Journal of Applied Analysis and Computation
Volume 9, Number 4, August 2019, 1193–1203

Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20180084

SEVERAL TYPES OF PERIODIC WAVE SOLUTIONS AND THEIR RELATIONS OF A FUJIMOTO–WATANABE EQUATION*

Lijuan Shi[1] and Zhenshu Wen[1,†]

Abstract In this paper, we study periodic wave solutions of a Fujimoto–Watanabe equation by exploiting the bifurcation method of dynamical systems. We obtain all possible bifurcations of phase portraits of the system in different regions of the parametric space, and then give the sufficient conditions to guarantee the existence of several types of periodic wave solutions. What's more, we present their exact expressions and reveal their inside relations as well as their relations with solitary wave solutions.

Keywords Fujimoto–Watanabe equation, dynamics, periodic wave solutions, solitary wave solutions, relations.

MSC(2010) 34C60, 35Q51, 35C05, 35C07, 35C08.

1. Introduction

Fujimoto and Watanabe [6] obtained a complete list of the third-order polynomial evolution equations of not normal type with nontrivial Lie-Bäcklund symmetries, called Fujimoto–Watanabe equations, among which there are two equations, respectively,

$$u_t = u^3 u_{xxx} + 3u^2 u_x u_{xx} + 3\alpha u^2 u_x, \tag{1.1}$$

$$u_t = u^3 u_{xxx} + 3u^2 u_x u_{xx} + 4\alpha u^3 u_x. \tag{1.2}$$

Sakovich [13] showed that Eq.(1.1) can be connected with the famous KdV equation. As the advent of Eq.(1.1), its solutions received considerate attention. In 2010, by using an irrational equation method, Du [5] obtained some implicit expressions of traveling wave solutions to Eq.(1.1). Further, in 2010, Liu [10] gave the classifications of traveling wave solutions of Eq.(1.1) through the method of complete discrimination system. Recently, Pan et al. [12] studied cuspons and periodic cuspons to Eq.(1.1) by exploiting the bifurcation method of dynamical systems. However, there has been little study about Eq.(1.2). Sakovich [14] related Qiao equation with the well-known modified KdV equation through a special form

†the corresponding author. Email address:wenzhenshu@hqu.edu.cn(Z. Wen)
1Fujian Province University Key Laboratory of Computational Science, School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China
*This work is partially supported by the National Natural Science Foundation of China under Grant No. 11701191, Program for Innovative Research Team in Science and Technology in Fujian Province University, Quanzhou High-Level Talents Support Plan under Grant 2017ZT012, and Subsidized Project for Cultivating Postgraduates' Innovative Ability in Scientific Research of Huaqiao University.

Eq.(1.2), and obtain the smooth soliton solutions of Qiao equation. In addition, for Eq.(1.2), note that its nonlinear structure is more complicated than that of Eq.(1.1) by substituting the term u^2u_x by u^3u_x. Hence, one may wonder how about the traveling wave solutions of Eq.(1.2) and their dynamics. Driven by these motivations, in this paper, we study the dynamics of periodic wave solutions to Eq.(1.2) and their relations from the perspective of the theory of dynamical systems [1–4, 7–9, 11, 15–28]. We obtain the sufficient conditions to guarantee the existence of several types of periodic wave solutions under different parameters conditions. Furthermore, we present their exact expressions and reveal their inside relations as well as their relations with solitary wave solutions.

2. Bifurcations of Phase Portraits

To present the bifurcations of phase portraits corresponding to Eq.(1.2), we first transform Eq.(1.2) into a planar system by substituting $u(x,t) = \varphi(\xi)$ with $\xi = x - ct$ into Eq.(1.2), and then obtain

$$-c\varphi' = (\varphi^3\varphi'')' + 4\alpha\varphi^3\varphi', \tag{2.1}$$

where the prime stands for the derivative with respect to ξ.

Integrating straightforwardly Eq.(2.1), it follows

$$\varphi^3\varphi'' = -\alpha\varphi^4 - c\varphi + g, \tag{2.2}$$

where g is the integral constant.

Letting $y = \varphi'$, we obtain a three-parameter planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{-\alpha\varphi^4 - c\varphi + g}{\varphi^3}, \end{cases} \tag{2.3}$$

with first integral

$$H(\varphi, y) = \frac{y^2}{2} + \frac{\alpha\varphi^2}{2} - \frac{c}{\varphi} + \frac{g}{2\varphi^2}. \tag{2.4}$$

Transformed by $d\xi = \varphi^3 d\tau$, system (2.3) becomes a regular system

$$\begin{cases} \frac{d\varphi}{d\tau} = \varphi^3 y, \\ \frac{dy}{d\tau} = -\alpha\varphi^4 - c\varphi + g. \end{cases} \tag{2.5}$$

System (2.5) has the same level curves as system (2.3). Therefore, we can analyze the phase portraits of system (2.3) from those of system (2.5).

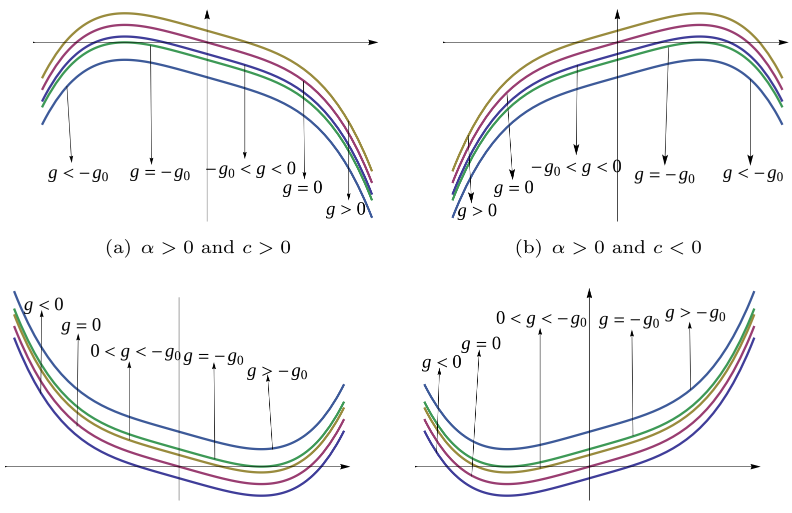
To study the singular points and their properties of system (2.5), let

$$f(\varphi) = -\alpha\varphi^4 - c\varphi + g. \tag{2.6}$$

We can easily obtain the graphics of the function $f(\varphi)$ in Figure 1 in corresponding regions of the parametric space, where $g_0 = \frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$.

Let $\lambda(\varphi^*, y)$ be the characteristic value of the linearized system of system (2.5) at the singular point (φ^*, y). We easily get

$$\lambda^2(\varphi^*, 0) = (\varphi^*)^3 f'(\varphi^*). \tag{2.7}$$



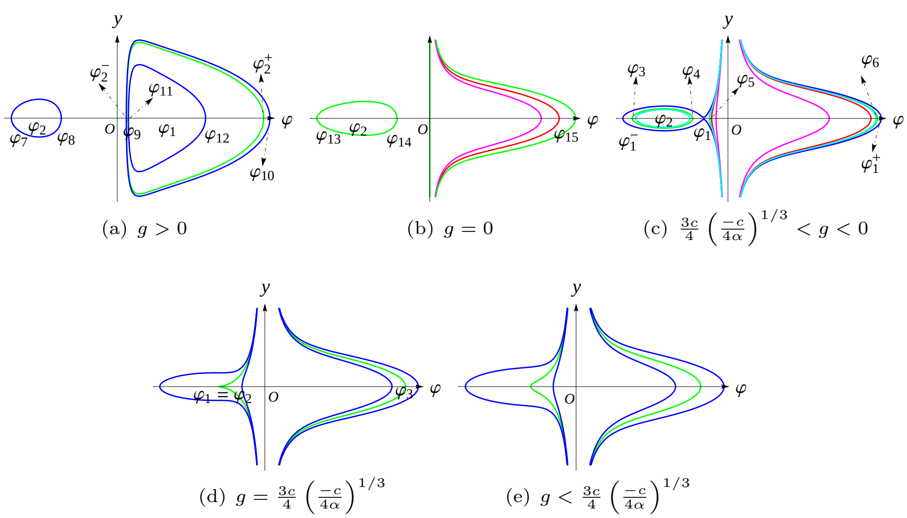
(a) $\alpha > 0$ and $c > 0$ (b) $\alpha > 0$ and $c < 0$

(c) $\alpha < 0$ and $c > 0$ (d) $\alpha < 0$ and $c < 0$

Figure 1. The graphics of the function $f(\varphi)$.

From (2.7) , we see that the sign of $f'(\varphi^*)$ and the relative position of the singular point $(\varphi^*, 0)$ with respect to the singular line $l : \varphi = 0$, can determine the dynamical properties (saddle, center and degenerate singular point) of the singular point $(\varphi^*, 0)$ according to the theory of planar dynamical systems.

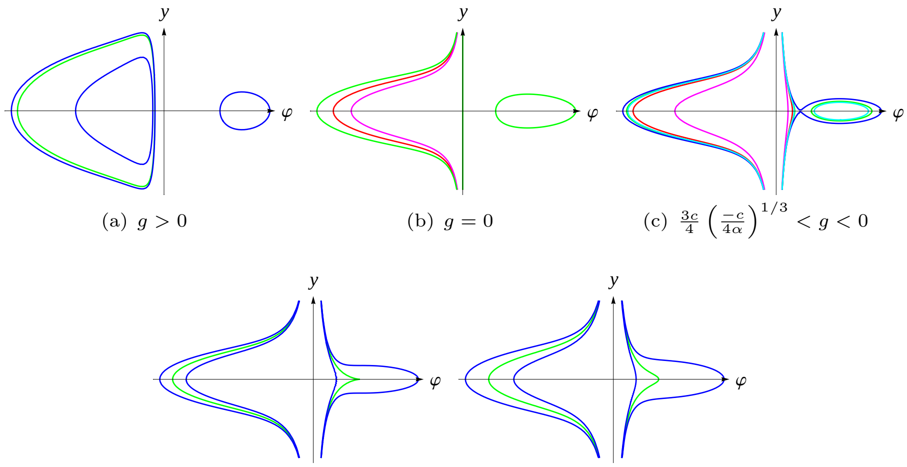
Therefore, we obtain all possible phase portraits of bifurcations of system (2.3) in Figures 2, 3, 4 and 5.



(a) $g > 0$ (b) $g = 0$ (c) $\frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3} < g < 0$

(d) $g = \frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$ (e) $g < \frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$

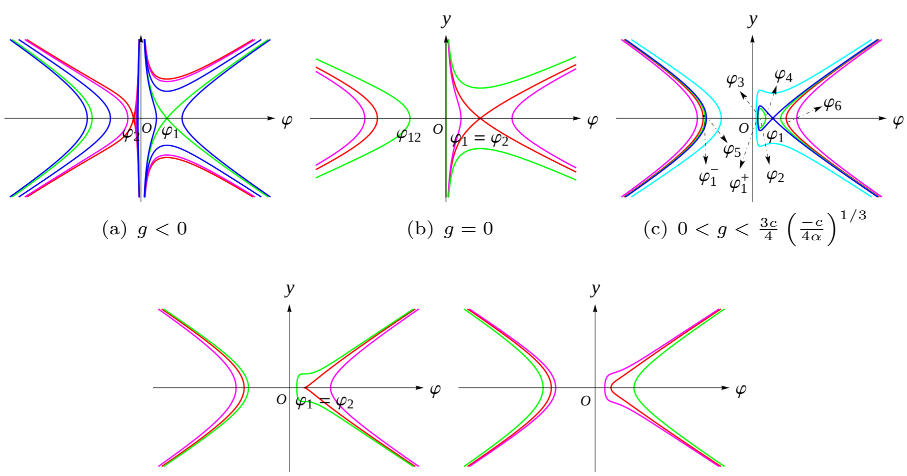
Figure 2. The phase portraits of system (2.3) when $\alpha > 0$ and $c > 0$.



(a) $g > 0$ (b) $g = 0$ (c) $\frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3} < g < 0$

(d) $g = \frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$ (e) $g < \frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$

Figure 3. The phase portraits of system (2.3) when $\alpha > 0$ and $c < 0$.



(a) $g < 0$ (b) $g = 0$ (c) $0 < g < \frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$

(d) $g = \frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$ (e) $g > \frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$

Figure 4. The phase portraits of system (2.3) when $\alpha < 0$ and $c > 0$.

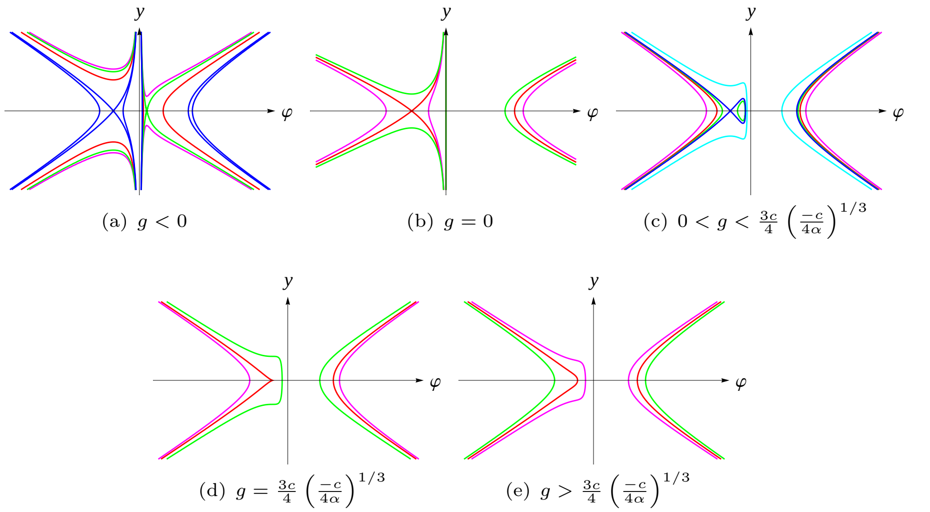


Figure 5. The phase portraits of system (2.3) when $\alpha<0$ and $c<0$.

3. Main results and the theoretic derivations of main results

To state conveniently, we introduce some marks, $h_1 = H(\varphi_1, 0)$, $h_2 = H(\varphi_2, 0)$, $\rho = (\varphi_1 - \varphi_1^-)(\varphi_1^+ - \varphi_1)$, $\sigma = (\varphi_2^+ - \varphi_2)(\varphi_2^- - \varphi_2)$, $\chi = (\varphi - \varphi_1^+)(\varphi - \varphi_1^-)$, where $\varphi_1, \varphi_2, \varphi_1^\pm, \varphi_2^\pm$ are given in the phase portraits and the proofs of theorems 3.1 and 3.2.

Our main results will be stated in the following theorems, and the proofs follow. Note that we only focus our attention on the two cases when $\alpha>0$, $c>0$ and $\alpha<0$, $c>0$ about the main results, because the other two cases when $\alpha>0$, $c<0$ and $\alpha<0$, $c<0$ can be considered similarly.

Theorem 3.1. *When $\alpha>0$ and $c>0$, Eq.(1.2) possesses the following periodic wave solutions and solitary wave solution under corresponding parameter conditions.*

1. When $g>0$, Eq.(1.2) has three types of periodic wave solutions.

(1) When $h_1<h<h_2$, there exist one family of periodic wave solutions to Eq.(1.2),

$$\frac{g_1(\varphi_{12}B+\varphi_{11}A)}{A-B}\left(\alpha_2 u_1+\frac{\alpha_1-\alpha_2}{1-\alpha_1^2}\left(\prod(u_1,\frac{\alpha_1^2}{\alpha_1^2-1},k_1)-\alpha_1 f_1\right)\right)=\sqrt{\alpha}|x-ct|, \tag{3.1}$$

where $c_1\overline{c_1}=\frac{g}{\alpha\varphi_{11}\varphi_{12}}$, $c_1+\overline{c_1}=-\varphi_{11}-\varphi_{12}$, $b_1=\frac{c_1+\overline{c_1}}{2}$, $a_1^2=-\frac{(c_1-\overline{c_1})^2}{4}$, $A^2=(\varphi_{12}-b_1)^2+{a_1}^2$, $B^2=(\varphi_{11}-b_1)^2+{a_1}^2$, $g_1=\frac{1}{\sqrt{AB}}$, $\alpha_1=\frac{A-B}{A+B}$, $\alpha_2=\frac{\varphi_{11}A-\varphi_{12}B}{\varphi_{12}B+\varphi_{11}A}$, $u_1=\frac{(\varphi_{12}-\varphi)B-(\varphi-\varphi_{11})A}{(\varphi_{12}-\varphi)B+(\varphi-\varphi_{11})A}$, $k_1^2=\frac{(\varphi_{12}-\varphi_{11})^2-(A-B)^2}{4AB}$, $f_1=\sqrt{\frac{1-\alpha_1^2}{k_1^2+{k_1'}^2\alpha_1^2}}\tan^{-1}\left[\sqrt{\frac{k_1^2+{k_1'}^2\alpha_1^2}{1-\alpha_1^2}}sdu_1\right]$, $k_1'=\sqrt{1-k_1^2}$, with c_1 and $\overline{c_1}$ are a pair of conjugated complex roots.

(2) When $h = h_2$, there exist a periodic wave solution to Eq.(1.2),

$$\begin{aligned}&\frac{\varphi_2}{\sqrt{\sigma}}\left[\arcsin\frac{(\varphi_2^+ + \varphi_2^- - 2\varphi_2)(\varphi - \varphi_2) - 2\sigma}{(\varphi - \varphi_2)(\varphi_2^+ - \varphi_2^-)} + \frac{\pi}{2}\right]\\&-\arcsin\frac{-2\varphi + \varphi_2^+ + \varphi_2^-}{\varphi_2^+ - \varphi_2^-} + \frac{\pi}{2} = \sqrt{\alpha}|x - ct|,\end{aligned} \tag{3.2}$$

where $\varphi_2^{\pm} = -\varphi_2 \pm \sqrt{\varphi_2^2 - \frac{g}{\alpha\varphi_2^2}}$.

(3) When $h > h_2$, there exist two families of periodic wave solutions to Eq.(1.2),

$$\varphi_9 g_2 \frac{\alpha_4^2}{\alpha_3^2}\left[u_2 + \frac{\alpha_3^2 - \alpha_4^2}{\alpha_4^2}\Pi(u_2, \alpha_3^2, k_2)\right] = \sqrt{\alpha}|x - ct|, \tag{3.3}$$

where $g_2 = \frac{2}{\sqrt{(\varphi_{10}-\varphi_8)(\varphi_9-\varphi_7)}}$, $\alpha_3^2 = \frac{\varphi_{10}-\varphi_9}{\varphi_{10}-\varphi_8}$, $\alpha_4^2 = \frac{\varphi_8(\varphi_{10}-\varphi_9)}{\varphi_9(\varphi_{10}-\varphi_8)}$, $k_2^2 = \frac{(\varphi_{10}-\varphi_9)(\varphi_8-\varphi_7)}{(\varphi_{10}-\varphi_8)(\varphi_9-\varphi_7)}$, *and* $u_2 = sn^{-1}\sqrt{\frac{(\varphi_{10}-\varphi_8)(\varphi-\varphi_9)}{(\varphi_{10}-\varphi_9)(\varphi-\varphi_8)}}$, *and*

$$-\varphi_7 g_3 \frac{\alpha_6^2}{\alpha_5^2}\left[u_3 + \frac{\alpha_5^2 - \alpha_6^2}{\alpha_6^2}\Pi(u_3, \alpha_5^2, k_3)\right] = \sqrt{\alpha}|x - ct|, \tag{3.4}$$

where $g_3 = g_2$, $\alpha_5^2 = \frac{\varphi_7-\varphi_8}{\varphi_{10}-\varphi_8}$, $\alpha_6^2 = \frac{\varphi_{10}(\varphi_7-\varphi_8)}{\varphi_7(\varphi_{10}-\varphi_8)}$, $k_3 = k_2$, *and* $u_3 = sn^{-1}\sqrt{\frac{(\varphi_{10}-\varphi_8)(\varphi-\varphi_7)}{(\varphi_8-\varphi_7)(\varphi_{10}-\varphi)}}$.

Additionally, when $\varphi_{12} \to \varphi_2^+$, periodic wave solutions (3.1) converges to periodic wave solution (3.2). Additionally, when $\varphi_{10} \to \varphi_2^+$, periodic wave solutions (3.3) converges to periodic wave solution (3.2).

2. When $\frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3} < g < 0$, Eq.(1.2) has a solitary wave solution and a family of periodic wave solutions, respectively,

$$\begin{aligned}&-\frac{\varphi_1}{\sqrt{\rho}}\ln\left|\frac{2\sqrt{\rho(\varphi - \varphi_1^-)(\varphi_1^+ - \varphi)} + (2\varphi_1 - \varphi_1^+ - \varphi_1^-)(\varphi_1 - \varphi) + 2\rho}{(\varphi_1^+ - \varphi_1^-)(\varphi_1 - \varphi)}\right|\\&-\arcsin\frac{-2\varphi + \varphi_1^+ + \varphi_1^-}{\varphi_1^+ - \varphi_1^-} + \frac{\pi}{2} = \sqrt{\alpha}|x - ct|,\end{aligned} \tag{3.5}$$

where $\varphi_1^{\pm} = -\varphi_1 \pm \sqrt{\varphi_1^2 - \frac{g}{\alpha\varphi_1^2}}$, *and*

$$-\varphi_4 g_4 \frac{\alpha_8^2}{\alpha_7^2}\left[u_4 + \frac{\alpha_7^2 - \alpha_8^2}{\alpha_8^2}\Pi(u_4, \alpha_7^2, k_4)\right] = \sqrt{\alpha}|x - ct|, \tag{3.6}$$

where $g_4 = \frac{2}{\sqrt{(\varphi_6-\varphi_4)(\varphi_5-\varphi_3)}}$, $\alpha_7^2 = \frac{\varphi_4-\varphi_3}{\varphi_5-\varphi_3}$, $\alpha_8^2 = \frac{\varphi_5(\varphi_4-\varphi_3)}{\varphi_4(\varphi_5-\varphi_3)}$, $k_4^2 = \frac{(\varphi_6-\varphi_5)(\varphi_4-\varphi_3)}{(\varphi_6-\varphi_4)(\varphi_5-\varphi_3)}$, *and* $u_4 = sn^{-1}\sqrt{\frac{(\varphi_5-\varphi_3)(\varphi_4-\varphi)}{(\varphi_4-\varphi_3)(\varphi_5-\varphi)}}$.

In addition, when $\varphi_4 \to \varphi_1$, periodic wave solutions (3.6) converges to solitary wave solution (3.5).

3. When $g = 0$, Eq.(1.2) has a family of periodic wave solutions,

$$-\frac{\varphi_{13} g_5}{\alpha_9^2}\left[k_5^2 u_5 + (\alpha_9^2 - k_5^2)\Pi(u_5, {\alpha_9}^2, k_5)\right] = \sqrt{\alpha}|x - ct|, \tag{3.7}$$

where $g_5 = \frac{2}{\sqrt{-\varphi_{13}(\varphi_{15}-\varphi_{14})}}$, ${\alpha_9}^2 = \frac{\varphi_{13}-\varphi_{14}}{\varphi_{15}-\varphi_{14}}$, $k_5^2 = -\frac{\varphi_{15}(\varphi_{14}-\varphi_{13})}{\varphi_{13}(\varphi_{15}-\varphi_{14})}$, *and* $u_5 = sn^{-1}\sqrt{\frac{(\varphi_{15}-\varphi_{14})(\varphi-\varphi_{13})}{(\varphi_{14}-\varphi_{13})(\varphi_{15}-\varphi)}}$.

Proof. 1. When $g > 0$, system (2.3) has three types of periodic orbits in Figure 2(a), which can be expressed as, respectively,

$$y = \pm\sqrt{\frac{\alpha(\varphi_{12}-\varphi)(\varphi-\varphi_{11})(\varphi-c_1)(\varphi-\overline{c_1})}{\varphi^2}}, \varphi_{11} < \varphi < \varphi_{12}, \tag{3.8}$$

when $h_1 < h < h_2$,

$$y = \pm\sqrt{\frac{\alpha(\varphi_2^+-\varphi)(\varphi-\varphi_2^-)(\varphi-\varphi_2)^2}{\varphi^2}}, \varphi_2 < \varphi_2^- < \varphi < \varphi_2^+, \tag{3.9}$$

when $h = h_2$, and

$$y = \pm\sqrt{\frac{\alpha(\varphi_{10}-\varphi)(\varphi-\varphi_9)(\varphi-\varphi_8)(\varphi-\varphi_7)}{\varphi^2}}, \varphi_7 < \varphi_8 < \varphi_9 < \varphi < \varphi_{10}, \quad \text{or } \varphi_7 < \varphi < \varphi_8 < \varphi_9 < \varphi_{10}, \tag{3.10}$$

when $h > h_2$.

Substituting (3.8), (3.9), and (3.10) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$, respectively, and integrating them along the perodic orbits, it follows that

$$\int_{\varphi_{11}}^{\varphi} \frac{s\mathrm{d}s}{\sqrt{(\varphi_{12}-s)(s-\varphi_{11})(s-c_1)(s-\overline{c_1})}} = \sqrt{\alpha}|\xi|, \tag{3.11}$$

$$\int_{\varphi_2^-}^{\varphi} \frac{s\mathrm{d}s}{(s-\varphi_2)\sqrt{(\varphi_2^+-s)(s-\varphi_2^-)}} = \sqrt{\alpha}|\xi|, \tag{3.12}$$

$$\int_{\varphi_9}^{\varphi} \frac{s\mathrm{d}s}{\sqrt{(\varphi_{10}-s)(s-\varphi_9)(s-\varphi_8)(s-\varphi_7)}} = \sqrt{\alpha}|\xi|, \tag{3.13}$$

and

$$\int_{\varphi_7}^{\varphi} \frac{-s\mathrm{d}s}{\sqrt{(\varphi_{10}-s)(\varphi_9-s)(\varphi_8-s)(s-\varphi_7)}} = \sqrt{\alpha}|\xi|. \tag{3.14}$$

From (3.11), (3.12), (3.13) and (3.14), we get the periodic wave solutions (3.1), (3.2), (3.3) and (3.4), respectively.

2. When $\frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3} < g < 0$, system (2.3) have one homoclinic orbit and a family of periodic orbits on the left side of line $\varphi = 0$ in Figure 2(c), which can be expressed as, respectively,

$$y = \pm\sqrt{\frac{\alpha(\varphi_1^+-\varphi)(\varphi-\varphi_1^-)(\varphi_1-\varphi)^2}{\varphi^2}}, \varphi_1^- < \varphi < \varphi_1 < \varphi_1^+, \tag{3.15}$$

and

$$y = \pm\sqrt{\frac{\alpha(\varphi-\varphi_3)(\varphi_4-\varphi)(\varphi_5-\varphi)(\varphi_6-\varphi)}{\varphi^2}}, \varphi_3 < \varphi < \varphi_4 < \varphi_5 < \varphi_6. \tag{3.16}$$

Substituting (3.15) and (3.16) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$ and integrating them along the orbit, it follows that

$$\int_{\varphi_1^-}^{\varphi} \frac{-s\mathrm{d}s}{(\varphi_1-s)\sqrt{(\varphi_1^+-s)(s-\varphi_1^-)}} = \sqrt{\alpha}|\xi|, \tag{3.17}$$

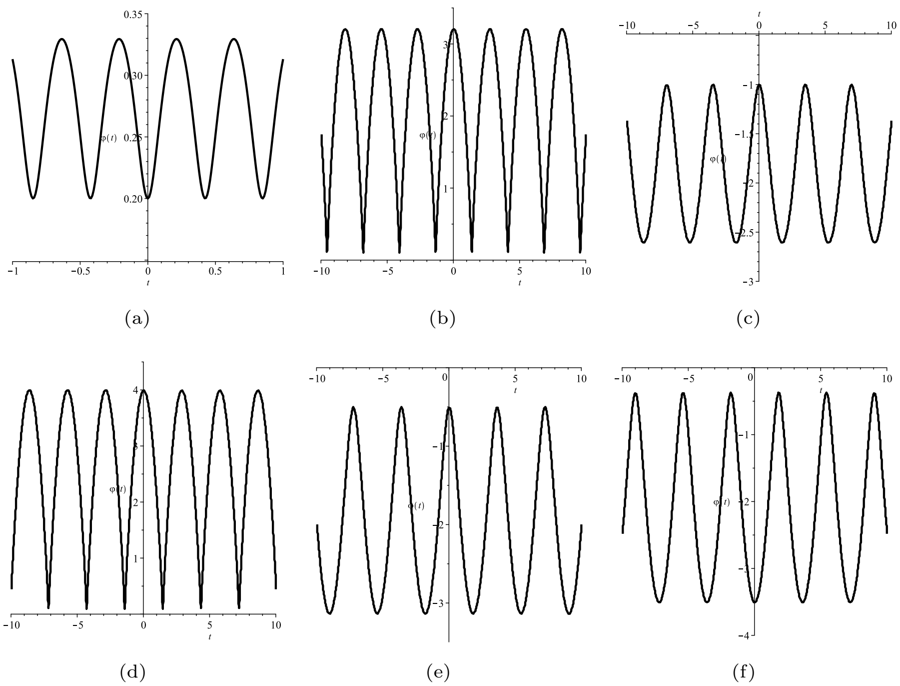


Figure 6. The profiles of periodic wave solutions (3.1), (3.2), (3.3), (3.4), (3.6), and (3.7), respectively.

and

$$\int_{\varphi}^{\varphi_4} \frac{-s\mathrm{d}s}{\sqrt{(s-\varphi_3)(\varphi_4-s)(\varphi_5-s)(\varphi_6-s)}} = \sqrt{\alpha}|\xi|. \tag{3.18}$$

From (3.17) and (3.18), we get the solitary wave solution (3.5) and periodic wave solutions (3.6), respectively.

3. When $g = 0$, system (2.3) has a family of periodic orbits on the left side of line $\varphi = 0$ in Figure 2(b), which can be expressed as

$$y = \pm\sqrt{\frac{-\alpha(\varphi-\varphi_{13})(\varphi_{14}-\varphi)(\varphi_{15}-\varphi)}{\varphi}}, \varphi_{13} < \varphi < \varphi_{14} < 0 < \varphi_{15}. \tag{3.19}$$

Substituting (3.19) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$ and integrating them along the orbit, it follows that

$$\int_{\varphi_{13}}^{\varphi} \frac{\sqrt{-s}\mathrm{d}s}{\sqrt{(s-\varphi_{13})(\varphi_{14}-s)(\varphi_{15}-s)}} = \sqrt{\alpha}|\xi|. \tag{3.20}$$

From (3.20), we get the family of periodic wave solutions (3.7). □

Here we illustrate the profiles of periodic wave solutions obtained in theorem 3.1 in Figure 6.

Theorem 3.2. *When $\alpha < 0$, $c > 0$, and $0 < g < \frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$, Eq.(1.2) has a solitary* *wave solution and a family of periodic wave solutions, respectively,*

$$\begin{aligned}&\frac{\varphi_1}{\sqrt{-\rho}}\ln\left|\frac{2\sqrt{-\rho\chi}+(\varphi_1^++\varphi_1^--2\varphi_1)(\varphi_1-\varphi)-2\rho}{(\varphi_1^+-\varphi_1^-)(\varphi_1-\varphi)}\right|\\&+\ln\frac{\varphi_1^+-\varphi_1^-}{2\sqrt{\chi}+2\varphi-\varphi_1^--\varphi_1^+}=\sqrt{-\alpha}|x-ct|,\end{aligned}\tag{3.21}$$

where $\varphi_1^\pm=-\varphi_1\pm\sqrt{\varphi_1^2-\frac{g}{\alpha\varphi_1^2}}$, *and*

$$\varphi_3 g_6\frac{\alpha_{11}^2}{\alpha_{10}^2}\left[u_6+\frac{\alpha_{10}^2-\alpha_{11}^2}{\alpha_{11}^2}\Pi(u_6,\alpha_{10}^2,k_6)\right]=\sqrt{-\alpha}|x-ct|,\tag{3.22}$$

where $g_6=\frac{2}{\sqrt{(\varphi_6-\varphi_3)(\varphi_4-\varphi_5)}}$, $\alpha_{10}^2=\frac{\varphi_4-\varphi_3}{\varphi_4-\varphi_5}$, $\alpha_{11}^2=\frac{\varphi_5(\varphi_4-\varphi_3)}{\varphi_3(\varphi_4-\varphi_5)}$, $k_6^2=\frac{(\varphi_4-\varphi_3)(\varphi_6-\varphi_5)}{(\varphi_6-\varphi_3)(\varphi_4-\varphi_5)}$, *and* $u_6=sn^{-1}\sqrt{\frac{(\varphi_4-\varphi_5)(\varphi-\varphi_3)}{(\varphi_4-\varphi_3)(\varphi-\varphi_5)}}$.

In addition, when $\varphi_4\to\varphi_1$, *periodic wave solutions* (3.22) *converges to solitary wave solution* (3.21).

Proof. When $0<g<\frac{3c}{4}\left(\frac{-c}{4\alpha}\right)^{1/3}$, system (2.3) have one homoclinic orbit and a family of periodic orbits on the right side of line $\varphi=0$ in Figure 4(c), which can be expressed as, respectively,

$$y=\pm\sqrt{\frac{-\alpha(\varphi-\varphi_1^+)(\varphi-\varphi_1^-)(\varphi_1-\varphi)^2}{\varphi^2}},\varphi_1^-<0<\varphi_1^+<\varphi<\varphi_1,\tag{3.23}$$

and

$$y=\pm\sqrt{\frac{-\alpha(\varphi-\varphi_3)(\varphi_4-\varphi)(\varphi-\varphi_5)(\varphi_6-\varphi)}{\varphi^2}},\varphi_5<\varphi_3<\varphi<\varphi_4<\varphi_6.\tag{3.24}$$

Substituting (3.23) and (3.24) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=y$ and integrating them along the orbit, them follow that

$$\int_{\varphi_1^+}^{\varphi}\frac{s\mathrm{d}s}{(\varphi_1-s)\sqrt{(s-\varphi_1^+)(s-\varphi_1^-)}}=\sqrt{-\alpha}|\xi|,\tag{3.25}$$

and

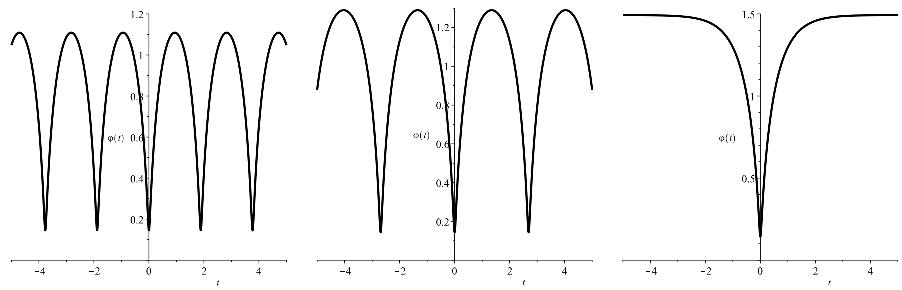
$$\int_{\varphi_3}^{\varphi}\frac{s\mathrm{d}s}{\sqrt{(s-\varphi_3)(\varphi_4-s)(s-\varphi_5)(\varphi_6-s)}}=\sqrt{-\alpha}|\xi|.\tag{3.26}$$

From (3.25) and (3.26), we get solitary wave solution (3.21) and the family of periodic wave solutions (3.22), respectively. □

Here we illustrate the process of periodic wave solutions (3.22) converging to solitary wave solution (3.21) in theorem 3.2 in Figure 7.

4. Conclusions

In this paper, we investigate the existence of several types of periodic wave solutions, present their exact expressions, and reveal their inside relations as well as their



(a) (b) (c)

Figure 7. The process of periodic wave solutions (3.22) converging to solitary wave solution (3.21).

relations with solitary wave solutions. The dynamical properties of these periodic wave solutions will greatly help us understand the structures and propagation of the nonlinear wave.

Acknowledgements. The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

References

[1] A. Biswas and M. Song, *Soliton solution and bifurcation analysis of the Zakharov-Kuznetsov-Benjamin-Bona-Mahoney equation with power law nonlinearity*, Commun. Nonlinear Sci. Numer. Simul., 2013, 18(7), 1676–1683.

[2] A. Biswas, M. Song, H. Triki et al., *Solitons, shock waves, conservation laws and bifurcation analysis of Boussinesq equation with power law nonlinearity and dual dispersion*, Appl. Math. Inform. Sci., 2014, 8(3), 949–957.

[3] A. Chen, S. Wen, S. Tang et al., *Effects of quadratic singular curves in integrable equations*, Studies in Applied Mathematics, 2015, 134(1), 24–61.

[4] Y. Chen, M. Song and Z. Liu, *Soliton and Riemann theta function quasi-periodic wave solutions for a (2+1)-dimensional generalized shallow water wave equation*, Nonlinear Dyn, 2015, 82(1–2), 333–347.

[5] X. Du, *An irrational trial equation method and its applications*, Journal of Physics, 2010, 75(3), 415–422.

[6] A. Fujimoto and Y. Watanabe, *Polynomial evolution equations of not normal type admitting nontrivial symmetries*, Physics Letters A, 1989, 136(6), 294–299.

[7] T. D. Leta and J. Li, *Various exact soliton solutions and bifurcations of a generalized Dullin-Gottwald-Holm Equation with a power law nonlinearity*, International Journal of Bifurcation and Chaos, 2017, 27(8), 1750129.

[8] J. Li, *Singular Nonlinear Travelling Wave Equations: Bifurcations and Exact Solutions*, Science Press, Beijing, 2013.

[9] J. Li and Z. Qiao, *Bifurcations and exact traveling wave solutions of the generalized two-component Camassa-Holm equation*, Int. J. Bifurcat. Chaos., 2012, 22(12), 1250305.

[10] C. Liu, *Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations*, Computer Physics Communications, 2010, 181(2), 317–324.

[11] Z. Liu and Y. Liang, *The explicit nonlinear wave solutions and their bifurcations of the generalized Camassa-Holm equation*, Int. J. Bifurcat. Chaos., 2011, 21(11), 3119–3136.

[12] C. Pan, L. Ling and Z. Liu, *A new integrable equation with cuspons and periodic cuspons*, Physica Scripta, 2014, 89(10), 105207.

[13] S. Sakovich, *Fujimoto-Watanabe equations and differential substitutions*, Journal of Physics A: Mathematical and General, 1991, 24(10), L519–L521.

[14] S. Sergei, *Smooth soliton solutions of a new integrable equation by Qiao*, Journal of Mathematical Physics, 2011, 52(2), 023509.

[15] L. Shi and Z. Wen, *Bifurcations and dynamics of traveling wave solutions to a Fujimoto-Watanabe equation*, Commun. Theor. Phys., 2018, 69(6), 631–636.

[16] L. Shi and Z. Wen, *Dynamics of traveling wave solutions to a highly nonlinear Fujimoto-Watanabe equation*, Chinese Physics B, 2019, 28(4), 040201.

[17] M. Song, *Nonlinear wave solutions and their relations for the modified Benjamin-Bona-Mahony equation*, Nonlinear Dyn, 2015, 80(1–2), 431–446.

[18] Z. Wen, *Bifurcation of solitons, peakons, and periodic cusp waves for $\theta-$equation*, Nonlinear Dyn, 2014, 77(1–2), 247–253.

[19] Z. Wen, *Several new types of bounded wave solutions for the generalized two-component Camassa-Holm equation*, Nonlinear Dyn, 2014, 77(3), 849–857.

[20] Z. Wen, *Bifurcations and nonlinear wave solutions for the generalized two-component integrable Dullin-Gottwald-Holm system*, Nonlinear Dyn, 2015, 82(1–2), 767–781.

[21] Z. Wen, *Extension on peakons and periodic cusp waves for the generalization of the Camassa-Holm equation*, Math Meth Appl Sci, 2015, 38(11), 2363–2375.

[22] Z. Wen, *Bifurcations and exact traveling wave solutions of a new two-component system*, Nonlinear Dynamics, 2017, 87(3), 1917–1922.

[23] Z. Wen, *Bifurcations and exact traveling wave solutions of the celebrated Green-Naghdi equations*, International Journal of Bifurcation and Chaos, 2017, 27(07), 1750114.

[24] Z. Wen and Z. Liu, *Bifurcation of peakons and periodic cusp waves for the generalization of the Camassa-Holm equation*, Nonlinear Anal., 2011, 12(3), 1698–1707.

[25] Z. Wen, Z. Liu and M. Song, *New exact solutions for the classical Drinfel'd-Sokolov-Wilson equation*, Appl. Math. Comput., 2009, 215(6), 2349–2358.

[26] Z. Wen and L. Shi, *Dynamical behaviors of traveling wave solutions to a Fujimoto-Watanabe equation*, Chinese Physics B, 2018, 27(9), 090201.

[27] Z. Wen and L. Shi, *Dynamics of bounded traveling wave solutions for the modified Novikov equation*, Dynam. Syst. Appl., 2018, 27(3), 581–591.

[28] L. Zhang, L. Chen and X. Huo, *The effects of horizontal singular straight line in a generalized nonlinear Klein-Gordon model equation*, Nonlinear Dyn, 2013, 72(4), 789–801.