcal{A}, P)$ on which there exists a standard Wiener process $w(t)$ and we call $\mathcal{F}^t = \sigma(w(s), s \leq t)$. The demand rate is described by

$$dD(t) = \nu(\varpi(t))dt + \sigma dw(t) \tag{2.1}$$

in which σ is the standard deviation of the random part. The average rate per unit of time depends on the decision $\varpi(t)$, which is the price. It is a control variable, depending on the information. We assume full information on the past and present time. Therefore $\varpi(t)$ is a stochastic process adapted to the filtration $\mathcal{F}^t$, which is positive. The function $\varpi \to \nu(\varpi)$ is defined from R^+ to R^+ and satisfies natural assumptions for a demand function. More specifically, we assume

$$\varpi \to \nu(\varpi) \text{ is decreasing, } \nu(0) = +\infty,\ \nu(+\infty) = 0, \tag{2.2}$$

$$\varpi \to \varpi\nu(\varpi) \text{ is decreasing, } \varpi\nu(\varpi)|_{\varpi=0} = +\infty,\ \varpi\nu(\varpi)|_{\varpi=+\infty} = 0. \tag{2.3}$$

The second condition expresses the fact, that not only the demand decreases with price, but also the sales. We next assume that

$$\varpi \to \nu(\varpi) \text{ is continuously differentiable,} \tag{2.4}$$

$$\varpi + \frac{\nu(\varpi)}{\nu'(\varpi)} \text{is monotone increasing, takes the value 0 at 0 and the value } +\infty \text{ at } +\infty.$$

As a template of function $\nu(\varpi)$ we shall use

$$\nu(\varpi) = \frac{1}{\varpi^{\gamma+1}}\, , \gamma > 0, \tag{2.5}$$

for which

$$\varpi + \frac{\nu(\varpi)}{\nu'(\varpi)} = \frac{\gamma}{\gamma+1}\varpi.$$

Besides the price $\varpi(t)$ there is a second control called V. It is an impulse control, consisting on an increasing sequence of stopping times θ_i, with respect to the filtration $\mathcal{F}^t$, and positive random variables ξ_i which are $\mathcal{F}^{\theta_i}$ measurable. We write

$$V = (\cdots, \theta_i, \xi_i, \cdots). \tag{2.6}$$

The stopping time θ_i represents the i^{th} ordering time and ξ_i the corresponding order amount. The state of the dynamic system $x(t)$ is the inventory level at time t . Its evolution is driven by the following equation

$$\begin{aligned} dx &= -\nu(\varpi(t))dt + \sum_i \xi_i \delta(t-\theta_i) - \sigma dw(t), \\ x(0) &= x, \end{aligned} \tag{2.7}$$

where $\delta(t)$ represents the Dirac measure at 0. The inventory jumps at time θ_i from the value $x(\theta_i - 0)$ to $x(\theta_i) = x(\theta_i - 0) + \xi_i$. So we can also write the piecewise evolution

$$\begin{aligned} dx &= -\nu(\varpi(t))dt - \sigma dw(t),\ \theta_i < t < \theta_{i+1}, \\ x(\theta_i) &= x(\theta_i - 0) + \xi_i. \end{aligned} \tag{2.8}$$

2.2. THE VALUE FUNCTION

The initial value of the state is a parameter, denoted by x. We define the profit of the policy $\varpi(t)$, V by

$$\begin{aligned} J_x(\varpi(.),\, V) =& E\left[\int_0^{+\infty} \exp(-\alpha t)\left((\varpi(t)\nu(\varpi(t)) - hx^+(t) - px^-(t))dt\right)\right] \\ &- E\left[\sum_{i=1}^{+\infty}(k + c\xi_i)\exp(-\alpha\theta_i)\right]. \end{aligned} \tag{2.9}$$

This expression is easy to figure out: per unit of time at time t, $\varpi(t)\nu(\varpi(t))$ represents the sales, $hx^+(t)$ the holding cost,$px^-(t)$ the shortage cost . The ordering costs occur at the ordering times θ_i and is composed of a fixed cost k and a variable cost c per unit amount ordered . The parameter α is the discount factor. A control is admissible if

$$E\left[\int_0^{+\infty} \exp(-\alpha t)\ \varpi(t)\nu(\varpi(t)dt\right] < +\infty. \tag{2.10}$$

This limitation is necessary to define the profit (2.9) without ambiguity. We accept the value $-\infty$, which of course cannot be optimal. Admissible controls exist of course. For example take $\varpi(t) = \varpi_0$. We next define the value function by

$$u(x) = \sup_{\varpi(.),V} J_x(\varpi(.),\, V). \tag{2.11}$$

The sup is taken over the set of admissible controls. This will be implicit from now on.

2.3. PROPERTIES OF THE VALUE FUNCTION

We state the

Proposition 2.1. *Assume (2.4), then $u(x)$ is increasing.*

Proof. To simplify the proof, we assume that there exists for any x an optimal policy $\hat{\varpi}_x(.)$, $\hat{V}_x$ such that

$$u(x) = J_x(\hat{\varpi}_x(.),\, \hat{V}_x).$$

We call $\hat{x}_x(t)$ the corresponding optimal trajectory, with initial state x. We shall write $\hat{\varpi}(.)$, $\hat{V}$ to simplify notation. We define for any t , $\varpi(t) > 0$ by the condition

$$\nu(\varpi(t)) - \nu(\hat{\varpi}(t)) = 1.$$

The price $\varpi(t)$ is uniquely defined from the assumptions on the function $\nu(.)$ and by the fact that $\varpi(t) < \hat{\varpi}(t)$. We next consider a policy $\tilde{\varpi}(.)$, $\tilde{V}$ defined by

$$\tilde{V} = \hat{V} \quad ; \tilde{\varpi}(t) = \begin{cases} \varpi(t),\ 0 < t < \epsilon, \\ \hat{\varpi}(t),\ t > \epsilon. \end{cases}$$

We consider the state $\tilde{x}(t)$ corresponding to the policy $\tilde{\varpi}(.)$, $\tilde{V}$ and the initial state $\tilde{x}(0) = x + \epsilon$. We see easily that

$$\begin{aligned} &\tilde{x}(t) = \hat{x}(t) + \epsilon - t,\ 0 < t < \epsilon, \\ &\tilde{x}(t) = \hat{x}(t),\ t \geq \epsilon. \end{aligned}$$

Therefore

$$\begin{aligned} &J_{x+\epsilon}(\tilde{\varpi}(.), \tilde{V}) - J_x(\hat{\varpi}(.), \hat{V}) \\ =&E\left[\int_0^\epsilon \exp(-\alpha t)[\varpi(t)\nu(\varpi(t)) - \hat{\varpi}(t)\nu(\hat{\varpi}(t)) - h(\tilde{x}^+(t) - \hat{x}^+(t)) - p(\tilde{x}^-(t) - \hat{x}^-(t))]dt\right] \\ =&E\left[\int_0^\epsilon \exp(-\alpha t)[\varpi(t)\nu(\varpi(t)) - \hat{\varpi}(t)\nu(\hat{\varpi}(t)) - h(\tilde{x}(t) - \hat{x}(t)) - (p+h)(\tilde{x}^-(t) - \hat{x}^-(t))]dt\right]. \end{aligned}$$

Since $\tilde{x}(t) > \hat{x}(t)$ we have $\tilde{x}^-(t) - \hat{x}^-(t) < 0$. Also $\varpi(t)\nu(\varpi(t)) > \hat{\varpi}(t)\nu(\hat{\varpi}(t))$. Therefore

$$\begin{aligned} J_{x+\epsilon}(\tilde{\varpi}(.), \tilde{V}) - J_x(\hat{\varpi}(.), \hat{V}) \geq & - hE\left[\int_0^\epsilon \exp -\alpha t(\tilde{x}(t) - \hat{x}(t))dt\right] \\ = & - h\int_0^\epsilon \exp -\alpha t(\epsilon - t)dt \geq -h\frac{\epsilon^2}{2} \end{aligned}$$

which implies

$$u(x+\epsilon) - u(x) \geq -h\frac{\epsilon^2}{2}.$$

It follows that $u(x)$ is monotone increasing. □

3. DYNAMIC PROGRAMMING

3.1. OPTIMALITY PRINCIPLE

Following the standard methodology of Dynamic Programming and optimality principle, we consider the consequence of not ordering on a small interval of time $(0, \epsilon)$ and fixing during this period the price at the level ϖ. After ϵ, we apply the optimal ordering and pricing policy corresponding to the state attained at time ϵ, which is $x - \nu(\varpi)\epsilon - \sigma w(\epsilon)$. Noting that the profit during the interval of time $(0, \epsilon)$ is approximately $\epsilon(\varpi\nu(\varpi) - hx^+ - px^-)$, we can write the inequality

$$\begin{aligned} u(x) \geq & \ \epsilon(\varpi\nu(\varpi) - hx^+ - px^-) \\ & + (1 - \alpha\epsilon)Eu(x - \nu(\varpi)\epsilon - \sigma w(\epsilon)). \end{aligned}$$

Assuming that $u(x)$ is smooth, and expanding on ϵ, we obtain the differential inequality

$$-\frac{1}{2}\sigma^2 u'' + \alpha u + (u' - \varpi)\nu(\varpi) + hx^+ + px^- \geq 0,$$

and since ϖ is an arbitrary positive number we can summarize all these inequalities for any $\varpi \geq 0$, by a single one

$$-\frac{1}{2}\sigma^2 u'' + \alpha u + \min_{\varpi \geq 0}[(u' - \varpi)\nu(\varpi)] + hx^+ + px^- \geq 0.$$

If we consider now the consequences of ordering at time 0, an amount $\xi > 0$, we can write

$$u(x) \geq u(x + \xi) - c\xi - k, \ \forall \xi > 0,$$

and since ξ is arbitrary

$$u(x) \geq \sup_{\xi > 0}(u(x + \xi) - c\xi) - k.$$

3.2. FUNCTION $\Phi(\lambda)$

We next introduce the function

$$\Phi(\lambda) = \min_{\varpi \geq 0}[(\lambda - \varpi)\nu(\varpi)]. \tag{3.1}$$

We have proved in our previous paper, the following properties of the function $\Phi(\lambda)$.The function $\Phi(\lambda)$ is monotone increasing, concave, continuously differentiable, and satisfies

$$\begin{aligned} &\Phi(\lambda) = -\infty, \qquad \text{if } \lambda \leq 0, \\ &-\infty < \Phi(\lambda) \ < 0, \text{ if } \lambda > 0, \\ &\Phi(+\infty) = 0. \end{aligned} \tag{3.2}$$

Moreover, if $\lambda > 0$, the minimum $\hat{\varpi}$=$\hat{\varpi}(\lambda)$ is uniquely defined. It is a monotone increasing function of λ and $\hat{\varpi}(0) = 0$, $\hat{\varpi}(+\infty) = +\infty$. We also assume

$$\nu(\varpi) \geq 0, \ 2 - \frac{\nu\nu}{(\nu')^2}(\varpi) \geq c_0 > 0, \ \forall \varpi \geq 0. \tag{3.3}$$

Noting that since $\hat{\varpi}(\lambda)$ is solution of

$$\lambda - \hat{\varpi} - \frac{\nu(\hat{\varpi})}{\nu'(\hat{\varpi})} = 0,$$

we have by differentiation

$$1 = \hat{\varpi}'(\lambda)[2 - \frac{\nu\nu''}{(\nu')^2}(\hat{\varpi}(\lambda))], \ \lambda > 0.$$

We get imediately $\frac{1}{2} \leq \ \hat{\varpi}'(\lambda) \leq \ \frac{1}{c_0}$. So $\hat{\varpi}(\lambda)$ is C^1 on $[0, +\infty)$ with bounded derivative. In our template (2.5) we have additionally that $\hat{\varpi}(\lambda) = \dfrac{\gamma + 1}{\gamma}\lambda, \Phi(\lambda) = -\dfrac{\gamma^\gamma}{(\gamma + 1)^{\gamma + 1}}\dfrac{1}{\lambda^\gamma}$, and $c_0 = \dfrac{\gamma}{\gamma + 1}$.

3.3. QUASI VARIATIONAL INEQUALITY

As we have done in the case without fixed cost, we associate to Dynamic Programming an analytic problem, which is a Q.V.I. The solution is a function $u(x)$, which is C^1, has bounded derivatives, and is a.e. twice differentiable. Eventually, it will be the value function, given by (2.11) , but at this stage it is introduced by itself, with no interpretation. This is why we use the same notation $u(x)$. The problem is the following

$$-\frac{1}{2}\sigma^2 u'' + \alpha u + \Phi(u') + hx^+ + px^- \geq 0, \text{ a.e. } x,$$

$$u(x) \geq \bar{M}(u)(x), \; \forall x \geq 0, \tag{3.4}$$

$$(u(x) - \bar{M}(u)(x))\left(-\frac{1}{2}\sigma^2 u'' + \alpha u + \Phi(u') + hx^+ + px^-\right) = 0, \text{ a.e. } x,$$

in which we have introduced the operator

$$\bar{M}(u)(x) = \sup_{\xi \geq 0}(u(x+\xi) - c\xi) - k, \tag{3.5}$$

which we call the Bellman Q.V.I. of problem (2.11). We also look for a solution such that

$$u'(x) > 0, \tag{3.6}$$

which corresponds to the property of the value function, see Proposition 2.1, but also to the fact that the function $\Phi(\lambda)$ is finite only when $\lambda > 0$.

3.4. TRANSFORMATIONS

We first begin with a simple transformation, setting $G(x) = u(x) - cx$. We obtain immediately

$$-\frac{1}{2}\sigma^2 G'' + \alpha G + \Phi(G' + c) + (h + \alpha c)x^+ + (p - \alpha c)x^- \geq 0, \text{ a.e. } x,$$

$$G(x) \geq M(G)(x), \tag{3.7}$$

$$(G(x) - M(G)(x))\left(-\frac{1}{2}\sigma^2 G + \alpha G + \Phi(G' + c) + (h + \alpha c)x^+ + (p - \alpha c)x^-\right) = 0,$$
$$\text{a.e. } x,$$

in which we have set

$$M(G)(x) = -k + \sup_{y \geq x} G(y). \tag{3.8}$$

We also have from (3.6)

$$G'(x) + c > 0. \tag{3.9}$$

We look for a solution of (3.7) as follows: Find s, and $G_s(x)$ such that

$$-\frac{1}{2}\sigma^2 G_s(x) + \alpha G_s(x) + \Phi(G_s'(x) + c) + (h + \alpha c)x^+ + (p - \alpha c)x^- = 0, x > s, \tag{3.10}$$

$$G_s'(s) = 0, \; G_s'(x) \text{ is bounded.}$$

For fixed s,this is a second order differential equation on $(s, +\infty)$, with two-point boundary conditions, of Neumann type (the derivative is given at s, and the boundedness acts as a condition at ∞). We then define s,by adding a condition, namely,

$$G_s(s) = M(G_s)(s), \tag{3.11}$$

and setting $G_s(x) = G_s(s)$, for $x < s$, we get a C^1 function on R. It is C^2 except at s, in which $G_s(x)$ has left and right limits. We need to find a function $G_s(x)$, such that

$$G'_s(x) + c > 0. \tag{3.12}$$

How does this problem (3.10), (3.11), (3.12) solves the original problem (3.7). Of course we define $G(x) = G_s(x)$. We note that the complentarity slackness condition is satisfied. The first inequality is satisfied when $x > s$ (it is an equality) , and the second inequality is satisfied when $x < s$ (it is also an equality). Therefore we need to check that the solution of (3.10) satisfies $G_s(x) \geq M(G_s)(x)$, $\forall x > s$. In that case the second condition in (3.7) is satisfied. There remains to satisfy the first inequality, when $x < s$. Since for $x < s$, we have $G(x) = G_s(x) = G_s(s)$, we get

$$\begin{aligned} &-\frac{1}{2}\sigma^2 G(x) + \alpha G(x) + \Phi(G'(x) + c) + (h + \alpha c)x^+ + (p - \alpha c)x^- \\ =&\alpha G_s(s) + \Phi(c) + (h + \alpha c)x^+ + (p - \alpha c)x^-, \end{aligned}$$

so we must have $\alpha G_s(s) + \Phi(c) + (h + \alpha c)x^+ + (p - \alpha c)x^- \geq 0$ for $x < s$. If we have $s < 0$, this condition will mean

$$\alpha G_s(s) + \Phi(c) - s(p - \alpha c) > 0.$$

So, if the pair made of s and the function $G_s(x)$ solution of (3.10) satisfies also (3.11) , (3.12), and

$$\begin{gathered} G_s(x) \geq M(G_s)(x),\ \forall x > s, \\ s < 0,\ \alpha G_s(s) + \Phi(c) - s(p - \alpha c) > 0, \end{gathered} \tag{3.13}$$

then the function $G(x) = G_s(x), x \geq s,\ G(x) = G_s(s),\ x \leq s$, is solution of the Q.V.I. (3.7).

4. SOLUTION OF THE INVENTORY CONTROL AND PRICING PROBLEM.

4.1. PRICING FEEDBACK.

Suppose we find a pair $s < 0$ $G_s(x)$, satisfying (3.6), (3.11), (3.12), (3.13). We set $H_s(x) = G'_s(x)$. We shall give below estimates on $H'_s(x)$ and prove that it is bounded as $x \to +\infty$. In fact, formally $H'_s(+\infty) = 0$, since $H_s(+\infty) + c = 0$. Also $H_s(x)$ is bounded. In fact we shall have

$$-c - \frac{h}{\alpha} < H_s(x) \leq -c + \frac{p}{\alpha}. \tag{4.1}$$

The fact that $H_s(x) + c \to 0$, as $x \to +\infty$ can be seen by contradiction. Otherwise $\Phi(H_s(x)+c)$ remains bounded as $x \to +\infty$. But then, considering (3.10) as a linear equation, in which $\Phi(H_s(x)+c)$ is given, we would have $H_s(x) \to -c-\dfrac{h}{\alpha}$ and then $\Phi(H_s(+\infty)+c) = -\infty$, which would be a contradiction.

We next define the feedback (it will be the optimal pricing feedback)

$$\hat{\pi}(x) = \hat{\varpi}(H_s(x)+c),\ x \geq s, \qquad (4.2)$$
$$\hat{\pi}(x) = \hat{\varpi}(c),\ x \leq s.$$

It takes the value 0, when $x = +\infty$. Moreover

$$\hat{\pi}'(x) = \hat{\varpi}'(H_s(x)+c))H_s'(x).$$

Therefore, from the property $\hat{\varpi}'(\lambda)$ bounded, see Section 3.2, the feedback $\hat{\pi}(x)$ is C^1, and has bounded derivative. From (4.2) we can also state that $\hat{\pi}(x)$ is bounded and away from 0, on compact sets of the argument x.

We shall also show that $H_s(x) > 0$, for x close to s, larger than s. Since $H_s(+\infty) = -c$, and $H_s(x)$ is continuous, there will exist $S(s) > s$, such that $H_s(S) = 0$. We shall see that $S(s)$ is uniquely defined, but at this stage, it is sufficient to define $S(s)$ as the smallest value $> s$, such that $H_s(S) = 0$. The pair s, S will form the s, S policy defining the inventory control.

4.2. INVENTORY ORDERING POLICY

To define the optimal inventory policy, we consider the function $\nu(\hat{\pi}(x))$. From the considerations of the previous section, we can claim that it is C^1 bounded with bounded derivatives on compact sets. We solve the stochastic differential equation with impulses

$$d\hat{x}(t) + \nu(\hat{\pi}(\hat{x}))dt + \sigma dw(t) = 0,\ \hat{\tau}_i < t < \hat{\tau}_{i+1}, \cdots \qquad (4.3)$$
$$\hat{x}(\hat{\tau}_i) = S, \text{ if, } i \geq 1,$$
$$\hat{\tau}_0 = 0, \hat{x}(0) = x,$$

$$\hat{\tau}_i = \inf\{t > \hat{\tau}_{i-1} | \hat{x}(t) = s\},\ i = 1, \cdots .$$

In the next section we shall solve (4.3) piecewise. We set next

$$\hat{\varpi}_x(t) = \hat{\pi}(\hat{x}(t)). \qquad (4.4)$$

Then this is the optimal joint inventory and pricing policy. The process $\hat{x}(t)$ is the optimal inventory.When the inventory is lower than s, we use the price $\hat{\pi}(s) = \hat{\varpi}(c)$. But then we jump immediately to S, by putting an order. The stock becomes S, and as long as the inventory is above s, we use the pricing $\hat{\pi}(\hat{x}(t))$. So we have the analogue of the s, S, p policy in discrete time. The feedback $\hat{\pi}(x)$ captures the pricing policy p. We shall study its properties below.

4.3. VERIFICATION THEOREM

In this section, we provide a verification theorem to check that the policy defined above is indeed optimal. We first study the stochastic equation (4.3). We have the

Proposition 4.1. *Assume (2.2), (2.3), (2.4) and (3.3). Then there exists one and only one solution of (4.3) such that*

$$E\left[\sup_{0\le t\le T}|\hat{x}(t)|^2\right] < c_0\max(S^2, x^2) + c_1 T,\ \forall T. \tag{4.5}$$

Proof. To simplify notation, we set $y(t) = \hat{x}(t)$, $\tau_i = \hat{\tau}_i$ and $a(x) = \nu(\hat{\pi}(x))$. So we have

$$\begin{aligned} dy + a(y)dt + \sigma dw(t) = 0, \\ y(\tau_i) = S,\ i \ge 1, \\ y(0) = x. \end{aligned} \tag{4.6}$$

$$\tau_i = \inf\{t > \tau_{i-1}|y(t) = s\},\ i = 1, \cdots .$$

The difficulty we have here is that the nonlinear function $a(x)$ is not globally Lipschitz. The derivative $a'(x)$ exists but it is not bounded. However it is bounded on any compact interval and $a(x)$ is positive. So in fact, we replace $a(x)$ by $a_M(x) = a(x \wedge M)$ which is also positive and globally Lipschitz and let M go to $+\infty$. If we solve (4.6) with a_M replacing a and obtain sufficient estimates to pass to the limit, we will solve (4.6). To shorten the proof, we shall obtain the a priori estimates directly on a solution of (4.6) assuming it exists and leave the details to the reader. We proceed as follows: Define the process $y^0(t)$ by solving

$$\begin{aligned} dy^0 + a(y^0)dt + \sigma dw(t) = 0, \\ y^0(0) = x. \end{aligned}$$

We then define

$$\tau_1 = \inf\{t|y^0(t) \le s\}.$$

For $i \ge 1$, if we have defined a stopping time τ_i , we define the process $y^i(t)$ by solving

$$\begin{aligned} dy^i + 1\!\!1_{t>\tau_i}(a(y^i)dt + \sigma dw(t)) = 0, \\ y^i(0) = S, \end{aligned}$$

and we define

$$\tau_{i+1} = \inf\{t|y^i(t) \le s\}.$$

We then set

$$y(t) = y^0(t)1\!\!1_{t<\tau_1} + \sum_{i=1}^{+\infty} y^i(t)1\!\!1_{\tau_i\le t<\tau_{i+1}}.$$

This process is the solution. We prove only the a priori estimates. We note that

$$y^2(t) = (y^0(t))^2 1\!\!1_{t<\tau_1} + \sum_{i=1}^{+\infty}(y^i(t))^2 1\!\!1_{\tau_i\le t<\tau_{i+1}}. \tag{4.7}$$

But for $i \ge 1$

$$\begin{aligned} d(y^i(t))^2 &= -2y^i(t)1\!\!1_{\tau_i<t}(a(y^i)dt + \sigma dw(t)) + \sigma^2 1\!\!1_{\tau_i<t}dt \\ &\le -2y^i(t)1\!\!1_{\tau_i<t}\sigma dw(t) + \sigma^2 1\!\!1_{\tau_i<t}dt \end{aligned}$$

hence

$$(y^i(t))^2 \leq S^2 - 2\int_0^t y^i(s)\mathbb{1}_{\tau_i<s}\sigma dw(s) + \sigma^2\int_0^t \mathbb{1}_{\tau_i<s}ds.$$

We also have

$$(y^0(t))^2 \leq x^2 - 2\int_0^t y^0(s)\sigma dw(s) + \sigma^2 t.$$

We can combine both inequalities by writing

$$(y^i(t))^2 \leq \max(S^2, x^2) - 2\int_0^t y^i(s)\mathbb{1}_{\tau_i<s}\sigma dw(s) + \sigma^2\int_0^t \mathbb{1}_{\tau_i<s}ds \tag{4.8}$$

for $i \geq 0$, with $\tau_0 = 0$. From (4.7) it follows

$$y^2(t) \leq \max(S^2, x^2) - 2\sum_{i=0}^{+\infty}\mathbb{1}_{\tau_i\leq t<\tau_{i+1}}\int_0^t y^i(s)\mathbb{1}_{\tau_i<s}\sigma dw(s) + \sigma^2\sum_{i=0}^{+\infty}\mathbb{1}_{\tau_i\leq t<\tau_{i+1}}\int_0^t \mathbb{1}_{\tau_i<s}ds.$$

It follows

$$y^2(t) \leq \max(S^2, x^2) + 2\sigma\sum_{i=0}^{+\infty}\mathbb{1}_{\tau_i\leq t<\tau_{i+1}}\left|\int_0^t y^i(s)\mathbb{1}_{\tau_i<s<\tau_{i+1}}dw(s)\right| + \sigma^2 t$$

and thus also

$$y^2(t) \leq \max(S^2, x^2) + \sigma^2 t + \frac{\sigma^2}{\epsilon} + \epsilon\sum_{i=0}^{+\infty}\left|\int_0^t y^i(s)\mathbb{1}_{\tau_i<s<\tau_{i+1}}dw(s)\right|^2, \tag{4.9}$$

$$\sup_{0\leq t\leq T} y^2(t) \leq \max(S^2, x^2) + \sigma^2 T + \frac{\sigma^2}{\epsilon} + \epsilon\sum_{i=0}^{+\infty}\sup_{0\leq t\leq T}\left|\int_0^t y^i(s)\mathbb{1}_{\tau_i<s<\tau_{i+1}}dw(s)\right|^2.$$

Taking the expectation, and using estimates on martingales we obtain

$$\begin{aligned} E\left[\sup_{0\leq t\leq T} y^2(t)\right] &\leq \max(S^2, x^2) + \sigma^2 T + \frac{\sigma^2}{\epsilon} + 4\epsilon\sum_{i=0}^{+\infty} E\left[\int_0^T (y^i(s))^2\mathbb{1}_{\tau_i<s<\tau_{i+1}}ds\right] \\ &\leq \max(S^2, x^2) + \sigma^2 T + \frac{\sigma^2}{\epsilon} + 4\epsilon E\left[\int_0^T y^2(s)ds\right] \\ &\leq \max(S^2, x^2) + \sigma^2 T + \frac{\sigma^2}{\epsilon} + 4\epsilon T\, E\left[\sup_{0\leq t\leq T} y^2(t)\right] \end{aligned}$$

and choosing $4\epsilon T = \frac{1}{2}$ we deduce easily

$$E\left[\sup_{0\leq t\leq T} y^2(t)\right] \leq 2\max(S^2, x^2) + 18\sigma^2 T. \tag{4.10}$$

So the estimate (4.5) has been obtained. The proof has been completed. □

We then can state the verification theorem

Theorem 4.1. *We make the assumptions of Proposition 4.1. and suppose that we can find* $s < 0$, $G_s(x)$ *satisfying (3.6),(3.9), (3.12), (3.13), (4.1). Also assume that* S *is the point of maximum of* $G_s(x)$*, then we can construct the pricing policy* $\hat{\varpi}_x(t)$ *by the feedback* $\hat{\pi}(x)$ *as explained in (4.2), (4.4). Define*

$$\hat{V}_x = (\hat{\tau}_1, S - \hat{x}(\hat{\tau}_1 - 0); \cdots ; \hat{\tau}_i, S - \hat{x}(\hat{\tau}_i - 0); \cdots)$$

with the notation of (4.3). Then the pair $\hat{\varpi}_x(.), \hat{V}_x$ *is optimal for the payoff (2.9), among all policies* $\varpi(.), V$ *such that*

$$E\left[\sup_{0\le t\le T} x(t)^2\right] < +\infty,\ \forall\, T,$$

$$E\left[\sup_{0\le t\le T} x(t)^2\right] \exp(-\alpha T) \to 0,\ as\, T \to +\infty. \tag{4.11}$$

Proof. We first note that, without loss of generality, we can assume that $\theta_i \to +\infty$ as $i \to +\infty$. Indeed, otherwise $\theta_i \uparrow \tau > 0$ on a set of positive measure. But then, from formula (2.9) we have

$$J_x(\varpi(.),\, V) \le \int_0^{+\infty} \exp(-\alpha t)\, \varpi(t)\nu(\varpi(t))dt - \sum_{i=0}^{+\infty} kE\left[\exp(-\alpha\tau)\right] = -\infty$$

since $E\left[\exp(-\alpha\tau)\right] > 0$. Such a policy cannot be optimal, so we can discard it. We note that $\hat{\tau}_i \to +\infty$, as $i \to +\infty$. Indeed, consider the differential equation

$$-\frac{1}{2}\sigma^2\varphi + \varphi'\nu(\hat{\pi}) + \alpha\varphi = 0,\ x > s,$$
$$\varphi(s) = 1, \varphi(+\infty) = 0.$$

We check easily the formula $\varphi(S)E\exp -\alpha\hat{\tau}_i = E\exp -\alpha\hat{\tau}_{i+1}$. It follows that $E\exp -\alpha\hat{\tau}_i = (\varphi(S))^i$. However $\hat{\tau}_i \uparrow \hat{\tau}$ and this $E\exp -\alpha\hat{\tau} = 0$. This implies $\hat{\tau} = +\infty$, a.s. Moreover, from Proposition 4.1, we see that the condition (4.11) is satisfied, see (4.10).

We have constructed a number a pair s, S and a function $u(x) = cx + G_s(x)$ which is C^1 , and C^2 except at point s. It has bounded derivative and satisfies

$$u(x) = -k + \sup_{\xi>0}(u(x+\xi) - c\xi)$$
$$= cx - k + u(S) - cS,\ x \le s,$$

$$-\frac{1}{2}\sigma^2 u + \alpha u + \Phi(u') + hx^+ + px^- = 0, x > s, \tag{4.12}$$
$$0 \le u',$$
$$-\frac{1}{2}\sigma^2 u + \alpha u + \Phi(u') + hx^+ + px^- \ge 0, x < s, \tag{4.13}$$
$$u(x) \ge -k + u(x+\xi) - c\xi, \forall\xi > 0, \forall x \ge s.$$

From the definition of Φ, we can write

$$-\frac{1}{2}\sigma^2 u + \alpha u + (u' - \varpi)\nu(\varpi) + hx^+ + px^- \ge 0,\ \forall x, \forall\varpi \ge 0. \tag{4.14}$$

Consider any admissible policy $\varpi(.)$, V and the corresponding inventory $x(t)$ defined by (2.7). We use Ito's formula to compute

$$d(u(x(t))\exp(-\alpha t)) = \exp(-\alpha t)[-\alpha u(x(t)) - u'(x(t))\nu(\varpi(t)) + \frac{1}{2}\sigma^2 u(x(t))]dt \\ - \sigma u'(x(t))\exp(-\alpha t)\,dw(t).$$

Applying (4.14) with $x = x(t)$ and $\varpi = \varpi(t)$ we obtain

$$d(u(x(t))\exp(-\alpha t)) \leq \exp(-\alpha t)[hx^+(t) + px^-(t) - \varpi(t)\nu(\varpi(t))]dt \\ - \sigma u'(x(t))\exp(-\alpha t)\,dw(t).$$

Integrating between $\theta_i \wedge T$ and $(\theta_{i+1} - 0) \wedge T$, and taking the mathematical expectation yields

$$\begin{aligned} &E\left[u(x(\theta_i \wedge T))\exp(-\alpha\theta_i \wedge T)\right] \\ \geq &E\left[u(x((\theta_{i+1} - 0) \wedge T))\exp(-\alpha\theta_{i+1} \wedge T)\right] \\ &+E[\int_{\theta_i \wedge T}^{\theta_{i+1}\wedge T} \exp{-\alpha t}(\varpi(t)\nu(\varpi(t)) - hx^+(t) - px^-(t))dt] \end{aligned} \tag{4.15}$$

also

$$\begin{aligned} E\left[u(x(\theta_i))\mathbb{1}_{\theta_i<T}\exp(-\alpha\theta_i)\right] \geq &E\left[u(x((\theta_{i+1} - 0)))\mathbb{1}_{\theta_{i+1}<T}\exp(-\alpha\theta_{i+1})\right] \\ &-E\left[u(x(T))\mathbb{1}_{\theta_i<T<\theta_{i+1}}\exp(-\alpha T)\right] \\ &+E[\int_{\theta_i \wedge T}^{\theta_{i+1}\wedge T} \exp{-\alpha t}(\varpi(t)\nu(\varpi(t)) - hx^+(t) - px^-(t))dt]. \end{aligned}$$

But $x(\theta_{i+1}) = x(\theta_{i+1} - 0) + \xi_{i+1}$ and from the 2nd inequality (4.13) we can write

$$u(x(\theta_{i+1} - 0)) \geq u(x(\theta_{i+1})) - k - c\xi_{i+1}$$

and thus

$$\begin{aligned} E\left[u(x(\theta_i))\mathbb{1}_{\theta_i<T}\exp(-\alpha\theta_i)\right] \geq &E\left[u(x(\theta_{i+1}))\mathbb{1}_{\theta_{i+1}<T}\exp(-\alpha\theta_{i+1})\right] \\ &-E\left[(k + c\xi_{i+1})\mathbb{1}_{\theta_{i+1}<T}\exp(-\alpha\theta_{i+1})\right] \\ &-E\left[u(x(T))\mathbb{1}_{\theta_i<T<\theta_{i+1}}\exp(-\alpha T)\right] \\ &+E[\int_{\theta_i \wedge T}^{\theta_{i+1}\wedge T} \exp{-\alpha t}(\varpi(t)\nu(\varpi(t)) - hx^+(t) - px^-(t))dt]. \end{aligned} \tag{4.16}$$

We set $\theta_0 = 0$. We then sum the inequalities (4.16) between $i = 0$ and $i = N$. We get

$$\begin{aligned} u(x) \geq &E\left[u(x(\theta_{N+1}))\mathbb{1}_{\theta_{N+1}<T}\exp(-\alpha\theta_{N+1})\right] \\ &- Eu(x(T))\exp(-\alpha T) \\ &- E\left[\sum_{i=1}^{N}(k + c\xi_i)\exp(-\alpha\theta_i)\right] \\ &+ E\left[\int_0^{\theta_{N+1}\wedge T} \exp(-\alpha t)(\varpi(t)\nu(\varpi(t)) - hx^+(t) - px^-(t))dt\right]. \end{aligned}$$

We first let $N \to +\infty$. We know that $\theta_{N+1} \to +\infty$. Since u has a bounded derivative, it has linear growth, then $|u(x(\theta_{N+1}))| \le C(1 + |x(\theta_{N+1}|)$. Therefore

$$\begin{aligned}
|Eu(x(\theta_{N+1}))\mathbb{1}_{\theta_{N+1}<T}| &\le CE(1 + |x(\theta_{N+1}|)\mathbb{1}_{\theta_{N+1}<T} \\
&\le CE(1 + \sup_{0<t<T} |x(t)|)\mathbb{1}_{\theta_{N+1}<T} \\
&\le C\sqrt{E(1 + \sup_{0<t<T} |x(t)|)^2}\sqrt{E\mathbb{1}_{\theta_{N+1}<T}} \\
&\to 0, \text{ as } N \to +\infty.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
u(x) \ge &- Eu(x(T))\exp(-\alpha T) - E\left[\sum_{i=1}^{\infty}(k + c\xi_i)\exp(-\alpha\theta_i)\right] \\
&+ E\left[\int_0^T \exp -\alpha t(\varpi(t)\nu(\varpi(t)) - hx^+(t) - px^-(t))dt\right].
\end{aligned}$$

We may then let $T \to +\infty$, and obtain $u(x) \ge J_x(\varpi(.), V)$, for all admissible policies. To prove the optimality of the pair $\hat{\varpi}_x(t)$, $\hat{V}_x$, it is thus sufficient to check that

$$u(x) = J_x(\hat{\varpi}_x(.), \hat{v}_x(.)). \tag{4.17}$$

We know that the pair $\hat{\varpi}_x(t)$, $\hat{V}_x$ is admissible. The proof has many similarities with the case of general policies, above except that inequalities will be replaced by equalities. We leave details to the reader. The proof of the result has been obtained. □

5. STUDY OF THE ANALYTIC PROBLEM

5.1. PRELIMINARIES

Our objective is now to study problem (3.10),(3.11),(3.12), (3.13), where the unknown is the pair s, G_s which allows to define the optimal policy by the verification theorem above. This will require also to check all the properties used in in this theorem, namely if $H_s(x) = G'_s(x)$ we need to check (4.1), the fact that $H'_s(x)$ is bounded , that it has a unique 0, called $S(s)$, and $H_s(x) > 0$, for $x \in (s, S)$.

5.2. APPROXIMATION

The major difficulty lies in in the function Φ. It is increasing, but the derivative is not bounded. We will need to proceed with an approximation. Define

$$\Phi_\epsilon(\lambda) = \Phi(\lambda^+ + \epsilon) \tag{5.1}$$

which is defined on R, and not just on R^+. We have $\Phi'_\epsilon(\lambda) = \Phi'(\lambda + \epsilon)\mathbb{1}_{\lambda>0}$. Hence $0 \le \Phi'_\epsilon(\lambda) \le \Phi'(\epsilon)$. Note that the function $\Phi_\epsilon(\lambda)$ remains concave on R^+, but is not globally concave. For any $s \le 0$, given, we consider the problem

$$-\frac{1}{2}\sigma^2 H_s(x) + \alpha H_s(x) + \frac{d}{dx}\Phi_\epsilon(H_s(x) + c) + (h + \alpha c)\mathbb{1}_{x>0} - (p - \alpha c)\mathbb{1}_{x<0} = 0,\ x > s,$$

$$H_s(s) = 0,\ H_s(+\infty) = -c - \frac{h}{\alpha}. \tag{5.2}$$

We simultaneously consider the problem

$$-\frac{1}{2}\sigma^2 G_s(x) + \alpha G_s(x) + \Phi_\epsilon(G_s'(x) + c) + hx^+ + px^- + \alpha cx = 0,\ x > s, \tag{5.3}$$
$$G_s'(s) = 0,\ G_s'(+\infty) = -c - \frac{h}{\alpha}.$$

The problems (5.2) and (5.3) are equivalent. If $G_s(x)$ is solution of (5.3), then $H_s(x) = G_s'(x)$ is solution of (5.2). Conversely, if $H_s(x)$ is solution of (5.2), then setting

$$G_s(s) = \frac{1}{\alpha}(\frac{1}{2}\sigma^2 H_s'(s) - \Phi_\epsilon(c) + s(p - \alpha c)), \tag{5.4}$$
$$G_s(x) = G_s(s) + \int_s^x H_s(\xi)d\xi$$

we obtain a solution of (5.3). Of course, the solution depends also on ϵ. But we consider first ϵ fixed. Many steps are identical to those in our previous paper [8]. We shall then state them without details. We shall restrict the interval for s, although it is not directly related to obtaining existence results. Eventually, it will be needed anyway. We introduce the number $\beta = \frac{\sqrt{2\alpha}}{\sigma}$. We have seen in our previous paper that (5.2) is equivalent to an integral equation as follows

$$\begin{aligned} H_s(x) =& \frac{2}{\sigma^2\beta}\int_s^x \exp(-\beta(x-\xi))\,(p - \alpha c - (p+h)\exp(-\beta\xi)^-)d\xi \\ &+ \frac{1}{\sigma^2}\int_s^x (\exp(-\beta(x-\xi)) + \exp(-\beta(x-s))\exp(-\beta(\xi-s)))\Phi_\epsilon(H_s(\xi)+c)d\xi \\ &+ \frac{1}{\sigma^2}\int_x^{+\infty} (\exp(-\beta(x-s))\exp(-\beta(\xi-s)) - v\exp(-\beta(\xi-x)))\Phi_\epsilon(H_s(\xi)+c)d\xi. \end{aligned} \tag{5.5}$$

We can compute

$$\begin{aligned} H_s'(x) =& \frac{2}{\sigma^2\beta}[p - \alpha c - (p+h)\exp(-\beta x^-) \\ &- \beta\int_s^x \exp(-\beta(x-\xi))\,(p - \alpha c - (p+h)\exp(-\beta\xi^-))d\xi] + \frac{2}{\sigma^2}\Phi_\epsilon(H_s(x)+c) \\ &- \frac{\beta}{\sigma^2}\int_s^x (\exp(-\beta(x-\xi)) + \exp(-\beta(x-s))\exp(-\beta(\xi-s)))\Phi_\epsilon(H_s(\xi)+c)d\xi \\ &- \frac{\beta}{\sigma^2}\int_x^{+\infty} (\exp(-\beta(x-s))\exp(-\beta(\xi-s)) + \exp(-\beta(\xi-x)))\Phi_\epsilon(H_s(\xi)+c)d\xi. \end{aligned} \tag{5.6}$$

In particular, for $x = s$, we get

$$\begin{aligned} H_s'(s) =& \frac{2}{\sigma^2\beta}(p - \alpha c - (p+h)\exp(-\beta s^-)) \\ &+ \frac{2}{\sigma^2}[\Phi_\epsilon(c) - \beta\int_s^{+\infty} \exp(-\beta(\xi - s))\Phi_\epsilon(H_s(\xi) + c)d\xi]. \end{aligned} \tag{5.7}$$

We now state the following

Theorem 5.1. *We make the assumptions of Proposition 4.1. For any $s \leq 0$, there exists one and only one function $H_s(x)$ solution of (5.2) which is $C^2(s, +\infty)$ and satisfies*

$$-c - \frac{h}{\alpha} \leq H_s(x) \leq -c + \frac{p}{\alpha}. \tag{5.8}$$

Alternatively, there exists one and only one solution $G_s(x)$ of (5.3) which is $C^3(s, +\infty)$. and has linear growth. The functions $H_s(x)$ and $G_s(x)$ are related by formulas (5.4).

Proof. Note that the two equations are equivalent. If $G_s(x)$ is solution of (5.3), then its derivative $H_s(x) = G_s'(x)$ is solution of (5.2). Conversely if $H_s(x)$ is a solution of (5.2) , the the function $G_s(x)$ defined by (5.4) is solution of (5.3). So it is sufficient to prove existence and uniqueness from one of them. We shall prove existence for $H_s(x)$ and uniqueness for $G_s(x)$. In fact we proved existence of a solution $H_s(x)$ in our previous paper. So we prove here only the uniqueness of $G_s(x)$. Suppose there are two solutions G^1 and G^2. We make explicit the definition of Φ_ϵ and thus can write

$$\begin{gathered}
-\frac{1}{2}\sigma^2(G^1)(x) + \alpha G^1(x) + \min_{\varpi \geq 0}[((G^1)'(x) + c)^+ + \epsilon - \varpi)\nu(\varpi)] \\
+ hx^+ + px^- + \alpha c x = 0,\ x > s, \\
(G^1)'(s) = 0,\ (G^1)'(+\infty) = -c - \frac{h}{\alpha},
\end{gathered}$$

$$\begin{gathered}
-\frac{1}{2}\sigma^2(G^2)(x) + \alpha (G^2)(x) + \min_{\varpi \geq 0}[((G^2)'(x) + c)^+ + \epsilon - \varpi)\nu(\varpi)] \\
+ hx^+ + px^- + \alpha c x = 0,\ x > s, \\
(G^2)'(s) = 0,\ (G^2)'(+\infty) = -c - \frac{h}{\alpha}.
\end{gathered}$$

We also call $\hat{\varpi}_\epsilon^1(x)$, $\hat{\varpi}_\epsilon^2(x)$ the optimal feedback. Clearly

$$\begin{gathered}
-\frac{1}{2}\sigma^2(G^1)(x) + \alpha (G^1)(x) + [((G^1)'(x) + c)^+ + \epsilon - \hat{\varpi}^1)\nu(\hat{\varpi}^1)] \\
+ hx^+ + px^- + \alpha c x = 0,\ x > s, \\
(G^1)'(s) = 0,\ (G^1)'(+\infty) = -c - \frac{h}{\alpha},
\end{gathered}$$

$$\begin{gathered}
-\frac{1}{2}\sigma^2(G^2)(x) + \alpha (G^2)(x) + [((G^2)'(x) + c)^+ + \epsilon - \hat{\varpi}^1)\nu(\hat{\varpi}^1)] \\
+ hx^+ + px^- + \alpha c x \geq 0,\ x > s, \\
(G^2)'(s) = 0,\ (G^2)'(+\infty) = -c - \frac{h}{\alpha}.
\end{gathered}$$

Define $\tilde{G}(x) = G^1(x) - G^2(x)$. We thus have

$$\begin{gathered}
-\frac{1}{2}\sigma^2(\tilde{G})(x) + \alpha\tilde{G}(x) + [((G^1)'(x) + c)^+ - ((G^2)'(x) + c)^+]\nu(\hat{\varpi}^1) \leq 0, \\
\tilde{G}'(s) = 0,\ \tilde{G}'(+\infty) = 0.
\end{gathered} \tag{5.9}$$

We claim that $\tilde{G}(x) \leq 0$. If not, there exist points where $\tilde{G}$ is strictly positive. But $\tilde{G}$ cannot attain its maximum at ∞, Otherwise from the equation, recalling that $\tilde{G}$ is C^3, $(\tilde{G})(+\infty)$¿0. This would imply $\tilde{G}'(+\infty) = +\infty$, which contradicts the boundary condition at $+\infty$.Similarly, a positive maximum cannot be attained at s, because we would have $(\tilde{G})(s)$¿0. This would contradict the fact that $\tilde{G}$ has a maximum at s. Finally a positive maximum cannot take place at a point inside the interval. This would contradict the maximum principle. So we must have $\tilde{G}(x) \leq 0$. But changing the role of G^1 and G^2 the reverse is also true. Therefore $\tilde{G}(x) = 0$. This completes the proof. □

5.3. FURTHER PROPERTIES

Since Φ_ϵ is monotone increasing , we obtain from (5.7) the estimate

$$H_s'(s) \geq \frac{2}{\sigma^2\beta}(p - \alpha c - (p+h)\exp -\beta s^-) + \frac{2}{\sigma^2}(\Phi_\epsilon(c) - \Phi_\epsilon(\frac{p}{\alpha})). \tag{5.10}$$

We state

$$\Phi_\epsilon(c) - \Phi_\epsilon(\frac{p}{\alpha}) = \Phi(c+\epsilon) - \Phi(\frac{p}{\alpha}+\epsilon) > \Phi(c) - \Phi(\frac{p}{\alpha})$$

by the concavity of Φ. Finally, from (5.10)

$$H_s'(s) > \frac{2}{\sigma^2\beta}(p - \alpha c - (p+h)\exp -\beta s^-) + \frac{2}{\sigma^2}(\Phi(c) - \Phi(\frac{p}{\alpha})) \tag{5.11}$$

the right hand side is independent of ϵ.

We make the fundamental assumption

$$\frac{1}{\beta}(p - \alpha c) + \Phi(c) - \Phi(\frac{p}{\alpha}) > 0. \tag{5.12}$$

Obviously, there is $s^* < 0$, such that

$$\frac{1}{\beta}(p - \alpha c - (p+h)\exp(\beta s*)) + \Phi(c) - \Phi(\frac{p}{\alpha}) = 0 \tag{5.13}$$

and

$$\frac{1}{\beta}(p - \alpha c - (p+h)\exp(\beta s)) + \Phi(c) - \Phi(\frac{p}{\alpha}) \geq 0,\ \forall s \leq s^*, \tag{5.14}$$

therefore $H_s'(s) > 0$, $\forall s \leq s^*$.

Proposition 5.1. *Assume $H_s'(s) > 0$. This is true as soon as $s \leq s^*$. Then the function $H_s(x)$ (a solution of (5.2)) has a unique zero, $S(s) > s$ as well as a unique maximum $\sigma(s) < 0$. Moreover $H_s'(x) > 0$, if $s \leq x < \sigma(s)$ and $H_s'(x) < 0$ if $x > \sigma(s)$. Also $H_s(x) > 0$, if $s < x < S$ and $H_s(x) < 0$ if $x > S$. If $H_s'(s) \leq 0$ then $H_s'(x) < 0$, $\forall x > s$. Also $H_s(x) < 0$, $\forall x > s$. Lastly we define $S(s) = s$.*

Proof. Since $H_s'(s) > 0$, $H_s(x)$ is strictly positive for $x > s$, close to s. Since $H_s(+\infty) = -c - \frac{h}{\alpha}$, and $H_s(x)$ is a continuous function, there is necessarily one zero. Let $S(s)$ be the smallest zero strictly larger than s. We first claim that $S(s) \geq s^*$. Suppose it is not true $S(s) < s^*$. Consider formula (5.5) for $s < x \leq S(s)$, We have $H_s(x) \geq 0$, hence $\Phi_\epsilon(H_s(x) + c) \geq \Phi_\epsilon(c)$. Moreover $H_s(x) + c \leq \frac{p}{\alpha}$ hence $\Phi_\epsilon(H_s(x) + c) \leq \Phi_\epsilon(\frac{p}{\alpha})$. Therefore we can majorize

$$\begin{aligned} H_s(x) \geq & \frac{2}{\sigma^2\beta} \int_s^x \exp(-\beta(x-\xi))\,(p - \alpha c - (p+h)\exp(\beta\xi))d\xi \\ & + \frac{1}{\sigma^2}[\Phi_\epsilon(c) \int_s^x (\exp(-\beta(x-\xi)) + \exp(-\beta(x-s))\exp(-\beta(\xi-s)))d\xi \\ & + \Phi_\epsilon(\frac{p}{\alpha}) \int_x^{+\infty} (\exp(-\beta(x-s))\exp(-\beta(\xi-s)) - \exp(-\beta(\xi-x)))d\xi \end{aligned}$$

and

$$\begin{aligned} H_s(x) \geq & \frac{2}{\sigma^2\beta^2}(p - \alpha c - (p+h)\exp(\beta x))(1 - \exp(-\beta(x-s))) \\ & + \frac{1}{\sigma^2\beta}(\Phi_\epsilon(c) - \Phi_\epsilon(\frac{p}{\alpha}))(1 - \exp(-2\beta(x-s))) \end{aligned}$$

and since $1 - \exp(-2\beta(x-s)) \leq 2(1 - \exp(-\beta(x-s)))$, we can state

$$\begin{aligned} H_s(x) \geq & \frac{2}{\sigma^2\beta^2}(1 - \exp(-\beta(x-s))) \\ & * [p - \alpha c - (p+h)\exp(\beta x) + \beta(\Phi_\epsilon(c) - \Phi_\epsilon(\frac{p}{\alpha}))]. \end{aligned} \tag{5.15}$$

Applied with $x = S(s)$ we get

$$H_s(S(s)) \geq \frac{2}{\sigma^2\beta^2}(1 - \exp(-\beta(S(s)-s)))[p - \alpha c - (p+h)\exp(\beta S(s)) + \beta(\Phi(c) - \Phi(\frac{p}{\alpha}))]$$

and since $S(s) < s^*$ we get $H_s(S(s)) > 0$, which contradicts the definition of $S(s)$. Let us check that there cannot be another zero $\tilde{S} > S$. Suppose there is such $\tilde{S}$, we claim that $\tilde{S} > 0$. This is obvious if $S \geq 0$. Suppose then that $S < 0$. The function $H_s(x)$ has clearly a negative minimum on $(S, \tilde{S})$. Such a minimum cannot occur on $(S, 0)$. It must take place on $[0, +\infty)$. This implies $\tilde{S} > 0$. Since $H_s(x)$ becomes strictly positive right after $\tilde{S}$, the function must have a local positive maximum on $(\tilde{S}, +\infty)$. This is impossible. Hence $\tilde{S}$ does not exist. Since $H_s'(s) > 0$, the function $H_s(x)$ will have a first local positive maximum $\sigma(s) < S(s)$. Necessarily also $\sigma(s) < 0$, from maximum principle considerations. Of course, $H_s'(x) > 0$, for $s \leq x < \sigma(s)$.Let us check that $H_s'(x) < 0$ for $x > \sigma(s)$. This will imply that $\sigma(s)$ is the maximum and the only one. If this is not true, $H_s'(x)$ will vanish at some point $x^* > \sigma(s)$ and x^* is a local minimum of $H_s(x)$. This point cannot be on $(\sigma(s), 0)$. Indeed, if it is then necessarily $H_s(x^*) > -c + \frac{p}{\alpha}$. On the other hand $H_s(\sigma(s)) < -c + \frac{p}{\alpha}$ and $H_s(\sigma(s)) > H_s(x^*)$, which leads to a contradiction. Suppose then that the first local minimum x^* after $\sigma(s)$ is positive. Since $H_s(x^*) \geq -c - \frac{h}{\alpha}$ and since $H_s(x)$ increases just after x^*, there must be a local maximum after

x^*. Indeed otherwise we could not have $H_s(+\infty) = -c - \dfrac{h}{\alpha}$. Let x_1 be this local maximum . We have by maximum principle considerations $H_s(x_1) < -c - \dfrac{h}{\alpha}$. But also $H_s(x_1) > H_s(x*) \geq -c - \dfrac{h}{\alpha}$, and we obtain again a contradiction. This proves that $H_s'(x) < 0$ for $x > \sigma(s)$.Naturally $H_s(x) > 0$ for $s < x < S$ and $H_s(x) > 0$, for $x > \sigma(s)$.

Assume now that $H_s'(s) \leq 0$, then $H_s(x)$ is strictly negative for $x > s$, close to s. This is obvious when $H_s'(s) < 0$. When we have $H_s'(s) = 0$, from the equation (5.2) , we obtain $H_s(s) < 0$, which will also imply $H_s(x)$ strictly negative for $x > s$, close to s. We then proceed as above. If $H_s'(x)$ has a zero at some point x^* larger than s, the function $H_s(x)$ has a local negative minimum at x^*. Such a point must be positive. But then, there will be a local maximum $x_1 > x^*$, which will lead to a contradiction. The proof is completed. □

We next provide estimates on the derivative of $H_s(x)$.

Proposition 5.2. *Assume $H_s'(s) > 0$. We have the estimates*

$$0 \leq H_s'(x) \leq \frac{2(p-\alpha c)}{\sigma\sqrt{\alpha}}, \text{ if } 0 \leq x \leq \sigma(s), \tag{5.16}$$

$$-\frac{\sqrt{2}}{\sigma\sqrt{\alpha}}(p+h) \leq H_s'(x) \leq 0, \text{ if } \sigma(s) \leq x < +\infty.$$

If $H_s'(s) \leq 0$, then the second estimate holds for $s \leq x < +\infty$. So in all cases

$$|H_s'(x)| \leq \frac{\sqrt{2}}{\sigma\sqrt{\alpha}} \max(\sqrt{2}(p-\alpha c), p+h). \tag{5.17}$$

Proof. We first deduce from (5.2) by multiplying by $H_s'(x)$

$$\begin{aligned} &-\frac{\sigma^2}{4}\frac{d}{dx}(H_s'(x))^2 + \Phi_\epsilon'(H_s(x)+c)(H_s'(x))^2 + \frac{\alpha}{2}\frac{d}{dx}(H_s(x))^2 \\ &= H_s'(x)(-(h+\alpha c)\mathbb{1}_{x>0} + (p-\alpha c)\mathbb{1}_{x<0}). \end{aligned} \tag{5.18}$$

Consider first the case $s < x < \sigma(s)$. We know that $H_s(x) > 0$, $H_s'(x) > 0$. Remembering that $\Phi_\epsilon' > 0$, we obtain

$$-\frac{\sigma^2}{4}\frac{d}{dx}(H_s'(x))^2 \leq (p-\alpha c)H_s'(x)$$

and integrationg between x and $\sigma(s)$ we obtain

$$\begin{aligned} \frac{\sigma^2}{4}(H_s'(x))^2 &\leq (p-\alpha c)(H_s(\sigma(s)) - H_s(x)) \\ &\leq (p-\alpha c)H_s(\sigma(s)) \leq \frac{(p-\alpha c)^2}{\alpha} \end{aligned}$$

from which we obtain immediately the first inequality (5.16). Assume then $x > \sigma(s)$, which implies $H_s'(x) < 0$. We deduce from (5.18)

$$-\frac{\sigma^2}{4}\frac{d}{dx}(H_s'(x))^2 + \frac{\alpha}{2}\frac{d}{dx}(H_s(x))^2 \leq -H_s'(x)(h+\alpha c)\mathbb{1}_{x>0}.$$

We integrate this inequality between x and $+\infty$, to obtain

$$\begin{aligned}\frac{\sigma^2}{4}(H_s'(x))^2 + \frac{\alpha}{2}((H_s(+\infty))^2 - (H_s(x))^2) &\le -(h+\alpha c)\int_{x^+}^{+\infty} H_s'(\xi)d\xi \\ &\le -(h+\alpha c)(H_s(+\infty) - H_s(x^+)) \\ &\le -(h+\alpha c)(H_s(+\infty) - H_s(x))\end{aligned}$$

and by easy calculations

$$\begin{aligned}\frac{\sigma^2}{4}(H_s'(x))^2 &\le \frac{\alpha}{2}(H_s(+\infty) - H_s(x))^2 \\ &= \frac{\alpha}{2}(H_s(x) + c + \frac{h}{\alpha})^2.\end{aligned}$$

We note that

$$0 \le H_s(x) + c + \frac{h}{\alpha} \le \frac{p+h}{\alpha}$$

and the 2nd estimate (5.16) follows immediately. When $H_s'(s) \le 0$, the proof of the 2nd estimate holds for $x \ge s$. The proof is completed. □

We also state the

Proposition 5.3. *The function $H_s(x)$ is continuous in s, and $H_s'(s)$ is also continuous.*

Proof. We complete $H_s(x)$ by 0 for $x \le s$. Let us consider a sequence $s_n \to s$. The sequence of functions $H_{s_n}(x)$ is uniformly bounded, as well as its derivative. Therefore, from Ascoli theorem, we can extract a subsequence such that $H_{s_n}(x) \to \zeta(x)$, $\forall x$. Considering the integral equation (5.5) it is easy to check that $\zeta(x) = H_s(x)$. In this identification, we have used the fact that the solution of (5.2) or (5.5) is unique. Now if we look at formula (5.7) which gives $H_s'(s)$, from the convergence $H_{s_n}(x) \to H_s(x)$, we see easily that $H_{s_n}'(s_n) \to H_s'(s)$. The proof is completed. □

Consider next equation (5.2) with $s = 0$. It writes

$$-\frac{1}{2}\sigma^2 H_0(x) + \alpha H_0(x) + \frac{d}{dx}\Phi_\epsilon(H_0(x) + c) + (h + \alpha c) = 0,\ x > 0, \tag{5.19}$$

$$H_0(0) = 0,\ H_0(+\infty) = -c - \frac{h}{\alpha}.$$

It is clear that $H_0(x)$ cannot have a positive local maximum, hence $H_0(x) \le 0$. Then $H_0'(0) \le 0$, and in fact $H_0'(0) < 0$. Indeed if $H_0'(0) = 0$, the equation leads to $H_0(0) > 0$, which will contradict the fact that 0 is the maximum.

Therefore we have

$$H_0'(0) < 0,\ H_s'(s) > 0, \forall s \le s^*. \tag{5.20}$$

From the continuity of the function $H_s'(s)$, there exists a point $\bar{s} \in (s^*, 0)$ such that $H_{\bar{s}}'(\bar{s}) = 0$. We can take the smallest one, so that $H_s'(s) > 0, \forall s < \bar{s}$. It follows that $S(s) > \max(s, s^*)$ for $s < \bar{s}$ and $S(\bar{s}) = \bar{s}$. Moreover the function $S(s)$ is continuous on $(-\infty, \bar{s})$. Indeed, if $s_n \to s$, and we note $S_n = S(s_n)$, then S_n is necessarily bounded. Otherwise there will be a subsequence, also denoted

$S_n \to +\infty$. From formula (5.5) , written with $x = S_n$ and the convergence of $H_{s_n}(x)$ to $H_s(x)$, we can check easily that $H_{s_n}(S_n) \to -c - \dfrac{h}{\alpha}$ which contradicts the fact that $H_{s_n}(S_n) = 0$. From that, the limit points of the sequence S_n reduce to $S(s)$. Therefore $S(s_n) \to S(s)$, which proves the continuity.

5.4. FINDING s

We obtain s by solving (3.11) which amounts to solving

$$k = \int_s^{S(s)} H_s(x)dx. \tag{5.21}$$

We have the

Proposition 5.4. *There exists a solution of equation (5.21), in the interval* $(-\infty, \bar{s})$. *We take the smallest value, in case there are several*

Proof. Define the function

$$\gamma(s) = \int_s^{S(s)} H_s(x)dx$$

then, from the continuity properties of $H_s(x)$ and $S(s)$, with respect to s, the function $\gamma(s)$ is continuous. Moreover $\gamma(\bar{s}) = 0$. Since $S(s) \geq s^*$ and $H_s(x) > 0$, for $x < S(s)$ we have

$$\gamma(s) \geq \int_s^{s \vee s^*} H_s(x)dx.$$

We next use (5.15) to state the majoration

$$\begin{aligned} H_s(x) \geq & \frac{2}{\sigma^2\beta^2}[p - \alpha c + \beta(\Phi(c) - \Phi(\frac{p}{\alpha})) - (p+h)\exp(\beta x) \\ & - (p - \alpha c)\exp(-\beta(x - s))] \end{aligned}$$

and thus for $s < s^*$ we obtain

$$\begin{aligned} \gamma(s) \geq & \frac{2}{\sigma^2\beta^2}\big[(s^* - s)(p - \alpha c + \beta(\Phi(c) - \Phi(\frac{p}{\alpha})) \\ & - \frac{p+h}{\beta}\exp(\beta s^*) - \frac{p - \alpha c}{\beta}\big] \end{aligned} \tag{5.22}$$

therefore $\gamma(s) \to +\infty$ as $s \to -\infty$. It follows that equation (5.21) has indeed a solution. This completes the proof. □

We conclude by giving estimates useful as we shall let ϵ go to 0. We keep s as the solution of (5.21). From (5.22) we have

$$k + \frac{4(p - \alpha c)}{\sigma^2\beta^3} \geq \frac{2}{\sigma^2\beta^2}(s^* - s)(p - \alpha c + \beta(\Phi(c) - \Phi(\frac{p}{\alpha}))). \tag{5.23}$$

We get also an estimate from above on $S = S(s)$. Suppose $S > 0$, then writing (5.2) for $x > 0$, yields

$$-\frac{1}{2}\sigma^2 H_s(x) + \alpha H_s(x) + \frac{d}{dx}\Phi_\epsilon(H_s(x) + c) + (h + \alpha c) = 0$$

and integrating between 0 and S, we obtain

$$(h+\alpha c)S+\alpha\int_0^S H_s(x)dx+\Phi_\epsilon(c)-\Phi_\epsilon(H_s(0)+c)-\frac{1}{2}\sigma^2 H_s'(S)+\frac{1}{2}\sigma^2 H_s'(0)=0.$$

We use $\Phi_\epsilon(c)-\Phi_\epsilon(H_s(0)+c)\geq\Phi_\epsilon(c)-\Phi_\epsilon(\frac{p}{\alpha})\geq\Phi(c)-\Phi(\frac{p}{\alpha})$, $\alpha\int_0^S H_s(x)dx>0$, $-\frac{1}{2}\sigma^2 H_s'(S)>0$ to obtain

$$(h+\alpha c)S+\Phi(c)-\Phi(\frac{p}{\alpha})+\frac{1}{2}\sigma^2 H_s'(0)<0$$

and using the estimate (5.16) it follows

$$(h+\alpha c)S\leq\Phi(\frac{p}{\alpha})-\Phi(c)+\frac{\sigma}{\sqrt{2\alpha}}(p+h) \tag{5.24}$$

and this estimate remains valid if $S\leq 0$. In the sequel we shall need the following property

$$0\geq -k+\int_x^y H_s(\xi)d\xi,\ \forall s\leq x\leq y. \tag{5.25}$$

This property is obvious if $x\geq S$, since $H_s(\xi)<0$ for $\xi>x$. So we may assume $x<S$. But then

$$\begin{aligned}-k+\int_x^y H_s(\xi)d\xi &\leq -k+\int_x^{y\wedge S} H_s(\xi)d\xi\\ &\leq -k+\int_s^S H_s(\xi)d\xi=0\end{aligned}$$

which implies (5.25).

5.5. MAIN RESULT

Our objective is to prove the existence of the pair s,$G_s(x)$ satisfying (3.6),(3.11),(3.12), (3.13). We have considered an approximation using the function $\Phi_\epsilon(\lambda)$. We will now let ϵ go to 0. We begin by collecting results. We denote by $s_\epsilon, S_\epsilon, H_\epsilon(x)$ the various quantities introduced above. So we have the following facts

$$s_\epsilon<0,\ H_\epsilon(x) \text{ is in } C^1(s_\epsilon,+\infty), \tag{5.26}$$

$$-\frac{1}{2}\sigma^2(H_\epsilon)(x)+\alpha H_\epsilon(x)+\frac{d}{dx}\Phi_\epsilon(H_\epsilon(x)+c)+(h+\alpha c)1\!\!1_{x>0}-(p-\alpha c)1\!\!1_{x<0}=0,\ x>s_\epsilon,$$

$$H_\epsilon(s_\epsilon)=0,\ H_\epsilon(+\infty)=-c-\frac{h}{\alpha},\ H_\epsilon'(+\infty)=0,$$

$$H_\epsilon(S_\epsilon)=0,\ k=\int_{s_\epsilon}^{S_\epsilon} H_\epsilon(x)dx,$$

$$H_\epsilon(x)>0\forall x\in(s_\epsilon,S_\epsilon),\ H_\epsilon(x)<0,\ \forall x>S_\epsilon,$$

$$H_\epsilon'(x)>0,\ s_\epsilon\leq x<\sigma_\epsilon<S_\epsilon,\ H_\epsilon'(x)<0,\ x>\sigma_\epsilon,$$

and also

$$- k + \int_x^y H_\epsilon(\xi)d\xi \leq 0, \ \forall s_\epsilon \leq x \leq y, \tag{5.27}$$

with the estimates

$$-c - \frac{h}{\alpha} \leq H_\epsilon(x) \leq -c + \frac{p}{\alpha}, \tag{5.28}$$
$$|H'_\epsilon(x)| \leq \frac{\sqrt{2}}{\sigma\sqrt{\alpha}} \max(\sqrt{2}(p - \alpha c), p + h).$$

On the pair s_ϵ,S_ϵ we have the estimates

$$k + \frac{4(p - \alpha c)}{\sigma^2\beta^3} \geq \frac{2}{\sigma^2\beta^2}(s^* - s_\epsilon)(p - \alpha c + \beta(\Phi(c) - \Phi(\frac{p}{\alpha}))), \tag{5.29}$$

$$S_\epsilon > s^*, \ (h + \alpha c)S_\epsilon \leq \Phi(\frac{p}{\alpha}) - \Phi(c) + \frac{\sigma}{\sqrt{2\alpha}}(p + h).$$

Moreover integrating (5.2) between s_ϵ and x, and using previously obtained estimates it follows

$$-\Phi_\epsilon(H^\epsilon(x) + c) \leq (p + h)x^+ + \sigma\sqrt{\frac{2}{\alpha}} \max(p + h, \sqrt{2}(p - \alpha c) - \Phi(c), \tag{5.30}$$
$$\forall x \geq s_\epsilon.$$

We can now state the main result

Theorem 5.2. *Assume (2.2),(2.3),(2.4),(3.3),(5.12). Then there exists a triplet* $(s, S, H(x))$ *satisfying*

$$s < 0, S > \max(s, s^*), H(x) \textit{ is Lipschitz continuous}, \tag{5.31}$$

$$-\frac{1}{2}\sigma^2 H(x) + \alpha H(x) + \frac{d}{dx}\Phi(H(x) + c) + (h + \alpha c)1\!\!1_{x>0} - (p - \alpha c)1\!\!1_{x<0} = 0, \ x > s,$$
$$H(s) = 0, \ H(+\infty) = -c, \ H'(+\infty) = 0,$$

$$H(S) = 0, \ k = \int_s^S H(x)dx,$$
$$H(x) > 0 \forall x \in (s, S), \ H(x) < 0, \ \forall x > S,$$

$$H'(x) > 0, \ s \leq x < \sigma < S, \ H'(x) < 0, \ x > \sigma,$$

and also

$$- k + \int_x^y H(\xi)d\xi \leq 0, \ \forall s \leq x \leq y, \tag{5.32}$$

with the estimates

$$-c - \frac{h}{\alpha} \leq H(x) \leq -c + \frac{p}{\alpha}, \tag{5.33}$$
$$|H'(x)| \leq \frac{\sqrt{2}}{\sigma\sqrt{\alpha}} \max(\sqrt{2}(p - \alpha c), p + h).$$

Proof. By the estimates (5.29) the numbers s_ϵ, S_ϵ remain bounded. So we can extract subsequences such that $s_\epsilon \to s$, $S_\epsilon \to S$. We have $s \le 0$, $S \ge \max(s, s^*)$. Consider the Hilbert space

$$L_1^2(R) = \{\varphi(.)| \int_{-\infty}^{+\infty} \frac{\varphi^2(x)}{1+x^2} dx < +\infty\}.$$

Define next a subset of $L_1^2(R)$ as follows

$$\Gamma = \{\varphi(x)| -\frac{h}{\alpha} - c \le \varphi(x) \le -c + \frac{p}{\alpha},\ |\varphi'(x)| \le \frac{\sqrt{2}}{\sigma\sqrt{\alpha}} \max(\sqrt{2}(p-\alpha c), p+h)\}$$

then Γ is a compact subset of $L_1^2(R)$, and $H_\epsilon(x)$ belongs to Γ. So we can extract a subsequence, still denoted $H_\epsilon(x)$ such that

$$\begin{gathered} H_\epsilon(x) \to H(x), \text{ pointwise}, \\ (H_\epsilon)'(x) \to H'(x) \text{ weakly in } L_1^2(R), \end{gathered} \tag{5.34}$$

and the limit $H(x)$ belongs to Γ. Consider $s-\delta$, $\delta > 0$ fixed. We can find $\epsilon(\delta)$ such that for $\epsilon < \epsilon(\delta)$, $s_\epsilon > s - \delta$. So for $x < s - \delta$, we have $H_\epsilon(x) = 0$, for $\epsilon < \epsilon(\delta)$. Therefore, necessarily $H(x) = 0$, for $x < S - \delta$. But $H(x)$ is continuous and δ is arbitrarily small. This implies $H(x) = 0$, $\forall x \le s$. Consider next $s + \delta$ and $\epsilon(\delta)$ such that for $\epsilon < \epsilon(\delta)$, $s_\epsilon < s + \delta$. We take $x > s + \delta$, the inequality (5.30) is valid for $\epsilon < \epsilon(\delta)$. Passing to the limit, we obtain

$$-\Phi((H(x)+c)^+) \le (p+h)x^+ + \sigma\sqrt{\frac{2}{\alpha}} \max(p+h, \sqrt{2}(p-\alpha c) - \Phi(c).$$

Since the right hand side is finite, necessarily $H(x) + c > 0$, for $x > S + \delta$. Since there is continuity at s, we have necessarily

$$\begin{gathered} H(x) + c \ge 0,\ -\Phi(H(x)+c) \le (p+h)x^+ \\ +\sigma\sqrt{\frac{2}{\alpha}} \max(p+h, \sqrt{2}(p-\alpha c) - \Phi(c),\ \forall x \ge s,\ x < +\infty. \end{gathered} \tag{5.35}$$

Take again $x > s + \delta$, and $\epsilon < \epsilon(\delta)$ so that $s_\epsilon < s + \delta$. Let $\varphi(x)$ be a smooth function with compact support in $[s+\delta, +\infty)$ and $\varphi(x) = 0$, for $x \le s + \delta$. We can write from the differential equation (5.26)

$$\frac{1}{2}\sigma^2 \int_{s+\delta}^{+\infty} H_\epsilon'\varphi' dx + \alpha \int_{s+\delta}^{+\infty} H_\epsilon \varphi dx - \int_{s+\delta}^{+\infty} \Phi_\epsilon(H_\epsilon(x)+c)\varphi' dx$$

$$+(h+c)\int_{s+\delta}^{+\infty} \mathbb{I}_{x>0}\varphi(x)dx - (p-\alpha c)\int_{s+\delta}^{+\infty} \mathbb{I}_{x<0}\varphi(x)dx.$$

Since $H_\epsilon' \to H'$ in $L_1^2(R)$ weakly, we can pass to the limit in this equality. Using the fact that δ is arbitrary, we obtain easily that $H(x)$ satisfies the differential equation (5.31) in the sense of distributions. Since $H(x) + c$ is strictly positive on any compact subset of $[s, +\infty)$, the function $\frac{d}{dx}\Phi(H(x)+c)$ is bounded on any compact subset of $[s, +\infty)$. Therefore also $H(x)$ is bounded on any compact subset of $[s,+\infty)$. It follows that $H(x)$ is C^1 on any compact subset of $[s,+\infty)$. Therefore $H'(s)$ is well defined. It is then easy to check that

$$H'(s)=\lim_{\epsilon\to 0}H'_\epsilon(s_\epsilon).$$

Then integrating the equation (5.31) between s and x, we obtain

$$-\frac{1}{2}\sigma^2H'(x)+\frac{1}{2}\sigma^2H'(s)+\alpha\int_s^x H(\xi)d\xi+\Phi(H(x)+c)-\Phi(c)$$
$$+(h+\alpha c)x^+-(p-\alpha c)(x^-+s)=0.$$

We know that $H(x)+c>0$. Consider $H(N)$ as $N\to+\infty$. Since $H(N)$ is bounded, we can extract a converging subsequence. If $H(N)$ does not tend to $-c$, the sequence $\Phi(H(N)+c)$ remains bounded, but then $\alpha\int_s^N H(\xi)d\xi+(h+\alpha c)N$ remains bounded. This implies

$$\alpha\frac{\int_s^N H(\xi)d\xi}{N}+h+\alpha c\to 0$$

and also $\alpha\lim H(N)+h+\alpha c\to 0$, which is impossible since $\lim H(N)+c\geq 0$. So $H(+\infty)=-c$. Similarly, $H'(N)$ is bounded. If we extract a converging subsequence and the limit is not 0,the function H cannot remain bounded. Therefore $H'(+\infty)=0$. Next $|H_\epsilon(S_\epsilon)-H_\epsilon(S)|\leq C|S_\epsilon-S|\to 0$, hence $H_\epsilon(S)\to 0$, which implies $H(S)=0$. We also check easily that $k=\int_s^S H(x)dx$, $H(x)>0,\forall x\in(s,S)$ and $H(x)<0$, $\forall x>S$. We have also $\sigma_\epsilon\to\sigma$, and by similar reasoning $H'(x)>0$, $\forall x\in(s,\sigma)$ and $H'(x)<0$, $\forall x>\sigma$. We also have (5.32) and (5.33). This concludes the proof. □

We next define

$$G_s(x)=G_s(s)+\int_s^x H_s(\xi)d\xi, \tag{5.36}$$
$$G_s(s)=\frac{1}{\alpha}(\frac{1}{2}\sigma^2H'_s(s)-\Phi(c)+s(p-\alpha c)),$$

then we see easily that it is a solution of (3.6),(3.11),(3.12), (3.13).Since $G_s(x)+cx$ is the value function $u(x)$ defined by (2.11), it is necessarily unique.

6. NUMERICAL WORK

6.1. INTRODUCTION

A computational procedure is presented in this section to find approximations of the solution pair $(s,H_s(x))$ by solving the associated epsilon problem (5.2) repeatedly to obtain the pair $(s_\epsilon,H_\epsilon(x))$ for decreasing epsilon values. The technique used is similar to what was applied for the Base-Stock case in [8]. In particular, in [8], the value for S_ϵ was found by adding the condition $H'_\epsilon(S_\epsilon)=0$ and in the current study the condition $k=\int_{s_\epsilon}^{S(s_\epsilon)}H_\epsilon(x)dx$ is imposed (see also (6.1) below). The specific case is considered in the calculations where the average demand function is defined by (2.5) similarly to [8]. The MATLAB boundary value solver (bvp5c), see [22], was used in the solution process..

6.2. SETUP AND NUMERICAL METHODOLOGY

We restate for convenience both the epsilon and base problems for the special case (2.5) and outline the numerical procedure used for their solutions.

6.2.1. Epsilon Problem

We will seek a solution triplet $s_\epsilon, S_\epsilon, H_\epsilon(x)$ that satisfies the non-linear two point boundary value problem with the extra condition $k = \int_{s_\epsilon}^{S(s_\epsilon)} H_\epsilon(x)dx$:

$$H_\epsilon''(x) = \frac{2}{\sigma^2}(h+\alpha c)\mathbf{1}_{x>0} - \frac{2}{\sigma^2}(p-\alpha c)\mathbf{1}_{x<0} + \frac{2\alpha}{\sigma^2}H_\epsilon(x) + \frac{2}{\sigma^2}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\left(\frac{1}{(H_\epsilon(x)+c)^+ + \epsilon}\right)^{\gamma+1} H_\epsilon'(x) \tag{6.1}$$

$$H_\epsilon(s_\epsilon) = 0,\ H_\epsilon(\infty) = -c - \frac{h}{\alpha},\ k = \int_{s_\epsilon}^{S(s_\epsilon)} H_\epsilon(x)dx,\ H_\epsilon(x) \text{ is in } C^1(s_\epsilon, \infty)$$

6.2.2. Base Problem

Our goal is to find a solution triplet $s, S, H_s(x)$ to the non-linear two point boundary value problem with the extra condition $k = \int_s^{S(s)} H_s(x)dx$:

$$H_s''(x) = \frac{2}{\sigma^2}(h+\alpha c)\mathbf{1}_{x>0} - \frac{2}{\sigma^2}(p-\alpha c)\mathbf{1}_{x<0} + \frac{2\alpha}{\sigma^2}H_s(x) + \frac{2}{\sigma^2}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\left(\frac{1}{H_s(x)+c}\right)^{\gamma+1} H_s'(x) \tag{6.2}$$

$$H_s(s) = 0,\ H_s(\infty) = -c,\ k = \int_s^{S(s)} H_s(x)dx,\ H_s(x) \text{ is Lipschitz continuous}$$

Note that the epsilon problem provides a regularization for the base problem as $x \to \infty$.

6.2.3. Numerical Methodology

We will briefly describe how (6.1) is solved using MATLAB. The solver bvp5c provides a C^1 solution to the general ordinary differential equation, $y'' = f(x, y)$, on the interval $[a, b]$ for given boundary conditions at a and b. In our problem we impose the conditions $H_\epsilon(s_\epsilon) = 0$ and $H_\epsilon(b) = -c - \frac{h}{\alpha}$ for fixed value of epsilon. In Subsection 5.4 we established a criteria for finding s_ϵ. Namely, we consider the condition

$$k = \int_{s_\epsilon}^{S(s_\epsilon)} H_\epsilon(x)dx. \tag{6.3}$$

We know $H_\epsilon(x) > 0$ when $x \in [s_\epsilon, S_\epsilon)$. From (5.13) and Proposition 5.1 we have an upper bound for s_ϵ, i.e., $s_\epsilon < s^*$. We initiate an iteration for s_ϵ by using the initial guess $s' = s^*$ and checking numerically if $k > \int_{s'}^{S(s')} H_s(x)dx$. If yes, we can decrease the value of s' systematically until we satisfy the condition $k = \int_{s'}^{S(s')} H_\epsilon(x)dx$

within tolerance level. We then have a good estimation for the value of s_ϵ. By locating x such that $H_\epsilon(x) = 0$ we have the associated value of S_{s_ϵ}.

Note that he original problem is defined on a semi-infinite interval which we truncate to a finite interval (s_ϵ, b) as follows: We set $b = 10$ and find the value of $s\epsilon$ that satisfies (6.3). We have our solution within tolerance on (s_ϵ, b), then increase the value of b until we see only small changes in the corresponding new $H_\epsilon(x)$ as well as we are able to avoid numerical instabilities.

6.3. RESULTS

For these specific results we used the following constants, $\gamma = 2$, $\alpha = .9, c = 3, p = 10$, $\sigma = 2, h = 1$, which satisfied the assumptions needed for a solution to exist. Namely, we have

$$\frac{p - \alpha c}{\beta} + \Phi(c) - \Phi(\frac{p}{\alpha}) = \frac{2694}{498} > 0.$$

The value of s^* which is the beginning point of our search algorithm is calculated from the equation

$$exp(\beta s^*) = \frac{p - \alpha c + \beta(\Phi(c) - \Phi(\frac{p}{\alpha}))}{p + h}. \tag{6.4}$$

Figure 1, together with Figure 2 and Figure 3 show the structure $H_\epsilon(x)$ for decreasing values of epsilon.

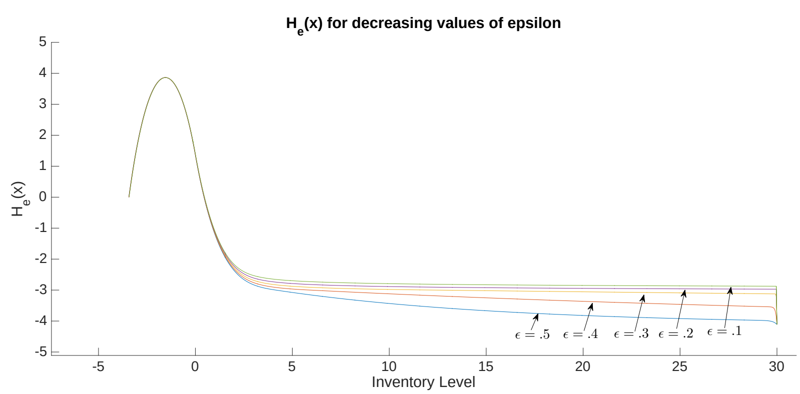

Figure 1. $H_\epsilon(x)$ for decreasing epsilon values

Our numerical results indicate that s_ϵ is increasing when ϵ is decreasing as is evident from Figure 2 (H_ϵ around s_ϵ)..

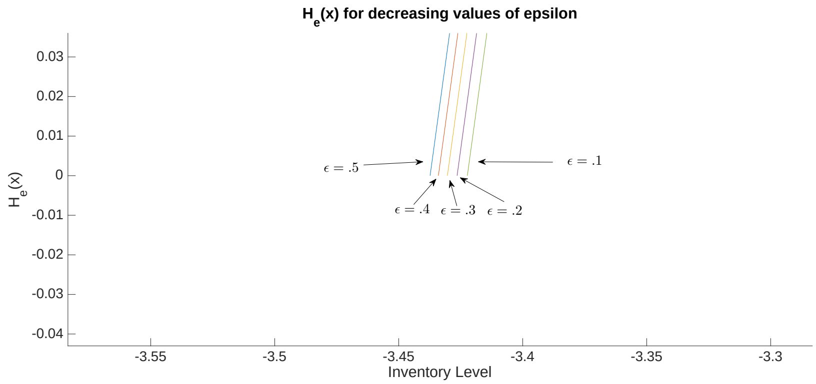

Figure 2. $H_\epsilon(x)$ for decreasing epsilon values around s_ϵ

We saw through theoretical results that we have a drop at $x = \infty$. We impose this boundary condition at $x = b$ and Figure 3 (H_ϵ around b) displays how the drop occurs for decreasing values of epsilon.

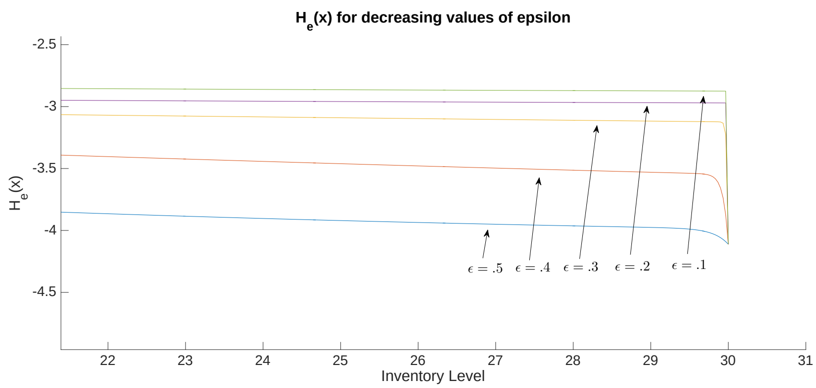

Figure 3. $H_\epsilon(x)$ for decreasing epsilon values around b

We know from [8] that when $k = 0$ we have $s_\epsilon = S_\epsilon$. Table 1 shows (for $\epsilon = 0.2$) for decreasing values of k that s_ϵ and S_ϵ gets closer and closer together.

Table 1. Values of s_ϵ and S_ϵ for decreasing k

k	s_ϵ	S_ϵ
10	-3.42647	0.46053
5	-2.56222	0.31192
3	-2.12918	0.20363
2	-1.86522	0.12492
1	-1.52656	0.01140
0	-0.55901	-0.55901

On Figure 4 we display H_ϵ for decreasing values of k.

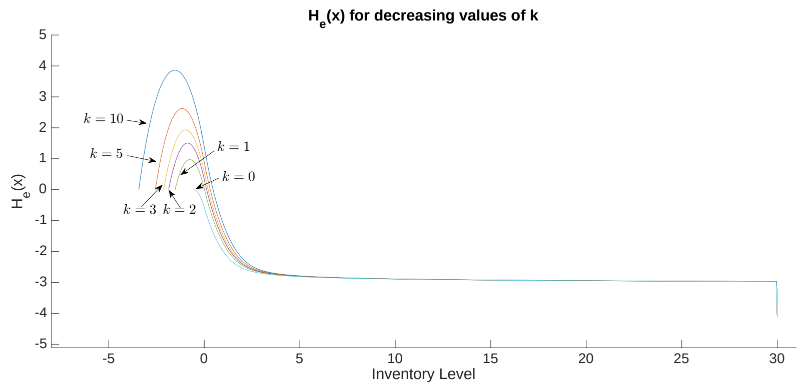

Figure 4. $H_\epsilon(x)$ for decreasing k

Lastly, we discuss how the analysis above enables us to solve the base problem. Using the theoretical convergence of $(s_\epsilon, H_\epsilon(x))$ to $(s, H_s(x))$ the same truncated interval is applied for the base problem. However, for the right boundary condition we cannot impose $H_s(b) = -c$ due to the singularity in (6.2) . Instead we use the information from the epsilon solution, with epsilon small, to acquire a suitable right boundary condition. In particular, we impose the condition $H_s'(b) = -\delta$ with $\delta = 0.1$. We display the solution, $H_s(x)$ on $[s, b]$ on Figure 5.

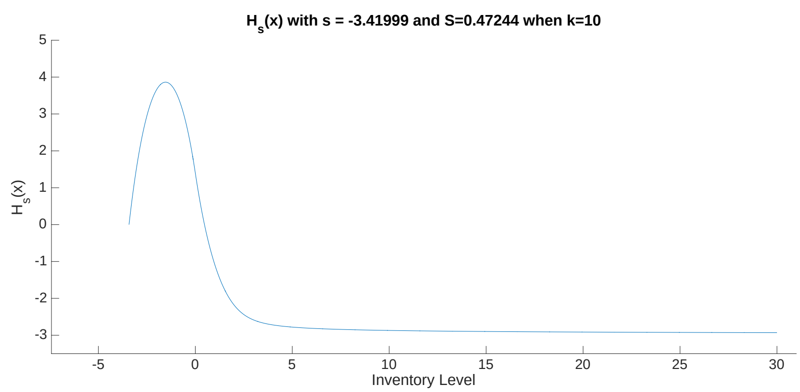

Figure 5. $H_s(x)$ when $k = 10$

We finish by noting that once $H_s(x)$ is known formula (4.2) gives the optimal pricing structure for our case study.

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