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RECENT DEVELOPMENT IN OSCILLATORY PROPERTIES OF CERTAIN DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we summarize some recent oscillation criteria for second order nonlinear differential equations and systems of differential equations, some known oscillation criteria for second order linear differential equations are also involved, and we point out the origin of theses criteria.

Keywords Oscillation, variational principle, linear transformation, interval type, Hamiltonian system.

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1. Introduction

Oscillation is one of the important qualitative properties of differential equations. There are thousands of papers majoring in this field since the foundation work of Sturm. The important contribution of Sturm is the comparison of solutions of second order linear differential equations which is known as Sturm's comparison theorem now. Let $p_i, q_i (i = 1, 2)$, be real-valued continuous functions on the interval $[a, b]$ and let

$$(r_1(t)y')' + q_1(t)y = 0, \tag{1.1}$$

$$(r_2(t)y')' + q_2(t)y = 0 \tag{1.2}$$

be two homogeneous linear second order differential equations in self-adjoint form with $0 < r_2(t) \leq r_1(t)$ and $q_1(t) \leq q_2(t)$.

Let u be a non-trivial solution of (1.1) with successive roots at t_1 and t_2 and let v be a non-trivial solution of (1.2). Then one of the following properties holds: there exists a t in (t_1, t_2) such that $v(t) = 0$; or there exists a λ in $\mathbb{R}$ such that $v(t) = \lambda u(t)$. The first part of the conclusion is due to Sturm [65] in 1836, while the second (alternative) part of the conclusion is due to Picone [58] in 1910 whose simple proof was given using his now famous Picone identity. He showed that for all t with $v(t) \neq 0$, the following identity holds

$$\left(\frac{u}{v}(r_1 u'v - r_2 uv')\right)' = (r_1 - r_2)\, u'^2 + r_2 \left(u' - v'\frac{u}{v}\right)^2 + (q_2 - q_1)\, u^2.$$

In the special case where both equations are identical, one obtains the Sturm separation theorem.

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There are many methods in investigating oscillation for ordinary differential equations, for example, comparison methods, Riccati transformation and integral average method, asymptotic analysis and inequalities methods, symplectic transformation methods (See [1]- [100]).

In this paper, we will major in oscillation of second order differential equations and first order systems of differential equations, oscillation of higher differential equations [20, 22, 24, 68, 91], oscillation of fractional differential equations [4, 8, 9, 19, 59, 60, 71, 86, 100], oscillation of dynamical equations on time scale [16, 23, 36, 55, 76, 85, 87], computation of oscillation solutions [79, 80, 88, 90] are not involved, the readers are advised to see the above papers and references cited therein. As we know, second order differential equations origin many problems in engineers and physics, a Newton mechanics process is characterized by a second order differential equation. Oscillatory property is one of the important properties of second order differential equations and it has been widely used in engineer and mechanics. On the other hand, for a second order differential equation, if it has a solution without any zero in some intervals, then Riccati transformation deduces it to a nonlinear differential equation of first order in the same intervals, then many of properties can be obtained by the Riccati equation. This transformation is of great important in oscillation theory since nonoscillation implies the existence of a solution which preserves its sign ultimately.

A nontrivial solution of the considered equation is called oscillatory if it has arbitrarily large zeros; otherwise, it is said to be non-oscillatory. The equation is said to be oscillatory if all its solutions are oscillatory.

For convenience, we list some function classes which will be used later.

We say that a function $H = H(t,s)$ belongs to a function class $\mathfrak{H}$, denoted by $H \in \mathfrak{H}$ if $H : D \equiv \{(t,s) : t \geq s \geq t_0\} \to \mathbb{R}$ is a continuous function which satisfies $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $t > s \geq t_0$, H has a continuous and non-positive partial derivative on $D_0 \equiv \{(t,s) : t > s \geq t_0\}$ with respect to the second variable. Moreover, let $h : D_0 \to \mathbb{R}$ be a continuous function with

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)}$$

for all $(t,s) \in D_0$.

A function $\Phi = \Phi(t,s,l)$ is said to belonging the function class $\mathfrak{Y}$, denoted by $\Phi \in \mathfrak{Y}$, if $\Phi \in C(E,\mathbb{R})$, where $E = \{(t,s,l) : t_0 \leq l \leq s \leq t < \infty\}$, which satisfies $\Phi(t,t,l) = 0, \Phi(t,l,l) = 0, \Phi(t,s,l) \neq 0$ for $l < s < t$, and has the partial derivative $\partial\Phi/\partial s$ on E such that $\partial\Phi/\partial s$ is locally integrable with respect to s in E and $\partial\Phi(t,s,l)/\partial s = \phi(t,s,l)\Phi(t,s,l)$.

For any given interval $[a,b]$, we define another function class as below

$$D(a,b) = \{u(t)|\ u \text{ and } u'(t) \in C^1[a,b], u(a) = u(b) = 0, u(t) \neq 0 \text{ in } (a,b)\}. \quad (1.3)$$

This paper is organized as follows: after this introduction part, we give some known oscillation criteria for second order linear differential equations in Section 2, and Section 3 is major in oscillation for second order nonlinear differential equations and Section 4 is major in oscillation for second order functional differential equations, in Section 5, we give oscillation for systems of differential equations of first order.

2. Oscillation for linear differential equations

For second order linear differential equations, the classical form is now called Hill equation of the form

$$y''(t) + q(t)y(t) = 0, \tag{2.1}$$

which has great physics background in mechanics systems and it is widely researched in many qualitative fields. It has many generalizations in format of equations. For example, if a damping term is added, we obtained the damped linear differential equation

$$y''(t) + p(t)y' + q(t)y(t) = 0, \tag{2.2}$$

this equation can be deduced to a formally self-adjoint equation which is called normal form of second order linear differential equation

$$(r(t)y'(t))' + q(t)y(t) = 0 \tag{2.3}$$

by multiplying an integral factor. If forcing term is added, we obtain the forced linear differential equation

$$(r(t)y'(t))' + q(t)y(t) = e(t). \tag{2.4}$$

Oscillation (and nonoscillation) for these linear equations and many kinds of generalizations are widely researched in past more than one hundred years. Here we list some known oscillation criteria for equation (2.1):

$$\limsup_{t\to\infty} t^2 q(t) > \frac{1}{4} \text{ (Kenser)}; \tag{2.5}$$

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)ds = \infty \text{ (Fite-Wintner-Leighton [39])}; \tag{2.6}$$

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds = \infty \text{ (Wintner [81])}; \tag{2.7}$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds < \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds \le \infty \text{ (Hartman)}. \tag{2.8}$$

In 1978, Kamenev [31] established a new oscillation criterion of differential equation (2.1), using integral average method, which has the result of Wintner as a particular case. The obtained result in [31] states that the condition

$$\limsup_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t} (t-s)^m q(s)ds = \infty$$

for some integer $m > 1$ is sufficient for the oscillation of (2.1).

In 1989, Philos [57] improved the Kamenev type criterion by defining a new class of functions $\mathfrak{H}$. He obtained the following results for (2.1).

Theorem 2.1. *Let $H \in \mathfrak{H}$. Then equation* (2.1) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} \left[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\right] ds = \infty.$$

Theorem 2.2. *Let $H \in \mathfrak{H}$. Suppose that*

$$0 < \inf_{s \ge t_0} \left[\liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right] \le \infty \tag{2.9}$$

and

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} h^2(t,s)ds < \infty.$$

Moreover, there exists a continuous function $A : [t_0, \infty) \to \mathbb{R}$ such that for every $T \ge t_0$,

$$\limsup_{t \to \infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)q(s) - \frac{1}{4}h^2(t,s) \right] ds \ge A(T),$$

where $A_+(t) = \max\{A(t), 0\}$. Then equation (2.1) is oscillatory provided

$$\int_{t_0}^{\infty} A_+^2(t)dt = \infty. \tag{2.10}$$

However, except Kenser's oscillation criterion, all these oscillation criteria involving integral forms can not determine the oscillation of Euler equation

$$y'' + \frac{\gamma}{t^2} y = 0, \qquad t \ge 1. \tag{2.11}$$

We know (2.11) is oscillation for $\gamma > \frac{1}{4}$, while nonoscillation for $0 < \gamma \le \frac{1}{4}$. In 1995, using a generalized Riccati transformation, Li [41] improved Kamenev oscillation criteria to grantee the oscillation of Euler equation (2.11).

Theorem 2.3. *Let $H \in \mathfrak{H}$. If there exists a C^1 function f such that*

$$\int_{t_0}^{t} a(s)r(s)h^2(t,s)ds < \infty$$

for all $t \ge t_0$ and

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)h^2(t,s) \right] ds = \infty,$$

where $a(t) = \exp\{-2\int^t f(s)ds\}, \psi(t) = a(t)[q(t) + r(t)f^2(t) - (r(t)f(t))']$. Then equation (2.3) is oscillatory.

In 1999, Kong [33] gave an interval criterion for oscillation of (2.3).

Theorem 2.4. *Let $H \in \mathfrak{H}$ with $\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)}$ for some function $h_1 \in D_0$. Then equation (2.3) is oscillatory if for all $T \ge t_0$, there exist $a, b, c \in \mathbb{R}$ with $T \le a < c < b$ such that*

$$\begin{aligned} &\frac{1}{H(c,a)} \int_a^c H(s,a)q(s)ds + \frac{1}{H(b,c)} \int_c^b H(b,s)q(s)ds \\ \ge &\frac{1}{4} \left[\frac{1}{H(c,a)} \int_a^c r(s)h_1^2(t,s)ds + \frac{1}{H(b,c)} \int_c^b r(s)h^2(b,s)ds \right] ds. \end{aligned}$$

In 2004, Sun et. al. [70] gave a new type oscillation criterion by defining a new type of kernel function $\mathfrak{Y}$, which is essential a product $H(t,s)H(s,l)$ for a kernel $H(t,s)$ of Philos type.

Theorem 2.5. *Equation* (2.3) *is oscillatory provided for each* $l \geq t_0$*, there exists a function* $\Phi \in \mathfrak{Y}$ *such that*

$$\lim_{t\to\infty} \int_l^t \Phi^2(t,s,l)[q(s) - r(s)\phi^2(t,s,l)]ds > 0.$$

As we know, oscillation is a property of interval type. We just need to pay attention to the intervals containing zero of a solution, i.e., if there exists a sequence of intervals $[a_i, b_i]$ of $[t_0, \infty)$, $a_i \to \infty$, such that for each i, there exists a solution of (2.3) which has at least one zero in $[a_i, b_i]$, then every solution of (2.3) is oscillatory, no matter how "bad" the equation (2.3) is on the remaining parts of $[t_0, \infty)$. Based on these facts, the classical Sturm comparison reflects the Leighton's variational property of the equation, which can be used to obtain oscillation criterion. Here we list the Leighton variation principle as follows:

Theorem 2.6 ([40]). *Let* $u \in D(a,b)$ *such that the quadratic functional satisfying*

$$\int_a^b (qu^2 - ru'^2)ds > 0.$$

Then a solution of equation (2.3) *with* $y(a) = 0$ *must have a zero in* (a,b).

In paper [32], Komkov gave a generalized Leighton's variational principle.

Theorem 2.7. *Suppose that there exist a* C^1 *function* $u(t)$ *defined on* $[s_1, t_1]$ *and a function* $G(u)$ *such that* $G(u(t))$ *is not constant on* $[s_1, t_1]$, $G(u(s_1)) = G(u(t_1)) = 0$, $g(u) = G'(u)$ *is continuous,*

$$\int_{s_1}^{t_1} [q(t)G(u(t)) - r(t)(u'(t))^2]dt > 0, \tag{2.12}$$

and $(g(u(t)))^2 \leq 4G(u(t))$ *for* $t \in [s_1, t_1]$. *Then each solution of equation* (2.3) *vanishes at least once on* $[s_1, t_1]$.

Based on the Leighton's variation principle and the "oscillatory interval" of forcing term, Wong [82] give a new type kind of oscillation criterion for the forced linear differential equation (2.4)

Theorem 2.8. *Suppose that for any* $T \geq t_0$*, there exist* $T \leq s_1 < t_1 \leq s_2 < t_2$ *such that*

$$(-1)^i e(t) \geq 0, \quad \textit{for} \;\; t \in [a_i, b_i]. \tag{2.13}$$

If there exists $u \in D(s_i, t_i)$ *such that*

$$Q_i(u) := \int_{s_i}^{t_i} [q(t)u^2(t) - r(t)(u'(t))^2]dt > 0, \quad i = 1, 2. \tag{2.14}$$

Then equation (2.4) *is oscillatory.*

This result is remarkable since it lays a road between variational principle and oscillation criteria of interval type using Riccati transformation. The detailed discussion of this theorem and its generalizations can be found in Kong and Pasic [34].

3. Oscillation for nonlinear differential equations

In recent years, there are a lot of papers majoring in oscillation criteria for some nonlinear differential equations, most of them are modeled the known results for linear differential equations list above, we note also that a summarized paper is published by Kong and Pasic [34]. We will give some further results in these years.

A natural generalization of Wong's criteria is oscillation for nonhomogeneous half-linear differential equations of the form

$$\big(r(t)|y'(t)|^{\alpha-1}y'(t)\big)' + q(t)|y(t)|^{\alpha-1}y(t) = e(t),\ t \geq t_0, \tag{3.1}$$

where α is a positive constant, $r, q, e \in C([t_0,\infty),\mathbb{R})$ with $r(t) > 0$, and second order forced quasi-linear differential equation

$$\big(r(t)|y'(t)|^{\alpha-1}y'(t)\big)' + q(t)|y(t)|^{\beta-1}y(t) = e(t), t \geq t_0, \tag{3.2}$$

where $r, q, e \in C([t_0,\infty),\mathbb{R})$ with $r(t) > 0$ and $0 < \alpha \leq \beta$ are constants. We note that when $\beta = \alpha$, (3.2) reduces to (3.1).

In 2002, Li and Cheng [44] obtained an oscillation criterion for (3.1) as follows:

Theorem 3.1. *Suppose that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that (2.13) holds. Let $D(s_i,t_i)$ be defined as (1.3) for $i = 1,2$. If there exist $H \in D(s_i,t_i)$ and a positive, nondecreasing function $\phi \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\int_{s_i}^{t_i} H^2(t)\phi(t)q(t)dt > \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \int_{s_i}^{t_i} \frac{r(t)\phi(t)}{|H(t)|^{\alpha-1}} \left(2|H'(t)| + |H(t)|\frac{\phi'}{\phi}\right)^{\alpha+1} dt \tag{3.3}$$

for $i = 1,2$. Then equation (3.1) is oscillatory.

However, the inequality (3.3) is no relation to the $(\alpha+1)-$degree functional

$$\int_{s_i}^{t_i} \left[q(t)H^{\alpha+1}(t) - r(t)|H'(t)|^{\alpha+1}\right] dt,$$

which is a natural generalization of quadratic functional to half-linear differential equation, and Theorem 3.1 cannot be applied when $\alpha > 1$, since $|H(t)|^{\alpha-1}$ is the denominator of the fraction on the right-side integral of (2.14), and $H(s_i) = H(t_i) = 0$. Hence, an improvement of Theorem 3.1 which conquered the deficiency to (3.2) was given as below (see [95]).

Theorem 3.2. *Assume that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that (2.13) holds. Let $D(s_i,t_i)$ be defined as (1.3) for $i = 1,2$. If there exist $H \in D(s_i,t_i)$ and a positive, nondecreasing function $\phi \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$Q_i^\phi(H) := \int_{s_i}^{t_i} \phi(t)\left[Q_e(t)H^{\alpha+1}(t) - r(t)\left(|H'(t)| + \frac{H(t)\phi'(t)}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right] dt > 0 \tag{3.4}$$

for $i = 1,2$. Then equation (3.2) is oscillatory, where

$$Q_e(t) = \alpha^{-\alpha/\beta}\beta(\beta-\alpha)^{(\alpha-\beta)/\beta}[q(t)]^{\alpha/\beta}|e(t)|^{(\beta-\alpha)/\beta} \tag{3.5}$$

with the convention that $0^0 = 1$.

The above obtained results can reduces to Wong's result when particular functions are selected. Some other generalized oscillation results align this line are generalization of the equations (3.1) and (3.2) to more general cases. For example, oscillation criteria of a damped half-linear differential equation of the form

$$\big(r(t)|y'(t)|^{\alpha-1}y'(t)\big)' + p(t)|y'(t)|^{\alpha-1}y'(t) + q(t)|y(t)|^{\alpha-1}y(t) = e(t),\ t \ge t_0,$$

and a more generalized nonlinear differential equation

$$\big(r(t)\Psi(y(t))|y'(t)|^{\alpha-1}y'(t)\big)' + q(t)f(y(t)) = e(t),\ t \ge t_0, \tag{3.6}$$

under some mild hypotheses such as

$$\text{(S1)}\quad 0 < \Psi(u) \le M, \text{and } f'(u) \ge K\,|f(u)|^{\frac{\beta-1}{\beta}} > 0 \text{ for } u \ne 0; \tag{3.7}$$

$$\text{(S2)}\quad \frac{f'(u)}{[\Psi(u)|f(u)|^{\beta-1}]^{1/\beta}} \ge \gamma > 0 \text{ for } u \ne 0; \tag{3.8}$$

$$\text{(S3)}\quad 0 < \Psi(u) \le M, \text{and } \frac{f(u)}{|u|^{\beta}\operatorname{sgn} u} \ge \delta > 0 \text{ for } u \ne 0, \tag{3.9}$$

here, $M, K > 0$, $0 < \alpha \le \beta$ and $\gamma, \delta > 0$ are constants can be obtained similarly (see [10, 11, 15, 27, 29, 54, 62, 63, 93] and references cited therein). Meanwhile, using Kokmov variational principle, oscillation criteria similar to Theorem 3.2 for second order forced nonlinear differential equations can be obtained easily.

Among the nonlinear differential equation (3.2), a particular kind of them are called mixed nonlinear differential equation of the form

$$\big(r(t)|y'(t)|^{\alpha-1}y'(t)\big)' + p(t)|y(t)|^{\alpha-1}y(t) + \sum_{j=1}^{m} q_j(t)|y(t)|^{\beta_j-1}y(t) = e(t), t \ge t_0, \tag{3.10}$$

where r, p, q_j $(1 \le j \le m), e \in C([t_0,\infty),\mathbb{R})$ with $r(t) > 0$ and $0 < \alpha < \beta_1 < \beta_2 < \cdots < \beta_m$ are real numbers, p, q_j $(1 \le j \le m)$ and e might change signs. The obtained results are a unification both of (3.1) and (3.2) (see [26, 94, 99] for details). We list the main results of Zheng, Wang and Han [99].

Theorem 3.3 ([99, Theorem 2.2])**.** *Assume that for any $T \ge t_0$, there exist $T \le s_1 < t_1 \le s_2 < t_2$ such that* (2.13) *holds. Let $D(s_i, t_i)$ be defined as* (1.3) *for $i = 1, 2$. If there exist $H \in D(s_i, t_i)$ and a positive, nondecreasing function $\phi \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\int_{s_i}^{t_i} \phi(t)\left[\left(p(t) + \sum_{j=1}^{m} Q_j(t)\right) H^{\alpha+1}(t) - r(t)\left(|H'(t)| + \frac{|H(t)\phi'(t)|}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right] dt > 0 \tag{3.11}$$

for $i = 1, 2$. Then equation (3.10) *is oscillatory, where*

$$Q_j(t) = \alpha^{-\alpha/\beta_j}\beta_j[m(\beta_j-\alpha)]^{(\alpha-\beta_j)/\beta_j}[q_j(t)]^{\alpha/\beta_j}|e(t)|^{(\beta_j-\alpha)/\beta_j}, 1 \le j \le m$$

with the convention that $0^0 = 1$.

Li, Rogovchenko and Tang [51] considered another differential equation of the form

$$\big(r(t)|y'(t)|^{\alpha-1}y'(t)\big)' + p(t)|y'(t)|^{\alpha-1}y'(t) + q(t)f(y(t)) = 0, t \ge t_0 \tag{3.12}$$

with $f(v)/v^{\gamma} \geq \mu$ for some positive number μ. We note that the results they obtained are the same as those of (3.6) and (3.10) essentially since they only added a damping term in the nonlinear equation.

For the nonlinear differential with mixed nonlinearity, we see that they are finite numbers of terms, and we know that integral is a particular kind of sum. Hence these equations are generalized to differential equations with integral terms. In 2011, Sun and Kong [66] considered the following differential equation

$$(r(t)y'(t))' + q(t)y(t) + \int_0^b g(t,s)|y(t)|^{\alpha(s)}\mathrm{sgn}y(t)\mathrm{d}\xi(s) = e(t). \tag{3.13}$$

They obtained another kind of oscillation criterion.

Theorem 3.4. *Suppose that for any $T > t_0$, there exist nontrivial subintervals $[a_1, b_1]$ and $[a_2, b_2]$ of $[T, +\infty)$ such that for $i = 1, 2$, (2.13) holds and*

$$g(t,s) \geq 0, \quad \textit{for} \quad (t,s) \in [a_i, b_i] \times [0, b).$$

Let $\delta \in \left(\left(\int_a^b \alpha^{-1}(s)\mathrm{d}\xi(s) \right) \left(\int_a^b \mathrm{d}\xi(s) \right)^{-1}, \left(\int_0^a \alpha^{-1}(s)\mathrm{d}\xi(s) \right) \left(\int_0^a \mathrm{d}\xi(s) \right)^{-1} \right)$ and $\eta \in L_\xi[0, b)$ satisfy $\int_0^b \alpha(s)\eta(s)\mathrm{d}\xi(s) = 1$, $\int_0^b \eta(s)\mathrm{d}\xi(s) = \delta$. We further assume that for $i = 1, 2$, there exists a function $u_i \in D(a_i, b_i)$ such that

$$\int_{a_i}^{b_i} [Q(t)u_i^2(t) - r(t)u_i'^2(t)]\mathrm{d}t > 0,$$

where

$$Q(t) = q(t) + \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} \exp\left(\int_0^b \eta(s) \ln \frac{g(t,s)}{\eta(s)} \mathrm{d}\xi(s) \right)$$

with the convention that $\ln 0 = -\infty$, $e^{-\infty} = 0$ and $(1-\delta)^{1-\delta} = 1$ for $\delta = 1$. Then equation (3.13) is oscillatory.

Liu and Meng [49] considered a more generalized differential equation of the form

$$(r(t)y'(t))' + q(t)y(t) + \int_a^b g(t,s)|y(t)|^{\gamma(t,s)+1-\beta(t)}\mathrm{sgn}y(t)\mathrm{d}\xi(s) = e(t), \tag{3.14}$$

they gave the following theorems:

Theorem 3.5. *Suppose that for any $T > t_0$, there exist nontrivial subintervals $[a_1, b_1]$ and $[a_2, b_2]$ of $[T, +\infty)$ such that for $i = 1, 2$, (2.13) holds and*

$$g(t,s) \geq 0, \quad \textit{for} \quad (t,s) \in [a_i, b_i] \times [a, b).$$

Let $\eta : [t_0, +\infty) \times [a, b) \to (0, +\infty)$ be a function in $L_\xi[a, b)$ such that

$$\int_a^b \gamma(t,s)\eta(t,s)\mathrm{d}\xi(s) = \beta^2(t), \qquad (t,s) \in [t_0, +\infty) \times [a, b),$$

and

$$\int_a^b \eta(t,s)\mathrm{d}\xi(s) = \beta(t), \qquad (t,s) \in [t_0, +\infty) \times [a, b).$$

We further assume that for $i = 1, 2$, there exists a function $v_i \in D(a_i, b_i)$ such that

$$\int_{a_i}^{b_i} [Q(t)v_i^2(t) - r(t)v_i'^2(t)]\mathrm{d}t > 0,$$

where

$$Q(t) = q(t) + \exp\left(\frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln\frac{\beta(t)g(t,s)}{\eta(t,s)}\mathrm{d}\xi(s)\right) \tag{3.15}$$

with the convention that $\ln 0 = -\infty$ and $e^{-\infty} = 0$. Then equation (3.14) is oscillatory.

Theorem 3.6. *Suppose that for any $T > 0$, there exist nontrivial subintervals $[a_1, b_1]$ and $[a_2, b_2]$ of $[T, +\infty)$ such that (2.13) holds for $i = 1, 2$. Let $\eta : [t_0, +\infty) \times [a, b) \to (0, +\infty)$ be defined as in Theorem 3.5. We further assume that for $i = 1, 2$, there exist a constant $c_i \in (a_i, b_i)$ and a function $H \in \mathbb{H}$ such that*

$$\frac{1}{H(c_i,a_i)}\int_{a_i}^{c_i}\Big[Q(t)H(t,a_i) - \frac{r(t)h_1^2(t,a_i)}{4}\Big]\mathrm{d}t + \frac{1}{H(b_i,c_i)}\int_{c_i}^{b_i}\Big[Q(t)H(b_i,t) - \frac{r(t)h_2^2(b_i,t)}{4}\Big]\mathrm{d}t > 0,$$

where $Q(t)$ is defined by (3.15). Then equation (3.14) is oscillatory.

Hassan and Kong [25, 26] considered the more general half-linear differential equation of the form

$$(r(t)|y'(t)|^{\alpha-1}y'(t))' + q_0(t)|y(t)|^{\alpha-1}y(t) + \int_0^b g(t,s)|y(t)|^{\alpha(s)-1}y(t)\mathrm{sgn}y(t)\mathrm{d}\xi(s) = e(t),$$

similar results were obtained as Theorem 3.4.

4. Oscillation for functional differential equations

Functional differential equations have similar oscillatory properties as linear equations in most cases. Under the hypothesis

$$\lim_{t\to\infty} R_1(t) = \lim_{t\to\infty}\int_{t_0}^t \frac{ds}{r(s)} = \infty.$$

Agarawal, Shieh and Yeh [3] obtained some oscillation criteria for the delay differential equation of the form

$$(r(t)y'(t))' + q(t)y(\tau(t)) = 0, \quad t \geq t_0; \tag{4.1}$$

while as

$$\lim_{t\to\infty} R_2(t) = \lim_{t\to\infty}\int_{t_0}^t \frac{ds}{r(s)^{1/\alpha}} = \infty, \tag{4.2}$$

oscillation criteria were obtained in papers [10, 14] for the half-linear differential equation with delay

$$(r(t)|y'(t)|^{\alpha-1}y'(t))' + q(t)|y(\tau(t))|^{\alpha-1}y(\tau(t)) = 0, \quad t \geq t_0. \tag{4.3}$$

Here we list the main results of Dzurina and Stavroulakis [14].

Theorem 4.1. *Suppose that $\alpha \geq 1$ and (4.2) holds. If for some $k \in (0,1)$,*

$$\int^{\infty} \left[R_2(\tau(t))^{\alpha} q(t) - \frac{\alpha \tau'(t)}{4k R_2(\tau(t)) r(\tau(t))^{1/\alpha}} \right] ds = \infty,$$

then equation (4.3) is oscillatory.

Theorem 4.2. *Suppose that $0 < \alpha \leq 1$ and (4.2) holds. If*

$$\int^{\infty} \left[R_2(\tau(t))^{\alpha} q(t) - \frac{\alpha \tau'(t)}{4 R_2(\tau(t))^{2-\alpha} r(\tau(t))^{2/\alpha - 1} \tilde{q}(t)} \right] ds = \infty,$$

where $\tilde{q}(t) = \left(\frac{1}{r(\tau(t))} \int_t^{\infty} q(s) ds \right)^{\frac{1-\alpha}{\alpha}}$, then equation (4.3) is oscillatory.

Using variation of Young inequality, an improvement of Theorem 4.2 is listed as follows (see [14]):

Theorem 4.3. *Suppose that (4.2) holds. If*

$$\int^{\infty} \left[R_2(\tau(t))^{\alpha} q(t) - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{\tau'(t)}{R_2(\tau(t)) r(\tau(t))^{1/\alpha}} \right] ds = \infty,$$

then equation (4.3) is oscillatory.

Based on oscillation criteria for linear and half-linear differential equations, there are many new oscillation criteria for more generalized functional differential equations. In paper [43], Li and Chen extended (4.1) to the delay differential equation of the form

$$(r(t)x'(t))' + q(t)x(t-\tau) + \sum_{i=1}^{n} q_i(t)|x(t-\tau)|^{\alpha_i} sgnx(t-\tau) = e(t), \quad t \geq 0,$$

oscillation of interval type similar to Kong's result was obtained, Liu and Bai [7] obtained oscillation criteria for the more general equation of the form

$$(r(t)x'(t))' + \sum_{i=1}^{n} p_i(t)x(t-\tau_i) + \sum_{i=1}^{n} q_i(t)|x(t-\tau_i)|^{\alpha_i} sgnx(t-\tau_i) = e(t), \quad t \geq 0, \tag{4.4}$$

where $r(t),\ p_i(t),\ q_i(t),\ e(t)$ are continuous functions defined on $[0,\infty)$ and $r(t) > 0$, $p'(t) \geq 0$, $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} \cdots > \alpha_n > 0$.

Theorem 4.4 ([7, Theorem 2.1])**.** *If for any $T \geq 0$, there exist a_1, b_1, c_1, a_2, b_2 and c_2 such that $T \leq a_1 < c_1 < b_1 \leq a_2 < c_2 < b_2$, and*

$$\begin{cases} p_i(t) \geq 0, \quad t \in [a_1 - \tau_i, b_1] \cup [a_2 - \tau_i, b_2],\ i = 1, 2, \cdots, n, \\ q_i(t) \geq 0, \quad t \in [a_1 - \tau_i, b_1] \cup [a_2 - \tau_i, b_2],\ i = 1, 2, \cdots, n, \\ e(t) \leq 0, \ \ t \in [a_1 - \tau_i, b_1], \quad e(t) \geq 0, \ \ t \in [a_2 - \tau_i, b_2]. \end{cases}$$

Let $D(a_j, b_j)$ be defined as (1.3), and there exist $H_j \in D(a_j, b_j)$ such that

$$\begin{aligned} &\frac{1}{H_j(c_j, a_j)} \int_{a_j}^{c_j} (Q_j(s) H_j(s, a_j) - r(s) h_{j1}^2(s, a_j)) ds \\ &+ \frac{1}{H_j(b_j, c_j)} \int_{c_j}^{b_j} (Q_j(s) H_j(b_j, s) - r(s) h_{j2}^2(b_j, s)) ds > 0, \end{aligned}$$

for $j = 1, 2$, *where* $\eta_1, \eta_2, \cdots, \eta_n$ *are positive constants,* $\eta_0 = 1 - \sum_{i=1}^{n} \eta_i$,

$$Q_j(t) = \sum_{i=1}^{n} p_i(t) \left(\frac{t - a_j}{t - a_j + \tau_i} \right) + (\eta_0^{-1} |e(t)|)^{\eta_0} \prod_{i=1}^{n} (\eta_i^{-1} q_i(t))^{\eta_i} \left(\frac{t - a_j}{t - a_j + \tau_i} \right)^{\alpha_i \eta_i},$$

then equation (4.4) *is oscillatory.*

Neutral delay differential equations draw many researchers' attention in recent years. In 2009, Ye and Xu [98] considered the following neutral differential equation

$$\left(r(t)\Psi(y(t)) \left| Z'(t) \right|^{\alpha-1} Z'(t) \right)' + q(t) f(y(\sigma(t))) = 0, \tag{4.5}$$

where $Z(t) = y(t) + p(t)y(\tau(t)), \alpha > 0, \tau(t) \leq t, \sigma(t) \leq t$. Under the hypothesis

$$\frac{f(u)}{|u|^{\alpha-1} u} \geq K, \Psi(u) \leq L^{-1} \text{ for } u \neq 0,$$

equation (4.5) is a generalization of half-linear differential equation of neutral type.

Theorem 4.5 ([98, Theorem 2.1]). *Let* (4.2) *hold. If there exists* $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ *such that*

$$\int^{\infty} \left[\rho Q - \frac{1}{LK(\alpha+1)^{\alpha+1}} \frac{(\rho'_+(t))^{\alpha+1} r(\sigma(t))}{\rho^{\alpha}(t)(\sigma'(t))^{\alpha}} \right] dt = \infty, \tag{4.6}$$

where $Q(t) = q(t)(1 - p(\sigma(t)))^{\alpha}$, *then equation* (4.5) *is oscillatory.*

Using method similar to Sun et. al [70], Liu and Bai [42] obtained an oscillation criterion of interval type for (4.5) with $\Psi \equiv 1$. They defined an operator $T[\cdot; l, t]$ by

$$T[g; l, t] = \int_{l}^{t} \phi(t, s, l) g(s) ds, \tag{4.7}$$

for $t \geq s \geq l \geq t_0$ and $g(s) \in C^1[t_0, \infty)$, where $\phi \in \mathfrak{Y}$.

Theorem 4.6 ([42, Theorem 2.1]). *Let* (4.2) *hold. Assume that there exist functions* $\phi \in \mathfrak{Y}$, $\rho(t) \in C^1([t_0, \infty), \mathbb{R}^+)$, *such that*

$$\limsup_{t \to \infty} T \left[Q_1(s) - \frac{r(\tau(s))\rho(s)}{(\tau'(s))^m} \left(\frac{\varphi(s) + \frac{\rho'(s)}{\rho(s)}}{\alpha+1} \right)^{\alpha+1} ; l, t \right] > 0,$$

where $Q_1(s) = \mu\rho(s)q(s)(1 - p(\tau(s)))^{\alpha}$, *the operator* T *is defined by* (4.7) *and* $\varphi = \varphi(t, s, l)$ *is defined by* $\partial\phi(t, s, l)/\partial s = \varphi(t, s, l)\phi(t, s, l)$. *Then equation* (4.5) *with* $\Psi \equiv 1$ *is oscillatory.*

Liu, Meng and Liu [50] considered the following nonlinear functional differential equation

$$\left(r(t) \left| Z'(t) \right|^{\alpha-1} Z'(t) \right)' + q(t) \left| y(\sigma(t)) \right|^{\beta-1} y(\sigma(t)) = 0, \tag{4.8}$$

where $Z(t) = y(t) + p(t)y(\tau(t)), \beta \geq \alpha > 0, \tau(t) \leq t, \sigma(t) \leq t$. They obtained the following theorem:

Theorem 4.7 ([50, Theorem 2.1]). *Let* (4.2) *hold,* $r'(t) \geqslant 0$ *and* $\sigma(t) > 0$. *If there exists a function* $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that for any positive number* M *and for some* $\theta \geqslant 1$,

$$\int^{\infty}\left(\rho(t)\overline{p}(t) - \frac{\theta\rho'^2(t)r^{\beta/\alpha}(\sigma(t))}{4\beta(\frac{\sigma(t)}{2})^{\beta-1}\sigma'(t)M^{\frac{\alpha-\beta}{\alpha}}\rho(t)}\right)\mathrm{d}t = \infty, \tag{4.9}$$

where $\overline{p}(t) = q(t)[1-p(\sigma(t))]^\beta$, *then equation* (4.8) *is oscillatory.*

We note that when $\beta = \alpha$, Theorem 4.7 reduces to Theorem 4.5. Similar results can be seen in Sun, Li, Han and Li [67] for the equation

$$\left(r(t)\left|Z'(t)\right|^{\alpha-1}Z'(t)\right)' + f(t, y(\sigma(t))) = 0$$

under $f(t,y)\mathrm{sgn}(y) \geq q(t)|y|^\alpha$ for some positive function $q(t)$, Li and Rogovchenko [52] for the equation

$$\left(r(t)\left|Z'(t)\right|^{\alpha-1}Z'(t)\right)' + q(t)f(y(t), y(\sigma(t))) = 0$$

under $f(x,y)/y^\alpha \geq K$ for some positive constant K, Tunc and Grace [75] for the equation with a damped term

$$\left(r(t)\left|Z'(t)\right|^{\alpha-1}Z'(t)\right)' + h(t)\left|Z'(t)\right|^{\alpha-1}Z'(t) + q(t)f(y(\sigma(t))) = 0.$$

All these three equations origin from half-linear differential equation of neutral type.

A more general case is considered in paper [2] by Arul and Shobha

$$(r(t)z'(t))' + q(t)y(\sigma(t)) = 0, \quad t \geq t_0; \tag{4.10}$$

where $z(t) = y(t) + a(t)y(t-\tau) + b(t)y(t+\delta)$. They showed that

$$\int_{t_0}^{\infty} Q_0(s)ds = \infty$$

is sufficient for oscillatory of equation (4.10), where $Q_0(t) = \min\{q(t), q(t-\tau), q(t+\delta)\}$. Moreover, they obtained an oscillation criterion of interval type.

Theorem 4.8 ([2, Theorem 2.3]). *Let* $\sigma(t) \leq t - \tau$. *Assume that there exist functions* $\phi \in Y$, $\rho(t) \in C^1([t_0,\infty),\mathbb{R}^+)$, *such that*

$$\limsup_{t\to\infty} T\left[\rho(s)Q_0(s) - \frac{(1+a+b)\left(\varphi(s)+\frac{\rho'(s)}{\rho(s)}\right)^2}{4\sigma'(s)}r(\sigma(s))\rho(s); l, t\right] > 0, \tag{2.1}$$

where the operator T *is defined by* (4.7) *and* $\varphi = \varphi(t,s,l)$ *is defined by* $\partial\phi(t,s,l)/\partial s = \varphi(t,s,l)\phi(t,s,l)$. *Then equation* (4.10) *is oscillatory.*

5. Oscillation for system of differential equations

In this section, we give some known results in oscillation criteria for linear system of differential equations mainly.

As a direct generalization, the equation (2.3) is generalized to the following system of differential equations

$$(P(t)Y'(t))' + Q(t)Y(t) = 0, \tag{5.1}$$

where $P(t) = P^*(t) > 0$, $Q(t) = Q^*(t)$, $Y(t)$ are $n \times n$ matrices of real valued continuous functions on the interval $[t_0, \infty)$. By M^* we mean the conjugate transpose of the matrix M, for any $n \times n$ Hermitian matrix M, it eigenvalues are real numbers, we always denote by $\lambda_1[M] \geq \lambda_2[M] \geq \cdots \geq \lambda_n[M]$. The trace of M is denoted by $\mathrm{tr}(M)$ and $\mathrm{tr}(M) = \sum_{k=1}^{n} \lambda_k(M)$.

A solution $Y(t)$ of (5.1) is said to be nontrivial solution if $\det Y(t) \neq 0$ for at least one point $t \in [t_0, \infty)$. In this paper, we say a nontrivial solution $Y(t)$ of (5.1) is prepared if for $t \in [t_0, \infty)$,

$$Y^*(t)P(t)Y'(t) - (Y^*(t))'P(t)Y(t) \equiv 0, \tag{5.2}$$

i.e., $Y^*(t)P(t)Y'(t)$ is symmetric. A prepared solution $Y(t)$ of (5.1) is called to be oscillatory, if $\det Y(t)$ has arbitrarily large zeros on $t \in [t_0, \infty)$. System (5.1) is said to be oscillatory on $[t_0, \infty)$ if every nontrivial prepared solution is oscillatory.

Etgen and Pawlowski [18] showed that system (5.1) is oscillatory provided the scalar linear differential equation

$$\Big(g[P(t)]y'\Big)' + g\Big[Q(t)\Big]y = 0$$

is oscillatory, where $g : \mathbb{R}^{n\times n} \to \mathbb{R}^+$ is a positive linear functional. So the very large number of well-known oscillation criteria for the above scalar linear differential equation can be used to determine associated oscillation criteria for system (5.1).

Other oscillation criteria related to (5.1) is using a nonlinear function such as the maximal eigenvalue functional $\lambda_1(\cdot)$ or negative-preserving functional, a nonlinear (and possibly discontinuous) functional $q : \mathbb{S} \to \mathbb{R}$ with $q(A) \leq 0$ whenever $A \leq 0$ is called negativity-preserving, the class of all such negativity-preserving functionals on $\mathbb{S}$ being denoted by $\mathcal{N}(\mathbb{S})$. The negativity-preserving functionals $\mathcal{N}(\mathbb{S})$ contain most known functionals used in oscillation, for example, $q(A) = \lambda_1(A)$; $q(A) = tr(A - P)$ where P is positive semi-definite and fixed, are of negativity-preserving functionals. In addition,

$$q(A) = \frac{\lambda_1(A)}{1 - \lambda_1(A)}; \qquad q(A) = a_{ii}, 1 \leq i \leq n,$$

are also negativity-preserving functionals. We also note that any positive linear functional is negativity-preserving. Thus, functionals in the class $\mathcal{N}(\mathbb{S})$ make up all the functionals being used in the current study of matrix oscillation theory. here we give a Kamenev type oscillation criteria for system (5.1).

Theorem 5.1 ([17]). *Let $H \in \mathfrak{H}$. Then system* (5.1) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \lambda_1 \left\{ \int_{t_0}^{t} \left[H(t,s)Q(s) - \frac{1}{4} h^2(t,s)P(s) \right] ds \right\} = \infty.$$

Another type of oscillation criteria are related to matrix differential system of the form

$$(P(t)Y'(t))' + R(t)Y'(t) + Q(t)Y(t) = 0, \tag{5.3}$$

which is modeled by the damped second order linear differential equations

$$(p(t)y')' + r(t)y + q(t)y = 0. \tag{5.4}$$

These results are obtained by new defined prepared solutions which satisfy both (5.2) and

$$Y^*(t)R(t)Y'(t) - (Y^*(t))'R(t)Y(t) \equiv 0. \tag{5.5}$$

In paper [72], by the function class $\mathfrak{Y}$, the authors defined a linear operator

$$T_a(D, E, F; r, t) = \int_r^t a(s)\left[\Phi^2(t, s, r)D(s) + \Phi(t, s, r)\Phi'_s(t, s, r)E(s) - \Phi'^2_s(t, s, r)F(s)\right] ds$$

for any $D(t), E(t)$ and $F(t)$ being $n \times n$ matrices of real valued continuous functions on the interval $[t_0, \infty)$, where a(t) is a positive and continuously differentiable function on $[t_0, \infty)$, $\Phi(t, s, l) \in \mathfrak{Y}$. They obtained the oscillation for system (5.3).

Theorem 5.2. *If there exist $\Phi \in \mathfrak{Y}$ and $f \in C^1[t_0, \infty)$ such that for each $r \geq t_0$,*

$$\limsup_{t\to\infty} \lambda_1[T_a(D, R, P; r, t)] > 0,$$

where $a(s) = \exp(-2\int^s f(s)ds), D(s) = (M - RP^{-1}R/4)(s)$, $M(s) = (Q + (f^2P) - (fP)' - fR)(s)$, then system (5.3) is oscillation.

Using Sturm's comparison theorem and a positive linear functions, Liu and Meng [48] obtained the following theorems for system (5.3).

Theorem 5.3. *Suppose equation (2.3) is oscillatory. If there exist a function $f(t) \in C^1[t_0, \infty)$ and a positive linear functional g such that*

$$\text{(i)} \quad g\Big[M(t) - q(t)I_n - \Big(\frac{\rho R P^{-1} R}{4}\Big)(t)\Big] \geqslant 0,$$

$$\text{(ii)} \quad g\Big[p(t)I_n - \rho P(t)\Big] \geqslant 0,$$

on $[t_1, \infty)$ for some $t_1 \geqslant t_0$, where $\rho(t) = \exp\Big(-2\int^t f(s)\mathrm{d}s\Big)$, and $M(t) = \rho(t)\Big[Q(t) + (f^2P)(t) - (fP)'(t) - \Big(\frac{R}{2}\Big)'(t)\Big]$, then system (5.3) is oscillatory.

Another system of differential equations is linear Hamiltonian system. The oscillatory properties for the linear Hamiltonian vector system

$$\begin{cases} x' = A(t)x + B(t)u, \\ u' = C(t)x - A^*(t)u, \quad t \geq t_0, \end{cases} \tag{5.6}$$

are investigated together with its corresponding matrix system

$$\begin{cases} X' = A(t)X + B(t)U, \\ U' = C(t)X - A^*(t)U, \quad t \geq t_0, \end{cases} \tag{5.7}$$

where $A(t), B(t), C(t)$ are real $n \times n$ matrix-valued functions, B, C are Hermitian, B is positive definite, $x, u \in \mathbb{R}^n$ and $X, U \in \mathbb{R}^{n\times n}$. For any two solutions $(X_1(t), U_1(t))$

and $(X_2(t), U_2(t))$ of system (5.7), the Wronski matrix $X_1^*(t)U_2(t) - U_1^*(t)X_2(t)$ is a constant matrix. In particular, for any solution $(X(t), U(t))$ of system (5.7), $X^*(t)U(t) - U^*(t)X(t)$ is a constant matrix.

A solution $(X(t), U(t))$ of system (5.7) is said to be nontrivial, if $\det X(t) \neq 0$ is fulfilled at least one $t \geq t_0$. A nontrivial solution $(X(t), U(t))$ of system (5.7) is said to be conjoined (prepared) if $X^*(t)U(t) - U^*(t)X(t) \equiv 0$, $t \geq t_0$. A conjoined solution $(X(t), U(t))$ of (5.7) is said to be a conjoined basis of (5.6) (or (5.7)) if the rank of the $2n \times n$ matrix $\begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ is n.

Two distinct points a, b in $[t_0, \infty)$ are said to be (mutually) conjugate with respect to (5.6) if there exists a solution $(x(t), u(t))$ of (5.6) with $x(a) = x(b) = 0$ and $x(t) \neq 0$ on the subinterval with end-points a and b. The system (5.6) is said to be disconjugate on a subinterval J of $[t_0, \infty)$ if no two distinct points are conjugate. If (5.6) is disconjugate on J and $(X(t), U(t))$ is the conjoined basis of (5.7) satisfying $X(a) = 0, U(a) = I$, the identity $n \times n$ matrix, $a \in J$, then $\det X(t) \neq 0$ for $t \in J$. A conjoined basis $(X(t), U(t))$ of system (5.7) is said to be oscillatory in case the determinant of $X(t)$ vanishes on $[T, \infty)$ for each $T \geq t_0$.

Let $\Phi(t)$ be a fundamental matrix for the linear system $v' = A(t)v$. The pair $(A(t), B(t))$ is called be controllable if the row of $\Phi^{-1}(t)B(t)$ are linearly independent over any subinterval of $[t_0, \infty)$. This definition is coincided with the following fact: if for any solution $(x(t), u(t))$ of (5.6), one have that $x(t) \equiv 0$ on any nondegenerate subinterval $J \subseteq [t_0, \infty)$ implies $x = u \equiv 0$ on $[t_0, \infty)$. Since $B(t) > 0$, we have the pair $(A(t), B(t))$ is controllable, suppose there exists an oscillatory conjoined basis of system (5.7), then by Sturm's separation theorem, we know that each conjoined basis of system (5.7) is oscillatory, so system (5.6) (or (5.7)) is called oscillatory. Now the definition of oscillation agrees with the non-disconjugacy of system (5.6) (or (5.7)) on any neighborhood of $+\infty$.

When considering the oscillatory properties for linear Hamiltonian system (5.6), one of the most useful methods is the so-called Reid's roundabout theorem. It gives an equivalence among disconjugacy; the existence of solution for Riccati differential equation and the positivity of corresponding quadratic functional on an given interval $[a, b]$.

Theorem 5.4 (Reid's Roundabout Theorem)**.** *The following statements are equivalent:*
(i) System (5.6) *is disconjugate in the interval* $[a, b]$.
(ii) The quadratic functional

$$\mathcal{F}(x, u; a, b) = \int_a^b [u^T(t)B(t)u(t) + x^T(t)C(t)x(t)]dt$$

is positive for every nontrivial admissible pairs x, u, *that is* $x' = A(t)x + B(t)u$ *and* $x(a) = 0 = x(b)$.
(iii) The solution (X, U) *of system* (5.7) *given by the initial condition* $X(a) = 0$, $U(a) = I$ *satisfies* $\det X(t) \neq 0$ *for* $t \in (a, b]$.
(iv) There exists a conjoined basis (X, U) *of system* (5.7) *such that* $X(t)$ *in nonsingular for* $t \in [a, b]$.
(v) There exists a symmetric matrix $Q(t)$ *which for* $t \in [a, b]$ *solves the Riccati*

matrix differential equation

$$Q'(t) - C(t) + A^T(t)Q(t) + Q(t)A(t) + Q(t)B(t)Q(t) = 0$$

related to system (5.7) *by the substitution* $Q = UX^{-1}$.

Using Theorem 5.4, one can obtain oscillation for linear Hamiltonian system (5.6) as follows: if for arbitrarily large $a < b$, there exists an admissible pair x, u such that $\mathcal{F}(x, u; a, b) < 0$, then system (5.6) is oscillatory. However, since x and u are n-dimensional vectors, it is difficult for one to find such an admissible pair x, u. It looks easier for us to obtain oscillation by scalar function, rather than vector function.

Some of oscillation criteria involve the fundamental matrix $\Phi(t)$ for the linear system $v' = A(t)v$. For example, using the fundamental matrix $\Phi(t)$, Kamenev type, Philos type and interval type oscillation criteria for Hamiltonian system (5.7) can be obtained in a similar manner. These results are generalized by Li, Meng and Zheng [46] by introducing a new parameter β.

Theorem 5.5. *Suppose that there exist three positive and real-valued functions* $\phi,\ \theta,\ k \in C^1[t_0, \infty)$*, such that, for some* $\beta \geq 1$*, and for some* $H \in \mathcal{W}$*,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\lambda_1\left\{\int_{t_0}^{t}\left[H(t,s)k(s)C_2(s) - \frac{\beta}{4}h^2(t,s)B_2^{-1}(s)\right]ds\right\} = \infty, \qquad (5.8)$$

where $C_2(t) = -\left[\Phi^*\frac{\theta}{\phi}\left\{C + \frac{\alpha}{\theta}(B^{-1}A + A^*B^{-1}) + (\frac{\alpha}{\theta}B^{-1})' - \frac{\alpha^2}{\theta^2}B^{-1}\right\}\Phi\right](t)$ *and* $B_2(t) = \frac{\phi(t)}{\theta(t)}\Phi^{-1}(t)B(t)\Phi^{*-1}(t)$*. Then system (1.2) is oscillatory.*

In paper [13], the authors obtain oscillation criteria with the fundamental matrix $\Phi(t)$ as follows:

Theorem 5.6. *System* (5.7) *is oscillatory provided for each* $T \geq t_0$*, there exist* $a, b \in \mathbb{R}$*,* $T \leq a < b$ *and* $u \in C^1[a,b]$ *satisfying* $u(a) = u(b) = 0$*, such that*

$$\lambda_1\left[\int_a^b \left(u^2(s)C_2(s) - (u'(s))^2B_2(s)\right)ds\right] = \infty,$$

where $C_2(s) = -\Phi^*(t)C(t)\Phi(t)$ *and* $B_2(s) = \Phi^*(t)B^{-1}(t)\Phi(t)$.

However, the use of the fundamental matrix $\Phi(t)$ eliminates the applications of these criteria, because such a system cannot be solved for variation of $A(t)$ in general. Moreover, by using the transformation

$$\begin{pmatrix}\overline{X}\\ \overline{U}\end{pmatrix} = \begin{pmatrix}\Phi^{-1}(t) & 0\\ 0 & \Phi^*(t)\end{pmatrix}\begin{pmatrix}X\\ U\end{pmatrix}, \qquad (5.9)$$

we can transforming (5.7) into the following Hamiltonian system

$$\begin{cases}\overline{X}' = \Phi^*(t)B(t)\Phi(t)\overline{U},\\ \overline{U}' = \Phi^{-1}(t)C(t)\Phi^{*-1}(t)\overline{X}, \quad t \geq t_0.\end{cases} \qquad (5.10)$$

We can easily transform (5.10) into (5.1) with $P(t) = \Phi^{-1}(t)B^{-1}(t)\Phi^{*-1}(t)$, $Q(t) = -\Phi^{-1}(t)C(t)\Phi^{*-1}(t)$. Thus these criteria are similar to that of system (5.1).

Similar to [70], by multiplying a ternary function $\phi(t,s,r) = (t-s)^2(s-r)^{2\xi}$, one can obtain oscillation for system (5.7). Here we list a result as follows.

Theorem 5.7. *System (5.7) is oscillatory provided for some $\xi > \frac{1}{2}$ and for each $r \ge t_0$,*

$$\limsup_{t\to\infty} \frac{1}{t^{2\xi+1}} \lambda_1 \left\{ \int_r^t (t-s)^2(s-r)^{2\xi} \left(D_0(t) + \frac{\xi t - (\xi+1)s + r}{(t-s)(s-r)} K(s) \right) ds \right\} > \frac{\xi}{(2\xi-1)(2\xi+1)},$$

where $D_0(s) = (-C - A^*B^{-1}A)(s)$ and $K(s) = (A^*B^{-1} + B^{-1}A)(s)$.

Dube and Minagrelli [13, 53] obtained some oscillation criteria for system (5.7) of interval type. Here we lists two of these theorems.

Theorem 5.8. *Suppose that there exists $f(t) \in C([t_0,\infty);\mathbb{R})$ such that fB^{-1} is differentiable, and $\alpha(t) = \exp\{-2\int^t f(s)ds\}$. If there exists $q \in \mathcal{N}(\mathbb{S})$, and for each $T \ge t_0$, there exist $a, b \in \mathbb{R}$, $T \le a < b$, $u \in Z[a,b]$ such that*

$$q\left\{ \int_a^b \left[u^2 D - \alpha\left(uA^* - u'I\right)B^{-1}\left(uA - u'I\right)\right](t)dt \right\} > 0, \tag{5.11}$$

where $D(t) = \{\alpha[-C - 2fK + f^2B^{-1} - (fB^{-1})']\}(t)$. Then system (5.7) is oscillatory.

Theorem 5.9. *Suppose that there exists $a(t) \in C([t_0,\infty);\mathbb{R}^+)$ such that $a(t)B^{-1}(t) \le I$ and $f(t)B^{-1}(t)$ is differentiable, where $f(t) = -\frac{a'(t)}{a(t)}$. If for each $r \ge t_0$ and for some $\mu, \nu > 1$,*

$$\limsup_{t\to\infty} \frac{1}{t^{\mu+\nu-1}} \lambda_1 \left\{ \int_r^t (t-s)^{\mu}(s-r)^{\nu} \left(D_1(s) + \frac{\nu t - (\mu+\nu)s + \mu r}{(t-s)(s-r)} K(s) \right) ds \right\} > \mu\nu(\mu+\nu-2)\frac{\Gamma(\mu-1)\Gamma(\nu-1)}{4\Gamma(\mu+\nu)}, \tag{5.12}$$

where where $D_1(t) = D(t) - (aA^*B^{-1}A)(t)$, $K(t) = \frac{a(t)}{2}(A^*B^{-1} + B^{-1}A)(t)$, and $D(t) = \{a[-C - 2fK + f^2B^{-1} - (fB^{-1})']\}(t)$. Then system (5.7) is oscillatory.

In 2009, Li, Meng and Zheng [47] obtained a new type of oscillation criteria for system (5.7) using a parameter $\beta \ge 1$.

Theorem 5.10 ([47, Theorem 2.1]). *Suppose there exist a function $f(t) \in C^1[t_0,\infty)$ and a positive linear functional g on $\mathcal{R}$, for some $\beta \ge 1$, such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t g\Big[-H(t,s)\left(C_1 + A^*B_1^{-1}A + (B_1^{-1}A)'\right)(s) - \frac{\beta}{4}h^2(t,s)B_1^{-1}(s)\Big] ds = \infty,$$

where

$$B_1(t) = a^{-1}(t)B(t), \qquad a(t) = \exp\left\{-2\int^t f(s)ds\right\}$$

and

$$C_1(t) = a(t)\left\{C(t) + f(t)[B^{-1}A + A^*B^{-1}](t) + [f(t)B^{-1}(t)]' - f^2(t)B^{-1}(t)\right\},$$

then system (5.7) *is oscillatory.*

Using positive linear functional, Yang, Mathsen and Zhu [96] obtained the following oscillation criteria for system (5.7).

Theorem 5.11. *Assume there exist a positive function $v \in \mathrm{C}^1([a,\infty),(0,\infty))$ and a positive linear functional satisfying g on the collections of real $n \times n$ matrix, such that*

$$\lim_{t\to\infty} \int_a^t \frac{1}{v(s)g(B^{-1}(s))} ds = \infty,$$

and

$$\lim_{t\to\infty} g(J_0(t)) = \infty,$$

where

$$\begin{aligned} J_0(t) = &\int_a^t \left[-v(A^*B^{-1}A + C) + \frac{v'}{2}(A^*B^{-1} + B^{-1}A) - \frac{v'^2}{4v}B^{-1}\right](s)ds \\ &- v(t)B^{-1}(t)A(t) + \frac{v'(t)}{2}B^{-1}(t). \end{aligned}$$

Then system (5.7) *is oscillatory.*

Since $J_0(t)$ is not symmetric, by defining a symmetric matrix $J(t) = (J_0(t) + J_0^*(t))/2$, one can obtain Hatman type oscillation criterion for Hamiltonian system (5.7) based on the above mentioned oscillation criterion and Hartman type oscillation criterion (2.8) of equation (2.3).

Theorem 5.12. *Assume that there exists a positive function $v \in \mathrm{C}^1([a,\infty),(0,\infty))$ satisfying $v(t) \le \lambda_n(B(t))$. Moreover,*

$$\liminf_{t\to\infty} \frac{1}{t}\int_a^t \mathrm{tr}J(s)ds = -\infty. \tag{5.13}$$

Then system (5.7) *is oscillatory if*

$$\lim \operatorname{approx} \sup_{t\to\infty} \lambda_n(J(t)) > -\infty; \tag{5.14}$$

Al-Dosary, Abdullah and Hussein [1] obtained some new oscillation criteria for the linear Hamiltonian matrix system of the form

$$\begin{cases} U' = A(t)U + B(t)V, \\ V' = C(t)U + (\mu I - A^*(t))V, \end{cases} \tag{5.15}$$

where $A(x), B(x) = B^*(x), B(x)$ is either positive definite or negative definite, $C(x) = C^*(x)$ are $n \times n$ matrices of real valued continuous functions on the interval $J = [t_0, \infty)$. A more generalized Hamiltonian system of the form

$$\begin{cases} U' = (A(t) - \lambda(t)I)U + B(t)V, \\ V' = C(t)U + (\mu(t)I - A^*(t))V, \quad t \ge t_0. \end{cases} \tag{5.16}$$

is considered in [69]. In fact, this system can be transformed into Hamiltonian system (5.7) by the transformation introduced by Shao, Meng and Zheng [69] as follows:

Lemma 5.1 ([69, Lemma 2.1]). *The transformation*

$$T = \begin{pmatrix} \alpha e^{\int \lambda} I & 0 \\ \left(e^{-\int \mu}\right) Q & \left(e^{-\int \mu}\right) I \end{pmatrix} \tag{5.17}$$

makes system (5.16) *to the Hamiltonian system* (5.7) *where*

$$\alpha = \begin{cases} 1, & if\ B\ is\ positive\ definite, \\ -1, & if\ B\ is\ negative\ definite, \end{cases}$$

and I is the identity matrix, 0 is the zero matrix, Q is any constant nonzero symmetric $n \times n$ matrix, $A_1 = A - BQ$, $B_1 = \alpha e^{\int(\lambda+\mu)} B$, $C_1 = \alpha e^{-\int(\lambda+\mu)}(QA - QBQ + C - (\lambda+\mu)Q + A^*Q)$ and $D_1 = A^* - QB$.

By introducing a new constant matrix Q, the authors in [69] obtained a new interval oscillation criterion for system (5.16), which improved Theorem 5.8.

Theorem 5.13. *Suppose that there exists $f(t) \in C([t_0, \infty); \mathbf{R})$ such that fB^{-1} is differentiable, and $\alpha(t) = \exp\{-2\int^t f(s)ds\}$. If there exists $q \in \mathcal{N}(\mathbf{S})$, and for each $T \geq t_0$, there exist $a, b \in \mathbf{R}$, $T \leq a < b$, $z \in Z[a,b]$ such that*

$$q\left\{\int_a^b \Big[(z^2 D)(t) - \alpha(t) e^{-\int_{t_0}^t (\lambda+\mu)(s)ds} \big[(z(A^* - QB) - z'I)B^{-1}(z(A - BQ) - z'I)\big](t)\Big] dt\right\} > 0,$$

where

$$D(t) = \Big\{\alpha \Big[-e^{-\int(\lambda+\mu)}(QA - QBQ + C - (\lambda+\mu)Q + A^*Q) - 2fK + e^{-\int(\lambda+\mu)} f^2 B^{-1} - \left(e^{-\int(\lambda+\mu)} f B^{-1}\right)'\Big]\Big\}(t),$$

$$K(t) = e^{-\int_{t_0}^t (\lambda+\mu)(s)ds}[\left(A^* B^{-1} + B^{-1} A\right)(t) - 2Q]$$

and Q is any constant nonzero symmetric $n \times n$ matrix, then system (5.7) *is oscillatory.*

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