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ON THE LIMIT CYCLES FOR A CLASS OF GENERALIZED KUKLES DIFFERENTIAL SYSTEMS

Amel Boulfoul[1,2,†], Amar Makhlouf[3] and Nawal Mellahi[1]

Abstract In this paper, we consider the limit cycles of a class of polynomial differential systems of the form $\dot{x} = -y, \quad \dot{y} = x - f(x) - g(x)y - h(x)y^2 - l(x)y^3$, where $f(x) = \epsilon f_1(x) + \epsilon^2 f_2(x)$, $g(x) = \epsilon g_1(x) + \epsilon^2 g_2(x)$, $h(x) = \epsilon h_1(x) + \epsilon^2 h_2(x)$ and $l(x) = \epsilon l_1(x) + \epsilon^2 l_2(x)$ where $f_k(x)$, $g_k(x)$, $h_k(x)$ and $l_k(x)$ have degree n_1, n_2, n_3 and n_4, respectively for each $k = 1, 2$, and ε is a small parameter. We obtain the maximum number of limit cycles that bifurcate from the periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$ using the averaging theory of first and second order.

Keywords Limit cycle, averaging theory, Kukles systems.

MSC(2010) 34C29, 34C25, 47H11.

1. Introduction

One of the main goals in the qualitative theory of real planar differential systems is the determination of their limit cycles, defined by Poincaré [19]. As we all know, this is a very difficult problem for a general polynomial systems. Therefore, many mathematicians study some systems with special conditions to obtain the number of limit cycles as many as possible for a planar differential systems see Han [12]. The knowledge of the existence or not of periodic solutions is very important for understanding the dynamics of the differential systems. There exist several methods to study the number of limit cycles that bifurcate from the periodic orbits of linear center such as the Poincaré return map, the inverse integrating factor and the averaging theory. The investigation of the existence of periodic orbits of differential systems via the averaging methods has a long history, see for instance Sanders and Verhulst [23], Verhulst [24], Marsden and McCracken [17], Han etc [10, 11], Buica etc [3, 4], Boulfoul and Makhlouf [2] and the references therein. The second part of the 16th Hilbert problem [13] proposes to find a uniform upper bound for the number of limit cycles that a planar polynomial vector field of degree n can have which only depends on the degree of the polynomial differential system.

[†]the corresponding author. Email address:a.boulfoul@univ-skikda.dz(A. Boulfoul)
[1]Department of Mathematics, 20 August 1955 University, BP26 El Hadaiek, 21000 Skikda, Algeria
[2]LAMAHIS Laboratory, 20 August 1955 University, BP26 El Hadaiek, 21000 Skikda, Algeria
[3]Departement of Mathematics, LMA Laboratory, Badji-Mokhtar University, BP12 El Hadjar, 23000 Annaba, Algeria

Consider the following Kukles polynomial differential system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + Q(x,y), \end{aligned} \tag{1.1}$$

where $Q(x,y)$ is polynomial with real coefficients of a given degree.

In [14], Kukles gave necessary and sufficient conditions in order that (1.1) with $n = 3$ has a center at the origin. This cubic system without the term y^3 is the so called reduced Kukles system. Christopher and Lloyd [6] presented some systems that yield at most five limit cycles bifurcating from the origin. In [9], Grin and Schneider studied the conditions of at most one limit cycle bifurcating from the origin for (1.1) with $n = 3$. Wu etc proved that the Kukles system with two fine foci can generate at least six limit cycles in [25]. In [22], Sadovskii solved the center-focus problem for this system with $a_2a_7 \neq 0$ and proved that it can have seven limit cycles. In [20] appears a description of the local bifurcations of critical periods in the neighborhood of a non-degenerate center of the reduced Kukles systems. Liu etc in [15] introduced a class of cubic systems (1.1) with an invariant parabola which coexists with a center under given parameters. In [5], Chavarriga etc described a cubic system (1.1) that has an invariant hyperbola to coexist with two limit cycles. Afterwards, the authors' interests converted to finding maximum number of small amplitude limit cycles coexisting with invariant ellipses. In [7], Giné studied the systems of the form (1.1). For $n = 4$ and $n = 5$, they obtained the maximum numbers of small-amplitude cycles using the method of calculation of Poincaré-Liapunov constants see [8]. Sáez and Szántó in [21] presented a class of quintic systems of the form (1.1) having an invariant ellipse with what small amplitude limit cycles bifurcating from the origin coexist. In [26], Zang etc studied the number and distribution of limit cycles for a class of reduced Kukles systems under cubic perturbation. In [16], Llibre and Mereu using the averaging theory studied the maximum number of limit cycles which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Kukles differential systems of the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - f(x) - g(x)y - h(x)y^2 - d_0y^3, \end{aligned} \tag{1.2}$$

where the polynomials $f(x)$, $g(x)$ and $h(x)$ have degree n_1, n_2 and n_3 respectively, $d_0 \neq 0$ is a real number.

In [18] Mellahi etc using the averaging theory studied the maximum number of limit cycles which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Kukles differential systems of the form

$$\begin{aligned} \dot{x} &= -y + l(x), \\ \dot{y} &= x - f(x) - g(x)y - h(x)y^2 - d_0y^3, \end{aligned} \tag{1.3}$$

where $l(x)$, $f(x)$, $g(x)$ and $h(x)$ have degree m, n_1, n_2 and n_3 respectively, $d_0 \neq 0$ is a real number.

In this work using the averaging theory we study the maximum number of limit cycles which can bifurcate from the periodic orbits of a linear center $\dot{x} = -y$, $\dot{y} = x$, perturbed inside of generalized polynomial Kukles differential systems

$$\dot{x} = -y, \tag{1.4}$$

$$\dot{y} = x - f(x) - g(x)y - h(x)y^2 - l(x)y^3,$$

where $f(x) = \varepsilon f_1(x) + \varepsilon^2 f_2(x)$, $g(x) = \varepsilon g_1(x) + \varepsilon^2 g_2(x)$, $h(x) = \varepsilon h_1(x) + \varepsilon^2 h_2(x)$ and $l(x) = \varepsilon l_1(x) + \varepsilon^2 l_2(x)$ where $f_k(x)$, $g_k(x)$, $h_k(x)$ and $l_k(x)$ have degree n_1, n_2, n_3 and n_4 respectively, for each $k = 1, 2$, and ε is a small parameter.

Our main results of system (1.4) are the following.

Theorem 1.1. *For $|\varepsilon| > 0$ sufficiently small, the maximum number of limit cycles of the generalized Kukles polynomial differential system (1.4) bifurcating from the periodic orbits of the linear centre $\dot{x} = -y$, $\dot{y} = x$ using the averaging theory of first order is*

$$\lambda_1^{'} = \max\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right] + 1\right\}.$$

Theorem 1.2. *For $|\varepsilon| > 0$ sufficiently small, the maximum number of limit cycles of the generalized Kukles polynomial differential system (1.4) bifurcating from the periodic orbits of the linear centre $\dot{x} = -y$, $\dot{y} = x$ using the averaging theory of second order is*

$$\lambda_2^{'} = \max\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right] + 1, \left[\frac{n_1}{2}\right] + \left[\frac{n_2-1}{2}\right], \left[\frac{n_1}{2}\right] + \left[\frac{n_4-1}{2}\right] + 1,\right.$$
$$\left[\frac{n_1-1}{2}\right] + \mu^{'}, \left[\frac{n_2-1}{2}\right] + \left[\frac{n_3}{2}\right] + 1, \left[\frac{n_4-1}{2}\right] + \left[\frac{n_3}{2}\right] + 2,$$
$$\left.\left[\frac{n_3-1}{2}\right] + \mu^{'} + 1\right\},$$

where $\mu^{'} = \min\left\{\left[\frac{n_2}{2}\right], \left[\frac{n_4}{2}\right] + 1\right\}$.

In Mellahi etc [18] it has been shown that there exists generalized Kukles equation (1.3), having at least $\lambda_2 = \max\left\{\left[\frac{n_1}{2}\right] + \left[\frac{n_2-1}{2}\right], \left[\frac{n_1}{2}\right] + \left[\frac{m}{2}\right] - 1, \left[\frac{n_1+1}{2}\right], \left[\frac{n_3+3}{2}\right], \left[\frac{n_3}{2}\right] + \left[\frac{m}{2}\right], \left[\frac{n_2+1}{2}\right] + \left[\frac{n_3}{2}\right], \left[\frac{n_2}{2}\right], \left[\frac{m-1}{2}\right], \left[\frac{n_1-1}{2}\right] + \mu, \left[\frac{n_3+1}{2}\right] + \mu, 1\right\}$ limit cycles.

The result in Theorem 1.2 improves this lower estimate ($\lambda_2^{'} > \lambda_2$ for all $n_1 \geq 1$, $n_2 \geq 1$, $n_3 \geq 1$, $m \geq 2$ and $n_4 \geq \max\{3, n_2, m-1\}$). For each fixed $n_1 \geq 1$, $n_2 \geq 1$ $n_3 \geq 1$ and $m \geq 2$ there exists $n_4^{'} \geq \max\{3, n_2, m-1\}$ such that $\lambda_2^{'} > \lambda_2$ for all $n_4 \geq n_4^{'}$.

Now we shall do some applications of Theorem 1.1 and Theorem 1.2.

Corollary 1.1. *Let*

$$f_1(x) = x + 2x^2 + x^3,\ g_1(x) = -2 + 3x - \tfrac{1}{3}x^2,\ h_1(x) = 1 - x,$$
$$l_1(x) = 5 + x - 16x^2 + 5x^3 + \tfrac{64}{9}x^4 - x^5.$$

Then the maximum number of limit cycles of system (1.4) is three using the averaging theory of first order.

Corollary 1.2. *Let*

$$f_1(x) = x + 2x^2 + 3x^3 - 2x^4,\ g_1(x) = 3x - 3x^2 - 5x^3,\ h_1(x) = 1 - x,$$
$$l_1(x) = 1 - 3x - \tfrac{10160}{3969}x^3 + \tfrac{64}{189}x^5,\ f_2(x) = x^4,\ g_2(x) = 2 - \tfrac{454}{15}x^2 + 3x^3,$$
$$h_2(x) = x,\ l_2(x) = 2 + \tfrac{262}{3}x^2 - \tfrac{475448}{11907}x^4 - x^5.$$

Then the maximum number of limit cycles of system (1.4) *is five using the averaging theory of second order.*

Note that we do not know if the upper bounds $\lambda_1^{'}$ and $\lambda_2^{'}$ are reached. In Corollary 1.1 and Corollary 1.2 we prove that the upper bounds $\lambda_1^{'}$ and $\lambda_2^{'}$ are reached.

2. Averaging theory of first and second order

In this section we summarize the main results of the averaging theory of first and second order for computing limit cycles of the generalized Kukles polynomial differential system (1.4).

Consider the differential system

$$X^{'}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{2.1}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}$. Assume that the following conditions hold.

(*i*) $F_1(t,.) \subset C^2(D), F_2(t,.) \subset C^1(D)$, for all $t \in \mathbb{R}, F_1, F_2, R, D_xF_1$ are locally Lipschitz with respect to x,and R is twice differentiable with respect to ε. We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)\, ds,$$

$$F_{20}(z) = \frac{1}{T}\int_0^T \left[D_xF_1(s,z)\, y_1(s,z) + F_2(s,z)\right] ds,$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)\, dt.$$

(*ii*) For $V \subset D$, an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(.,\varepsilon)$ of the system (2.1) such that $\varphi(0,\varepsilon) \to a_\varepsilon$ when $\varepsilon \to 0$.

The expression $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} : V \to \mathbb{R}$ at the fixed point a_ε is not zero. A sufficient condition for this inequality to be true is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{10} for ϵ sufficiently small. In this case the previous result provides the averaging theory of first order.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the averaging theory of second order.

3. Proof of Theorem 1.1

This proof is based on the first order averaging theory, by using the coordinates (θ, r) with

$$x = r\cos\theta, \quad y = r\sin\theta \ \text{ and } \ r > 0. \tag{3.1}$$

In this context, we take

$$f_1(x) = \sum_{i=0}^{n_1} a_i x^i,\ g_1(x) = \sum_{i=0}^{n_2} b_i x^i,\ h_1(x) = \sum_{i=0}^{n_3} c_i x^i \text{ and } \ l_1(x) = \sum_{i=0}^{n_4} d_i x^i, \tag{3.2}$$

then system (1.4) with $k = 1$ can be written as

$$\begin{cases} \dot{r} = & -\epsilon\left(\sum_{i=0}^{n_1} a_i R_i(\theta) r^i + \sum_{i=0}^{n_2} b_i T_i(\theta) r^{i+1} + \sum_{i=0}^{n_3} c_i S_i(\theta) r^{i+2} + \sum_{i=0}^{n_4} d_i U_i(\theta) r^{i+3}\right), \\ \dot{\theta} = & 1 - \frac{\epsilon}{r}\left(\sum_{i=0}^{n_1} a_i \cos^{i+1}\theta r^i + \sum_{i=0}^{n_2} b_i R_{i+1}(\theta) r^{i+1} + \sum_{i=0}^{n_3} c_i T_{i+1}(\theta) r^{i+2}\right. \\ & \left. + \sum_{i=0}^{n_4} d_i r^{i+3} S_{i+1}(\theta)\right), \end{cases} \tag{3.3}$$

where

$$\begin{aligned} R_j(\theta) &= \cos^j\theta\sin\theta, \\ T_j(\theta) &= \cos^j\theta\sin^2\theta = \cos^j\theta - \cos^{j+2}\theta, \\ S_j(\theta) &= \cos^j\theta\sin^3\theta = \cos^j\theta\sin\theta - \cos^{j+2}\theta\sin\theta, \\ U_j(\theta) &= \cos^j\theta\sin^4\theta = \cos^j\theta - 2\cos^{j+2}\theta + \cos^{j+4}\theta. \end{aligned}$$

Now taking θ as the new independent variable, system (3.3) becomes

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \epsilon F_1(\theta, r) + O(\epsilon^2),$$

where

$$F_1(\theta, r) = -\sum_{i=0}^{n_1} a_i R_i(\theta) r^i - \sum_{i=0}^{n_2} b_i T_i(\theta) r^{i+1} - \sum_{i=0}^{n_3} c_i S_i(\theta) r^{i+2} - \sum_{i=0}^{n_4} d_i U_i(\theta) r^{i+3}. \tag{3.4}$$

Hence

$$F_{10}(r) = -\frac{1}{2\pi}\left(\sum_{i=0}^{n_1} a_i J_k(2\pi) r^i + \sum_{i=0}^{n_2} b_i \tilde{I}_k(2\pi) r^{i+1} + \sum_{i=0}^{n_3} c_i \tilde{J}_k(2\pi) r^{i+2} + \sum_{i=0}^{n_4} d_i \tilde{\tilde{I}}_k(2\pi) r^{i+3}\right),$$

where

$$\begin{aligned} &J_k(2\pi) = \int_0^{2\pi} R_k(\theta)\, d\theta,\ \tilde{I}_k(2\pi) = \int_0^{2\pi} T_k(\theta)\, d\theta,\ \tilde{J}_k(2\pi) = \int_0^{2\pi} S_k(\theta)\, d\theta, \\ &\tilde{\tilde{I}}_k(2\pi) = \int_0^{2\pi} U_k(\theta)\, d\theta. \end{aligned}$$

Now using the expressions of the integrals in appendix A, we obtain

$$F_{10}(r) = -\frac{r}{2\pi}\left(\sum_{i=0}^{[\frac{n_2}{2}]}\tilde{I}_{2i}(2\pi)b_{2i}r^{2i} + \sum_{i=0}^{[\frac{n_4}{2}]}\tilde{\tilde{I}}_{2i}(2\pi)d_{2i}r^{2i+2}\right)$$
$$= -r\left(\sum_{i=0}^{[\frac{n_2}{2}]}\frac{\alpha_i}{2^{i+1}(i+1)!}b_{2i}r^{2i} + \sum_{i=0}^{[\frac{n_4}{2}]}\frac{3\alpha_i}{2^{i+2}(i+2)!}d_{2i}r^{2i+2}\right), \quad (3.5)$$

where

$$\alpha_k = 3\cdot 5\cdots(2k-1),\ \alpha_{k+1} = (2k+1)\alpha_k.$$

Then the polynomial $F_{10}(r)$ has at most $\max\{[\frac{n_2}{2}], [\frac{n_4}{2}]+1\}$ positive roots. Hence, Theorem 1.1 is proved.

4. Proof of Theorem 1.2

We write $f_1(x)$, $g_1(x)$, $h_1(x)$, and $l_1(x)$ as in (3.2), and

$$f_2(x) = \sum_{i=0}^{n_1} p_i x^i,\ g_2(x) = \sum_{i=0}^{n_2} q_i x^i,\ h_2(x) = \sum_{i=0}^{n_3} s_i x^i,\ l_2(x) = \sum_{i=0}^{n_4} w_i x^i,$$

then system (1.4) in polar coordinates (r, t) with $r > 0$ becomes

$$\begin{cases} \dot{r} = & -\epsilon\left(\sum_{i=0}^{n_1} a_i R_i(\theta)r^i + \sum_{i=0}^{n_2} b_i T_i(\theta)r^{i+1} + \sum_{i=0}^{n_3} c_i S_i(\theta)r^{i+2} + \sum_{i=0}^{n_4} d_i r^{i+3}U_i(\theta)\right) \\ & -\epsilon^2\left(\sum_{i=0}^{n_1} p_i R_i(\theta)r^i + \sum_{i=0}^{n_2} q_i T_i(\theta)r^{i+1} + \sum_{i=0}^{n_3} s_i S_i(\theta)r^{i+2} + \sum_{i=0}^{n_4} w_i r^{i+3}U_i(\theta)\right), \\ \dot{\theta} = & 1 - \frac{\epsilon}{r}\left(\sum_{i=0}^{n_1} a_i \cos^{i+1}\theta r^i + \sum_{i=0}^{n_2} b_i R_{i+1}(\theta)r^{i+1} + \sum_{i=0}^{n_3} c_i T_{i+1}(\theta)r^{i+2}\right. \\ & \left. + \sum_{i=0}^{n_4} d_i r^{i+3}S_{i+1}(\theta)\right) - \frac{\epsilon^2}{r}\left(\sum_{i=0}^{n_1} p_i \cos^{i+1}\theta r^i + \sum_{i=0}^{n_2} q_i R_{i+1}(\theta)r^{i+1}\right. \\ & \left. + \sum_{i=0}^{n_3} s_i T_{i+1}(\theta)r^{i+2} + \sum_{i=0}^{n_4} w_i r^{i+3}S_{i+1}(\theta)\right). \end{cases} \quad (4.1)$$

Taking θ as the new independent variable, system (4.1) becomes

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \epsilon F_1(r,\theta) + \epsilon^2 F_2(r,\theta) + O(\epsilon^3), \quad (4.2)$$

where

$$F_1(r,\theta) = -\sum_{i=0}^{n_1} a_i R_i(\theta)r^i - \sum_{i=0}^{n_2} b_i T_i(\theta)r^{i+1} - \sum_{i=0}^{n_3} c_i S_i(\theta)r^{i+2} - \sum_{i=0}^{n_4} d_i r^{i+3}U_i(\theta), \quad (4.3)$$

$$F_2(r,\theta) = -\sum_{i=0}^{n_1} p_i R_i(\theta)r^i - \sum_{i=0}^{n_2} q_i T_i(\theta)r^{i+1} - \sum_{i=0}^{n_3} s_i S_i(\theta)r^{i+2} - \sum_{i=0}^{n_4} w_i r^{i+3}U_i(\theta)$$
$$-\frac{1}{r}\left(\sum_{i=0}^{n_1} a_i R_i(\theta)r^i + \sum_{i=0}^{n_2} b_i T_i(\theta)r^{i+1} + \sum_{i=0}^{n_3} c_i S_i(\theta)r^{i+2}\right.$$

$$+\sum_{i=0}^{n_4} d_i r^{i+3} U_i(\theta)\Bigg) \times \Bigg(\sum_{i=0}^{n_1} a_i \cos^{i+1}\theta r^i + \sum_{i=0}^{n_2} b_i R_{i+1}(\theta) r^{i+1}$$
$$+\sum_{i=0}^{n_3} c_i T_{i+1}(\theta) r^{i+2} + \sum_{i=0}^{n_4} d_i r^{i+3} S_{i+1}(\theta)\Bigg). \tag{4.4}$$

In order to compute $F_{20}(r)$, we need that $F_{10}(r)$ be identically zero. Then from (3.5),

$$\begin{cases} b_{2k} = \dfrac{-3}{2k-1} d_{2k-2} \quad 0 \le k \le \mu^{'}, \\ b_0 = b_{2k} = d_{2k-2} = 0, \quad \mu^{'} + 1 \le k \le \lambda_1^{'}, \end{cases} \tag{4.5}$$

where

$$\mu^{'} = \min\{[\frac{n_2}{2}], [\frac{n_4}{2}] + 1\}, \quad \lambda_1^{'} = \max\{[\frac{n_2}{2}], [\frac{n_4}{2}] + 1\}.$$

First, using (4.5) and, by substituting in (4.3) we obtain

$$F_1(r,\theta) = -\sum_{i=0}^{n_1} r^i a_i R_i(\theta) - \sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+2} b_{2k+1} T_{2k+1}(\theta) - \sum_{i=0}^{n_3} c_i r^{i+2} S_i(\theta)$$
$$-\sum_{k=0}^{[\frac{n_4}{2}]} r^{2k+4} d_{2k+1} U_{2k+1}(\theta) - \sum_{k=0}^{\mu} r^{2k+1} d_{2k-2}\Bigg(\cos^{2k-2}\theta - \frac{4k+1}{2k-1}$$
$$\times \cos^{2k}\theta + \frac{2k+2}{2k-1}\cos^{2k+2}\theta\Bigg). \tag{4.6}$$

Then

$$\frac{dF_1(r,\theta)}{dr} = -\sum_{i=0}^{n_1} i a_i r^{i-1} R_i(\theta) - \sum_{k=0}^{[\frac{n_2-1}{2}]} (2k+2) b_{2k+1} r^{2k+1} T_{2k+1}(\theta) - \sum_{i=0}^{n_3} (i+2) c_i$$
$$\times r^{i+1} S_i(\theta) - \sum_{k=0}^{[\frac{n_4}{2}]} (2k+4) r^{2k+3} d_{2k+1} U_{2k+1}(\theta) - \sum_{k=0}^{\mu} (2k+1) r^{2k} d_{2k-2}$$
$$\times \Bigg(\cos^{2k-2}\theta - \frac{4k+1}{2k-1}\cos^{2k}\theta + \frac{2k+2}{2k-1}\cos^{2k+2}\theta\Bigg). \tag{4.7}$$

Again, using the integrals of appendix A, we obtain

$$y(r,\theta) = \int_0^\theta F_1(s,r) ds$$
$$= -\sum_{i=0}^{n_1} r^i a_i J_i(\theta) - \sum_{i=0}^{[\frac{n_2-1}{2}]} r^{2i+2} b_{2i+1} \sum_{l=0}^{i+1} \tilde{\gamma}_{i,l} \sin(2l+1)\theta - \sum_{i=0}^{n_3} c_i r^{i+2} \tilde{J}_i(\theta)$$
$$-\sum_{i=0}^{[\frac{n_4-1}{2}]} r^{2i+4} d_{2i+1} \sum_{l=0}^{i+2} \tilde{\tilde{\gamma}}_{i,l} \sin(2l+1)\theta - \sum_{i=0}^{\mu} r^{2i+1} d_{2i-2} \sum_{l=1}^{i+1} \tilde{\beta}_{i,l} \sin(2l)\theta, \tag{4.8}$$

where

$$\tilde{\gamma}_{i,l}=\begin{cases}\gamma_{i,l}-\gamma_{i+1,l}, & 0\le l\le i,\\ -\gamma_{i+1,i+1}, & l=i+1.\end{cases},\ \tilde{\tilde{\gamma}}_{i,l}=\begin{cases}\gamma_{i,l}-2\gamma_{i+1,l}+\gamma_{i+2,l}, & 0\le l\le i,\\ -2\gamma_{i+1,i+1}+\gamma_{i+2,i+1}, & l=i+1,\\ \gamma_{i+2,i+2}, & l=i+2,\end{cases}$$

and $$\tilde{\beta}_{i,l}=\begin{cases}\beta_{i-1,l}-\frac{4i+1}{2i-1}\beta_{i,l}+\frac{2i+2}{2i-1}\beta_{i+1,l}, & 0\le l\le i-1,\\ -\frac{4i+1}{2i-1}\beta_{i,i}+\frac{2i+2}{2i-1}\beta_{i+1,i}, & l=i,\\ \frac{2i+2}{2i-1}\beta_{i+1,i+1}, & l=i+1.\end{cases}$$

Now, we determine the corresponding function $F_{20}(r)=F_{20}^1(r)+F_{20}^2(r)$, with

$$F_{20}^1(r)=\frac{1}{2\pi}\int_0^{2\pi}\frac{dF_1(r,\theta)}{dr}.y(r,\theta)d\theta,$$
$$F_{20}^2(r)=\frac{1}{2\pi}\int_0^{2\pi}F_2(\theta,r)d\theta.$$

In the following Lemmas we obtain some results of the integrals $F_{20}^1(r)$ and $F_{20}^2(r)$.

Lemma 4.1. *The integral $F_{20}^1(r)$ is a polynomial in the variable r given by*

$$\begin{aligned}F_{20}^1(r)=&\sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]}a_{2s}b_{2k+1}r^{2s+2k+1}M_1^{s,k}+\sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]}c_{2s}b_{2k+1}r^{2s+2k+3}M_2^{s,k}\\&+\sum_{s=0}^{[\frac{n_1-1}{2}]}\sum_{k=0}^{\mu}a_{2s+1}d_{2k-2}r^{2s+2k+1}M_3^{s,k}+\sum_{s=0}^{[\frac{n_3-1}{2}]}\sum_{k=0}^{\mu}c_{2s+1}d_{2k-2}r^{2s+2k+3}M_4^{s,k}\\&+\sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]}a_{2s}d_{2k+1}r^{2s+2k+3}M_5^{s,k}+\sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]}c_{2s}d_{2k+1}r^{2s+2k+5}M_6^{s,k},\end{aligned}\tag{4.9}$$

where

$$M_1^{s,k}=s\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}C_{s,l}-\frac{(k+1)\alpha_{k+s+1}}{2^{k+s+1}(2s+1)(k+s+2)!},$$
$$M_2^{s,k}=(s+1)\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}K_{s,l}-\frac{(k+1)(4k+10s+15)\alpha_{k+s+1}}{2^{k+s+2}(2s+1)(2s+3)(k+s+3)!},$$
$$M_3^{s,k}=\frac{2s+1}{2}\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\tilde{C}_{s,l}+\frac{3(2k+1)\alpha_{k+s+1}}{2^{k+s+2}(2k-1)(k+s+2)!},$$
$$M_4^{s,k}=\frac{2s+3}{2}\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\tilde{K}_{s,l}+\frac{15(2k+1)\alpha_{k+s}}{2^{k+s+3}(2k-1)(k+s+3)!},$$
$$M_5^{s,k}=s\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}C_{s,l}-\frac{3(k+2)\alpha_{k+s+1}}{2^{k+s+2}(2s+1)(k+s+3)!},$$
$$M_6^{s,k}=(s+1)\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}K_{s,l}-\frac{3(k+2)(4k+14s+21)\alpha_{k+s+1}}{2^{k+s+3}(2s+1)(2s+3)(k+s+4)!}.$$

Proof. From (4.7) and (4.8) we have

$$F_{20}^1(r) = N_1(r) + N_2(r) + N_3(r) + N_4(r) + N_5(r),$$

where

$$\begin{aligned}
N_1(r) =& \frac{-1}{2\pi}\int_0^{2\pi}\sum_{i=0}^{n_1} i a_i r^{i-1} R_i(\theta) y_1(\theta, r)\, d\theta,\\
N_2(r) =& \frac{-1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{[\frac{n_2-1}{2}]} (2k+2) b_{2k+1} r^{2k+1} T_{2k+1}(\theta) y_1(\theta, r)\, d\theta,\\
N_3(r) =& \frac{-1}{2\pi}\int_0^{2\pi}\sum_{i=0}^{n_3} (i+2) c_i r^{i+1} S_i(\theta) y_1(\theta, r)\, d\theta,\\
N_4(r) =& \frac{-1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{[\frac{n_4}{2}]} (2k+4) r^{2k+3} d_{2k+1} U_{2k+1}(\theta) y_1(\theta, r)\, d\theta,\\
N_5(r) =& \frac{-1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{\mu} (2k+1) r^{2k} d_{2k-2}\Big(\cos^{2k-2}\theta - \frac{4k+1}{2k-1}\cos^{2k}\theta\\
&+\frac{2k+2}{2k-1}\cos^{2k+2}\theta\Big) y_1(\theta, r)\, d\theta.
\end{aligned}$$

For simplifying the expression of the polynomial $N_1(r)$, using the integrals of appendix A, we have

$$(\mathbf{a}_1) = \int_0^{2\pi}\left(\sum_{i=0}^{n_1} i a_i r^{i-1} R_i(\theta)\right)\left(-\sum_{j=0}^{n_1} r^j a_j J_j(\theta)\right) = 0.$$

$$\begin{aligned}
(\mathbf{b}_1) =& \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_1} i a_i r^{i-1} R_i(\theta)\right)\left(-\sum_{j=0}^{[\frac{n_2-1}{2}]} r^{2j+2} b_{2j+1}\sum_{l=0}^{j+1}\tilde{\gamma}_{j,l}\sin(2l+1)\theta\right)\\
=& \sum_{s=0}^{\left[\frac{n_1}{2}\right]}\sum_{k=0}^{\left[\frac{n_2-1}{2}\right]} a_{2s} b_{2k+1} s \sum_{l=0}^{k+1}\tilde{\gamma}_{k,l} C_{s,l} r^{2i+2j+1}.
\end{aligned}$$

$$(\mathbf{c}_1) = \int_0^{2\pi}\left(\sum_{i=0}^{n_1} i a_i r^{i-1} R_i(\theta)\right)\left(-\sum_{j=0}^{n_3} c_j r^{j+2}\tilde{J}_j(\theta)\right) = 0.$$

$$\begin{aligned}
(\mathbf{d}_1) =& \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_1} i a_i r^{i-1} R_i(\theta)\right)\left(-\sum_{j=0}^{[\frac{n_4-1}{2}]} r^{2j+4} d_{2j+1}\sum_{l=0}^{j+2}\tilde{\tilde{\gamma}}_{j,l}\sin(2l+1)\theta\right)\\
=& \sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]} a_{2s} d_{2k+1} s \sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l} C_{s,l} r^{2s+2k+3}.
\end{aligned}$$

$$\begin{aligned}
(\mathbf{e}_1) =& \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_1} i a_i r^{i-1} R_i(\theta)\right)\left(-\sum_{i=0}^{\mu} r^{2i+1} d_{2i-2}\sum_{l=1}^{i+1}\tilde{\beta}_{i,l}\sin(2l)\theta\right)\\
=& \sum_{s=0}^{[\frac{n_1-1}{2}]}\sum_{k=0}^{\mu} a_{2s+1} d_{2k-2}\frac{2s+1}{2}\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\tilde{C}_{s,l} r^{2s+2k+1}.
\end{aligned}$$

We have that the sum of the integrals (a_1), (b_1), (c_1), (d_1) and (e_1) is the polynomial $N_1(r)$.

From the integrals of appendix A, we have

$$
\begin{aligned}
(\mathbf{a}_2) \;&= \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_2-1}{2}]}(2k+2)b_{2k+1}r^{2k+1}T_{2k+1}(\theta)\right)\left(-\sum_{j=0}^{n_1}r^j a_j J_j(\theta)\right)\\
&= \sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]} a_{2s}b_{2k+1}\frac{-(k+1)\alpha_{k+s+1}}{2^{k+s+1}(2s+1)(k+s+2)!}r^{2s+2k+1}.
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{b}_2) \;&= \int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_2-1}{2}]}(2k+2)b_{2k+1}r^{2k+1}T_{2k+1}(\theta)\right)\\
&\times\left(-\sum_{j=0}^{[\frac{n_2-1}{2}]}r^{2j+2}b_{2j+1}\sum_{l=0}^{j+1}\tilde{\gamma}_{j,l}\sin(2l+1)\theta\right) = 0.
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{c}_2) \;&= \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_2-1}{2}]}(2k+2)b_{2k+1}r^{2k+1}T_{2k+1}(\theta)\right)\left(-\sum_{i=0}^{n_3}c_i r^{i+2}\tilde{J}_i(\theta)\right)\\
&= \sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]} c_{2s}b_{2k+1}\frac{-(k+1)(4k+10s+15)\alpha_{k+s+1}}{2^{k+s+2}(2s+1)(2s+3)(k+s+3)!}r^{2s+2k+3}.
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{d}_2) \;&= \int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_2-1}{2}]}(2k+2)b_{2k+1}r^{2k+1}T_{2k+1}(\theta)\right)\\
&\times\left(-\sum_{j=0}^{[\frac{n_4-1}{2}]}r^{2j+4}d_{2j+1}\sum_{l=0}^{j+2}\tilde{\tilde{\gamma}}_{j,l}\sin(2l+1)\theta\right) = 0.
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{e}_2) \;&= \int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_2-1}{2}]}(2k+2)b_{2k+1}r^{2k+1}T_{2k+1}(\theta)\right)\\
&\times\left(-\textstyle\sum_{i=0}^{\mu}r^{2i+1}d_{2i-2}\sum_{l=1}^{i+1}\tilde{\beta}_{i,l}\sin(2l)\theta\right) = 0.
\end{aligned}
$$

The sum of the integrals (a_2), (b_2), (c_2), (d_2) and (e_2) is the polynomial $N_2(r)$.

For finding the expression of the polynomial $N_3(r)$, using the integrals of appendix A, we have

$$
(\mathbf{a}_3) \;= \int_0^{2\pi}\left(\sum_{i=0}^{n_3}(i+2)c_i r^{i+1}S_i(\theta)\right)\left(-\sum_{j=0}^{n_1}r^j a_j J_j(\theta)\right) = 0.
$$

$$
\begin{aligned}
(\mathbf{b}_3) \;&= \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_3}(i+2)c_i r^{i+1}S_i(\theta)\right)\left(-\sum_{k=0}^{[\frac{n_2-1}{2}]}r^{2k+2}b_{2k+1}\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}\sin(2l+1)\theta\right)\\
&= \sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]}(s+1)\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}K_{s,l}r^{2s+2k+3}.
\end{aligned}
$$

$$(\mathbf{c}_3)\ = \int_0^{2\pi}\left(\sum_{i=0}^{n_3}(i+2)c_i r^{i+1}S_i(\theta)\right)\left(-\sum_{j=0}^{n_3}c_j r^{j+2}\tilde{J}_j(\theta)\right) = 0.$$

$$(\mathbf{d}_3)\ \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_3}(i+2)c_i r^{i+1}S_i(\theta)\right)\left(-\sum_{k=0}^{[\frac{n_4-1}{2}]} r^{2k+4}d_{2k+1}\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}\sin(2l+1)\theta\right)$$
$$= \sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]} c_{2s}d_{2k+1}(s+1)\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}K_{s,l}r^{2s+2k+5}.$$

$$(\mathbf{e}_3)\ = \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_3}(i+2)c_i r^{i+1}S_i(\theta)\right)\left(-\sum_{k=0}^{\mu} r^{2k+1}d_{2k-2}\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\sin(2l)\theta\right)$$
$$= \sum_{s=0}^{[\frac{n_3-1}{2}]}\sum_{k=0}^{\mu} c_{2s+1}d_{2k-2}\frac{2s+3}{2}\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\tilde{K}_{s,l}r^{2s+2k+3}.$$

We have that the sum of the integrals (a_3), (b_3), (c_3), (d_3) and (e_3) is the polynomial $N_3(r)$.

By using the integrals of appendix A, we have

$$(\mathbf{a}_4)\ = \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4}{2}]}(2k+4)r^{2k+3}d_{2k+1}U_{2k+1}(\theta)\right)\left(-\sum_{j=0}^{n_1}r^j a_j J_j(\theta)\right)$$
$$= \sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]} a_{2s}d_{2k+1}\frac{-3(k+2)\alpha_{k+s+1}}{2^{k+s+2}(2s+1)(k+s+3)!}r^{2s+2k+3}.$$

$$(\mathbf{b}_4)\ = \int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4}{2}]}(2k+4)r^{2k+3}d_{2k+1}U_{2k+1}(\theta)\right)$$
$$\times\left(-\sum_{j=0}^{[\frac{n_2-1}{2}]} r^{2j+2}b_{2j+1}\sum_{l=0}^{j+1}\tilde{\gamma}_{j,l}\sin(2l+1)\theta\right) = 0.$$

$$(\mathbf{c}_4)\ = \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4}{2}]}(2k+4)r^{2k+3}d_{2k+1}U_{2k+1}(\theta)\right)\left(-\sum_{i=0}^{n_3}c_i r^{i+2}\tilde{J}_i(\theta)\right)$$
$$= \sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]} c_{2s}d_{2k+1}\frac{-3(k+2)(4k+14s+21)\alpha_{k+s+1}}{2^{k+s+3}(2s+1)(2s+3)(k+s+4)!}r^{2s+2k+5}.$$

$$(\mathbf{d}_4)\ = \int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4}{2}]}(2k+4)r^{2k+3}d_{2k+1}U_{2k+1}(\theta)\right)$$
$$\times\left(-\sum_{j=0}^{[\frac{n_4-1}{2}]} r^{2j+4}d_{2j+1}\sum_{l=0}^{j+2}\tilde{\tilde{\gamma}}_{j,l}\sin(2l+1)\theta\right) = 0.$$

$$(\mathbf{e}_4)\ = \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4}{2}]}(2k+4)r^{2k+3}d_{2k+1}U_{2k+1}(\theta)\right)\left(-\sum_{i=0}^{\mu} r^{2i+1}d_{2i-2}\sum_{l=1}^{i+1}\tilde{\beta}_{i,l}\sin(2l)\theta\right)$$
$$= 0.$$

We have that the sum of the integrals (a_4), (b_4), (c_4), (d_4) and (e_4) is the polynomial $N_4(r)$.

Finally, for computing the polynomial $N_5(r)$, using the integrals of appendix A, we have

$$
\begin{aligned}
(\mathbf{a}_5) \;&= \frac{-1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{\mu}(2k+1)r^{2k}d_{2k-2}\left(\cos^{2k-2}\theta-\frac{4k+1}{2k-1}\cos^{2k}\theta+\frac{2k+2}{2k-1}\cos^{2k+2}\theta\right)\right)\\
&\quad\times\left(-\sum_{j=0}^{n_1} r^j a_j J_j(\theta)\right)\\
&= \sum_{s=0}^{[\frac{n_1-1}{2}]}\sum_{k=0}^{\mu} a_{2s+1}d_{2k-2}\frac{3(2k+1)\alpha_{k+s+1}}{2^{k+s+2}(2k-1)(k+s+2)!}r^{2s+2k+1}.
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{b}_5) \;&= \int_0^{2\pi}\left(\sum_{k=0}^{\mu}(2k+1)r^{2k}d_{2k-2}\left(\cos^{2k-2}\theta-\frac{4k+1}{2k-1}\cos^{2k}\theta+\frac{2k+2}{2k-1}\cos^{2k+2}\theta\right)\right)\\
&\quad\times\left(-\sum_{j=0}^{[\frac{n_2-1}{2}]} r^{2j+2}b_{2j+1}\sum_{l=0}^{j+1}\tilde{\gamma}_{j,l}\sin(2l+1)\theta\right)=0.
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{c}_5) \;&= \frac{-1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{\mu}(2k+1)r^{2k}d_{2k-2}\left(\cos^{2k-2}\theta-\frac{4k+1}{2k-1}\cos^{2k}\theta+\frac{2k+2}{2k-1}\cos^{2k+2}\theta\right)\\
&\quad\times\left(-\sum_{i=0}^{n_3} c_i r^{i+2}\tilde{J}_i(\theta)\right)\\
&= \sum_{s=0}^{[\frac{n_3-1}{2}]}\sum_{k=0}^{\mu} c_{2s+1}d_{2k-2}\frac{15(2k+1)\alpha_{k+s}}{2^{k+s+3}(2k-1)(k+s+3)!}r^{2s+2k+3}.
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{d}_5) \;&= \int_0^{2\pi}\left(\sum_{k=0}^{\mu}(2k+1)r^{2k}d_{2k-2}\left(\cos^{2k-2}\theta-\frac{4k+1}{2k-1}\cos^{2k}\theta+\frac{2k+2}{2k-1}\cos^{2k+2}\theta\right)\right)\\
&\quad\times\left(-\sum_{j=0}^{[\frac{n_4-1}{2}]} r^{2j+4}d_{2j+1}\sum_{l=0}^{j+2}\tilde{\tilde{\gamma}}_{j,l}\sin(2l+1)\theta\right)=0.
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{e}_5) \;&= \int_0^{2\pi}\left(\sum_{k=0}^{\mu}(2k+1)r^{2k}d_{2k-2}\left(\cos^{2k-2}\theta-\frac{4k+1}{2k-1}\cos^{2k}\theta+\frac{2k+2}{2k-1}\cos^{2k+2}\theta\right)\right)\\
&\quad\times\left(-\sum_{i=0}^{\mu} r^{2i+1}d_{2i-2}\sum_{l=1}^{i+1}\tilde{\beta}_{i,l}\sin(2l)\theta\right)=0.
\end{aligned}
$$

We have that the sum of the integrals (a_5), (b_5), (c_5), (d_5) and (e_5) is the polynomial $N_5(r)$. Hence Lemma 4.1 is proved. □

Lemma 4.2. *The integral $F_{20}^2(r)$ is a polynomial in the variable r given by*

$$
\begin{aligned}
F_{20}^2(r) = &-\sum_{s=0}^{[\frac{n_2}{2}]} r^{2s+1}\frac{\alpha_s}{2^{s+1}(s+1)!}q_{2s}-\sum_{s=0}^{[\frac{n_4}{2}]} r^{2s+3}\frac{3\alpha_s}{2^{s+2}(s+2)!}w_{2s}-\sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]} a_{2s}b_{2k+1}\\
&\times r^{2s+2k+1}\frac{\alpha_{s+k+1}}{2^{s+k+1}(s+k+2)!}-\sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]} c_{2s}b_{2k+1}r^{2s+2k+3}\frac{3\alpha_{s+k+1}}{2^{s+k+2}(s+k+3)!}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{k=0}^{[\frac{n_1}{2}]}\sum_{s=0}^{[\frac{n_4-1}{2}]} a_{2k}d_{2s+1}r^{2s+2k+3}\frac{3\alpha_{s+k+1}}{2^{s+k+2}(s+k+3)!}+\sum_{k=0}^{[\frac{n_1-1}{2}]}\sum_{s=0}^{\mu} a_{2k+1}d_{2s-2}\\
&\times r^{2s+2k+1}\frac{3(k+1)\alpha_{s+k}}{2^{s+k}(2s-1)(s+k+2)!}-\sum_{k=0}^{[\frac{n_3}{2}]}\sum_{s=0}^{[\frac{n_4-1}{2}]} c_{2k}d_{2s+1}r^{2s+2k+3}\frac{15}{2^{s+k+3}}\\
&\times\frac{\alpha_{s+k+1}}{(s+k+4)!}+\sum_{k=0}^{[\frac{n_3-1}{2}]}\sum_{s=0}^{\mu} c_{2k+1}d_{2s-2}r^{2s+2k+3}\frac{3(3k-2s+4)\alpha_{s+k}}{2^{s+k+1}(2s-1)(s+k+3)!}.
\end{aligned}
\tag{4.10}
$$

Proof. Using (4.5) and, substituting in (4.4) we have

$$
\begin{aligned}
F_2(r,\theta) =& -\sum_{i=0}^{n_1} p_i r^i R_i(\theta) - \sum_{i=0}^{n_2} q_i r^{i+1}T_i(\theta) - \sum_{i=0}^{n_3} s_i r^{i+2}S_i(\theta) - \sum_{i=0}^{n_4} w_i r^{i+3}U_i(\theta)\\
&-\Bigg(\sum_{i=0}^{n_1} a_i r^{i-1}R_i(\theta) + \sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+1}b_{2k+1}T_{2k+1}(\theta) + \sum_{i=0}^{n_3} c_i r^{i+1}S_i(\theta)\\
&+\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1}r^{2k+3}U_{2k+1}(\theta) + \sum_{k=0}^{\mu} d_{2k-2}r^{2k}(T_{2k-2}(\theta) - \frac{2k+2}{2k-1}T_{2k}(\theta))\Bigg)\\
&\times\Bigg(\sum_{i=0}^{n_1} a_i r^i \cos^{i+1}(\theta) + \sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+2}b_{2k+1}R_{2k+2}(\theta) + \sum_{i=0}^{n_3} c_i r^{i+2}T_{i+1}(\theta)\\
&+\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1}r^{2k+4}S_{2k+2}(\theta) + \sum_{k=0}^{\mu} d_{2k-2}r^{2k+1}(R_{2k-1}(\theta) - \frac{2k+2}{2k-1}R_{2k+1}(\theta))\Bigg).
\end{aligned}
$$

For an explicit expression of the polynomial $F_{20}^2(r)$, using the integrals of appendix A we have

$$
\Lambda_1 = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_1} a_i r^{i-1}R_i(\theta)\right)\left(\sum_{j=0}^{n_1} a_j r^j \cos^{j+1}(\theta)\right) = 0.
$$

$$
\begin{aligned}
\Lambda_2 &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_1} a_i r^{i-1}R_i(\theta)\right)\left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+2}b_{2k+1}R_{2k+2}(\theta)\right)\\
&= \sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]}\frac{\alpha_{k+s+1}}{2^{k+s+2}(k+s+2)!}a_{2s}b_{2k+1}r^{2s+2k+1}.
\end{aligned}
$$

$$
\Lambda_3 = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_1} a_i r^{i-1}R_i(\theta)\right)\left(\sum_{i=0}^{n_3} c_i r^{i+2}T_{i+1}(\theta)\right) = 0.
$$

$$
\Lambda_4 = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_1} a_i r^{i-1}R_i(\theta)\right)\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1}r^{2k+4}S_{2k+2}(\theta)\right)
$$

$$= \sum_{s=0}^{[\frac{n_1}{2}]} \sum_{k=0}^{[\frac{n_4-1}{2}]} \frac{3\alpha_{k+s+1}}{2^{k+s+3}(k+s+3)!} a_{2s} d_{2k+1} r^{2s+2k+3}.$$

$$\Lambda_5 = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{n_1} a_i r^{i-1} R_i(\theta)\right)\left(\sum_{s=0}^{\mu} d_{2s-2} r^{2s+1}(R_{2s-1}(\theta) - \frac{2s+2}{2s-1} R_{2s+1}(\theta))\right)$$

$$= \sum_{k=0}^{[\frac{n_1-1}{2}]} \sum_{s=0}^{\mu} a_{2k+1} d_{2s-2} r^{2s+2k+1} \frac{-3(k+1)\alpha_{s+k}}{2^{s+k+1}(2s-1)(s+k+2)!}.$$

$$\Lambda_6 = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+1} b_{2k+1} T_{2k+1}(\theta)\right)\left(\sum_{j=0}^{n_1} a_j r^j \cos^{j+1}(\theta)\right)$$

$$= \sum_{s=0}^{[\frac{n_1}{2}]} \sum_{k=0}^{[\frac{n_2-1}{2}]} \frac{\alpha_{k+s+1}}{2^{k+s+2}(k+s+2)!} a_{2s} b_{2k+1} r^{2s+2k+1}.$$

$$\Lambda_7 = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+1} b_{2k+1} T_{2k+1}(\theta)\right)\left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+2} b_{2k+1} R_{2k+2}(\theta)\right) = 0.$$

$$\Lambda_8 = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+1} b_{2k+1} T_{2k+1}(\theta)\right)\left(\sum_{i=0}^{n_3} c_i r^{i+2} T_{i+1}(\theta)\right)$$

$$= \sum_{s=0}^{[\frac{n_3}{2}]} \sum_{k=0}^{[\frac{n_2-1}{2}]} \frac{3\alpha_{k+s+1}}{2^{k+s+3}(k+s+3)!} c_{2s} b_{2k+1} r^{2s+2k+3}.$$

$$\Lambda_9 = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+1} b_{2k+1} T_{2k+1}(\theta)\right)\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1} r^{2k+4} S_{2k+2}(\theta)\right) = 0.$$

$$\Lambda_{10} = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+1} b_{2k+1} T_{2k+1}(\theta)\right)$$

$$\times \left(\sum_{k=0}^{\mu} d_{2k-2} r^{2k+1}(R_{2k-1}(\theta) - \frac{2k+2}{2k-1} R_{2k+1}(\theta))\right) = 0.$$

$$\Lambda_{11} = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{n_3} c_i r^{i+1} S_i(\theta)\right)\left(\sum_{j=0}^{n_1} a_j r^j \cos^{j+1}(\theta)\right) = 0.$$

$$\Lambda_{12} = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{n_3} c_i r^{i+1} S_i(\theta)\right)\left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+2} b_{2k+1} R_{2k+2}(\theta)\right)$$

$$= \sum_{s=0}^{[\frac{n_3}{2}]} \sum_{k=0}^{[\frac{n_2-1}{2}]} \frac{3\alpha_{k+s+1}}{2^{k+s+3}(k+s+3)!} c_{2s} b_{2k+1} r^{2s+2k+3}.$$

$$\Lambda_{13} = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i=0}^{n_3} c_i r^{i+1} S_i(\theta)\right)\left(\sum_{i=0}^{n_3} c_i r^{i+2} T_{i+1}(\theta)\right) = 0.$$

$$\Lambda_{14} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_3} c_i r^{i+1} S_i(\theta)\right)\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1} r^{2k+4} S_{2k+2}(\theta)\right)$$

$$= \sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]} \frac{15\alpha_{k+s+1}}{2^{k+s+4}(k+s+4)!} c_{2s} d_{2k+1} r^{2s+2k+3}.$$

$$\Lambda_{15} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n_3} c_i r^{i+1} S_i(\theta)\right)\left(\sum_{s=0}^{\mu} d_{2s-2} r^{2s+1}(R_{2s-1}(\theta) - \frac{2s+2}{2s-1} R_{2s+1}(\theta))\right)$$

$$= \sum_{k=0}^{[\frac{n_3-1}{2}]}\sum_{s=0}^{\mu} c_{2k+1} d_{2s-2} r^{2s+2k+3} \frac{-3(3k-2s+4)\alpha_{s+k}}{2^{s+k+2}(2s-1)(s+k+3)!}.$$

$$\Lambda_{16} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1} r^{2k+3} U_{2k+1}(\theta)\right)\left(\sum_{j=0}^{n_1} a_j r^j \cos^{j+1}(\theta)\right)$$

$$= \sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]} \frac{3\alpha_{k+s+1}}{2^{k+s+3}(k+s+3)!} a_{2s} d_{2k+1} r^{2s+2k+3}.$$

$$\Lambda_{17} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1} r^{2k+3} U_{2k+1}(\theta)\right)\left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+2} b_{2k+1} R_{2k+2}(\theta)\right) = 0.$$

$$\Lambda_{18} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1} r^{2k+3} U_{2k+1}(\theta)\right)\left(\sum_{i=0}^{n_3} c_i r^{i+2} T_{i+1}(\theta)\right)$$

$$= \sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]} \frac{15\alpha_{k+s+1}}{2^{k+s+4}(k+s+4)!} c_{2s} d_{2k+1} r^{2s+2k+3}.$$

$$\Lambda_{19} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1} r^{2k+3} U_{2k+1}(\theta)\right)\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1} r^{2k+4} S_{2k+2}(\theta)\right) = 0.$$

$$\Lambda_{20} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1} r^{2k+3} U_{2k+1}(\theta)\right)$$

$$\times\left(\sum_{k=0}^{\mu} d_{2k-2} r^{2k+1}(R_{2k-1}(\theta) - \frac{2k+2}{2k-1} R_{2k+1}(\theta))\right) = 0.$$

$$\Lambda_{21} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{s=0}^{\mu} d_{2s-2} r^{2s}(T_{2s-2}(\theta) - \frac{2s+2}{2s-1} T_{2s}(\theta))\right)\left(\sum_{j=0}^{n_1} a_j r^j \cos^{j+1}(\theta)\right)$$

$$= \sum_{k=0}^{[\frac{n_1-1}{2}]}\sum_{s=0}^{\mu} a_{2k+1} d_{2s-2} r^{2s+2k+1} \frac{-3(k+1)\alpha_{s+k}}{2^{s+k+1}(2s-1)(s+k+2)!}.$$

$$\Lambda_{22} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{\mu} d_{2k-2} r^{2k}(T_{2k-2}(\theta) - \frac{2k+2}{2k-1} T_{2k}(\theta))\right)$$

$$\times\left(\sum_{k=0}^{[\frac{n_2-1}{2}]} r^{2k+2}b_{2k+1}R_{2k+2}(\theta)\right)=0.$$

$$\Lambda_{23}=\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{s=0}^{\mu} d_{2s-2}r^{2s}(T_{2s-2}(\theta)-\frac{2s+2}{2s-1}T_{2s}(\theta))\right)\left(\sum_{i=0}^{n_3} c_i r^{i+2}T_{i+1}(\theta)\right)$$
$$=\sum_{k=0}^{[\frac{n_3-1}{2}]}\sum_{s=0}^{\mu} c_{2k+1}d_{2s-2}r^{2s+2k+3}\frac{-3(3k-2s+4)\alpha_{s+k}}{2^{s+k+2}(2s-1)(s+k+3)!}.$$

$$\Lambda_{24}=\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{\mu} d_{2k-2}r^{2k}(T_{2k-2}(\theta)-\frac{2k+2}{2k-1}T_{2k}(\theta))\right)$$
$$\times\left(\sum_{k=0}^{[\frac{n_4-1}{2}]} d_{2k+1}r^{2k+4}S_{2k+2}(\theta)\right)=0.$$

$$\Lambda_{25}=\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{\mu} d_{2k-2}r^{2k}(T_{2k-2}(\theta)-\frac{2k+2}{2k-1}T_{2k}(\theta))\right)$$
$$\times\left(\sum_{k=0}^{\mu} d_{2k-2}r^{2k+1}(R_{2k-1}(\theta)-\frac{2k+2}{2k-1}R_{2k+1}(\theta))\right)=0.$$

We have that the sum of the integrals from Λ_1 to Λ_{25} is the polynomial (4.10). Hence Lemma 4.2 is proved. □

By Lemma 4.1 and Lemma 4.2, we have

$$F_{20}(r)=F_{20}^1(r)+F_{20}^2(r)$$
$$=-\sum_{s=0}^{[\frac{n_2}{2}]} r^{2s+1}\frac{\alpha_s}{2^{s+1}(s+1)!}q_{2s}-\sum_{s=0}^{[\frac{n_4}{2}]} r^{2s+3}\frac{3\alpha_s}{2^{s+2}(s+2)!}w_{2s}+\sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]} a_{2s}b_{2k+1}$$
$$\times Ar^{2s+2k+1}+\sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_2-1}{2}]} c_{2s}b_{2k+1}Br^{2s+2k+3}+\sum_{s=0}^{[\frac{n_1}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]} a_{2s}d_{2k+1}Cr^{2s+2k+3}$$
$$+\sum_{s=0}^{[\frac{n_1-1}{2}]}\sum_{k=0}^{\mu} a_{2s+1}d_{2k-2}Dr^{2s+2k+1}+\sum_{s=0}^{[\frac{n_3}{2}]}\sum_{k=0}^{[\frac{n_4-1}{2}]} c_{2s}d_{2k+1}Er^{2s+2k+5}$$
$$+\sum_{s=0}^{[\frac{n_3-1}{2}]}\sum_{k=0}^{\mu} c_{2s+1}d_{2k-2}Fr^{2s+2k+3},$$

where

$$A=s\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}C_{s,l}-\frac{(2s+k+2)\alpha_{k+s+1}}{2^{k+s+1}(2s+1)(k+s+2)!},$$

$$B=(s+1)\sum_{l=0}^{k+1}\tilde{\gamma}_{k,l}K_{s,l}-\frac{(12s^2+4k^2+34s+10sk+19k+24)\alpha_{k+s+1}}{2^{k+s+2}(2s+1)(2s+3)(k+s+3)!},$$

$$C=s\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}C_{s,l}-\frac{3(2s+k+3)\alpha_{k+s+1}}{2^{k+s+2}(2s+1)(k+s+3)!},$$

$$D = \frac{(2s+1)}{2}\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\tilde{C}_{s,l} + \frac{3(4s+2k+5)\alpha_{k+s}}{2^{k+s+2}(2k-1)(k+s+2)!},$$

$$E = (s+1)\sum_{l=0}^{k+2}\tilde{\tilde{\gamma}}_{k,l}K_{s,l} - \frac{(20s^2+68s+57+4k^2+14sk+29k)3\alpha_{k+s+1}}{2^{k+s+3}(2s+1)(2s+3)(k+s+4)!},$$

$$F = \frac{(2s+3)}{2}\sum_{l=1}^{k+1}\tilde{\beta}_{k,l}\tilde{K}_{s,l} - \frac{3(12s+2k+21)\alpha_{k+s}}{2^{k+s+3}(2k-1)(k+s+3)!}.$$

We conclude that F_{20} has at most $\max\{[\frac{n_2}{2}], [\frac{n_4}{2}]+1, [\frac{n_1}{2}]+[\frac{n_2-1}{2}], [\frac{n_1}{2}]+[\frac{n_4-1}{2}]+1, [\frac{n_1-1}{2}]+\mu', [\frac{n_2-1}{2}]+[\frac{n_3}{2}]+1, [\frac{n_4-1}{2}]+[\frac{n_3}{2}]+2, [\frac{n_3-1}{2}]+\mu'+1\}$ positive roots. Hence the Theorem 1.2 follows.

5. Applications

In this section we shall prove the corollaries.

Proof of Corollary 1.1. We consider system (1.4), where

$$f_1(x) = x+2x^2+x^3,\ g_1(x) = -2+3x-\tfrac{1}{3}x^2,\ h_1(x) = 1-x,$$
$$l_1(x) = 5+x-16x^2+5x^3+\tfrac{64}{9}x^4-x^5,$$

we have $n_1 = 3$, $n_2 = 2$, $n_3 = 1$ and $n_4 = 5$.

The function of averaging theory of first order (3.5) is

$$F_{10}(r) = -\frac{r}{6}\left(r^6-6r^4+11r^2-6\right)$$

that has exactly three positive zeros. In [18], this system when $d_0 \neq 0$ and $m = 6$ has at most two positive zeros. □

Proof of Corollary 1.2. We consider system (1.4), where

$$f_1(x) = x+2x^2+3x^3-2x^4,\ g_1(x) = 3x-3x^2-5x^3,\ h_1(x) = 1-x,$$
$$l_1(x) = 1-3x+\tfrac{10160}{3969}x^3+\tfrac{64}{189}x^5,\ f_2(x) = x^4,\ g_2(x) = 2-\tfrac{454}{15}x^2+3x^3,$$
$$h_2(x) = x,\ l_2(x) = 2+\tfrac{262}{3}x^2-\tfrac{475448}{11907}x^4-x^5,$$

we have $n_1 = 4$, $n_2 = 3$, $n_3 = 1$ and $n_4 = 5$.

The function of averaging theory of first order (3.5) is $F_{10}(r) = 0$. By using the averaging theory of second order, then from (4.11) we obtain

$$F_{20}(r) = \frac{r}{120}\left(r^{10}-15r^8+85r^6-225r^4+274r^2-120\right)$$

that has exactly five positive zeros. In [18], this system when $d_0 \neq 0$ and $m = 6$ has at most four positive zeros.

Appendix A. Formulae

In this appendix we recall some formulae that will be used during the paper, see for more details [1]. For $i \geq 0$ we have

$$J_i(\theta) = \int_0^\theta \cos^i t \sin t dt = \frac{1}{i+1}(1-\cos^{i+1}\theta).$$

$$\tilde{J}_i(\theta) = \int_0^\theta \cos^i t \sin^3 t dt = \frac{2}{(i+1)(i+3)} - \frac{1}{i+1}\cos^{i+1}\theta + \frac{1}{i+3}\cos^{i+3}\theta.$$

$$J_i(2\pi) = \tilde{J}_i(2\pi) = 0.$$

$$I_i(\theta) = \int_0^{2\pi} \cos^i \theta d\theta = \begin{cases} \sum_{l=0}^{k} \gamma_{k,l}\sin(2l+1)\theta, & \text{if i=2k+1;} \\ \delta_k + \sum_{l=1}^{k} \beta_{k,l}\sin(2l\theta), & \text{if i=2k,} \end{cases}$$

where

$$\delta_i = \frac{1}{2^{2i}}\binom{2i}{i}\theta, \quad \gamma_{i,l} = \frac{1}{2^{2i}}\binom{2i+1}{i-l}\frac{1}{2l+1}, \quad \beta_{i,l} = \binom{2i}{i+l}\frac{1}{l}.$$

$$I_i(2\pi) = \begin{cases} 0, & \text{if i=2k+1,} \\ \dfrac{\pi\alpha_k}{2^{k-1}k!}, & \text{if i=2k,} \end{cases}$$

where

$$\alpha_k = 3\cdot 5\cdot \cdots \cdot (2k-1), \ \ \alpha_{k+1} = (2k+1)\alpha_k$$

and

$$I_{2k+2}(2\pi) = \frac{2k+1}{2k+2}I_{2k}(2\pi).$$

$$\tilde{I}_{2k}(2\pi) = I_{2k}(2\pi) - I_{2k+2}(2\pi).$$

$$\tilde{\tilde{I}}_{2k}(2\pi) = I_{2k}(2\pi) - 2I_{2k+2}(2\pi) + I_{2k+4}(2\pi).$$

$$\int_0^{2\pi} \cos^i\theta \sin^j\theta \sin(2l+1)\theta d\theta \neq 0, \text{if i even and j odd,}$$

$$\int_0^{2\pi} \cos^i\theta \sin^j\theta \sin(2l+1)\theta d\theta = \begin{cases} 0, & \text{if i odd or j even;} \\ \pi C_{k,l}, & \text{i=2k, j=1 and l} \geq 0, \\ \pi K_{k,l}, & \text{i=2k, j=3 and l} \geq 0, \end{cases}$$

where $C_{k,l}$, $K_{k,l}$ are non-zero constants.

$$\int_0^{2\pi} \cos^i\theta \sin^j\theta \sin(2l\theta) d\theta \neq 0, \text{ if } i \text{ and } j \text{ odd,}$$

$$\int_0^{2\pi} \cos^i\theta \sin^j\theta \sin(2l\theta)d\theta \begin{cases} 0, & \text{if i odd or j even;} \\ \pi\tilde{C}_{i,l}, & \text{if i odd j=1 and } l \geq 0, \\ \pi\tilde{K}_{i,l}, & \text{if i odd j=3 and } l \geq 0, \end{cases}$$

where $\tilde{C}_{k,l}$, $\tilde{K}_{k,l}$ are non-zero constants.

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