INVARIANT MEASURES FOR SYSTEMS OF KOLMOGOROV EQUATIONS*

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To Claude-Michel Brauner on the occasion of his 70th birthday.[‡]

Abstract In this paper we provide sufficient conditions which guarantee the existence of a system of invariant measures for semigroups associated to systems of parabolic differential equations with unbounded coefficients. We prove that these measures are absolutely continuous with respect to the Lebesgue measure and study some of their main properties. Finally, we show that they characterize the asymptotic behaviour of the semigroup at infinity.

Keywords Systems of Kolmogorov equations, unbounded coefficients, gradient estimates, systems of invariant measures, asymptotic behaviour.

MSC(2010) 35B40, 35B41, 35K45, 37L40, 47A35.

1. Introduction

In this paper we prove the existence of systems $\{\mu_i : i = 1, \ldots, m\}$ of finite signed Borel measures which satisfy the equality

$$\sum_{i=1}^{m} \int_{\mathbb{R}^d} (\mathbf{T}(t)\mathbf{f})_i d\mu_i = \sum_{i=1}^{m} \int_{\mathbb{R}^d} f_i d\mu_i, \qquad i = 1, \ldots, m, \tag{1.1}$$

for any $t \geq 0$ and $\mathbf{f} \in B_b(\mathbb{R}^d; \mathbb{R}^m)$ (the space of vector-valued bounded Borel functions $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^m$) or, equivalently, for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ (the subspace of $B_b(\mathbb{R}^d; \mathbb{R}^m)$ consisting of continuous functions). Here, $\{\mathbf{T}(t)\}_{t \geq 0}$ (from now on simply denoted by $\mathbf{T}(t)$) is the semigroup of bounded linear operators on $B_b(\mathbb{R}^d; \mathbb{R}^m)$, associated to the vector-valued differential operator $\boldsymbol{\mathcal{A}}$, defined on smooth functions $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_m)$ by

$$(\boldsymbol{\mathcal{A}}\boldsymbol{\zeta})_j(x) := \sum_{h,k=1}^{d} q_{hk}(x) \frac{\partial^2 \zeta_j}{\partial x_h x_k}(x) + \sum_{k=1}^{d} b_k(x) \frac{\partial \zeta_j}{\partial x_k}(x) + \sum_{k=1}^{d} \sum_{i=1}^{m} (B_k(x))_{ji} \frac{\partial \zeta_i}{\partial x_k}(x) \tag{1.2}$$

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[‡]L.L.: I met Claude-Michel for the first time when I was preparing my Ph.D. thesis. He always encouraged me and stimulated my research. In the years he shared and still shares with me his deep knowledge and passion for mathematics. Merci mon ami Claude-Michel!

for any $x \in \mathbb{R}^d$ and $j = 1, \ldots, m$, under suitable assumptions on its coefficients. Formula (1.1) seems the natural vector-valued counterpart of the invariant measure of the scalar case. A probability measure is called invariant for a Markov semigroup $\{T(t)\}_{t \geq 0}$ (from now on simply denoted by $T(t)$) associated in $B_b(\mathbb{R}^d)$ to an elliptic operator (with unbounded coefficients) $\mathcal{A}$ if

$$\int_{\mathbb{R}^d} T(t)f d\mu = \int_{\mathbb{R}^d} f d\mu, \qquad t > 0, \ f \in B_b(\mathbb{R}^d).$$

For this reason, we call *system of invariant measures* for $\mathbf{T}(t)$ any family $\{\mu_i : i = 1, \ldots, m\}$ of finite measures which satisfy (1.1).

In the scalar case, under quite mild (algebraic) conditions on the coefficients of the operator $\mathcal{A}$, there exists a unique invariant measure μ and μ is equivalent to the Lebesgue measure (see Lorenzi [16] and Metafune etc [23]). This measure plays an essential role in the analysis of the semigroup $T(t)$. Indeed, if the coefficients of $\mathcal{A}$ are unbounded, then the L^p-spaces with respect to the measure μ (say $L^p(\mathbb{R}^d, \mu)$) are the best L^p-setting where to study the semigroup $T(t)$ (see e.g., Angiuli etc [4,5], Davies etc [9], Farkas etc [11], Gross [12], Lorenzi etc [17], Lunardi [19], Metafune etc [22], Röckner etc [26], Wang [27]). As it is shown in Prüss etc [25], the usual Lebesgue L^p-spaces are not, in general, a good setting for the semigroup $T(t)$, unless (restrictive) assumptions are prescribed on the coefficients of the associated elliptic operator (see also [6] for the vector-valued case). As a matter of fact, the measure μ is not explicit in general. In any case, both local and global regularity results for its density ρ with respect to the Lebesgue measure are known in many cases (see e.g., Bogachev etc [7], Lorenzi [16] and Metafune etc [21]). The typical way to study the regularity of ρ is to see it as a distributional solution of the adjoint equation $\mathcal{A}^* \rho = 0$.

The relevance of the invariant measure μ lies also on the fact that it allows to characterize the asymptotic behaviour, as t tends to $+\infty$, of the semigroup they are associated to. Indeed, for any $f \in L^p(\mathbb{R}^d, \mu)$, $T(t)f$ converges in $L^p(\mathbb{R}^d, \mu)$ to the average of f with respect to the measure μ, as t tends to $+\infty$ (see e.g., Da Prato etc [8] and Lorenzi [16]). The convergence is also local uniform in $\mathbb{R}^d$ if f is bounded and continuous (see e.g., Lorenzi etc [18]).

For semigroups associated to systems of elliptic equations, as the case that we are considering here, the situation is much more complicated and the picture is still far to be completely clear. One of the most typical feature of the scalar case is the positivity of the semigroup $T(t)$, which follows from a variant of the classical maximum principle. This property and the ergodicity of $T(t)$ imply in a straightforward way that, whenever it exists, μ is a positive measure since

$$\int_{\mathbb{R}^d} f d\mu = \lim_{t \to +\infty} \frac{1}{t} \int_0^t (T(s)f)(0)ds, \qquad f \in C_b(\mathbb{R}^d).$$

As it is well known, already in the case of bounded and smooth enough coefficients, in general the semigroups associated to systems of elliptic operators do not preserve the cone of functions $\mathbf{f}$ with all nonnegative components. Indeed, Otsuka [24] shows that the semigroup associated to the operator $\mathcal{A}_0$ (with smooth and bounded coefficients), defined on smooth functions ζ by

$$(\mathcal{A}_0 \zeta)_j(x) := \sum_{h,k=1}^{d} q_{hk}(x) \frac{\partial^2 \zeta_j}{\partial x_h x_k}(x) + \sum_{k=1}^{d} \sum_{i=1}^{m} (B_k(x))_{ji} \frac{\partial \zeta_i}{\partial x_k}(x) + \sum_{i=1}^{m} C_{ji} \zeta_i(x)$$

for any $x \in \mathbb{R}^d$ and $j = 1, \dots, m$, is positive if and only if the drift terms of $\boldsymbol{\mathcal{A}}_0$ are diagonal, i.e., $(B_k) = b_k I_m$ for any $k = 1, \dots, d$, and the potential matrix C has nonnegative elements outside the main diagonal (see also Maz'ia etc [15]).

To the best of our knowledge, the first paper which deals with systems of invariant measures for systems of Kolmogorov equations is Addona etc [2], where the case of *weakly coupled* systems with a potential term is considered, i.e., in that situation the operator $\boldsymbol{\mathcal{A}}$ is defined, on smooth enough functions $\boldsymbol{\zeta}$, by

$$(\boldsymbol{\mathcal{A}}\boldsymbol{\zeta})_j(x) = \sum_{h,k=1}^d q_{hk}(x)\frac{\partial^2 \zeta_j(x)}{\partial x_h x_k} + \sum_{k=1}^d b_k(x)\frac{\partial \zeta_j}{\partial x_k}(x) + \sum_{i=1}^m (C(x))_{ji}\zeta_i(x)$$

for any $j = 1, \dots, m$, $x \in \mathbb{R}^d$. Under suitable conditions on C which, in particular, imply that the associated semigroup $\mathbf{T}(t)$ is bounded, in Addona etc [2] we prove that the semigroup $\mathbf{T}(t)$ also preserves the cone of nonnegative functions and this makes the analysis easier. In particular, we are able to characterize all the systems of invariant measures in terms of the invariant measure of the scalar semigroup $T(t)$ associated to the operator $\mathcal{A} = \mathrm{Tr}(QD^2) + \langle \mathbf{b}, \nabla \rangle$ in the space of bounded and continuous functions over $\mathbb{R}^d$.

This paper represents the second step in a better understanding of systems of invariant measures, its analogies and differences with the invariant measure of the scalar case. Motivated by the scalar case and also by the results in Addona etc [2] we would like to define a system of invariant measure through the limit (in a suitable sense)

$$\lim_{t\to 0} \frac{1}{t} \int_0^t (\mathbf{T}(s)\mathbf{f})ds, \tag{1.3}$$

when $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^m$ is an arbitrary bounded and continuous function. The first problem that we have to face is that, in the scalar case, $T(t)$ is a bounded semigroup. In general, this is no more true for semigroups associated to systems of Kolmogorov equations coupled up to the first-order (see Remark 2.1). As it is shown in Addona etc [1] the semigroup $\mathbf{T}(t)$ admits the integral representation

$$(\mathbf{T}(t)\mathbf{f})_i(x) = \sum_{j=1}^m \int_{\mathbb{R}^d} f_j(y)p_{ij}(t,x,dy), \qquad \mathbf{f} \in B_b(\mathbb{R}^d; \mathbb{R}^m), \ i = 1, \dots, m,$$

for any $x \in \mathbb{R}^d$, where each $p_{ij}(t, x, dy)$ is a signed measure. In Proposition 2.1 we show that the boundedness of $\mathbf{T}(t)$ is equivalent to the boundedness of the family of measures $\{|p_{ij}|(t,x,dy) : t > 0, x \in \mathbb{R}^d\}$ for any $i, j = 1, \dots, m$, where $|p_{ij}|(t,x,dy)$ denotes the total variation of the measure $p_{ij}(t,x,dy)$. Nevertheless, even if this condition is not satisfied, under suitable conditions and using the pointwise gradient estimate in Proposition 2.4 we prove that, for each $x_0 \in \mathbb{R}^d$ and each $\mathbf{f} \in B_b(\mathbb{R}^d; \mathbb{R}^m)$, the function $(\mathbf{T}(\cdot)\mathbf{f})(x_0)$ is bounded in $(0, +\infty)$ (see Theorem 2.4). This fundamental result allows us to prove that the limit in (1.3) exists in the sense that the function $x \mapsto \frac{1}{t} \int_0^t (\mathbf{T}(s)\mathbf{f})(x)ds$ converges locally uniformly in $\mathbb{R}^d$ as t tends to $+\infty$. The limit $\mathbf{g_f}$, which is a continuous, but *a priori* an *unbounded* function, has a controlled growth at infinity, and this property allows us to apply the semigroup $\mathbf{T}(t)$ to such a function $\mathbf{g_f}$. It turns out that $\mathbf{T}(t)\mathbf{g_f} = \mathbf{g_f}$ for any $t > 0$, i.e., $\mathbf{g_f}$ is a fixed point of the semigroup $\mathbf{T}(t)$. Using again the pointwise gradient estimates, we can then conclude that $\mathbf{g_f}$ is a constant function. This allows us to define m

systems of invariant measures for the semigroup $\mathbf{T}(t)$ (say, $\{\mu_j^i : j = 1, \ldots, m\}$, $i = 1, \ldots, m$).

We then exploit some properties of the above systems of invariant measures. We show that each measure μ_j^i is absolutely continuous with respect to the Lebesgue measure and prove some regularity and integrability properties of the density of their total variations with respect to the Lebesgue measure (see Proposition 3.2 and Theorem 3.3). We also prove that a suitable unbounded function ζ (which is a power of the Lyapunov function of the scalar operator $\mathcal{A} = \text{Tr}(QD^2) + \langle \mathbf{b}, \nabla \rangle$, see Hypothesis 2.1(iii)) is summable with respect to all the measures $|\mu_j^i|$. This gives a first partial information on the structure of the spaces $L^p(\mathbb{R}^d, |\mu_j^i|)$ and, combined with Theorem 2.4(i), shows that $|\mathbf{T}(t)\mathbf{f}|$ is in $L^p(\mathbb{R}^d, |\mu_j^i|)$ for any $\mathbf{f} \in B_b(\mathbb{R}^d, \mathbb{R}^m)$ and $p \leq p_*$ for a suitable exponent p_* (explicitly computable). Then, in Theorem 3.2 we characterize all the systems of invariant measures $\{\mu_j : j = 1, \ldots, m\}$ such that the above function ζ belongs to $L^1(\mathbb{R}^d, |\mu_j|)$ for any j: they are linear combinations of the measures μ_j^i in the sense that there exist real constants $c_1, \ldots, c_m$ such that $\mu_j = \sum_{i=1}^m c_i \mu_j^i$ for any $j = 1, \ldots, m$. This result shows, in particular, that systems of invariant measures are infinitely many. Among all the systems of invariant measures, the m systems $\{\mu_j^i : j = 1, \ldots, m\}$ have a very relevant peculiarity: as the invariant measure of the scalar case, they are related to the long-time behaviour of the function $\mathbf{T}(t)\mathbf{f}$. More precisely, in Theorem 3.5 we show that $(\mathbf{T}(t)\mathbf{f})_i$ converges to $\sum_{j=1}^m \int_{\mathbb{R}^d} f_j d\mu_j^i$ for any $i = 1, \ldots, m$ and any bounded and continuous function $\mathbf{f}$, locally uniformly in $\mathbb{R}^d$ as t tends to $+\infty$.

Finally, we confine to a particular case where the invariant measure of the scalar operator $\mathcal{A}$ is explicit and provide a sufficient condition for a system of measures, absolutely continuous with respect to the Lebesgue measure, to be a system of invariant measures for $\mathbf{T}(t)$. Based on this result, we provide some concrete examples of systems of invariant measures, which consist of signed measures.

The paper is organized as follows. In Section 2 we first introduce the main assumptions on the coefficients of the operator $\boldsymbol{\mathcal{A}}$ that we use in the paper. In particular, these assumptions guarantee the existence of both the semigroups $T(t)$ and $\mathbf{T}(t)$ and the invariant measure μ of the semigroup $T(t)$. We also provide a class of elliptic systems which satisfies our assumptions. Then, we prove that the so-called *weak generator* can be applied to the semigroup $\mathbf{T}(t)$ and characterize its domain. Subsection 2.3 is devoted to pointwise gradient estimates, which relate the jacobian matrix of $\mathbf{T}(t)\mathbf{f}$ to the scalar semigroup $T(t)$ applied to the jacobian matrix of $\mathbf{f}$ or to $\mathbf{f}$ itself. This kind of estimates have been already proved in [6] when the semigroup $\mathbf{T}(t)$ is associated to an elliptic operator with a nontrivial potential term. In that case the presence of the potential term was crucial to deduce the estimates. To conclude, in Subsection 2.4, we prove some further relevant properties of the semigroup $\mathbf{T}(t)$ that we need in the paper. In Section 3, the main body of the paper, we prove the existence of systems of invariant measures for the semigroup $\mathbf{T}(t)$, study their main properties as well as the asymptotic behaviour of the function $\mathbf{T}(t)\mathbf{f}$ when t tends to $+\infty$ and $\mathbf{f}$ is bounded and continuous over $\mathbb{R}^d$. Finally, in Appendix A we collect elliptic and parabolic *a priori* estimates which we use in the paper.

Notation. Vector-valued functions are displayed in bold style. Given a function $\mathbf{f}$ (resp. a sequence $(\mathbf{f}_n)$) as above, we denote by f_i (resp. $f_{n,i}$) its i-th component

(resp. the i-th component of the function $\mathbf{f}_n$). By $B_b(\mathbb{R}^d; \mathbb{R}^m)$ we denote the set of all the bounded Borel measurable functions $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^m$, where $\|\mathbf{f}\|_\infty^2 = \sum_{k=1}^m \sup_{x \in \mathbb{R}^d} |f_k(x)|^2$. For any $k \geq 0$, $C_b^k(\mathbb{R}^d; \mathbb{R}^m)$ is the space of all $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^m$ whose components belong to $C_b^k(\mathbb{R}^d; \mathbb{R})$ (throughout the paper simply denoted by $C_b^k(\mathbb{R}^d)$), where "b" stays for bounded. Similarly, the subscripts "c", "0" and "loc" stands for *compactly supported*, *vanishing at infinity* and *locally*, respectively. The symbols $D_t f$, $D_i f$ and $D_{ij} f$, respectively, denote the time derivative, the first-order spatial derivative with respect to the i-th variable and the second-order spatial derivative with respect to the i-th and j-th variables. We write $J_x \mathbf{u}$ for the Jacobian matrix of $\mathbf{u}$ with respect to the spatial variables, omitting the subscript x when no confusion may arise. By $\boldsymbol{e}_j$ we denote the j-th vector of the Euclidean basis of $\mathbb{R}^m$. Finally, throughout the paper we denote by c a positive constant, which may vary from line to line and, if not otherwise specified, may depend at most on d, m. We write c_δ when we want to stress that the constant depends on δ.

2. Assumptions, preliminary results and gradient estimates

2.1. Assumptions and preliminary results

We consider the following standing assumptions on the coefficients of the operator $\boldsymbol{\mathcal{A}}$ defined in (1.2) which we split into the sum of two differential operators as follows:

$$(\boldsymbol{\mathcal{A}}\boldsymbol{\zeta})_j(x) =: (\mathcal{A}\zeta_j)(x) + \sum_{k=1}^d \sum_{i=1}^m (B_k(x))_{ji} D_k \zeta_i(x), \qquad j = 1, \ldots, m,$$

for any $x \in \mathbb{R}^d$ and any smooth enough function $\boldsymbol{\zeta} : \mathbb{R}^d \to \mathbb{R}^m$.

Hypotheses 2.1.   (i) The coefficients $q_{ij} = q_{ji}$ belong to $C_{\mathrm{loc}}^{1+\alpha}(\mathbb{R}^d)$, for any $i, j = 1, \ldots, d$ and some $\alpha \in (0, 1)$. Moreover, $\lambda_0 := \inf_{x \in \mathbb{R}^d} \lambda_Q(x) > 0$ where $\lambda_Q(x)$ denotes the minimum eigenvalue of the matrix $Q(x)$;

(ii) the coefficients b_i and the entries of the matrices B_i ($i = 1, \ldots, d$) belong to $C_{\mathrm{loc}}^{1+\alpha}(\mathbb{R}^d)$. Moreover, there exists a nonnegative function ψ such that $|(B_i)_{jk}| \leq \psi$ in $\mathbb{R}^d$, for any $j, k = 1, \ldots, m, i = 1, \ldots, d$ and $\xi := \sup_{\mathbb{R}^d} \lambda_Q^{-\frac{1}{2}} \psi < +\infty$;

(iii) there exist $a_* \in \mathbb{R}$, $c_* > 0$ and a (Lyapunov) function $1 \leq \varphi \in C^2(\mathbb{R}^d)$, blowing up as $|x|$ tends to $+\infty$, such that $\mathcal{A}\varphi \leq a_* - c_* \varphi$, where $\mathcal{A} = \mathrm{Tr}(QD^2) + \langle \mathbf{b}, \nabla \rangle$, and $\mathbf{b} = (b_1, \ldots, b_d)$;

(iv) there exists $p_0 \in (1, 2]$ such that

$$0 > \sigma_{p_0} := \sup_{x \in \mathbb{R}^d} \left[\Lambda_{J_x \mathbf{b}}(x) + \left(\sum_{i,j=1}^d |D_j B_i(x)|^2 \right)^{\frac{1}{2}} + \frac{d(m\xi + dk(x)(\lambda_Q(x))^{-\frac{1}{2}})^2}{4 \min\{1, p_0 - 1\}} \right],$$

where $\Lambda_{J_x \mathbf{b}}(x)$ is any function which bounds from above the quadratic form associated to $(J_x \mathbf{b})(x)$ and $k(x) = \max_{1 \leq i,j,h \leq d} |D_h q_{ij}(x)|$ for any $x \in \mathbb{R}^d$;

(v) there exist two constants $\gamma > 2$ and $c_\gamma > 0$ such that $\psi^\gamma \leq c_\gamma \varphi$ in $\mathbb{R}^d$.

Example 2.1. Let $\mathcal{A}$ be the second order elliptic differential operator defined in (1.2) with

$$Q(x) = (1 + |x|^2)^p Q^0(x), \quad b(x) = -b_0 x (1 + |x|^2)^r, \quad B_i(x) = (1 + |x|^2)^{s_i} B_i^0(x),$$

for any $x \in \mathbb{R}^d$, where $Q^0(x)$ is a positive definite $d \times d$-matrix for any $x \in \mathbb{R}^d$, $b_0 > 0$, $B_i^0(x)$ are $m \times m$-matrices for any $x \in \mathbb{R}^d$, the entries of the matrix-valued functions Q^0 and B_i^0 $(i = 1, \ldots, d)$ belong to $C_b^{1+\alpha}(\mathbb{R}^d)$ for some $\alpha \in (0, 1)$, and $p, r, s_i \geq 0$ satisfy the conditions $s_{\max} := \max\{s_1, \ldots, s_d\} \leq p/2$ and $r > p$. Clearly, Hypothesis 2.1(i) and (ii) are satisfied. In particular, $\lambda_Q(x) \geq \lambda_1 (1 + |x|^2)^p$ for any $x \in \mathbb{R}^d$ and $\xi \leq \xi_0 := \lambda_1^{-1/2} \max_{i=1,\ldots,d} \|B_i^0\|_\infty$, where λ_1 denotes the infimum over $\mathbb{R}^d$ of the minimum eigenvalue of the matrix $Q^0(x)$. Moreover, for any choice of $h \in \mathbb{N}$ the function $\varphi(x) = (1 + |x|^2)^h$ is a Lyapunov function for the operator $\mathcal{A} = \mathrm{Tr}(QD^2) + \langle \mathbf{b}, \nabla \rangle$, so that Hypotheses 2.1(iii) and (v) are trivially satisfied, this latter one for any choice of $\gamma > 2$. Finally, since $k(x) \leq c_0 (1 + |x|^2)^p$ for any $x \in \mathbb{R}^d$ and some positive constant c_0, we obtain that, if there exists $p_0 \in (1, 2]$ such that

$$\left(\sum_{i,j=1}^d (2 s_i \|B_i^0\|_\infty + \|D_j B_i^0\|_\infty)^2 \right)^{\frac{1}{2}} + \frac{d(m\xi_0 + dc_0 \lambda_1^{-\frac{1}{2}})^2}{4 \min\{1, p_0 - 1\}} < b_0, \qquad (2.1)$$

then Hypothesis 2.1(iv) is fulfilled. Indeed,

$$\Lambda_{J_x \mathbf{b}}(x) + \left(\sum_{i,j=1}^d |D_j B_i^0(x)|^2 \right)^{\frac{1}{2}} + \frac{d(m\xi + dk(x)(\lambda_Q(x))^{-\frac{1}{2}})^2}{4 \min\{1, p_0 - 1\}}$$

$$\leq (1 + |x|^2)^r \left[(1 + |x|^2)^{s_{\max} - r} \left(\sum_{i,j=1}^d (2 s_i (1 + |x|^2)^{-\frac{1}{2}} \|B_i^0\|_\infty + \|D_j B_i^0\|_\infty)^2 \right)^{\frac{1}{2}} - b_0 \right.$$

$$\left. + \frac{d(m\xi_0 + dc_0 \lambda_1^{-\frac{1}{2}} (1 + |x|^2)^{-\frac{p}{2}})^2}{4 \min\{1, p_0 - 1\}} (1 + |x|^2)^{p-r} \right]$$

for any $x \in \mathbb{R}^d$. Due to our choice of the parameters p, r and s, the function in square brackets assume its maximum value at $x = 0$, which is negative when (2.1) is satisfied.

Under (a part of) Hypotheses 2.1 the following result holds true.

Theorem 2.2 (Section 4, Metafune etc [23])**.** *Assume that Hypotheses 2.1(i)-(iii) are satisfied. Then, for any $f \in C_b(\mathbb{R}^d)$ the Cauchy problem*

$$\begin{cases} D_t u(t, x) = \mathcal{A} u(t, x), & t \in (0, +\infty), x \in \mathbb{R}^d, \\ u(0, x) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

admits a unique classical solution $u_f \in C_b([0, +\infty) \times \mathbb{R}^d) \cap C_{\mathrm{loc}}^{1+\alpha/2, 2+\alpha}((0, +\infty) \times \mathbb{R}^d)$ satisfying $\|u_f(t, \cdot)\|_\infty \leq \|f\|_\infty$ for any $t \geq 0$. Moreover, if we set $T(t)f := u_f(t, \cdot)$ for any $t > 0$ and $f \in C_b(\mathbb{R}^d)$, and $T(0) = Id$, then $T(t)g_n$ converges to $T(t)g$ locally uniformly as n tends to $+\infty$, for any $t > 0$ and any bounded sequence $(g_n) \subset C_b(\mathbb{R}^d)$ converging pointwise in $\mathbb{R}^d$ to $g \in C_b(\mathbb{R}^d)$.

The semigroup $T(t)$ admits the following integral representation:

$$(T(t)f)(x) = \int_{\mathbb{R}^d} f(y)p(t,x,dy), \qquad t > 0, \ x \in \mathbb{R}^d, \ f \in C_b(\mathbb{R}^d), \qquad (2.2)$$

where the $p(t,x,dy)$'s are probability Borel measures, each of them equivalent to the Lebesgue measure. In addition, Hypothesis 2.1(iii) implies that $T(t)$ admits a unique invariant measure μ, that is a Borel probability measure which satisfies the condition

$$\int_{\mathbb{R}^d} T(t)f d\mu = \int_{\mathbb{R}^d} f d\mu, \quad t \geq 0,$$

for any $f \in C_b(\mathbb{R}^d)$. This result is due to Has'minskii (see Has'minskii [13, Chapter 3, Theorem 5.1]). Hypothesis 2.1(iii) and the result in Kunze etc [14], (see Lemma 5.3), imply that $\varphi \in L^1(\mathbb{R}^d, \mu)$, that one can apply the scalar semigroup $T(t)$ to the function φ and

$$(T(t)\varphi)(x) \leq c_*^{-1} a_* + \varphi(x), \qquad t > 0, \ x \in \mathbb{R}^d, \qquad (2.3)$$

where the constants a_* and c_* are the same as in the quoted hypothesis.

Hypotheses 2.1(i)-(iii) ensure that also the vector-valued Cauchy problem associated to the operator $\mathcal{A}$ admits a unique classical solution.

Theorem 2.3 (Theorem 2.9, Addona etc [1]). *Let Hypotheses* 2.1 *be satisfied. Then, for any* $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ *the Cauchy problem*

$$\begin{cases} D_t \mathbf{u}(t,x) = \mathcal{A}\mathbf{u}(t,x), & t \in (0, +\infty), x \in \mathbb{R}^d, \\ \mathbf{u}(0,x) = \mathbf{f}(x), & x \in \mathbb{R}^d, \end{cases} \qquad (2.4)$$

admits a unique locally in time bounded classical solution $\mathbf{u}$. *Moreover,* $\mathbf{u}$ *belongs to* $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}((0, +\infty) \times \mathbb{R}^d; \mathbb{R}^m)$ *and*

$$|\mathbf{u}(t,x)|^2 \leq e^{2\beta t}(T(t)|\mathbf{f}|^2)(x), \qquad (t,x) \in [0, +\infty) \times \mathbb{R}^d, \qquad (2.5)$$

where $\beta = 4^{-1}m^2 d\xi^2$ *and* ξ *is defined in Hypothesis* 2.1(ii). *As a byproduct,*

$$|\mathbf{u}(t,x)| \leq e^{\beta t}\|\mathbf{f}\|_\infty, \qquad (t,x) \in [0, +\infty) \times \mathbb{R}^d, \ \mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m). \qquad (2.6)$$

Remark 2.1. In the scalar case, the semigroup associated to an elliptic operator $\mathcal{A}$ with zero potential term is always contractive as a straightforward consequence of the maximum principle. In the vector-valued case, the maximum principle does not hold if the elliptic operator is coupled at the first-order as our operator is. This shows why we cannot expect (2.5) with $\beta = 0$. On the other hand, we can expect neither the semigroup $\mathbf{T}(t)$ to be bounded. Indeed, consider the two-dimensional elliptic operator $\mathcal{A}$ defined on smooth functions $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$ by

$$\mathcal{A}\boldsymbol{\zeta} = (D_{xx}\zeta_1 - D_x\zeta_1 + D_x\zeta_2, D_{xx}\zeta_2 - 5D_x\zeta_1 + D_x\zeta_2).$$

A straightforward computation shows that, if $\mathbf{f}(x) = (\cos(x), 2\sin(x) + \cos(x))$ for any $x \in \mathbb{R}$, then $(\mathbf{T}(t)\mathbf{f})(x) = (e^t \cos(x), e^t(2\sin(x) + \cos(x)))$ for any $t > 0$ and $x \in \mathbb{R}$. As a consequence, $\|\mathbf{T}(t)\mathbf{f}\|_\infty \geq e^t$ for any $t > 0$.

Theorem 2.3 allows to introduce the vector-valued semigroup $\mathbf{T}(t)$ of bounded linear operators in $C_b(\mathbb{R}^d; \mathbb{R}^m)$ by setting $(\mathbf{T}(t)\mathbf{f})(x) := \mathbf{u}(t, x)$, for any $t \geq 0$ and $x \in \mathbb{R}^d$, where $\mathbf{u}$ is the classical solution to the Cauchy problem (2.4). By Addona etc [1, Theorem 3.3], the following integral representation formula

$$(\mathbf{T}(t)\mathbf{f})_i(x) = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j(y) p_{ij}(t, x, dy), \qquad \mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m), \ i = 1, \ldots, m, \quad (2.7)$$

holds true for any $t > 0$, where $\{p_{ij}(t, x, dy) : t > 0, \ x \in \mathbb{R}^d, \ i, j = 1, \ldots, m\}$ is a family of finite signed Borel measures, which are absolutely continuous with respect to the Lebesgue measure. Formula (2.7) allows to extend easily the semigroup $\mathbf{T}(t)$ to $B_b(\mathbb{R}^d; \mathbb{R}^m)$, by approximating any $\mathbf{f} \in B_b(\mathbb{R}^d; \mathbb{R}^m)$ by a bounded sequence $(\mathbf{f}_n) \subset C_b(\mathbb{R}^d; \mathbb{R}^m)$, which converges to $\mathbf{f}$ almost everywhere (with respect to each measure $p_{ij}(t, x, dy)$) in $\mathbb{R}^d$. We can also use this formula to characterize the boundedness of the semigroup $\mathbf{T}(t)$ in terms of the boundedness of the total variations $|p_{ij}|(t, x, dy)$ of the measures $p_{ij}(t, x, dy)$ $(t > 0, x \in \mathbb{R}^d)$.

Proposition 2.1. *The semigroup $\mathbf{T}(t)$ is bounded in $C_b(\mathbb{R}^d; \mathbb{R}^m)$ if and only if the family of measures $\{|p_{ij}|(t, x, dy) : t > 0, x \in \mathbb{R}^d\}$ is bounded for any $i, j = 1, \ldots, m$.*

Proof. Suppose that the semigroup $\mathbf{T}(t)$ is bounded and fix $t > 0$, $x \in \mathbb{R}^d$ and $i, j \in \{1, \ldots, m\}$. By the Hahn decomposition theorem, there exist two disjoint Borel sets P_x^{ij} and N_x^{ij}, whose union is $\mathbb{R}^d$, which are, respectively, the supports of the positive part $p_{ij}^+(t, x, dy)$ and of the negative part $p_{ij}^-(t, x, dy)$ of the measure $p_{ij}(t, x, dy)$.

Since each measure $p_{ij}(t, x, dy)$ is absolutely continuous with respect to the Lebesgue measure, we can determine two bounded sequences $(f_n), (g_n) \subset C_b(\mathbb{R}^d)$ converging to $\chi_{P_x^{ij}}$ and $\chi_{N_x^{ij}}$, respectively, almost everywhere in $\mathbb{R}^d$ (with respect to the measure $p_{ij}(t, x, dy)$). We set $\mathbf{f}_n = f_n \mathbf{e}_j$ and $\mathbf{g}_n = g_n \mathbf{e}_j$ and observe that

$$p_{ij}^+(t, x, \mathbb{R}^d) = p_{ij}(t, x, P_x^{ij}) = \int_{\mathbb{R}^d} \chi_{P_x^{ij}}(y) p_{ij}(t, x, dy)$$

$$= \lim_{n \to +\infty} \int_{\mathbb{R}^d} f_n(y) p_{ij}(t, x, dy) = \lim_{n \to +\infty} (\mathbf{T}(t)\mathbf{f}_n)_i(x).$$

From the boundedness of each operator $\mathbf{T}(t)$ and the previous formula we deduce that $\sup_{(t,x) \in (0,+\infty) \times \mathbb{R}^d} p_{ij}^+(t, x, \mathbb{R}^d) < +\infty$. Replacing $(\mathbf{f}_n)$ with the sequence $(\mathbf{g}_n)$ and arguing similarly, we conclude that $\sup_{(t,x) \in (0,+\infty) \times \mathbb{R}^d} p_{ij}^-(t, x, \mathbb{R}^d) < +\infty$ Thus, the family $\{|p_{ij}|(t, x, dy) : t > 0, x \in \mathbb{R}^d\}$ is bounded for any $i, j = 1, \ldots, m$.

The other part of the statement follows trivially from the representation formula (2.7). Indeed, let $M > 0$ be any constant such that $|p_{ij}|(t, x, \mathbb{R}^d) \leq M$ for any $t > 0$, $x \in \mathbb{R}^d$, $i, j = 1, \ldots, m$. Then, we can estimate

$$|(\mathbf{T}(t)\mathbf{f})_i(x)| \leq \left| \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j(y) p_{ij}(t, x, dy) \right| \leq \sum_{j=1}^{m} \int_{\mathbb{R}^d} |f_j(y)| |p_{ij}|(t, x, dy)$$

$$\leq M \sum_{j=1}^{m} \|f_j\|_\infty \leq M \sqrt{m} \|\mathbf{f}\|_\infty$$

for any $t > 0$, $x \in \mathbb{R}^d$, $i = 1, \ldots, m$, $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and we are done. $\square$

2.2. The weak generator of $\mathbf{T}(t)$

As it is known, (in general) semigroups associated with scalar elliptic operators with unbounded coefficients are not strongly continuous in $C_b(\mathbb{R}^d)$. However, it is possible to associate the so called *weak* generator to them, in two different ways (see, e.g., Lorenzi [16] and Metafune etc [23]). This approach has been already extended to vector-valued weakly coupled elliptic operators with unbounded coefficients in Delmonte etc [10]. We show that it works also in our case.

The first approach considered leads to the definition of the resolvent of the generator by means of the Laplace transform. Indeed, thanks to estimate (2.6), the function $t \mapsto e^{-\lambda t}(\mathbf{T}(t)\mathbf{f})(x)$ is continuous and belongs to $L^1((0, +\infty))$ for any $\lambda > \beta$ and $x \in \mathbb{R}^d$. Hence, we can define bounded operators $\mathbf{R}(\lambda)$ in $C_b(\mathbb{R}^d; \mathbb{R}^m)$ for $\lambda > \beta$ through the formula

$$(\mathbf{R}(\lambda)\mathbf{f})(x) = \int_0^{+\infty} e^{-\lambda t}(\mathbf{T}(t)\mathbf{f})(x)dt, \qquad x \in \mathbb{R}^d,\ \mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m). \qquad (2.8)$$

The family $\{\mathbf{R}(\lambda) : \lambda > \beta\}$ satisfies the resolvent identity and every operator $\mathbf{R}(\lambda)$ is injective in $C_b(\mathbb{R}^d; \mathbb{R}^m)$, so that there exists a unique closed operator $(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{D}})$ such that $\mathbf{R}(\lambda) = (\lambda - \widehat{\boldsymbol{A}})^{-1}$ for any $\lambda > \beta$, i.e., $\lambda - \widehat{\boldsymbol{A}}$ is bijective from $\widehat{\boldsymbol{D}}$ onto $C_b(\mathbb{R}^d; \mathbb{R}^m)$ for any $\lambda > \beta$ (see e.g., Yosida [28, Section 8.4]).

On the other hand, we can define the weak generator of $\mathbf{T}(t)$ mimicking the classical definition of infinitesimal generator of a strongly continuous semigroup, by setting

$$\widetilde{\boldsymbol{D}} = \left\{ \mathbf{u} \in C_b(\mathbb{R}^d; \mathbb{R}^m) : \sup_{t>0} \left\| \frac{\mathbf{T}(t)\mathbf{u} - \mathbf{u}}{t} \right\|_\infty < +\infty \text{ and } \exists \mathbf{g} \in C_b(\mathbb{R}^d; \mathbb{R}^m) \right.$$
$$\left. \text{such that } \lim_{t \to 0^+} \frac{(\mathbf{T}(t)\mathbf{u})(x) - \mathbf{u}(x)}{t} = \mathbf{g}(x)\ \forall x \in \mathbb{R}^d \right\},$$

$$\widetilde{\boldsymbol{A}}\mathbf{u} = \mathbf{g}, \quad \mathbf{u} \in \widetilde{\boldsymbol{D}}.$$

The same arguments used in the scalar case (see Metafune etc [23, Section 5]) show that $\widehat{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{A}}$ actually coincide. Thus, we can set $\boldsymbol{A} := \widehat{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}$, $\boldsymbol{D} := \widehat{\boldsymbol{D}} = \widetilde{\boldsymbol{D}}$ and prove the following characterization for the weak generator $(\boldsymbol{A}, \boldsymbol{D})$.

Proposition 2.2. *The weak generator $(\boldsymbol{A}, \boldsymbol{D})$ of the semigroup $\mathbf{T}(t)$ coincides with the operator $(\boldsymbol{\mathcal{A}}, D_{\max}(\boldsymbol{\mathcal{A}}))$, where $D_{\max}(\boldsymbol{\mathcal{A}})$ denotes the domain of the maximal realization of the operator $\boldsymbol{\mathcal{A}}$ in $C_b(\mathbb{R}^d; \mathbb{R}^m)$, i.e.,*

$$D_{\max}(\boldsymbol{\mathcal{A}}) = \left\{ \mathbf{u} \in C_b(\mathbb{R}^d; \mathbb{R}^m) \cap \bigcap_{1 \leq p < \infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^m) : \boldsymbol{\mathcal{A}}\mathbf{u} \in C_b(\mathbb{R}^d; \mathbb{R}^m) \right\}.$$

Moreover, for any $\mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}})$ and $t > 0$ the function $\mathbf{T}(t)\mathbf{f}$ belongs to $D_{\max}(\boldsymbol{\mathcal{A}})$ and $\mathbf{T}(t)\boldsymbol{\mathcal{A}}\mathbf{f} = \boldsymbol{\mathcal{A}}\mathbf{T}(t)\mathbf{f}$ for any $t > 0$. Finally, for any $\mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}})$ and $x \in \mathbb{R}^d$, the function $(\mathbf{T}(\cdot)\mathbf{f})(x)$ is continuously differentiable in $[0, +\infty)$ with

$$\frac{d}{dt}(\mathbf{T}(t)\mathbf{f})(x) = (\mathbf{T}(t)\boldsymbol{\mathcal{A}}\mathbf{f})(x), \qquad t \geq 0. \qquad (2.9)$$

Proof. Fix $\mathbf{u} \in \boldsymbol{D}$, $\lambda > \beta$ and let $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ be such that $\mathbf{u} = \mathbf{R}(\lambda, \boldsymbol{A})\mathbf{f}$. For any $n \in \mathbb{N}$ let the function $\mathbf{u}_n$ be defined by

$$\mathbf{u}_n(x) = \int_{1/n}^n e^{-\lambda t}(\mathbf{T}(t)\mathbf{f})(x)dt, \qquad x \in \mathbb{R}^d.$$

Taking estimate (2.6) into account we deduce that $\sup_{n \in \mathbb{N}} \|\mathbf{u}_n\|_\infty < +\infty$ and $\mathbf{u}_n$ converges to $\mathbf{u}$ uniformly in $\mathbb{R}^d$ as n tends to $+\infty$. Moreover,

$$\boldsymbol{A}\mathbf{u}_n(x) = \int_{1/n}^n e^{-\lambda t}(\boldsymbol{A}\mathbf{T}(t)\mathbf{f})(x)dt = \int_{1/n}^n e^{-\lambda t}(D_t\mathbf{T}(t)\mathbf{f})(x)dt$$
$$= e^{-\lambda n}(\mathbf{T}(n)\mathbf{f})(x) - e^{-\lambda/n}(\mathbf{T}(n^{-1})\mathbf{f})(x) + \lambda\mathbf{u}_n, \qquad (2.10)$$

whence $\sup_{n \in \mathbb{N}} \|\boldsymbol{A}\mathbf{u}_n\|_\infty < +\infty$ as well. Estimate (A.1) yields that $\|u_{n,k}\|_{W^{2,p}(B(0,r))} \leq c_{p,r}$ for any $n \in \mathbb{N}$, $k = 1, \ldots, m$, $r > 0$ and $p \in (1, +\infty)$. By compactness, there exist $\bar{u}_1, \ldots, \bar{u}_m \in W^{2,p}_{\text{loc}}(\mathbb{R}^d)$ such that $u_{n,k}$ converges to $\bar{u}_k$ ($k = 1, \ldots, m$) strongly in $W^{1,p}(B(0,r))$ and weakly in $W^{2,p}(B(0,r))$ for any $r > 0$. Thus, we infer that $\bar{u}_k = u_k$, whence $u_k \in W^{2,p}_{\text{loc}}(\mathbb{R}^d)$ for any $k = 1, \ldots, m$ and $p \in [1, +\infty)$.

Since $\mathbf{u}_n$ converges to $\mathbf{u}$ weakly in $W^{2,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$, $\boldsymbol{A}\mathbf{u}_n$ converges to $\boldsymbol{A}\mathbf{u}$ weakly in $L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$. On the other hand formula (2.10) implies that $\boldsymbol{A}\mathbf{u}_n$ converges to $\lambda\mathbf{u} - \mathbf{f}$ locally uniformly in $\mathbb{R}^d$ as n tends to $+\infty$. As a byproduct, we conclude that $\boldsymbol{A}\mathbf{u} = \lambda\mathbf{u} - \mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$, i.e., $\boldsymbol{A}\mathbf{u} = \boldsymbol{A}\mathbf{u} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$. This yields that $\boldsymbol{D} \subset D_{\max}(\boldsymbol{A})$ and $\boldsymbol{A}\mathbf{u} = \boldsymbol{A}\mathbf{u}$ for any $\mathbf{u} \in \boldsymbol{D}$.

From the definition of $\boldsymbol{D}$ and the inclusion $\boldsymbol{D} \subset D_{\max}(\boldsymbol{A})$, it follows that $\boldsymbol{D} = D_{\max}(\boldsymbol{A})$ if and only if $\lambda - \boldsymbol{A}$ is injective on $D_{\max}(\boldsymbol{A})$. To prove the injectivity of $\lambda - \boldsymbol{A}$ on $D_{\max}(\boldsymbol{A})$ we show that $\mathbf{u} = 0$ is the unique solution to the equation $\lambda\mathbf{u} - \boldsymbol{A}\mathbf{u} = 0$ in $D_{\max}(\boldsymbol{A})$. To this aim, let $\mathbf{u} \in D_{\max}(\boldsymbol{A})$ solve such an equation. A straightforward computation reveals that $|\mathbf{u}|^2$

$$\lambda|\mathbf{u}|^2 - \frac{1}{2}\mathcal{A}|\mathbf{u}|^2 = -\sum_{i,j=1}^d \sum_{k=1}^m q_{ij} D_i u_k D_j u_k + \sum_{i=1}^d \sum_{j,k=1}^m (B_i)_{kj} u_k D_i u_j. \qquad (2.11)$$

Using Hypotheses 2.1(i)-(ii) we can estimate the term in the right-hand side of (2.11) as follows

$$-\sum_{i,j=1}^d \sum_{k=1}^m q_{ij} D_i u_k D_j u_k + \sum_{i=1}^d \sum_{j,k=1}^m (B_i)_{kj} u_k D_i u_j \leq -\lambda_Q |J_x \mathbf{u}|^2 + m\sqrt{d}\psi|\mathbf{u}||J_x\mathbf{u}|$$
$$\leq (\varepsilon\psi^2 - \lambda_Q)|J_x\mathbf{u}|^2 + \frac{m^2 d}{4\varepsilon}|\mathbf{u}|^2$$
$$\leq \lambda_Q(\varepsilon\xi^2 - 1)|J_x\mathbf{u}|^2 + \frac{m^2 d}{4\varepsilon}|\mathbf{u}|^2.$$

Choosing $\varepsilon = \xi^{-2}$ we obtain that $\lambda|\mathbf{u}|^2 - \hat{\mathcal{A}}_0|\mathbf{u}|^2 \leq 0$ where $\hat{\mathcal{A}}_0 = \frac{1}{2}\mathcal{A} + \frac{m^2 d\xi^2}{4}$. Hypothesis 2.1(iii) and the maximum principle in Lorenzi [16, Theorem 3.1.6] yield that $\mathbf{u} = 0$.

To complete the proof, let us fix $\mathbf{f} \in D_{\max}(\boldsymbol{A})$ and $t > 0$. Estimate (2.5) and the semigroup law show that

$$\left| \frac{\mathbf{T}(h) - I}{h}\mathbf{T}(t)\mathbf{f} - \mathbf{T}(t)\boldsymbol{A}\mathbf{f} \right|^2 = \left| \mathbf{T}(t)\left(\frac{\mathbf{T}(h) - I}{h}\mathbf{f} - \boldsymbol{A}\mathbf{f} \right) \right|^2$$

$$\leq e^{2\beta t}T(t)\left(\left|\frac{\mathbf{T}(h)-I}{h}\mathbf{f}-\boldsymbol{\mathcal{A}}\mathbf{f}\right|^2\right). \qquad (2.12)$$

Since $\mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}}) = \boldsymbol{D}$ it holds that $\sup_{h\in(0,1)} \|(\mathbf{T}(h)\mathbf{f}-\mathbf{f})h^{-1}\|_\infty < +\infty$ and $(\mathbf{T}(h)\mathbf{f}-\mathbf{f})h^{-1}$ converges pointwise to $\boldsymbol{\mathcal{A}}\mathbf{f}$ as h tends to 0^+. Thanks to Theorem 2.2, the right-hand side of (2.12) vanishes locally uniformly as h tends to 0^+. Thus, $(\mathbf{T}(h)\mathbf{T}(t)\mathbf{f}-\mathbf{T}(t)\mathbf{f})h^{-1}$ converges to $\mathbf{T}(t)\boldsymbol{\mathcal{A}}\mathbf{f}$ locally uniformly as h tends to 0^+. Moreover, the semigroup law, estimate (2.6) and the fact that $\sup_{h\in(0,1)} \|(\mathbf{T}(h)\mathbf{f}-\mathbf{f})h^{-1}\|_\infty < +\infty$ imply that $\sup_{h\in(0,1)} \|(\mathbf{T}(h)\mathbf{T}(t)\mathbf{f}-\mathbf{T}(t)\mathbf{f})h^{-1}\|_\infty < +\infty$; whence we deduce that $\mathbf{T}(t)\mathbf{f} \in \boldsymbol{D} = D_{\max}(\boldsymbol{\mathcal{A}})$ and $\boldsymbol{\mathcal{A}}\mathbf{T}(t)\mathbf{f} = \mathbf{T}(t)\boldsymbol{\mathcal{A}}\mathbf{f}$.

To show formula (2.9), we fix $x \in \mathbb{R}^d$, $\mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}})$ and observe that estimate (2.12) implies that the right-derivative of the function $(\mathbf{T}(\cdot)\mathbf{f})(x)$ exists in $[0,+\infty)$ and coincide with the function $(\mathbf{T}(\cdot)\boldsymbol{\mathcal{A}}\mathbf{f})(x)$. Since this function is continuous in $[0,+\infty)$, formula (2.9) follows at once. □

2.3. Pointwise gradient estimates

In this subsection we provide sufficient conditions which ensure pointwise gradient estimates for the vector-valued semigroup $\mathbf{T}(t)$. As a by product, under additional assumptions we show that the function $\mathbf{T}(\cdot)\mathbf{f}$ is uniformly bounded in $[0,+\infty) \times B(0,r)$ for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and $r > 0$.

Proposition 2.3. *Under Hypotheses 2.1(i)-(iv), for any $p \geq p_0$ and any $\mathbf{f} \in C_b^1(\mathbb{R}^d; \mathbb{R}^m)$ it holds that*

$$|(J_x\mathbf{T}(t)\mathbf{f})(x)|^p \leq e^{p\sigma_{p_0}t}(T(t)|J\mathbf{f}|^p)(x), \qquad t > 0, \ x \in \mathbb{R}^d, \qquad (2.13)$$

where σ_{p_0} is defined in Hypothesis 2.1(iv).

Proof. Let $\mathbf{f}$ and p be as in the statement. We claim that

$$|J_x\mathbf{u}_n(t,x)|^p \leq e^{p\sigma_{p_0}t}(T_n^{\mathcal{N}}(t)|J\mathbf{f}|^p)(x), \qquad t > 0, \ x \in B(0,n), \qquad (2.14)$$

where $\mathbf{u}_n$ is the unique classical solution to the homogeneous Neumann-Cauchy problem associated to the equation $D_t\mathbf{u} = \boldsymbol{\mathcal{A}}\mathbf{u}$ in $B(0,n)$ and $T_n^{\mathcal{N}}(t)$ denotes the semigroup associated to the realization of the operator $\mathcal{A}$ in $C_b(\overline{B(0,n)})$ with homogeneous Neumann boundary conditions. Once (2.14) is proved, (2.13) will follow simply observing that $\mathbf{u}_n$ converges to $\mathbf{T}(\cdot)\mathbf{f}$ in $C^{1,2}(K)$ for any compact set $K \subset (0,+\infty) \times \mathbb{R}^d$ (see Addona etc [1, Remark 2.10]) and $T_n^{\mathcal{N}}(\cdot)|J\mathbf{f}|^p$ converges to $T(\cdot)|J\mathbf{f}|^p$ for any K as above.

So, let us prove (2.14). Fix $\varepsilon > 0$ and for any $n \in \mathbb{N}$ set $v_n := (|J_x\mathbf{u}_n|^2 + \varepsilon_p)^{p/2}$, where $\varepsilon_p = \varepsilon > 0$ if $p \in (p_0, 2)$ and $p_0 < 2$ and $\varepsilon_p = 0$ otherwise (i.e., $p \geq 2$). By classical results, v_n belongs to $C^{1,2}((0,T) \times B(0,n)) \cap C_b([0,T] \times \overline{B(0,n)})$ for any $T > 0$ and

$$\begin{cases} D_t v_n = \mathcal{A}v_n + pv_n^{1-\frac{2}{p}}(\psi_1 + \psi_2) + pv_n^{1-\frac{4}{p}}\psi_3, & \text{in } (0,+\infty) \times B(0,n), \\ \dfrac{\partial v_n}{\partial \nu}(t,x) \leq 0, & \text{in } (0,+\infty) \times \partial B(0,n), \\ v_n(0,\cdot) = (|J\mathbf{f}|^2 + \varepsilon_p)^{\frac{p}{2}}, & \text{in } B(0,n), \end{cases}$$

where

$$\psi_1 = \sum_{k=1}^{m} \langle (J_x\mathbf{b})\nabla_x u_{n,k}, \nabla_x u_{n,k}\rangle - \sum_{i=1}^{d}\sum_{k=1}^{m} |\sqrt{Q}\nabla_x D_i u_{n,k}|^2,$$

$$\psi_2 = \sum_{i,j=1}^{d} \langle D_i B_j D_j \mathbf{u}_n, D_i \mathbf{u}_n\rangle + \sum_{i,j=1}^{d} \langle B_j D_{ij}\mathbf{u}_n, D_i\mathbf{u}_n\rangle + \sum_{i,j,h=1}^{d} D_h q_{ij}\langle D_{ij}\mathbf{u}_n, D_h\mathbf{u}_n\rangle,$$

$$\psi_3 = (2-p)\sum_{k=1}^{m} |\sqrt{Q}D_x^2 u_{n,k}\nabla_x u_{n,k}|^2.$$

We can estimate ψ_1 and ψ_2 as follows:

$$\psi_1 \leq \Lambda_{Jb}|J_x\mathbf{u}_n|^2 - \sum_{i=1}^{d}\sum_{k=1}^{m} |\sqrt{Q}\nabla_x D_i u_{n,k}|^2$$

$$\leq \Lambda_{Jb}|J_x\mathbf{u}_n|^2 - \lambda_Q|D_x^2\mathbf{u}_n|^2 \tag{2.15}$$

and

$$|\psi_2| \leq \left(\sum_{i,j=1}^{d} |D_j B_i|^2\right)^{\frac{1}{2}}|J_x\mathbf{u}_n|^2 + (\sqrt{d}m\xi\sqrt{\lambda_Q} + d^{\frac{3}{2}}k)|J_x\mathbf{u}_n||D_x^2\mathbf{u}_n|$$

$$\leq \left[\left(\sum_{i,j=1}^{d} |D_j B_i|^2\right)^{\frac{1}{2}} + \frac{1}{4a}(\sqrt{d}m\xi + d^{\frac{3}{2}}k\lambda_Q^{-\frac{1}{2}})^2\right]|J_x\mathbf{u}_n|^2 + a\lambda_Q|D_x^2\mathbf{u}_n|^2 \tag{2.16}$$

for any $a > 0$. Note that if $p \geq 2$ then $\psi_3 \leq 0$. In this case, the second part of estimate (2.15) and estimate (2.16) with $a = 1$ yield that $D_t v_n - \mathcal{A}v_n \leq p\sigma_{p_0} v_n$. Otherwise if $p_0 < 2$ and $p \in (p_0, 2)$ then the function ψ_3 is nonnegative and can be estimated as follows:

$$(2-p)^{-1}\psi_3 \leq \sum_{i,j=1}^{d}\sum_{k=1}^{m} |\sqrt{Q}\nabla_x D_i u_{n,k}||\sqrt{Q}\nabla_x D_j u_{n,k}||D_i u_{n,k}||D_j u_{n,k}|$$

$$= \sum_{k=1}^{m}\left(\sum_{i=1}^{d} |\sqrt{Q}\nabla_x D_i u_{n,k}||D_i u_{n,k}|\right)^2$$

$$\leq \sum_{k=1}^{m}\sum_{i=1}^{d} |\sqrt{Q}\nabla_x D_i u_{n,k}|^2|\nabla_x u_{n,k}|^2$$

$$\leq |J_x\mathbf{u}_n|^2 \sum_{k=1}^{m}\sum_{i=1}^{d} |\sqrt{Q}\nabla_x D_i u_{n,k}|^2. \tag{2.17}$$

Summing up, from (the first part of) (2.15), (2.16) and (2.17), we obtain that

$$D_t v_n - \mathcal{A}v_n \leq p v_n^{1-\frac{2}{p}}|J_x\mathbf{u}_n|^2\left[\Lambda_{Jb} + \left(\sum_{i,j=1}^{d} |D_j B_i|^2\right)^{\frac{1}{2}} + \frac{1}{4a}(\sqrt{d}m\xi + d^{\frac{3}{2}}k\lambda_Q^{-\frac{1}{2}})^2\right]$$

$$+ p(1-p)v_n^{1-\frac{2}{p}}\sum_{k=1}^{m}\sum_{i=1}^{d}\langle Q\nabla D_i u_{n,k}, \nabla D_i u_{n,k}\rangle + a p v_n^{1-\frac{2}{p}}\lambda_Q|D_x^2\mathbf{u}_n|^2$$

$$\leq p v_n^{1-\frac{2}{p}} |J_x \mathbf{u}_n|^2 \left[\Lambda_{Jb} + \left(\sum_{i,j=1}^{d} |D_j B_i|^2 \right)^{\frac{1}{2}} + \frac{1}{4a} (\sqrt{d} m \xi + d^{\frac{3}{2}} k \lambda_Q^{-\frac{1}{2}})^2 \right]$$

$$+ p v_n^{1-\frac{2}{p}} [(1 - p + a) \lambda_Q |D_x^2 \mathbf{u}_n|^2].$$

Thus, choosing $a = p - 1$, the coefficient in front of $|D_x^2 \mathbf{u}_n|^2$ vanishes and the estimate becomes $D_t v_n - \mathcal{A} v_n \leq p \sigma_{p_0} v_n - p \sigma_{p_0} \varepsilon_{p_0} v_n^{1-2/p} \leq p \sigma_{p_0} v_n - p \sigma_{p_0} \varepsilon_p^{p/2}$.

Now, the procedure is the same in the two cases considered: we set $w_n(t, \cdot) := v_n(t, \cdot) - \varepsilon_p^{p/2} - e^{p \sigma_{p_0} t} T_n^{\mathcal{N}}(t) ((|\nabla \mathbf{f}|^2 + \varepsilon_p)^{p/2})$ for any $t \geq 0$ and observe that

$$\begin{cases} D_t w_n - (\mathcal{A} + p \sigma_{p_0}) w_n \leq 0, & \text{in } (0, +\infty) \times B(0, n), \\ \dfrac{\partial w_n}{\partial \nu} \leq 0, & \text{in } (0, +\infty) \times \partial B(0, n), \\ w_n(0, \cdot) = -\varepsilon_p^{\frac{p}{2}}, & \text{in } B(0, n). \end{cases}$$

The classical maximum principle yields that $w_n \leq 0$ in $(0, +\infty) \times B(0, n)$, i.e., $v_n(t, \cdot) \leq e^{p \sigma_{p_0} t} T_n^{\mathcal{N}}(t) ((|\nabla \mathbf{f}|^2 + \varepsilon_p)^{p/2}) + \varepsilon_p^{p/2}$ for any $t > 0$ and this concludes the proof if $p \geq 2$. Otherwise, we let ε_p tend to 0^+ and again we conclude the proof. $\square$

Proposition 2.4. *Under Hypotheses 2.1(i)-(iv), for any $p \geq p_0$ there exists a positive constant C_p such that*

$$|(J_x \mathbf{T}(t) \mathbf{f})(x)|^p \leq C_p e^{p \sigma_{p_0} t} (1 \vee t^{-\frac{p}{2}}) (T(t) |\mathbf{f}|^p)(x), \qquad t > 0, \ x \in \mathbb{R}^d, \quad (2.18)$$

for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$.

Proof. Here, we take advantage of the notation and the results in the proof of Proposition 2.3. We actually reduce ourselves to proving that for any $p \geq p_0$ there exists a positive constant k_p such that

$$|J_x \mathbf{u}_n(t, x)|^p \leq k_p t^{-\frac{p}{2}} (T_n^{\mathcal{N}}(t) |\mathbf{f}|^p)(x), \qquad t \in (0, 1], \ x \in B(0, n), \quad (2.19)$$

for any $n \in \mathbb{N}$ and $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$. Once (2.19) is proved, letting n tend to $+\infty$ we obtain

$$|(J_x \mathbf{T}(t) \mathbf{f})(x)|^p \leq k_p t^{-\frac{p}{2}} (T(t) |\mathbf{f}|^p)(x), \qquad t \in (0, 1], \ x \in B(0, n). \quad (2.20)$$

Finally, estimate (2.18) will follow using the semigroup rule, estimates (2.13), (2.20) and the positivity of the scalar semigroup $T(t)$. Indeed, if $t > 1$ then

$$|(J_x \mathbf{T}(t) \mathbf{f})(x)|^p = |(J_x \mathbf{T}(t-1) \mathbf{T}(1) \mathbf{f})(x)|^p \leq e^{p \sigma_{p_0} (t-1)} (T(t-1) |J_x \mathbf{T}(1) \mathbf{f}|^p)(x)$$

$$\leq k_p e^{p \sigma_{p_0} (t-1)} (T(t) |\mathbf{f}|^p)(x)$$

for any $x \in \mathbb{R}^d$, and (2.18) follows with $C_p = k_p e^{p|\sigma_{p_0}|}$.

So, let us prove estimate (2.19). First, we set

$$v_n(t, x) := (|\mathbf{u}_n(t, x)|^2 + \gamma t |J_x \mathbf{u}_n(t, x)|^2 + \varepsilon_p)^{\frac{p}{2}}$$

for any $t \in (0, 1]$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, where ε_p is as in the proof of Proposition 2.3 and γ is a positive constant which will be fixed later. For any $n \in \mathbb{N}$, the function

v_n belongs to $C^{1,2}((0,+\infty) \times \overline{B(0,n)}) \cap C([0,+\infty) \times \overline{B(0,n)})$, is bounded in each strip $[0,T] \times \overline{B(0,n)}$ and

$$\begin{cases} D_t v_n = \mathcal{A}v_n + pv_n^{1-\frac{2}{p}}(\widetilde{\psi}_1 + \widetilde{\psi}_2 + \widetilde{\psi}_3) + pv_n^{1-\frac{4}{p}}\widetilde{\psi}_4, & \text{in } (0,1] \times B(0,n), \\ \dfrac{\partial v_n}{\partial \nu}(t,x) \le 0, & \text{in } (0,1] \times \partial B(0,n), \\ v_n(0,\cdot) = (|\mathbf{f}|^2 + \varepsilon_p)^{\frac{p}{2}}, & \text{in } B(0,n), \end{cases}$$

with

$$\widetilde{\psi}_1(t,\cdot) = -\sum_{k=1}^{m} |\sqrt{Q}\nabla_x u_{n,k}(t,\cdot)|^2 + \gamma t \psi_1(t,\cdot),$$

$$\widetilde{\psi}_2(t,\cdot) = \gamma t \psi_2(t,\cdot),$$

$$\widetilde{\psi}_3(t,\cdot) = \frac{\gamma}{2}|J_x \mathbf{u}_n(t,\cdot)|^2 + \sum_{j=1}^{d} \langle \mathbf{u}_n(t,\cdot), B_j D_j \mathbf{u}_n(t,\cdot)\rangle,$$

$$\widetilde{\psi}_4(t,\cdot) = (2-p)\sum_{h,k=1}^{m} \langle Q\zeta_h(t,\cdot), \zeta_k(t,\cdot)\rangle,$$

where $\zeta_j = u_j \nabla_x u_{n,j} + \gamma t D_x^2 u_{n,j} \nabla_x u_{n,j}$ for $j = 1, \dots, m$ and the functions ψ_1 and ψ_2 are defined in the proof of Proposition 2.3.

Using Hypothesis 2.1(i)-(ii) and the Young inequality we estimate $\widetilde{\psi}_i$, $i = 1, 2, 3$ in the following way:

$$\widetilde{\psi}_1(t,\cdot) \le \gamma t \Lambda_{Jb}|J_x \mathbf{u}_n(t,\cdot)|^2 - \sum_{k=1}^{m}|\sqrt{Q}\nabla_x u_{n,k}(t,\cdot)|^2 - \gamma t \sum_{i=1}^{d}\sum_{k=1}^{m}|\sqrt{Q}\nabla_x D_i u_{n,k}(t,\cdot)|^2$$

$$=: \gamma t \Lambda_{Jb}|J_x \mathbf{u}_n(t,\cdot)|^2 - \mathcal{I}_1(t,\cdot) - \gamma t \mathcal{I}_2(t,\cdot)$$

$$\le \gamma t \Lambda_{Jb}|J_x \mathbf{u}_n(t,\cdot)|^2 - \lambda_Q |J_x \mathbf{u}_n(t,\cdot)|^2 - \gamma t \lambda_Q |D_x^2 \mathbf{u}_n(t,\cdot)|^2, \tag{2.21}$$

$$\widetilde{\psi}_2(t,\cdot) \le \gamma t \left[\left(\sum_{i,j=1}^{d}|D_j B_i|^2\right)^{\frac{1}{2}} + \frac{(\sqrt{d}m\xi + d^{\frac{3}{2}}k\lambda_Q^{-\frac{1}{2}})^2}{4a}\right]|J_x \mathbf{u}_n(t,\cdot)|^2$$

$$+ a\gamma t \lambda_Q |D_x^2 \mathbf{u}_n(t,\cdot)|^2, \tag{2.22}$$

$$\widetilde{\psi}_3(t,\cdot) \le \frac{\gamma}{2}|J_x \mathbf{u}_n(t,\cdot)|^2 + \left(\sum_{j=1}^{d}|B_j|^2\right)^{\frac{1}{2}}|\mathbf{u}_n(t,\cdot)||J_x \mathbf{u}_n(t,\cdot)|$$

$$\le \frac{\gamma}{2}|J_x \mathbf{u}_n(t,\cdot)|^2 + \sqrt{d}m\xi\lambda_Q^{\frac{1}{2}}|\mathbf{u}_n(t,\cdot)||J_x \mathbf{u}_n(t,\cdot)|$$

$$\le \frac{dm^2\xi^2}{4\varepsilon}|\mathbf{u}_n(t,\cdot)|^2 + \left(\frac{\gamma}{2} + \varepsilon\lambda_Q\right)|J_x \mathbf{u}_n(t,\cdot)|^2 \tag{2.23}$$

for any $t \in (0,1]$ and $a, \varepsilon > 0$, where ξ is defined in Hypothesis 2.1(ii). We distinguish two cases. If $p \ge p_0 \vee 2$ then $\widetilde{\psi}_4 \le 0$. Thus, using the previous estimates with $a = 1$, $\varepsilon = 2^{-1}$ and $\gamma = \lambda_0$ and Hypothesis 2.1(iv) we obtain immediately that $D_t v_n - \mathcal{A}v_n \le 2^{-1}pdm^2\xi^2 v_n^{1-\frac{2}{p}}|\mathbf{u}_n|^2 \le h_p v_n$, where $h_p = 2^{-1}pdm^2\xi^2$. On the other

hand, if $p_0 < 2$ and $p \in (p_0, 2)$ then we need to estimate $\widetilde{\psi}_4$ too. We obtain

$$
\begin{aligned}
\widetilde{\psi}_4(t, \cdot) \leq & \left(\sum_{h=1}^{m} |u_{n,h}(t, \cdot)| |\sqrt{Q} \nabla u_{n,h}(t, \cdot)| \right)^2 \\
& + 2\gamma t \sum_{h=1}^{m} |u_{n,h}(t, \cdot)| |\sqrt{Q} \nabla u_{n,h}(t, \cdot)| \sum_{h=1}^{m} |\sqrt{Q} D_x^2 u_{n,h}(t, \cdot) \nabla_x u_{n,h}(t, \cdot)| \\
& + \gamma^2 t^2 \left(\sum_{h=1}^{m} |\sqrt{Q} D_x^2 u_{n,h}(t, \cdot) \nabla_x u_{n,h}(t, \cdot)| \right)^2 \\
\leq & \left[|\mathbf{u}_n(t, \cdot)| \left(\sum_{h=1}^{m} |\sqrt{Q} \nabla_x u_{n,h}(t, \cdot)|^2 \right)^{\frac{1}{2}} + \gamma t |J_x \mathbf{u}_n| \left(\sum_{h=1}^{m} |\sqrt{Q} D_x^2 u_h(t, \cdot)|^2 \right)^{\frac{1}{2}} \right]^2 \\
= & \left[|\mathbf{u}_n(t, \cdot)| \sqrt{\mathcal{I}_1(t, \cdot)} + \gamma t |J_x \mathbf{u}_n(t, \cdot)| \sqrt{\mathcal{I}_2(t, \cdot)} \right]^2 \\
\leq & (|\mathbf{u}_n(t, \cdot)|^2 + \gamma t |J_x \mathbf{u}_n(t, \cdot)|^2)(\mathcal{I}_1(t, \cdot) + \gamma t \mathcal{I}_2(t, \cdot)) \\
\leq & (v_n(t, \cdot))^{\frac{2}{p}} (\mathcal{I}_1(t, \cdot) + \gamma t \mathcal{I}_2(t, \cdot)) \tag{2.24}
\end{aligned}
$$

for any $t \in (0, 1]$, where $\mathcal{I}_1$ and $\mathcal{I}_2$ have been defined in (2.21). Thus, choosing $a = p - 1$, $\gamma = (p-1)\lambda_0^{-1}$ and $\varepsilon = (p-1)/2$, from (2.21)-(2.24) we obtain that

$$
D_t v_n - \mathcal{A} v_n \leq \frac{pdm^2\xi^2}{2(p-1)} |\mathbf{u}_n(t, \cdot)|^2 v_n^{1-\frac{2}{p}} \leq \frac{pdm^2\xi^2}{2(p-1)} v_n =: \widetilde{h}_p v_n.
$$

Now, the procedure is the same in the two cases and, arguing as in the last part of Proposition 2.3, we conclude that

$$
t^{\frac{p}{2}} |J_x \mathbf{u}_n(t, \cdot)|^p \leq k_p T_n^{\mathcal{N}}(t)((|\mathbf{f}|^2 + \varepsilon_p)^{\frac{p}{2}})
$$

in $\mathbb{R}^d$, for any $t \in (0, 1]$, where $k_p = 2^{-1}(p \wedge 2 - 1)^{-1} pdm^2\xi^2$. Thus, (letting ε_p tend to 0^+ if $p \in (p_0, 2)$ and $p_0 < 2$), we deduce (2.19) and the proof is so completed. $\square$

2.4. Further properties of the semigroup

As we have already stressed, for each $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and $t > 0$, the function $\mathbf{T}(t)\mathbf{f}$ is bounded on $\mathbb{R}^d$ and estimate (2.5) holds true. For our purpose, we need to slightly improve Theorem 2.3, showing the global in time and local in space boundedness of the function $\mathbf{T}(\cdot)\mathbf{f}$.

For notational convenience, for each $\sigma > 0$ we denote by $B_\sigma(\mathbb{R}^d; \mathbb{R}^m)$ (resp. $C_\sigma(\mathbb{R}^d; \mathbb{R}^m)$) the set of all measurable (resp. continuous) vector-valued functions $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^m$ such that $\mathbf{f}\varphi^{-\sigma}$ is bounded in $\mathbb{R}^d$, where φ is the Lyapunov function in Hypothesis 2.1(iii). It is a Banach space when endowed with the norm $\|\mathbf{f}\|_{B_\sigma(\mathbb{R}^d; \mathbb{R}^m)} = \operatorname{esssup}_{x \in \mathbb{R}^d} |(\varphi(x))^{-\sigma} \mathbf{f}(x)|$ (resp. $\|\mathbf{f}\|_{C_\sigma(\mathbb{R}^d; \mathbb{R}^m)} = \sup_{x \in \mathbb{R}^d} |(\varphi(x))^{-\sigma} \mathbf{f}(x)|$).

Theorem 2.4. *Let Hypotheses* 2.1 *hold true. Then,*

(i) *there exists a positive constant $C_0 \geq 1$ such that*

$$
|(\mathbf{T}(t)\mathbf{f})(x)| \leq C_0 \|\mathbf{f}\|_\infty (\varphi(x))^{\frac{1}{\gamma}}, \qquad t > 0, \ x \in \mathbb{R}^d, \tag{2.25}
$$

for any $\mathbf{f} \in B_b(\mathbb{R}^d; \mathbb{R}^m)$, where γ is defined in Hypothesis 2.1(v);

(ii) *for any $\sigma \in (0, 1/2]$, $\mathbf{T}(t)$ can be extended to $C_\sigma(\mathbb{R}^d; \mathbb{R}^m)$ with a semigroup. More precisely, there exists a positive constant $C_1 \geq 1$ such that*

$$\|\mathbf{T}(t)\mathbf{f}\|_{C_\sigma(\mathbb{R}^d;\mathbb{R}^m)} \leq C_1 e^{\beta t}\|\mathbf{f}\|_{C_\sigma(\mathbb{R}^d;\mathbb{R}^m)}, \qquad t > 0, \ x \in \mathbb{R}^d, \qquad (2.26)$$

for any $\mathbf{f} \in C_\sigma(\mathbb{R}^d; \mathbb{R}^m)$, where β is the constant in Theorem 2.3. Finally, for any $0 \leq \delta \leq \gamma_0 := \min\{1 - 1/\gamma, 1/p_0\}$ there exists a constant $C_2 = C_2(\delta, \gamma) \geq 1$ such that

$$|(\mathbf{T}(t)\mathbf{f})(x)| \leq C_2\|\mathbf{f}\|_{B_\delta(\mathbb{R}^d;\mathbb{R}^m)}(\varphi(x))^{\delta + \frac{1}{\gamma}}, \qquad t > 0, \ x \in \mathbb{R}^d, \qquad (2.27)$$

for any $\mathbf{f} \in B_\delta(\mathbb{R}^d; \mathbb{R}^m)$.

Proof. (i) Since it is rather long, we split the proof into some steps.

Step 1. As a starting point, we prove that, if $v \in C^{1,2}((0, +\infty) \times \mathbb{R}^d) \cap C([0, +\infty) \times \mathbb{R}^d)$ is bounded in each strip $[0, T] \times \mathbb{R}^d$ and solves the Cauchy problem

$$\begin{cases} D_t v(t, x) = \mathcal{A}v(t, x) + g(t, x), & t > 0, x \in \mathbb{R}^d, \\ v(0, x) = f_0(x), & x \in \mathbb{R}^d, \end{cases} \qquad (2.28)$$

for some functions $f_0 \in C_b(\mathbb{R}^d)$ and g such that the function $(s, x) \mapsto \sqrt{s}g(s, x)$ is bounded and continuous in $[0, T] \times \mathbb{R}^d$, then

$$v(t, x) = (T(t, \cdot)f_0)(x) + \int_0^t (T(t - s)g(s, \cdot))(x)ds, \qquad t > 0, \ x \in \mathbb{R}^d.$$

For this purpose, we observe that Hypothesis 2.1(iii) yields a maximum principle for solutions to the Cauchy problem (2.28) which belong to $C^{1,2}((0, +\infty) \times \mathbb{R}^d) \cap C([0, +\infty) \times \mathbb{R}^d)$ and are bounded in each strip $[0, T] \times \mathbb{R}^d$. Hence, v is the unique solution to problem (2.28). Up to splitting g into its positive and negative part, we can assume that g is nonnegative on $(0, T] \times \mathbb{R}^d$. By classical results, the Cauchy-Dirichlet problem

$$\begin{cases} D_t v(t, x) = \mathcal{A}v(t, x) + g(t, x), & t > 0, x \in B(0, n), \\ v(t, x) = 0, & t > 0, x \in \partial B(0, n), \\ v(0, x) = f_0(x), & x \in B(0, n), \end{cases}$$

admits, for any $n \in \mathbb{N}$, a unique solution $v_n \in C^{1,2}((0, +\infty) \times B(0, n))$ which is bounded and continuous in $([0, +\infty) \times \overline{B(0, n)}) \setminus (\{0\} \times \partial B(0, n))$. In particular, each function v_n is nonnegative in $(0, +\infty) \times B(0, n)$. Hence, applying the classical maximum principle to the function $v_{n+1} - v_n$, we easily deduce that the sequence (v_n) is pointwise increasing in $B(0, n)$. Moreover, since

$$v_n(t, x) = (T_n(t, \cdot)f_0)(x) + \int_0^t (T_n(t - s)g(s, \cdot))(x)ds, \qquad t > 0, \ x \in B(0, n),$$

where $T_n(t)$ is the analytic semigroup of contractions in $C_b(B(0, n))$ associated with the realization of the operator $\mathcal{A}$ with homogeneous Dirichlet boundary conditions, we can estimate

$$|v_n(t, x) - (T_n(t)f_0)(x)| \leq 2\sqrt{t} \sup_{s \in (0, t]} \sqrt{s}\|g(s, \cdot)\|_\infty$$

for any $t > 0$ and $n \in \mathbb{N}$. Clearly, the function v_0, which is the pointwise limit of the sequence (v_n), fulfills the same estimate, so that

$$|v_0(t,x)| \leq \|f_0\|_\infty + 2\sqrt{T} \sup_{s \in (0,T]} \sqrt{s}\|g(s,\cdot)\|_\infty$$

for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $T > 0$, and

$$|v_0(t,x) - (T(t)f_0)(x)| \leq 2\sqrt{t} \sup_{s \in (0,1]} \sqrt{s}\|g(s,\cdot)\|_\infty$$

for any $(t,x) \in [0,1] \times \mathbb{R}^d$. Since the function $T(\cdot)f_0$ is continuous in $[0,+\infty) \times \mathbb{R}^d$, the above estimate shows that v_0 can be extended by continuity in $\{0\} \times \mathbb{R}^d$, where it equals function f_0. To identify v_0 with v we use the interior Schauder estimates in Theorem A.2 and the uniform L^∞-boundedness of v_n, to infer that the sequence (v_n) is bounded in $C^{1+\alpha/2,2+\alpha}(K)$ for any compact set $K \subset (0,+\infty) \times \mathbb{R}^d$. Arzelà-Ascoli theorem and the pointwise convergence of v_n to v_0 show that v_n converges to v_0 in $C^{1,2}(K)$ for any K as above, so that, in particular, $v_0 \in C^{1,2}((0,+\infty) \times \mathbb{R}^d)$ and solves the Cauchy problem (2.28). Thus, $v = v_0$.

Step 2. Here, based on Step 1, we show that

$$(\mathbf{T}(t)\mathbf{f})_i(x) = (T(t)f_i)(x) + \int_0^t (T(t-s)w_i(s,\cdot))(x)ds, \qquad (2.29)$$

with $w_i = \sum_{j=1}^d \sum_{h=1}^m (B_j)_{ih} D_j u_h$ for any $(t,x) \in [0,+\infty) \times \mathbb{R}^d$, $i = 1, \ldots, m$. For this purpose, we fix a sequence (ϑ_n) of odd and smooth enough functions such that, for any $n \in \mathbb{N}$, $\vartheta_n(t) = t$ if $0 \leq t \leq n$, $\vartheta_n(t) = n + 1/2$ if $t \geq n + 1$, $0 \leq \vartheta'_n \leq 1$ in $\mathbb{R}$ and $\vartheta''_n \leq 0$ in $[0,+\infty)$. Then, we consider the Cauchy problem (2.4), where now the operator $\mathbf{\mathcal{A}}$ is replaced by the operator $\mathbf{\mathcal{A}}_n$ defined as $\mathbf{\mathcal{A}}$, with the matrices B_i being replaced by the matrices $B_{i,n}$, with entries $(B_{i,n})_{hk} = \vartheta_n \circ (B_{i,n})_{hk}$. Clearly, $|(B_{i,n})_{hk}| \leq |(B_i)_{hk}| \leq \xi\sqrt{\lambda_Q}$ in $\mathbb{R}^d$, for any $n \in \mathbb{N}$, $i = 1, \ldots, d$ and $h, k = 1, \ldots, m$, so that Theorem 2.9 in Addona etc [1] applies and shows that, for any $n \in \mathbb{N}$, there exists a unique function $\mathbf{u}_n \in C([0,+\infty) \times \mathbb{R}^d;\mathbb{R}^m) \cap C^{1,2}((0,+\infty) \times \mathbb{R}^d;\mathbb{R}^m)$, which is bounded in each strip $[0,T] \times \mathbb{R}^d$, solves the equation $D_t\mathbf{u}_n = \mathbf{\mathcal{A}}_n\mathbf{u}_n$ on $(0,+\infty) \times \mathbb{R}^d$ and agrees with the function $\mathbf{f}$ on $\{0\} \times \mathbb{R}^d$. As a first step, we observe that, up to a subsequence, $\mathbf{u}_n$ converges to a function $\mathbf{v}$ in $C^{1,2}(K)$ for any compact set $K \subset (0,+\infty) \times \mathbb{R}^d$. Indeed, by Theorem 2.3, the sequence $(\mathbf{u}_n)$ is bounded in $[0,T] \times \mathbb{R}^d$ for any $T > 0$ and, thus, the interior Schauder estimates in Theorem A.2 show that it is bounded in $C^{1+\alpha/2,2+\alpha}(K)$ for any K as above. Hence, we can argue as in the last part of Step 1. In particular, it turns out that the function $\mathbf{v}$ solves the differential equation $D_t\mathbf{v} = \mathbf{\mathcal{A}}\mathbf{v}$ in $(0,+\infty) \times \mathbb{R}^d$ and is bounded in each strip $[0,T] \times \mathbb{R}^d$.

Next, we observe that, by Proposition 2.4, which can be applied also in this situation since $|D_j B_{i,n}| \leq \|\vartheta'_n\|_\infty |D_j B_i| \leq |D_j B_i|$ for any $i, j = 1, \ldots, d$ and $n \in \mathbb{N}$, we deduce that

$$|J_x\mathbf{u}_n(t,x)|^2 \leq c_T e^{2\sigma_{p_0}t} t^{-1}((T(t)|\mathbf{f}|^{p_0})(x))^{\frac{2}{p_0}}$$

for any $t \in (0,T]$, $x \in \mathbb{R}^d$, $T > 0$ and some positive constant c_T depending also on p_0.

In view of the previous estimate and Step 1, we can write

$$u_{n,i}(t,x) = (T(t,\cdot)f_i)(x) + \int_0^t (T(t-s)w_{n,i}(s,\cdot))(x)ds$$

for any $t > 0$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $i = 1, \ldots, m$, where $w_{n,i}$ $(i = 1, \ldots, d)$ is defined as w_i, with u_h and the matrices B_j being replaced, respectively, by $u_{n,h}$ and $B_{j,n}$ $(n \in \mathbb{N})$. Clearly, the function $(r, s, x) \mapsto (T(r)w_{n,j}(s, \cdot))(x)$ is continuous on $(0, +\infty) \times (0, +\infty) \times \mathbb{R}^d$ and, in view of Hypothesis 2.1(ii), we can estimate

$$|w_{n,i}(s,x)| \leq c_T s^{-\frac{1}{2}} \|\mathbf{f}\|_\infty \psi(x), \qquad s \in (0, T], \ x \in \mathbb{R}^d, \qquad (2.30)$$

for some positive constant c_T depending also on p_0. Hypothesis 2.1(v), the Hölder inequality, formula (2.2) and estimate (2.3) show that

$$(T(t)\psi)(x) = \int_{\mathbb{R}^d} \psi p(t,x,dy) \leq \left(\int_{\mathbb{R}^d} \psi^\gamma p(t,x,dy) \right)^{\frac{1}{\gamma}} \leq c_\gamma^{\frac{1}{\gamma}} \left(\int_{\mathbb{R}^d} \varphi p(t,x,dy) \right)^{\frac{1}{\gamma}}$$

$$= c_\gamma^{\frac{1}{\gamma}} ((T(t)\varphi)(x))^{\frac{1}{\gamma}} \leq c_\gamma^{\frac{1}{\gamma}} \left(\frac{a_*}{c_*} + \varphi(x) \right)^{\frac{1}{\gamma}} \qquad (2.31)$$

for any $t > 0$ and $x \in \mathbb{R}^d$. Taking into account that $\varphi \geq 1$ in $\mathbb{R}^d$, we conclude that $T(\cdot)\psi \leq c\varphi^{1/\gamma}$ in $(0, +\infty) \times \mathbb{R}^d$. In particular, $T(\cdot)\psi$ is bounded in $(0, +\infty) \times B(0, r)$ for any $r > 0$. Hence, taking also Theorem 2.2 into account, we can apply twice the dominated convergence theorem to show, first, that $T(t - \cdot)w_{n,i}$ pointwise converges to $T(t - \cdot)\widetilde{w}_i$ (where $\widetilde{w}_i$ is defined as w_i $(i = 1, \ldots, m)$ with $\mathbf{u}$ being replaced by $\mathbf{v}$) and then that

$$\lim_{n \to +\infty} \int_0^t (T(t-s)w_{n,i}(s, \cdot))(x)ds = \int_0^t (T(t-s)\widetilde{w}_i(s, \cdot))(x)ds$$

for any $t > 0$ and $x \in \mathbb{R}^d$. It thus follows that

$$v_i(t,x) = (T(t)f_i)(x) + \int_0^t (T(t-s)\widetilde{w}_i(s, \cdot))(x)ds, \qquad t > 0, \ x \in \mathbb{R}^d, \ i = 1, \ldots, m.$$

Using estimates (2.30) and (2.31) we conclude that

$$|v_i(t,x) - f_i(x)| \leq |(T(t)f_i)(x) - f_i(x)| + c\|\mathbf{f}\|_\infty \int_0^t s^{-\frac{1}{2}} (T(t-s)\psi)(x)ds$$

$$\leq |(T(t)f_i)(x) - f_i(x)| + c\|\mathbf{f}\|_\infty (\varphi(x))^{\frac{1}{\gamma}} \sqrt{t}$$

for any $t \in (0, 1]$, $x \in \mathbb{R}^d$, $i = 1, \ldots, m$ and some positive constant c depending on d, m and p_0. From this chain of inequalities we easily deduce that $\mathbf{v}$ is continuous on $\{0\} \times \mathbb{R}^d$, where it equals the function $\mathbf{f}$. Summing up, we have shown that $\mathbf{v} \in C^{1,2}((0, +\infty) \times \mathbb{R}^d; \mathbb{R}^m) \cap C([0, +\infty) \times \mathbb{R}^d; \mathbb{R}^m)$, solves the differential equation $D_t\mathbf{v} = \mathcal{A}\mathbf{v}$ in $(0, +\infty) \times \mathbb{R}^d$ and $\mathbf{v}(0, \cdot) = \mathbf{f}$. By Theorem 2.3, we conclude that $\mathbf{v} = \mathbf{u}$ and formula (2.29) follows.

Step 3. Using (2.13) and (2.29), we can estimate

$$w_i(s,x) \leq \sqrt{dm}\psi(x)|(J_x\mathbf{T}(s)\mathbf{f})(x)| \leq c_2(s^{-\frac{1}{2}} \vee 1)e^{\sigma_2 s}\|\mathbf{f}\|_\infty \psi(x)$$

for any $s > 0$, $x \in \mathbb{R}^d$, $i = 1, \ldots, m$. Hence, for $t > 0$, $x \in \mathbb{R}^d$ and $i = 1, \ldots, m$ we get

$$|u_i(t,x)| \leq |(T(t)f_i)(x)| + \int_0^t |(T(t-s)w_i(s, \cdot))(x)|ds$$

$$\leq \|f_i\|_\infty + c\|\mathbf{f}\|_\infty (\varphi(x))^{\frac{1}{\gamma}} \int_0^t (s^{-\frac{1}{2}} \vee 1) e^{\sigma_{p_0} s} ds$$

$$\leq \|f_i\|_\infty + c\|\mathbf{f}\|_\infty (\varphi(x))^{\frac{1}{\gamma}}.$$

Estimate (2.25) follows at once for functions in $C_b(\mathbb{R}^d; \mathbb{R}^m)$.

Suppose now that $\mathbf{f} \in B_b(\mathbb{R}^d; \mathbb{R}^m)$ and let $(\mathbf{f}_n) \subset C_b(\mathbb{R}^d; \mathbb{R}^m)$ be a bounded sequence converging to $\mathbf{f}$ almost everywhere in $\mathbb{R}^d$, with respect to the Lebesgue measure, and such that $\|\mathbf{f}_n\|_\infty \leq \|\mathbf{f}\|_\infty$ for any $n \in \mathbb{N}$. Then, Addona etc [1, Corollary 3.4] shows that $\mathbf{T}(\cdot)\mathbf{f}_n$ converges to $\mathbf{T}(\cdot)\mathbf{f}$ pointwise in $(0, +\infty) \times \mathbb{R}^d$, as n tends to $+\infty$. Writing (2.25) with $\mathbf{f}$ being replaced by $\mathbf{f}_n$ and letting n tend to $+\infty$, we complete the proof of (2.25).

(ii) Fix $\sigma \in (0, 1/2]$. Without loss of generality, we can assume that all the components of $\mathbf{f} \in C_\sigma(\mathbb{R}^d; \mathbb{R}^m)$ are nonnegative since the general case then will follow splitting $\mathbf{f} = \mathbf{f}^+ - \mathbf{f}^-$, where the i-th component of $\mathbf{f}^+$ (resp. $\mathbf{f}^-$) is the positive part of f_i (resp. $-f_i$).

For any $n \in \mathbb{N}$, we set $\mathbf{f}_n := \vartheta_n \mathbf{f}$, where (ϑ_n) is a sequence of smooth enough functions satisfying $\chi_{B(0,n)} \leq \vartheta_n \leq \chi_{B(0,2n)}$. We also fix $i, j \in \{1, \ldots, m\}$, $t > 0$, $x \in \mathbb{R}^d$ and denote by $P = P_{ij}^{t;x}$ the positive set of the Hahn decomposition of $p_{ij}(t, x, dy)$. Since each sequence $(f_{n,j})$ is weakly increasing, by monotone convergence we can infer that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^d} f_{n,j}(y) \chi_P(y) p_{ij}(t, x, dy) = \int_{\mathbb{R}^d} f_j(y) \chi_P(y) p_{ij}(t, x, dy).$$

Moreover, as it has been noticed in Section 2, the semigroup $\mathbf{T}(t)$ can be extended to $B_b(\mathbb{R}^d; \mathbb{R}^m)$ through formula (2.7) and $|\mathbf{T}(t)\mathbf{f}| \leq e^{\beta t}(T(t)|\mathbf{f}|^2)^{1/2}$, pointwise in $\mathbb{R}^d$ for any $\mathbf{f} \in B_b(\mathbb{R}^d; \mathbb{R}^m)$, where β is the constant in Theorem 2.3. In particular, since

$$\int_{\mathbb{R}^d} f_{n,j}(y) \chi_P(y) p_{ij}(t, x, dy) = |(\mathbf{T}(t)(f_{n,j} \chi_P \mathbf{e}_j))_i(x)| \leq e^{\beta t}(T(t)|\vartheta_n f_j|^2(x))^{\frac{1}{2}}$$

$$\leq e^{\beta t}\left[\left(T(t)\left(\frac{|\vartheta_n f_j|^2}{\varphi^{2\sigma}}\right)^{\frac{1}{1-2\sigma}}\right)(x)\right]^{\frac{1}{2}-\sigma}((T(t)\varphi)(x))^\sigma$$

$$\leq e^{\beta t}\|\mathbf{f}\|_{C_\sigma(\mathbb{R}^d;\mathbb{R}^m)}((T(t)\varphi)(x))^\sigma$$

$$\leq e^{\beta t}\|\mathbf{f}\|_{C_\sigma(\mathbb{R}^d;\mathbb{R}^m)}\left(\frac{a_*}{c_*} + \varphi(x)\right)^\sigma$$

$$\leq c e^{\beta t}\|\mathbf{f}\|_{C_\sigma(\mathbb{R}^d;\mathbb{R}^m)}(\varphi(x))^\sigma,$$

we conclude that $\int_{\mathbb{R}^d} f_j \chi_P p_{ij}(t, x, dy)$ is real and

$$\int_{\mathbb{R}^d} f_j(y) \chi_P(y) p_{ij}(t, x, dy) \leq c e^{\beta t}\|\mathbf{f}\|_{C_\sigma(\mathbb{R}^d;\mathbb{R}^m)}(\varphi(x))^\sigma. \qquad (2.32)$$

The same arguments can be applied to show that

$$\int_{\mathbb{R}^d} f_j \chi_N p_{ij}(t, x, dy) \leq c e^{\beta t}\|\mathbf{f}\|_{C_\sigma(\mathbb{R}^d;\mathbb{R}^m)}(\varphi(x))^\sigma, \qquad (2.33)$$

where $N = N_{ij}^{t;x}$ is the negative set of the Hahn decomposition of the measure $p_{ij}(t, x, dy)$. In particular, the interior Schauder estimates in Theorem A.2 can be

used to prove that the function $\mathbf{T}(\cdot)\mathbf{f}$ is continuous in $(0, +\infty) \times \mathbb{R}^d$ and, together with (2.32) and (2.33), they allow to conclude that each operator is bounded from $C_\sigma(\mathbb{R}^d; \mathbb{R}^m)$ into itself and estimate (2.26) holds true. To prove the semigroup rule, we observe that $\mathbf{T}(t)\mathbf{f}_n = \mathbf{T}(t - s)\mathbf{T}(s)\mathbf{f}_n$ in $\mathbb{R}^d$ for any $n \in \mathbb{N}$ and $0 < s < t$. Moreover, since $|\mathbf{f}_n| + |\mathbf{T}(s)\mathbf{f}_n| \leq c\varphi^\sigma$ in $\mathbb{R}^d$, for any $n \in \mathbb{N}$, $s > 0$, by the dominated convergence theorem we conclude that $\mathbf{T}(t)\mathbf{f} = \mathbf{T}(t - s)\mathbf{T}(s)\mathbf{f}$.

Finally, estimate (2.27) can be obtained adapting the arguments used in the proof of (i), taking the positivity of $T(t)$ into account. More precisely, using (2.18) we can estimate

$$
\begin{aligned}
|u_{n,i}(t, \cdot)| \leq & |T(t)f_i| + \int_0^t (T(t - s)|w_{n,i}(s, \cdot)|)(\cdot)ds \\
\leq & \|\mathbf{f}\|_{C_\delta(\mathbb{R}^d; \mathbb{R}^m)} T(t)\varphi^\delta \\
& + c_{p_0} \int_0^t (s^{-\frac{1}{2}} \vee 1) e^{\sigma_{p_0} s} [T(t - s)(\psi(T(s)|\mathbf{f}|^{p_0})^{\frac{1}{p_0}})](\cdot)ds
\end{aligned}
$$

in $\mathbb{R}^d$ for any $\mathbf{f} \in C_\delta(\mathbb{R}^d; \mathbb{R}^m)$. Observe that for any $s > 0$

$$
T(s)|\mathbf{f}|^{p_0} \leq \|\mathbf{f}\|_{C_\delta(\mathbb{R}^d; \mathbb{R}^m)}^{p_0} T(s)\varphi^{\delta p_0} \leq c\|\mathbf{f}\|_{C_\delta(\mathbb{R}^d; \mathbb{R}^m)}^{p_0} \varphi^{\delta p_0},
$$

where in the last inequality we used the fact that $\delta p_0 \leq 1$ and $T(t)\varphi^\eta \leq (T(t)\varphi)^\eta \leq (c_*^{-1}a_* + \varphi)^\eta \leq \varphi^\eta$ for any $t \geq 0$ and $\eta \leq 1$. Hence, using the previous estimates, Hypothesis 2.1(v), again the positivity of $T(t)$ and estimate (2.3), we can infer that

$$
\begin{aligned}
|u_{n,i}(t, \cdot)| \leq & \|\mathbf{f}\|_{C_\delta(\mathbb{R}^d; \mathbb{R}^m)} T(t)\varphi^\delta \\
& + c_{p_0} \|\mathbf{f}\|_{C_\delta(\mathbb{R}^d; \mathbb{R}^m)} \int_0^t (s^{-\frac{1}{2}} \vee 1) e^{\sigma_{p_0} s} T(t - s)\varphi^{\delta + \frac{1}{\gamma}} ds \\
\leq & c_{p_0} \|\mathbf{f}\|_{C_\delta(\mathbb{R}^d; \mathbb{R}^m)} \left(\varphi^\delta + \varphi^{\delta + \frac{1}{\gamma}} \int_0^t (s^{-\frac{1}{2}} \vee 1) e^{\sigma_{p_0} s} ds \right)
\end{aligned}
$$

for any $t > 0$. Letting n tend to $+\infty$, estimate (2.27) follows for functions $\mathbf{f} \in C_\delta(\mathbb{R}^d; \mathbb{R}^m)$.

If $\mathbf{f} \in B_\delta(\mathbb{R}^d; \mathbb{R}^m)$ is not continuous, then it suffices to approximate[§] it with a sequence $(\mathbf{f}_n)$ of continuous functions, converging to $\mathbf{f}$ almost everywhere in $\mathbb{R}^d$ and such that $\sup_{x \in \mathbb{R}^d} |\mathbf{f}_n(x)(\varphi(x))^{-\delta}| \leq \|\mathbf{f}\|_{B_\delta(\mathbb{R}^d; \mathbb{R}^m)}$, and use the dominated convergence together with the above result which shows that $\int_{\mathbb{R}^d} \varphi^{1/\gamma}|p_{ij}(t, x, dy)| < +\infty$ for any $t > 0$, $x \in \mathbb{R}^d$, $i, j = 1, \ldots, m$, to infer that $\mathbf{T}(\cdot)\mathbf{f}_n$ converges to $\mathbf{T}(\cdot)\mathbf{f}$ as n tends to $+\infty$ pointwise in $(0, +\infty) \times \mathbb{R}^d$. Writing (2.27) with $\mathbf{f}_n$ replacing $\mathbf{f}$ and letting n tend to $+\infty$, (2.27) follows in its full generality. $\qquad\square$

Proposition 2.5. *Let $(\mathbf{f}_n)$ be a bounded sequence in $C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)$ which converges to a function $\mathbf{f} \in C(\mathbb{R}^d; \mathbb{R}^m)$, locally uniformly in $\mathbb{R}^d$. Then, $(\mathbf{T}(\cdot)\mathbf{f}_n)$ converges uniformly in $(0, +\infty) \times B(0, R)$ to $\mathbf{T}(\cdot)\mathbf{f}$, for any $R > 0$.*

Proof. We set $\mathbf{g}_n := \mathbf{f}_n - \mathbf{f}$ and notice that $\mathbf{f}, \mathbf{g}_n \in C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)$ for any $n \in \mathbb{N}$. By Theorem 2.4, the functions $\mathbf{T}(t)\mathbf{f}$ and $\mathbf{T}(t)\mathbf{g}_n$ are well defined for any $t > 0$ and

---

[§]This can be easily done, approximating the bounded function $\mathbf{f}/\varphi^\delta$ with a bounded sequence $(\widetilde{\mathbf{f}}_n) \subset C_b(\mathbb{R}^d; \mathbb{R}^m)$ converging to $\mathbf{f}/\varphi^\delta$ almost everywhere in $\mathbb{R}^d$ with respect to the Lebesgue measure and, hence, with respect to each measure $p(t, x, dy)$. Setting $\mathbf{f}_n = \widetilde{\mathbf{f}}_n \varphi^\delta$ we obtain the sought for sequence.

$n \in \mathbb{N}$. Moreover, the arguments in Step 2 of the proof of Theorem 2.4 can be easily adapted to prove, by an approximation argument, that

$$(\mathbf{T}(t)\mathbf{g})_i(x) = (T(t)g_i)(x) + \int_0^t \left(T(t-s) \sum_{j=1}^d \sum_{h=1}^m (B_j)_{ih} D_j (\mathbf{T}(s)\mathbf{g})_h \right)(x)ds,$$

for any $t > 0$, $i = 1, \ldots, m$, $x \in \mathbb{R}^d$ and $\mathbf{g} \in C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)$. Applying this formula with $\mathbf{g} = \mathbf{g}_n$ and using (2.18) with[¶] $p = 2$, we can infer that

$$|(\mathbf{T}(t)\mathbf{f}_n)_i(x) - (\mathbf{T}(t)\mathbf{f})_i(x)|$$

$$\leq |(T(t)g_{n,i})(x)| + c \int_0^t (s^{-\frac{1}{2}} \vee 1)e^{\sigma_2 s}[T(t-s)(\psi(T(s)|\mathbf{g}_n|^2)^{\frac{1}{2}})](x)ds$$

$$\leq |(T(t)g_{n,i})(x)| + c|(T(t)|\mathbf{g}_n|^2)(x)|^{\frac{1}{2}} \int_0^t (s^{-\frac{1}{2}} \vee 1)e^{\sigma_2 s}|(T(t-s)\varphi)(x)|^{\frac{1}{2}}ds$$

$$\leq |(T(t)g_{n,i})(x)| + c|(T(t)|\mathbf{g}_n|^2)(x)|^{\frac{1}{2}} \sqrt{\varphi(x)} \int_0^{+\infty} (s^{-\frac{1}{2}} \vee 1)e^{\sigma_2 s}ds$$

for any $t > 0$ and $x \in \mathbb{R}^d$. Now, we fix $R > 0$, $x \in B(0, R)$ and for any $r > 0$ we split (see (2.2))

$$(T(t)|\mathbf{g}_n|^2)(x) = \int_{B(0,r)} |\mathbf{g}_n|^2 p(t, x, dy) + \int_{\mathbb{R}^d \setminus B(0,r)} |\mathbf{g}_n|^2 p(t, x, dy)$$

$$\leq \|\mathbf{g}_n\|_{C_b(B(0,r))}^2 + \sup_{n \in \mathbb{N}} \|\mathbf{g}_n\|_{C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)}^2 \int_{\mathbb{R}^d \setminus B(0,r)} \varphi^{\frac{2}{\gamma}} p(t, x, dy)$$

$$\leq \|\mathbf{g}_n\|_{C_b(B(0,r))}^2 + \frac{\sup_{n \in \mathbb{N}} \|\mathbf{g}_n\|_{C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)}^2}{\inf_{\mathbb{R}^d \setminus B(0,r)} \varphi^{1-\frac{2}{\gamma}}} \int_{\mathbb{R}^d \setminus B(0,r)} \varphi p(t, x, dy)$$

$$\leq \|\mathbf{g}_n\|_{C_b(B(0,r))}^2 + \frac{c}{\inf_{\mathbb{R}^d \setminus B(0,r)} \varphi^{1-\frac{2}{\gamma}}} \left(\frac{a_*}{c_*} + \sup_{B(0,R)} \varphi \right)$$

for any $t > 0$ and $n \in \mathbb{N}$. Letting first n and then r tend to $+\infty$ in the first- and last-side of the previous chain of inequalities, taking into account that φ blows up as $|x|$ tends to $+\infty$, we easily conclude that $T(t)|\mathbf{g}_n|^2$ vanishes uniformly in $(0, +\infty) \times B(0, R)$ for any $R > 0$.

Finally, since

$$|(T(t)g_{n,i})(x)| \leq (T(t)|\mathbf{g}_n|)(x) = \int_{\mathbb{R}^d} |\mathbf{g}_n| p(t, x, dy)$$

$$\leq \left(\int_{\mathbb{R}^d} |\mathbf{g}_n|^2 p(t, x, dy) \right)^{\frac{1}{2}} (p(t, x, \mathbb{R}^d))^{\frac{1}{2}} = |(T(t)|\mathbf{g}_n|^2)(x)|^{\frac{1}{2}}$$

for any $t > 0$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $i = 1, \ldots, m$, where we have taken into account that the $p(t, x, dy)$'s are probability measures, we also conclude that $T(\cdot)|\mathbf{g}_n|$ vanishes uniformly in $(0, +\infty) \times B(0, R)$ for any $R > 0$. $\square$

[¶]Note that such an inequality can be extended to functions in $C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)$ by a density argument, approximating any such function $\mathbf{h}$ with a sequence of bounded and continuous functions, which is bounded in $C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)$ and converges to $\mathbf{h}$ locally uniformly in $\mathbb{R}^d$.

3. Systems of invariant measures

Definition 3.1. A family of signed finite Borel measures on $\mathbb{R}^d$ $\{\mu_j : j = 1, \ldots, m\}$ is a system of invariant measures for $\mathbf{T}(t)$ if for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and $t > 0$ it holds that

$$\sum_{j=1}^{m} \int_{\mathbb{R}^d} (\mathbf{T}(t)\mathbf{f})_j \, d\mu_j = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j \, d\mu_j. \tag{3.1}$$

The following proposition gives an equivalent characterization of systems of invariant measures.

Proposition 3.1. *A family $\{\mu_j : j = 1, \ldots, m\}$ of (signed) finite measures is a system of invariant measures for $\mathbf{T}(t)$ if and only if*

$$\sum_{i=1}^{m} \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{f})_i \, d\mu_i = 0, \qquad \mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}}). \tag{3.2}$$

Proof. First, we suppose that $\{\mu_i : i = 1, \ldots, m\}$ is a system of invariant measures for $\mathbf{T}(t)$ and fix $\mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}})$. The invariance property of the system $\{\mu_j : j = 1, \ldots, m\}$ implies that

$$\sum_{j=1}^{m} \int_{\mathbb{R}^d} \frac{(\mathbf{T}(t)\mathbf{f})_j - f_j}{t} \, d\mu_j = 0, \qquad t > 0. \tag{3.3}$$

By Proposition 2.2 we know that, for any $j = 1, \ldots, m$, $t^{-1}((\mathbf{T}(t)\mathbf{f})_j - f_j)$ converges to $(\boldsymbol{\mathcal{A}}\mathbf{f})_j$ pointwise in $\mathbb{R}^d$ as t tends to 0^+. Moreover, $\sup_{t \in (0,1]} t^{-1}|(\mathbf{T}(t)\mathbf{f})_j - f_j|$ is a bounded function in $\mathbb{R}^d$, thanks to Proposition 2.2. Since each μ_j is a finite measure, we can let t tend to 0^+ in both sides of (3.3) and obtain (3.2).

Let us now assume that (3.2) holds true in $D_{\max}(\boldsymbol{\mathcal{A}})$ and fix $\mathbf{f}$ in such a space. Then,

$$(\mathbf{T}(t)\mathbf{f})_i(x) - f_i(x) = \int_0^t (\mathbf{T}(s)\boldsymbol{\mathcal{A}}\mathbf{f})_i(x) \, ds, \qquad t > 0, \ x \in \mathbb{R}^d, \ i = 1, \ldots, m.$$

Therefore, integrating again in $\mathbb{R}^d$ with respect to the measure μ_i, summing over i from 1 to m and applying the Fubini's theorem, we deduce that

$$\begin{aligned}
\sum_{i=1}^{m} \int_{\mathbb{R}^d} ((\mathbf{T}(t)\mathbf{f})_i - f_i) \, d\mu_i &= \sum_{i=1}^{m} \int_{\mathbb{R}^d} d\mu_i \int_0^t (\mathbf{T}(s)\boldsymbol{\mathcal{A}}\mathbf{f})_i \, ds \\
&= \int_0^t \left(\sum_{i=1}^{m} \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{T}(s)\mathbf{f})_i \, d\mu_i \right) ds = 0,
\end{aligned}$$

and this completes the proof. $\qquad\qquad\square$

Under Hypotheses 2.1 we prove that there exist m-systems of invariant measures for $\mathbf{T}(t)$.

The following result shows that the average in $(0, t)$ of any component of the function $\mathbf{T}(t)\mathbf{f}$ converges as t tends to $+\infty$. As in the scalar case, this convergence allows us to define the systems of invariant measures associated to $\mathbf{T}(t)$ (see Lorenzi [16, Prop. 8.1.13]).

Theorem 3.1. *Assume that Hypotheses* 2.1 *hold true. Then, there exist at least* m *systems* $\{\mu_j^i : j = 1, \ldots, m\}$, $i = 1, \ldots, m$, *of invariant measures associated to the semigroup* $\mathbf{T}(t)$ *in* $C_b(\mathbb{R}^d; \mathbb{R}^m)$.

Proof.   We split the proof into several steps.

Step 1. Here, we introduce the sequence $(\mathcal{R}_n)$ of bounded linear operators in $C_b(\mathbb{R}^d; \mathbb{R}^m)$ defined by

$$(\mathcal{R}_n \mathbf{f})(x) = \frac{1}{n} \sum_{k=0}^{n-1} (\mathbf{T}(k)\mathbf{f})(x), \qquad x \in \mathbb{R}^d, \ \mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m),$$

and prove that, for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$, $\mathcal{R}_n \mathbf{f}$ converges locally uniformly in $\mathbb{R}^d$ as n tends to $+\infty$ to a constant function. We fix any such function $\mathbf{f}$ and we first show that a suitable subsequence of $(\mathcal{R}_n \mathbf{f})$ converges locally uniformly in $\mathbb{R}^d$. For this purpose, we observe that $\mathcal{R}_n \mathbf{f} = n^{-1}\mathbf{f} + \mathbf{T}(1)((1 - n^{-1})\mathcal{R}_{n-1}\mathbf{f})$ in $\mathbb{R}^d$. By Theorem 2.4(i), the sequence $(\mathcal{R}_{n-1}\mathbf{f})$ is bounded in $C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)$. Hence, we can determine a subsequence $(\mathbf{T}(1)((1 - n_k^{-1})\mathcal{R}_{n_k-1}\mathbf{f}))$, which converges locally uniformly in $\mathbb{R}^d$ to a function $\mathbf{g} \in C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)$. Indeed, Theorem 2.4(i) and the interior Schauder estimates in Theorem A.2 show that

$$\|\mathbf{T}(1)((1 - n^{-1})\mathcal{R}_{n-1}\mathbf{f})\|_{C^{2+\alpha}(B(0,R);\mathbb{R}^m)} \leq c_R \|(1 - n^{-1})\mathcal{R}_{n-1}\mathbf{f}\|_{C_b(B(0,R+1);\mathbb{R}^m)}$$
$$\leq c_R \|\mathbf{f}\|_\infty \sup_{B(0,R+1)} \varphi^{\frac{1}{\gamma}}$$

for any $R > 0$. Thus, the Arzelà-Ascoli theorem and a compactness argument allow us to extract a subsequence of $(\mathcal{R}_n \mathbf{f})$ converging locally uniformly in $\mathbb{R}^d$ to a function $\mathbf{g} \in C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)$.

To prove that all the sequence $(\mathcal{R}_n \mathbf{f})$ converges to $\mathbf{g}$ locally uniformly in $\mathbb{R}^d$, we observe that[||]

$$\mathbf{f} - \mathbf{g} = \mathbf{f} - \lim_{k \to +\infty} \mathcal{R}_{n_k} \mathbf{f} = \lim_{k \to +\infty} \frac{1}{n_k} \sum_{j=1}^{n_k-1} (I - (\mathbf{T}(1))^j)\mathbf{f} = \lim_{k \to +\infty} (I - \mathbf{T}(1))\boldsymbol{\zeta}_k,$$

where $\boldsymbol{\zeta}_k = n_k^{-1} \sum_{j=1}^{n_k-1} \sum_{h=0}^{j-1} \mathbf{T}(h)\mathbf{f}$ ($k \in \mathbb{N}$) is a bounded and continuous function and the sequence $((I - \mathbf{T}(1))\boldsymbol{\zeta}_k)$ is bounded in $C_{\gamma^{-1}}(\mathbb{R}^d; \mathbb{R}^m)$. Moreover, $\mathcal{R}_n(I - \mathbf{T}(1))\boldsymbol{\zeta}_k = n^{-1}(\boldsymbol{\zeta}_k - \mathbf{T}(n)\boldsymbol{\zeta}_k)$ for any $k, n \in \mathbb{N}$. Combining the last two formulas, we can estimate

$$\|\mathcal{R}_n(\mathbf{f} - \mathbf{g})\|_{C_b(B(0,r);\mathbb{R}^m)}$$
$$\leq \|\mathcal{R}_n[\mathbf{f} - \mathbf{g} - (I - \mathbf{T}(1))\boldsymbol{\zeta}_k]\|_{C_b(B(0,r);\mathbb{R}^m)} + \|\mathcal{R}_n(I - \mathbf{T}(1))\boldsymbol{\zeta}_k\|_{C_b(B(0,r);\mathbb{R}^m)}$$
$$\leq \|\mathbf{T}(\cdot)[\mathbf{f} - \mathbf{g} - (I - \mathbf{T}(1))\boldsymbol{\zeta}_k]\|_{C_b((0,+\infty) \times B(0,r);\mathbb{R}^m)} + \frac{1}{n}\|\boldsymbol{\zeta}_k - \mathbf{T}(n)\boldsymbol{\zeta}_k\|_{C_b(B(0,r);\mathbb{R}^m)}$$

for any $k, n \in \mathbb{N}$ and $r > 0$. Now, letting first n and then k tend to $+\infty$, taking Proposition 2.5 into account, from the above chain of inequalities we can infer that $\mathcal{R}_n(\mathbf{f} - \mathbf{g})$ vanishes locally uniformly in $\mathbb{R}^d$ as n tends to $+\infty$. The convergence of $\mathcal{R}_n \mathbf{g}$ is easier to prove since $\mathcal{R}_n \mathbf{g} = \mathbf{g}$ in $\mathbb{R}^d$ for any $n \in \mathbb{N}$. Indeed, since $\mathcal{R}_{n_k} \mathbf{f} - \mathbf{T}(1)\mathcal{R}_{n_k}\mathbf{f} = n_k^{-1}(\mathbf{f} - \mathbf{T}(n_k)\mathbf{f})$ in $\mathbb{R}^d$, letting k tend to $+\infty$, the last side

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[||]The below limits are all local uniform in $\mathbb{R}^d$.

of the previous equality vanishes locally uniformly in $\mathbb{R}^d$. Moreover, since $\mathcal{R}_{n_k}\mathbf{f}$ converges to $\mathbf{g}$ locally uniformly in $\mathbb{R}^d$, by Proposition 2.5 $\mathbf{T}(1)\mathcal{R}_{n_k}\mathbf{f}$ converges to $\mathbf{T}(1)\mathbf{g}$, locally uniformly in $\mathbb{R}^d$. Thus, we conclude that $\mathbf{g} = \mathbf{T}(1)\mathbf{g}$ in $\mathbb{R}^d$. Using the semigroup rule in Theorem 2.4(ii), we deduce that $\mathbf{T}(k)\mathbf{g} = \mathbf{g}$ in $\mathbb{R}^d$ for any $k \in \mathbb{N}$, which implies the claim.

Finally, we prove that $\mathbf{g}$ is a constant function. For this purpose, we approximate $\mathbf{g}$ locally uniformly on $\mathbb{R}^d$ by a sequence $(\mathbf{g}_n)$ of bounded and continuous functions such that $|\mathbf{g}_n| \le |\mathbf{g}|$ in $\mathbb{R}^d$ for any $n \in \mathbb{N}$. Thanks to the interior Schauder estimates in Theorem A.2 and Theorem 2.4(i) we conclude that the sequence $(\mathbf{T}(\cdot)\mathbf{g}_n)$ is bounded in $C^{1+\alpha/2,2+\alpha}(\mathcal{K})$ for any compact set $\mathcal{K} \subset (0,+\infty) \times \mathbb{R}^d$. Hence, up to a subsequence, $\mathbf{T}(\cdot)\mathbf{g}_n$ converges in $C^{1,2}(\mathcal{K})$, for any $\mathcal{K}$ as above, to a function $\zeta \in C_{\mathrm{loc}}^{1+\alpha/2,2+\alpha}((0,+\infty) \times \mathbb{R}^d)$. On the other hand, by Proposition 2.5, $\mathbf{T}(\cdot)\mathbf{g}_n$ converges to $\mathbf{T}(\cdot)\mathbf{g}$ uniformly in $(0,+\infty) \times B(0,R)$, for any $R > 0$. We conclude, in particular, that $J_x\mathbf{T}(k)\mathbf{g}_n$ converges to $J_x\mathbf{T}(k)\mathbf{g}$ locally uniformly in $\mathbb{R}^d$, for any $k \in \mathbb{N}$. We are almost done. Indeed, using Proposition 2.4 we can now estimate

$$|J\mathbf{g}|^2 = |J_x\mathbf{T}(k)\mathbf{g}|^2 = \lim_{n\to+\infty} |J_x\mathbf{T}(k)\mathbf{g}_n|^2 \le ce^{2\sigma_2 k}(1 \vee k^{-1})T(k)|\mathbf{g}_n|^2$$

$$\le ce^{2\sigma_2 k}(1 \vee k^{-1})T(k)|\mathbf{g}|^2 \le ce^{2\sigma_2 k}\left(\frac{a_*}{c_*} + \varphi\right),$$

where the convergence is local uniform in $\mathbb{R}^d$ and we have used (2.3) in the last step of the previous chain of inequalities. We have so shown that $\||J\mathbf{g}|\|_{C(\overline{B(0,R)};\mathbb{R}^m)} \le c_R e^{\sigma_2 k}$. Since $\sigma_2 < 0$, letting k tend to $+\infty$, we conclude that $J\mathbf{g} \equiv 0$ on $B(0,R)$ and, hence, on $\mathbb{R}^d$. This shows that $\mathbf{g}$ is a constant function as claimed.

Step 2. Here, we prove that there exist m systems $\{\mu_j^i : i,j = 1,\ldots,m\}$ ($i = 1,\ldots,m$) of Radon measures such that

$$\lim_{t\to+\infty}(\boldsymbol{\mathcal{P}}_t\mathbf{f})_i := \lim_{t\to+\infty}\frac{1}{t}\int_0^t (\mathbf{T}(s)\mathbf{f})_i ds = \sum_{j=1}^m \int_{\mathbb{R}^d} f_j d\mu_j^i, \qquad (3.4)$$

locally uniformly in $\mathbb{R}^d$ for any $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and $i = 1,\ldots,m$. For this purpose, we note that $\boldsymbol{\mathcal{P}}_t\mathbf{f} = t^{-1}[t]\mathcal{R}_{[t]}\boldsymbol{\mathcal{P}}_1\mathbf{f} + t^{-1}\{t\}\boldsymbol{\mathcal{P}}_{\{t\}}\mathbf{T}([t])\mathbf{f}$ in $\mathbb{R}^d$, for any $t > 1$ and $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$, where $[t]$ and $\{t\}$ denote respectively the integer and the fractional part of t. Since $|\boldsymbol{\mathcal{P}}_{\{t\}}\mathbf{T}([t])\mathbf{f}| \le c\|\mathbf{f}\|_\infty \varphi^{1/\gamma}$ in $\mathbb{R}^d$, for any $t > 0$, due to Theorem 2.4(i), letting t tend to $+\infty$ in the above estimate we obtain that $\boldsymbol{\mathcal{P}}_t\mathbf{f}$ converges locally uniformly on $\mathbb{R}^d$ for every $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and, in view of Step 1, the limit $\boldsymbol{\mathcal{P}}_*\mathbf{f}$ is a constant function in $\mathbb{R}^d$. Thus, it follows that $\boldsymbol{\mathcal{P}}_*\mathbf{f} = \sum_{i=1}^m (\mathcal{M}_i\mathbf{f})\mathbf{e}_i$ for any $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and some linear operators $\mathcal{M}_j : C_b(\mathbb{R}^d;\mathbb{R}^m) \to \mathbb{R}$, $j = 1,\ldots,m$. Note that these operators are bounded. Indeed, using (2.25), we can estimate

$$|(\boldsymbol{\mathcal{P}}_t\mathbf{f})(0)| \le \frac{1}{t}\int_0^t |(\mathbf{T}(s)\mathbf{f})(0)|ds \le c\|\mathbf{f}\|_\infty (\varphi(0))^{\frac{1}{\gamma}}, \qquad t > 0.$$

Since $(\boldsymbol{\mathcal{P}}_t\mathbf{f})(0)$ converges to $\sum_{i=1}^m (\mathcal{M}_i\mathbf{f})\mathbf{e}_i$ as t tends to $+\infty$, we conclude that $|\mathcal{M}_i\mathbf{f}| \le c(\varphi(0))^{1/\gamma}\|\mathbf{f}\|_\infty$ for any $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and $i = 1,\ldots,m$. In particular, each operator $\mathcal{M}_i$ is an element of $(C_0(\mathbb{R}^d;\mathbb{R}^m))'$ and the Riesz representation theorem shows that there exists a family $\{\mu_j^i : i,j = 1,\ldots,m\}$ of finite Radon measures on $\mathbb{R}^d$ such that

$$\mathcal{M}_i\mathbf{f} = \sum_{j=1}^m \int_{\mathbb{R}^d} f_j d\mu_j^i, \qquad \mathbf{f} \in C_0(\mathbb{R}^d;\mathbb{R}^m). \qquad (3.5)$$

To complete the proof of (3.4), we begin by observing that each operator $\mathcal{M}_j$ is well defined and bounded in $C_b(\mathbb{R}^d; \mathbb{R}^m)$. Moreover, if $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ then we can fix a bounded sequence $(\mathbf{f}_n) \subset C_0(\mathbb{R}^d; \mathbb{R}^m)$, converging to $\mathbf{f}$ locally uniformly in $\mathbb{R}^d$ as $n \to +\infty$, and (taking into account that $|\boldsymbol{\mathcal{P}}_t(\mathbf{f}_n - \mathbf{f})| \leq \sup_{t \geq 0} |\mathbf{T}(t)(\mathbf{f}_n - \mathbf{f})|$) estimate

$$\left| \boldsymbol{\mathcal{P}}_t \mathbf{f} - \sum_{i=1}^m (\mathcal{M}_i \mathbf{f}) \mathbf{e}_i \right| \leq |\boldsymbol{\mathcal{P}}_t \mathbf{f}_n - \boldsymbol{\mathcal{P}}_* \mathbf{f}_n| + \sum_{i,j=1}^m \int_{\mathbb{R}^d} |f_{n,j} - f_j| d|\mu_j^i| + \sup_{t \geq 0} |\mathbf{T}(t)(\mathbf{f}_n - \mathbf{f})|$$

in $\mathbb{R}^d$, for any $t > 0$ and $n \in \mathbb{N}$, letting first t and then n tend to $+\infty$, we conclude that (3.5) holds true also for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$.

Step 3. Now, we can complete the proof showing that, for any $i = 1, \ldots, m$, the family $\{\mu_j^i : j = 1, \ldots, m\}$ is a system of invariant measures for $\mathbf{T}(t)$. For this purpose, we fix $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$, $\tau > 0$, $x \in \mathbb{R}^d$, $i \in \{1, \ldots, m\}$ and observe that

$$\begin{aligned}
(\boldsymbol{\mathcal{P}}_t \mathbf{T}(\tau) \mathbf{f})(x) &= \frac{1}{t} \int_0^t (\mathbf{T}(s) \mathbf{T}(\tau) \mathbf{f})(x) ds = \frac{1}{t} \int_0^t (\mathbf{T}(s + \tau) \mathbf{f})(x) ds \\
&= (\boldsymbol{\mathcal{P}}_t \mathbf{f})(x) + \frac{1}{t} \int_t^{t+\tau} (\mathbf{T}(s) \mathbf{f})(x) ds - \frac{1}{t} \int_0^\tau (\mathbf{T}(s) \mathbf{f})(x) ds. \quad (3.6)
\end{aligned}$$

By Theorem 2.4(i), the second and third terms in the right-hand side of (3.6) can be estimated from above by $t^{-1} c \|\mathbf{f}\|_\infty (\varphi(x))^{1/\gamma} \tau$. Hence, letting t tend to $+\infty$ in both sides of (3.6), we deduce that $(\boldsymbol{\mathcal{P}}_* \mathbf{T}(\tau) \mathbf{f})(x) = (\boldsymbol{\mathcal{P}}_* \mathbf{f})(x)$ or, equivalently, that

$$\sum_{i,j=1}^m \left(\int_{\mathbb{R}^d} (\mathbf{T}(\tau) \mathbf{f})_j d\mu_j^i \right) \mathbf{e}_i = \sum_{i,j=1}^m \left(\int_{\mathbb{R}^d} f_j d\mu_j^i \right) \mathbf{e}_i$$

and the assertion follows at once. $\square$

3.1. Properties of systems of invariant measures

To begin with, we observe that $\mu_j^i(\mathbb{R}^d) = \delta_{ij}$ for any $i, j = 1, \ldots, m$, where δ_{ij} is the Kronecker delta. Indeed, fix $i, j \in \{1, \ldots, m\}$ and set $\mathbf{f} := \mathbf{e}_j$. Then, using the invariance property of the system $\{\mu_j^i : j = 1, \ldots, m\}$ we deduce that

$$\mu_j^i(\mathbb{R}^d) = \int_{\mathbb{R}^d} f_j \mu_j^i = \sum_{k=1}^m \int_{\mathbb{R}^d} f_k d\mu_k^i = \lim_{t \to +\infty} \frac{1}{t} \int_0^t (\mathbf{T}(s) \mathbf{f})_i(x) ds = \delta_{ij},$$

since $\mathbf{T}(\cdot) \mathbf{f} = \mathbf{e}_j$ in $[0, +\infty) \times \mathbb{R}^d$.

Next, we prove that the total variation of the measures μ_j^i is absolutely continuous with respect to the Lebesgue measure and that the function φ^{γ_0} (see Theorem 2.4(ii)) is integrable with respect to the measure $|\mu_j^i|$ for any $i, j = 1, \ldots, m$.

Proposition 3.2. *Each measure $|\mu_j^i|$ is absolutely continuous with respect to the Lebesgue measure. Moreover, for any $i, j = 1, \ldots, m$, $\varphi^{\gamma_0} \in L^1(\mathbb{R}^d, |\mu_j^i|)$ and $\|\varphi^{\gamma_0}\|_{L^1(\mathbb{R}^d; |\mu_j^i|)} \leq C_2 (\inf_{\mathbb{R}^d} \varphi)^{\gamma_0 + 1/\gamma}$, where γ_0 and C_2 are defined in the statement of Theorem 2.4(ii).*

Proof.   We fix $i, j \in \{1, \ldots, m\}$ and split the proof into two steps. In the first one we prove the absolutely continuity of $|\mu_j^i|$ with respect to the Lebesgue measure. Then, in Step 2, we prove that the function φ^{γ_0} is in $L^1(\mathbb{R}^d, |\mu_j^i|)$.

Step 1. We denote by $x_0 \in \mathbb{R}^d$ the point where φ attains its minimum value and introduce the family of measures $\{r_{ij}(t, x_0, dy) : t > 0\}$, defined by

$$r_{ij}(t, x_0, B) := \frac{1}{t} \int_0^t p_{ij}(s, x_0, B) ds \qquad (3.7)$$

for any $t > 0$, any Borel set $B \subset \mathbb{R}^d$ and $i, j = 1, \ldots, m$. Note that $p_{ij}(s, x_0, B) = (\mathbf{T}(s)(\chi_B \mathbf{e}_j))_i(x_0)$ for any s, i, j and B as above. Since the semigroup $\mathbf{T}(t)$ is strong Feller, the function $p_{ij}(\cdot, x_0, B)$ is continuous in $(0, +\infty)$ and bounded, due to estimate (2.25). Hence, the integral in the right-hand side of (3.7) is well defined. Moreover, each $r_{ij}(t, x_0, dy)$ is a finite measure. Indeed, we can write

$$|p_{ij}|(s, x_0, \mathbb{R}^d) = \sup \left\{ \int_{\mathbb{R}^d} \zeta p_{ij}(s, x_0, dy) : \zeta \in C_c(\mathbb{R}^d), \|\zeta\|_\infty \leq 1 \right\}$$

(see e.g., Ambrosio etc [3, Proposition 1.43]) and, again by (2.25), the function $s \mapsto \int_{\mathbb{R}^d} \zeta p_{ij}(s, x_0, dy) = (\mathbf{T}(s)(\zeta \mathbf{e}_j))_i(x_0)$ is bounded in $(0, +\infty)$. Therefore,

$$|r_{ij}|(t, x_0, \mathbb{R}^d) \leq \frac{1}{t} \int_0^t |p_{ij}|(s, x_0, \mathbb{R}^d) ds \leq C_0(\varphi(x_0))^{\frac{1}{\gamma}}, \qquad t > 0.$$

In view of Theorem 3.1, for any $t > 0$ and $f \in C_b(\mathbb{R}^d)$ it holds that

$$\int_{\mathbb{R}^d} f r_{ij}(t, x_0, dy) = \frac{1}{t} \int_0^t (\mathbf{T}(s) f \mathbf{e}_j)_i(x_0) ds$$

and the right-hand side of the previous formula converges to $\int_{\mathbb{R}^d} f d\mu_j^i$ as t tends to $+\infty$. Hence, $r_{ij}(t, x_0, dy)$ weakly* converges to μ_j^i as t tends to ∞.

Now, we claim that $|r_{ij}|(t, x_0, \Omega)$ converges to $|\mu_j^i|(\Omega)$ as t tends to $+\infty$, for any open set $\Omega \subset \mathbb{R}^d$. For this purpose, we fix a sequence (t_n) diverging to $+\infty$ such that $|r_{ij}|(t_n, x_0, \Omega)$ admits limit as n tends to $+\infty$. Again by Ambrosio etc [3, Proposition 1.43], we can determine a sequence $(\zeta_n) \subset C_c(\Omega)$ with $\|\zeta_n\|_\infty \leq 1$ such that

$$|r_{ij}|(t_n, x_0, \Omega) \leq \int_{\mathbb{R}^d} \zeta_n(y) r_{ij}(t_n, x_0, dy) + \frac{1}{n} = \frac{1}{t_n} \int_0^{t_n} (\mathbf{T}(s)(\zeta_n \mathbf{e}_j))_i(x_0) ds + \frac{1}{n} \qquad (3.8)$$

for any $n \in \mathbb{N}$. Now, we observe that if $t_n > 1$ then we can write

$$\begin{aligned} \frac{1}{t_n} \int_0^{t_n} (\mathbf{T}(s)(\zeta_n \mathbf{e}_j))_i(x_0) ds =& \frac{1}{t_n} \int_0^1 (\mathbf{T}(s)(\zeta_n \mathbf{e}_j))_i(x_0) ds \\ &+ \frac{1}{t_n} \int_0^{t_n} (\mathbf{T}(s)\mathbf{T}(1)(\zeta_n \mathbf{e}_j))_i(x_0) ds \\ &- \frac{1}{t_n} \int_{t_n-1}^{t_n} (\mathbf{T}(s+1)(\zeta_n \mathbf{e}_j))_i(x_0) ds. \qquad (3.9) \end{aligned}$$

Since $\|\mathbf{T}(s)(\zeta_n \mathbf{e}_j)\|_\infty \leq e^\beta \|\zeta_n\|_\infty \leq e^\beta$ for any $s \in [0, 1]$, the first term in the right-hand side of (3.9) vanishes as n tends to $+\infty$. Similarly, $|(\mathbf{T}(s+1)(\zeta_n \mathbf{e}_j))_i(x_0)| \leq$

$C\sqrt{\varphi(x_0)}$ for any $s > 0$, by (2.25). Hence, also the third term in the right-hand side of (3.9) vanishes as n tends to $+\infty$. As far as the second term in the right-hand side of (3.9) is concerned, we observe that the interior Schauder estimates in Theorem A.2 show that there exists an increasing sequence $(n_k) \subset \mathbb{N}$ such that $\mathbf{T}(1)(\zeta_{n_k}\mathbf{e}_j)$ converges locally uniformly in $\mathbb{R}^d$ to a bounded and continuous function $\mathbf{g}$. This result and Proposition 2.5 imply that $(\mathbf{T}(s)(\mathbf{T}(1)(\zeta_{n_k}\mathbf{e}_j) - \mathbf{g}))_i(x_0)$ converges to 0 uniformly in $(0, +\infty)$ as k tends to $+\infty$. Therefore,

$$\lim_{k\to+\infty} \frac{1}{t_{n_k}} \int_0^{t_{n_k}} (\mathbf{T}(s)(\mathbf{T}(1)(\zeta_{n_k}\mathbf{e}_j) - \mathbf{g}))_i(x_0)ds = 0$$

and from (3.9) we conclude that

$$\lim_{k\to+\infty} \frac{1}{t_{n_k}} \int_0^{t_{n_k}} (\mathbf{T}(s)(\zeta_{n_k}\mathbf{e}_j))_i(x_0)ds = \lim_{k\to+\infty} \frac{1}{t_{n_k}} \int_0^{t_{n_k}} (\mathbf{T}(s)\mathbf{g})_i(x_0)ds$$
$$= \sum_{j=1}^m \int_{\mathbb{R}^d} g_j d\mu_j^i.$$

We claim that $\sum_{j=1}^m \int_{\mathbb{R}^d} g_j d\mu_j^i \leq |\mu_j^i|(\Omega)$. For this purpose we use the invariance property of the family $\{\mu_j^i : j = 1, \ldots, m\}$ to write

$$\sum_{h=1}^m \int_{\mathbb{R}^d} (\mathbf{T}(1)(\zeta_{n_k}\mathbf{e}_j))_h d\mu_h^i = \int_{\mathbb{R}^d} \zeta_{n_k} d\mu_j^i$$

for any $k \in \mathbb{N}$. Hence, by dominated convergence we obtain

$$\sum_{h=1}^m \int_{\mathbb{R}^d} g_h d\mu_h^i = \lim_{k\to+\infty} \sum_{h=1}^m \int_{\mathbb{R}^d} (\mathbf{T}(1)(\zeta_{n_k}\mathbf{e}_j))_h d\mu_h^i = \lim_{k\to+\infty} \int_{\mathbb{R}^d} \zeta_{n_k} d\mu_j^i$$
$$\leq \limsup_{k\to+\infty} \int_{\mathbb{R}^d} |\zeta_{n_k}| d|\mu_j^i| \leq |\mu_j^i|(\Omega).$$

Now, we are almost done. Indeed, writing (3.8) with n_k replacing n and letting k tend to $+\infty$, we deduce that

$$\lim_{n\to+\infty} |r_{ij}|(t_n, x_0, \Omega) = \lim_{k\to+\infty} |r_{ij}|(t_{n_k}, x_0, \Omega) \leq |\mu_j^i|(\Omega).$$

The arbitrariness of the sequence (t_n) yields that $\limsup_{t\to+\infty} |r_{ij}|(t, x_0, \Omega) \leq |\mu_j^i|(\Omega)$. On the other hand, $|\mu_j^i|(\Omega) \leq \liminf_{t\to+\infty} |r_{ij}|(t, x_0, \Omega)$. Indeed, since $r_{ij}(t, x_0, dy)$ weakly* converges to μ_j^i as t tends to $+\infty$, we can write

$$\int_{\mathbb{R}^d} \zeta d\mu_j^i = \lim_{t\to+\infty} \int_{\mathbb{R}^d} \zeta r_{ij}(t, x_0, dy) \leq \liminf_{t\to+\infty} |r_{ij}|(t, x_0, \Omega),$$

due to the fact that $\int_{\mathbb{R}^d} \zeta r_{ij}(t, x_0, dy) \leq |r_{ij}|(t, x_0, \Omega)$ for any $t > 0$. We have so proved that $\limsup_{t\to+\infty} |r_{ij}|(t, x_0, \Omega) \leq |\mu_j^i|(\Omega) \leq \liminf_{t\to+\infty} |r_{ij}|(t, x_0, \Omega)$ i.e., $|r_{ij}|(t, x_0, \Omega)$ converges to $|\mu_j^i|(\Omega)$ as $t \to +\infty$.

It is now straightforward to show that $|r_{ij}|(t, x_0, C)$ converges to $|\mu_j^i|(C)$ as t tends to $+\infty$ also when C is a closed set.

Let us prove that each measure μ_j^i is absolutely continuous with respect to the Lebesgue measure. For this purpose, we fix a Borel set $B \subset \mathbb{R}^d$ with null Lebesgue measure and, for any $\varepsilon > 0$, we denote by $K_\varepsilon \subset B$ a compact set such that $|\mu_j^i|(B \setminus K_\varepsilon) \leq \varepsilon$ (see e.g., Maggi [20, Theorem 2.8]). Since each measure $p_{ij}(t, 0, dy)$ is absolutely continuous with respect to the Lebesgue measure (see Addona etc [1, Theorem 3.3]), $r_{ij}(t, x_0, dy)$ is absolutely continuous with respect to the Lebesgue measure, as well, for any $t > 0$. Hence, $|r_{ij}|(t, x_0, K_\varepsilon) = 0$ for any $t > 0$ and $|\mu_j^i|(K_\varepsilon) = 0$. Splitting B into the union of K_ε and $B \setminus K_\varepsilon$, we thus conclude that $|\mu_j^i|(B) \leq \varepsilon$ and the arbitrariness of $\varepsilon > 0$ shows that $|\mu_j^i|(B) = 0$ and we are done.

Step 2. To begin with, we claim that (3.4) can be extended to any bounded Borel measurable function $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^m$. For this purpose, we approximate any such function $\mathbf{f}$ by a bounded sequence $(\mathbf{f}_n) \subset C_b(\mathbb{R}^d; \mathbb{R}^m)$ which converges to $\mathbf{f}$ almost everywhere in $\mathbb{R}^d$. By the proof of Theorem 2.4, $(\mathbf{T}(\cdot)\mathbf{f}_n)$ converges to $\mathbf{T}(\cdot)\mathbf{f}$ pointwise in $\mathbb{R}^d$. Since the measures $|\mu_j^i|$ are absolutely continuous with respect to the Lebesgue measure, writing (3.1) with $\mathbf{f}$ and μ_j being replaced by $\mathbf{f}_n$ and μ_j^i, respectively, and letting n tend to $+\infty$ (taking (2.6) into account), we get (3.1) in its full generality. Now, as in (3.9) we write

$$\frac{1}{t} \int_0^t (\mathbf{T}(s)\mathbf{f})_i(x_0) ds = o(1) + \frac{1}{t} \int_0^t (\mathbf{T}(s)\mathbf{T}(1)\mathbf{f})_i(x_0) ds$$

as t tends to $+\infty$. The strong Feller property of the semigroup $\mathbf{T}(t)$ and (3.4) yield that

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t (\mathbf{T}(s)\mathbf{f})_i(x_0) ds = \sum_{j=1}^m \int_{\mathbb{R}^d} (\mathbf{T}(1)\mathbf{f})_j d\mu_j^i = \sum_{j=1}^m \int_{\mathbb{R}^d} f_j d\mu_j^i$$

which proves the claim.

By Riesz's theorem (see e.g., Maggi [20, Theorem 4.7]), there exists a measurable function g_{ij} such that $|g_{ij}| = 1$ everywhere in $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} f d\mu_j^i = \int_{\mathbb{R}^d} f g_{ij} d|\mu_j^i|, \qquad f \in C_c(\mathbb{R}^d).$$

Since $|\mu_j^i|$ is absolutely continuous with respect the Lebesgue measure, using the dominated convergence theorem we can extend the above equality to any $f \in B_b(\mathbb{R}^d)$. Equivalently, we can write

$$\int_{\mathbb{R}^d} \frac{f}{g_{ij}} d\mu_j^i = \int_{\mathbb{R}^d} f d|\mu_j^i|, \qquad f \in B_b(\mathbb{R}^d).$$

From all above and (2.27), we deduce that

$$\int_{\mathbb{R}^d} \vartheta_n \varphi^{\gamma_0} d|\mu_j^i| = \int_{\mathbb{R}^d} \frac{\vartheta_n \varphi^{\gamma_0}}{g_{ij}} d\mu_j^i$$
$$= \lim_{t \to +\infty} \frac{1}{t} \int_0^t \left(\mathbf{T}(s) \left(\frac{\vartheta_n \varphi^{\gamma_0}}{g_{ij}} \mathbf{e}_j \right) \right)_i (x_0) ds$$
$$\leq \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \left| \left(\mathbf{T}(s) \left(\frac{\vartheta_n \varphi^{\gamma_0}}{g_{ij}} \mathbf{e}_j \right) \right)_i (x_0) \right| ds \leq C_2 (\varphi(x_0))^{\gamma_0 + \frac{1}{\gamma}},$$

where (ϑ_n) is a standard sequence of cut-off functions. Thus, Fatou lemma yields the assertion. This concludes the proof. $\qquad \square$

Remark 3.1. In view of Proposition (3.2), using the continuity properties of the semigroup $\mathbf{T}(t)$ proved in Addona etc [1, Corollary 3.4] and the dominated convergence theorem, it can be easily proved that formula (3.1) holds true for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ if and only if it is satisfied by any $\mathbf{f} \in B_b(\mathbb{R}^d; \mathbb{R}^m)$.

As an important consequence of the previous proposition we can prove the following characterization of the evolution systems of measures $\{\mu_j : j = 1, \ldots, m\}$ such that $\varphi^{\gamma_0} \in L^1(\mathbb{R}^d, |\mu_j|)$ for any $j = 1, \ldots, m$.

Theorem 3.2. *Let* $\{\mu_j : j = 1, \ldots, m\}$ *be a family of Borel measures such that* $\varphi^{\gamma_0} \in L^1(\mathbb{R}^d, |\mu_j|)$ *for any* $j = 1, \ldots, m$. *Then* $\{\mu_j : j = 1, \ldots, m\}$ *is a system of invariant measures for* $\mathbf{T}(t)$ *if and only if there exist real constants* $c_1, \ldots, c_m$ *such that*

$$\mu_j = \sum_{i=1}^{m} c_i \mu_j^i, \qquad j = 1, \ldots, m. \tag{3.10}$$

Proof. To begin with, we observe that if the measures μ_j $(j = 1, \ldots, m)$ are defined by (3.10) for some real constants c_i $(i = 1, \ldots, m)$, then $\{\mu_j : j = 1, \ldots, m\}$ is a system of invariant measures of $\mathbf{T}(t)$. Indeed, for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and $t \geq 0$

$$\sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\mu_j = \sum_{i,j=1}^{m} c_i \int_{\mathbb{R}^d} f_j d\mu_j^i = \sum_{i,j=1}^{m} c_i \int_{\mathbb{R}^d} (\mathbf{T}(t)f)_j d\mu_j^i = \sum_{j=1}^{m} \int_{\mathbb{R}^d} (\mathbf{T}(t)f)_j d\mu_j.$$

Let us now suppose that $\{\mu_j : j = 1, \ldots, m\}$ is a system of invariant measures of $\mathbf{T}(t)$. Then,

$$\sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\mu_j = \sum_{j=1}^{m} \int_{\mathbb{R}^d} (\mathbf{T}(t)\mathbf{f})_j d\mu_j, \qquad t > 0, \ \mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m). \tag{3.11}$$

Integrating both the sides of (3.11) between 0 and t and then dividing by t, we get

$$\sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\mu_j = \sum_{j=1}^{m} \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{P}}_t \mathbf{f})_j d\mu_j, \tag{3.12}$$

where the operator $\boldsymbol{\mathcal{P}}_t$ has been introduced in (3.4). Since $\gamma_0 > 1/\gamma$, by Theorem 2.4(i) we can estimate $|(\boldsymbol{\mathcal{P}}_t \mathbf{f})(x)| \leq C_0(\varphi(x))^{\gamma_0}$ for any $t > 0$ and $x \in \mathbb{R}^d$, where C_0 is the constant in (2.25). Since $\varphi^{\gamma_0} \in L^1(\mathbb{R}^d, |\mu_j|)$ for any $j = 1, \ldots, m$ and $(\boldsymbol{\mathcal{P}}_t \mathbf{f})_j$ converges to $\sum_{k=1}^{m} \int_{\mathbb{R}^d} f_k d\mu_k^j$ as t tends to $+\infty$, we can let t tend to $+\infty$ in both sides of (3.12) and conclude that

$$\sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\mu_j = \sum_{k,j=1}^{m} \int_{\mathbb{R}^d} f_k \mu_j(\mathbb{R}^d) d\mu_k^j = \sum_{k=1}^{m} \int_{\mathbb{R}^d} f_k d\lambda_k,$$

where $d\lambda_k := \sum_{j=1}^{m} \mu_j(\mathbb{R}^d) d\mu_k^j$ for any $k = 1, \ldots, m$. Fix $j \in \{1, \ldots, m\}$. Taking $\mathbf{f} = f \mathbf{e}_j$ in the above formula, we get

$$\int_{\mathbb{R}^d} f_j d\mu_j = \int_{\mathbb{R}^d} f_j d\lambda_j,$$

which means that the measures λ_j and μ_j coincide in $C_b(\mathbb{R}^d)$, for any $j = 1, \ldots, m$. Riesz theorem implies that λ_j and μ_j coincide on the Borel sets of $\mathbb{R}^d$, for any $j = 1, \ldots, m$. Hence, formula (3.10) holds true with $c_i = \mu_i(\mathbb{R}^d)$. $\qquad\square$

Proposition 3.2 and the equivalence between μ and the Lebesgue measure yield immediately that $|\mu_j^i|$ is absolutely continuous with respect to μ for any $i, j = 1, \ldots, m$. Next theorem provides a more refined result on the density of $|\mu_j^i|$ with respect to the measure μ.

Theorem 3.3. *For any $i, j = 1, \ldots, m$, let g_{ij} be the density of the measure $|\mu_j^i|$ with respect to the measure μ. Then, $g_{ij} \in L^{r_0}(\mathbb{R}^d, \mu) \cap W_{\mathrm{loc}}^{1,q}(\mathbb{R}^d)$ for any $q < +\infty$, where $r_0 = \min\{\gamma, p_0'\}$, γ is the constant appearing in Hypothesis 2.1(v) and p_0' is the exponent conjugate to p_0.*

Proof. To begin with, let us prove that each function g_{ij} belongs to $L^{r_0}(\mathbb{R}^d, \mu)$. For this purpose, we fix $i, j \in \{1, \ldots, m\}$, $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and recall that, up to a subsequence, $\mathbf{T}(\cdot)\mathbf{f}$ is the pointwise limit of the sequence $(\mathbf{u}_n)$, where $\mathbf{u}_n$ is implicitly defined by the equation

$$u_{n,i}(t, x) = (T(t)f_i)(x) + \int_0^t (T(t - s)w_{n,i}(s, \cdot))(x)ds \qquad (3.13)$$

for any $t > 0$ and $x \in \mathbb{R}^d$, and $w_{n,i} = \sum_{j=1}^d \sum_{h=1}^m (B_j)_{ih} D_j u_{n,h}$ (see the proof of Theorem 2.4). In particular, the sequence $(\mathbf{u}_n)$ is bounded in each strip $[0, T] \times \mathbb{R}^d$.

Integrating both sides of (3.13) in $\mathbb{R}^d$ with respect to μ we get

$$\int_{\mathbb{R}^d} u_{n,i}(t, \cdot)d\mu = \int_{\mathbb{R}^d} T(t)f_i d\mu + \int_{\mathbb{R}^d} d\mu \int_0^t T(t - s)w_{n,i}(s, \cdot)ds$$

$$= \int_{\mathbb{R}^d} f_i d\mu + \int_0^t ds \int_{\mathbb{R}^d} T(t - s)w_{n,i}(s, \cdot)d\mu$$

$$= \int_{\mathbb{R}^d} f_i d\mu + \int_0^t ds \int_{\mathbb{R}^d} w_{n,i}(s, \cdot)d\mu. \qquad (3.14)$$

Here, we have used the continuity of the function $(s, x) \mapsto (T(t - s)w_{n,i}(s, \cdot))(x)$, together with the estimate $|(T(t-s)w_{n,i}(s, \cdot))(x)| \leq c_n s^{-1/2}\|f\|_\infty$ for any $s \in (0, t)$, $x \in \mathbb{R}^d$, to change the order of integration, and the invariance property of the measure μ.

Now, we distinguish the cases $\gamma > p_0'$ and $\gamma \leq p_0'$. In the first case, we use Hypothesis 2.1(ii), estimate (2.18) and the invariance property of μ to deduce that

$$\int_{\mathbb{R}^d} |w_{n,i}(s, \cdot)|d\mu \leq c \int_{\mathbb{R}^d} \psi|J_x\mathbf{T}(s)\mathbf{f}|d\mu$$

$$\leq c\|\psi\|_{L^{p_0'}(\mathbb{R}^d,\mu)}\||J_x\mathbf{T}(s)\mathbf{f}|\|_{L^{p_0}(\mathbb{R}^d,\mu)}$$

$$\leq ce^{\sigma_{p_0}s}\|\varphi\|_{L^1(\mathbb{R}^d,\mu)}^{\frac{p_0-1}{p_0}}(1 \vee s^{-\frac{1}{2}})\|T(s)|\mathbf{f}|^{p_0}\|_{L^1(\mathbb{R}^d,\mu)}^{\frac{1}{p_0}}$$

$$= ce^{\sigma_{p_0}s}(1 \vee s^{-\frac{1}{2}})\||\mathbf{f}|\|_{L^{p_0}(\mathbb{R}^d,\mu)} \qquad (3.15)$$

for any $s > 0$. On the other hand, if $\gamma \leq p_0'$, arguing similarly, we estimate

$$\int_{\mathbb{R}^d} |w_{n,i}(s, \cdot)|d\mu \leq c\|\psi\|_{L^\gamma(\mathbb{R}^d,\mu)}\||J_x\mathbf{T}(s)\mathbf{f}|\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^d,\mu)}$$

$$\leq c e^{\sigma_{p_0} s}(1 \vee s^{-\frac{1}{2}}) \|\|\mathbf{f}\|\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^d, \mu)} \qquad (3.16)$$

for any $s > 0$. From (3.14)-(3.16) we can infer that

$$\int_{\mathbb{R}^d} u_{n,i}(t, \cdot) d\mu \leq \int_{\mathbb{R}^d} f_i d\mu + c\|\|\mathbf{f}\|\|_{L^{r'_0}(\mathbb{R}^d, \mu)} \int_0^{+\infty} e^{\sigma_{p_0} s}(1 \vee s^{-\frac{1}{2}}) ds. \qquad (3.17)$$

Letting n tend to $+\infty$ in (3.17) we conclude that

$$\int_{\mathbb{R}^d} (\mathbf{T}(t)\mathbf{f})_i d\mu \leq \int_{\mathbb{R}^d} f_i d\mu + c\|\|\mathbf{f}\|\|_{L^{r'_0}(\mathbb{R}^d, \mu)}. \qquad (3.18)$$

Now, we let t tend to $+\infty$ in (3.18). Taking (2.25) and the forthcoming Theorem 3.5 into account, we can apply the dominated convergence theorem and deduce that

$$\sum_{j=1}^m \int_{\mathbb{R}^d} f_j d\mu_j^i = \int_{\mathbb{R}^d} \left(\sum_{j=1}^m \int_{\mathbb{R}^d} f_j d\mu_j^i \right) d\mu \leq \int_{\mathbb{R}^d} f_i d\mu + c\|\|\mathbf{f}\|\|_{L^{r'_0}(\mathbb{R}^d, \mu)}. \qquad (3.19)$$

To go further, we extend (3.19) to any $\mathbf{f} \in B_b(\mathbb{R}^d; \mathbb{R}^m)$ by approximating any such function $\mathbf{f}$ by a bounded sequence $(\mathbf{f}_n) \subset C_b(\mathbb{R}^d; \mathbb{R}^m)$ which converges to $\mathbf{f}$ almost everywhere (with respect to the Lebesgue measure and, hence, with respect to the measures μ_j^i and μ) in $\mathbb{R}^d$. Writing (3.19) with $\mathbf{f}$ being replaced by the function $\mathbf{f}_n$ and letting n tend to $+\infty$, by dominated convergence we obtain that $\mathbf{f}$ satisfies (3.19) as well.

Now, we are almost done. Indeed, take $f \in B_b(\mathbb{R}^d)$ and let A_{ij}^+ be the set where the positive part of μ_j^i is concentrated. Writing (3.19) with $\mathbf{f} = f\chi_{A_{ij}^+}\mathbf{e}_j$ gives

$$\int_{\mathbb{R}^d} f d(\mu_j^i)^+ \leq \delta_{ij} \int_{A_{ij}^+} f d\mu + c\|f\chi_{A_{ij}^+}\|_{L^{r'_0}(\mathbb{R}^d, \mu)}$$

$$\leq \int_{\mathbb{R}^d} |f| d\mu + c\|f\|_{L^{r'_0}(\mathbb{R}^d, \mu)} \leq (1 + c)\|f\|_{L^{r'_0}(\mathbb{R}^d, \mu)}.$$

This shows that the operator $f \mapsto \int_{\mathbb{R}^d} f d(\mu_j^i)^+$ can be (uniquely) extended to a bounded linear operator on $L^{r'_0}(\mathbb{R}^d, \mu)$ and the Riesz's representation theorem implies that there exists a nonnegative function $\phi_{ij}^+ \in L^{r_0}(\mathbb{R}^d, \mu)$ such that

$$\int_{\mathbb{R}^d} f d(\mu_j^i)^+ = \int_{\mathbb{R}^d} f \phi_{ij}^+ d\mu, \qquad f \in L^{r'_0}(\mathbb{R}^d, \mu).$$

Repeating the same arguments with A_{ij}^+ being replaced by the set A_{ij}^-, where the negative part of μ_j^i is concentrated, we can show that

$$\int_{\mathbb{R}^d} f d(\mu_j^i)^- = \int_{\mathbb{R}^d} f \phi_{ij}^- d\mu, \qquad f \in L^{r'_0}(\mathbb{R}^d, \mu)$$

for some nonnegative function $\phi_{ij}^- \in L^{r_0}(\mathbb{R}^d, \mu)$. Since $g_{ij} = \phi_{ij}^+ + \phi_{ij}^-$, we immediately conclude that $g_{ij} \in L^{r_0}(\mathbb{R}^d, \mu)$.

To conclude the proof, let us show that the function g_{ij} belongs to $W_{\text{loc}}^{1,q}(\mathbb{R}^d)$ for any $i, j = 1, \ldots, m$ and any $q < +\infty$. For this purpose, we use a bootstrap argument. We fix $r > 0$, $\eta \in C_c^\infty(B(0, r))$ and $i, j \in \{1, \ldots, m\}$. Choosing $\mathbf{f} = \eta \mathbf{e}_j$

in (3.2) and observing that $d\mu_j^i = (\phi_{ij}^+ - \phi_{ij}^-)d\mu := h_{ij}d\mu$ for any $i, j = 1, \ldots, m$, we get

$$\int_{\mathbb{R}^d} h_{ij}\rho\mathcal{A}\eta dx = -\sum_{h=1}^d \sum_{k=1}^m \int_{\mathbb{R}^d} (B_h)_{kj} D_h\eta(\phi_{ik}^+ - \phi_{ik}^-)\rho dx.$$

Let us estimate the right-hand side of the previous formula, which we denote by $\mathcal{I}_{ij}$. From the first part of the proof, we know that $\phi_{hk}^\pm \in L^2(\mathbb{R}^d, \mu)$ for any $h, k = 1, \ldots, m$. Moreover, [7, Corollary 2.9] implies that $\rho \in W_{\mathrm{loc}}^{1,p}(\mathbb{R}^d)$ for any $p \geq 1$ and so, in particular, ρ is locally Hölder continuous. Since the entries of the matrices B_h $(h = 1, \ldots, d)$ are locally bounded, we get

$$|\mathcal{I}_{ij}| \leq c \max_{\substack{h=1,\ldots,d \\ k=1,\ldots,m}} \|(B_h)_{kj}\sqrt{\rho}\|_{L^\infty(B(0,r))} \|h_{ik}\|_{L^2(B(0,r),\mu)} \|\nabla\eta\|_{L^2(B(0,r))}$$

and, therefore,

$$\left| \int_{\mathbb{R}^d} \mathcal{A}\eta d\mu_i^j \right| \leq c_r \|\nabla\eta\|_{L^2(B(0,r))}. \tag{3.20}$$

Now, we fix $f \in C_c^\infty(\mathbb{R}^d)$ and a smooth function ψ_r such that $\chi_{B(0,r/2)} \leq \psi_r \leq \chi_{B(0,r)}$. Since $\mathcal{A}(\zeta_1\zeta_2) = \zeta_1\mathcal{A}\zeta_2 + \zeta_2\mathcal{A}\zeta_1 + 2\langle Q\nabla\zeta_1, \nabla\zeta_2\rangle$ for any pair of smooth functions ζ_1, ζ_2, we can estimate

$$\left| \int_{\mathbb{R}^d} (\mathcal{A}f)\psi_r h_{ik}\rho dx \right| = \left| \int_{\mathbb{R}^d} (\mathcal{A}f)\psi_r d\mu_j^i \right|$$
$$\leq \left| \int_{\mathbb{R}^d} \mathcal{A}(\psi_r f)d\mu_j^i \right| + \left| \int_{\mathbb{R}^d} \mathcal{A}\psi_r f d\mu_j^i \right| + 2\left| \int_{\mathbb{R}^d} \langle Q\nabla f, \nabla\psi_r\rangle d\mu_j^i \right|$$
$$=: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.$$

We claim that

$$\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \leq c_r \|f\|_{W^{1,2}(\mathbb{R}^d)}. \tag{3.21}$$

Estimate (3.20) shows that $\mathcal{J}_1 \leq c_r \|\nabla(\psi_r f)\|_{L^2(B(0,r))} \leq c_r \|f\|_{W^{1,2}(\mathbb{R}^d)}$. As far as $\mathcal{J}_2$ and $\mathcal{J}_3$ are concerned, arguing as above we deduce that

$$\mathcal{J}_2 \leq \|(\mathcal{A}\psi_r)\sqrt{\rho}\|_{L^\infty(B(0,r))} \|h_{ij}\|_{L^2(B(0,r),\mu)} \|f\|_{L^2(\mathbb{R}^d)},$$
$$\mathcal{J}_3 \leq 2\||Q\nabla\psi_r|\sqrt{\rho}\|_{L^\infty(B(0,r))} \|h_{ij}\|_{L^2(B(0,r),\mu)} \|\nabla f\|_{L^2(\mathbb{R}^d)}.$$

Estimate (3.21) is so proved and, from [16, Theorem D.1.4(ii)], we deduce that $h_{ij}\rho\psi_r \in W_{\mathrm{loc}}^{1,2}(\mathbb{R}^d)$ for any $i, j = 1, \ldots, m$. Thus, $h_{ij}\rho$ belongs to $W^{1,2}(B(0,r/2))$. The arbitrariness of $r > 0$ yields immediately that $h_{ij}\rho \in W_{\mathrm{loc}}^{1,2}(\mathbb{R}^d)$. Since $\rho \in W_{\mathrm{loc}}^{1,p}(\mathbb{R}^d)$ for any $p \geq 1$ and $\inf_{\mathbb{R}^d} \rho$ is positive, we can infer that h_{ij} belongs to $W_{\mathrm{loc}}^{1,2}(\mathbb{R}^d)$.

Now, we can make the bootstrap argument work in the case when $d \geq 3$, since the remaining cases are similar and even easier. By the Sobolev embedding theorem, h_{ij} belongs to $L_{\mathrm{loc}}^{2^*}(\mathbb{R}^d)$ (and, hence, to $L_{\mathrm{loc}}^{2^*}(\mathbb{R}^d, \mu)$), where $1/2^* = 1/2 - 1/d$. Thus, arguing as above, replacing $\|h_{ij}\|_{L^2(B(0,r),\mu)}$ by $\|h_{ij}\|_{L^{2^*}(B(0,r),\mu)}$ and $\|f\|_{W^{1,2}(\mathbb{R}^d)}$ by $\|f\|_{W^{1,(2^*)'}(\mathbb{R}^d)}$ in the estimate (3.21) and applying again [16, Theorem D.1.4(ii)]

we can infer that g_{ij} belongs to $W^{1,2^*}_{\mathrm{loc}}(\mathbb{R}^d)$. Iterating this procedure, in a finite number of steps we get that $h_{ij} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p > d$. We are almost done. Indeed, again by the Sobolev embedding theorem we deduce that $g_{ij} \in L^q_{\mathrm{loc}}(\mathbb{R}^d)$ for any $q \leq +\infty$. Hence, we can write estimate (3.21) with $\|f\|_{W^{1,2}(\mathbb{R}^d)}$ being replaced by $\|f\|_{W^{1,q'}(\mathbb{R}^d)}$, for any $q' < +\infty$, and [16, Theorem D.1.4(ii)] allows us to conclude that h_{ij} belongs to $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^d)$ for any $q < +\infty$.

Finally, we observe that ϕ^+_{ij} coincides with the positive part of h_{ij} (and, hence, ϕ^-_{ij} coincides with the negative part of h_{ij}) as it is immediately checked recalling that ϕ^+_{ij} and ϕ^-_{ij} are nonnegative functions with disjoint supports. Since the positive and negative parts of a function in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ belong to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$, we immediately conclude that $g_{ij} = \phi^+_{ij} + \phi^-_{ij}$ belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ as well. $\square$

To conclude this subsection, we consider the particular case where the measure μ is symmetrizing for the scalar semigroup $T(t)$, i.e.,

$$\int_{\mathbb{R}^d} \mathcal{A}f \, g d\mu = - \int_{\mathbb{R}^d} \langle Q\nabla f, \nabla g\rangle d\mu. \tag{3.22}$$

for any $f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$, $g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d)$ such that f or g has compact support.

Remark 3.2. Sufficient conditions for (3.22) to hold are provided in Lorenzi etc [17] under the following additional assumptions on the coefficients q_{ij} and b_j $(i, j = 1, \ldots, d)$: there exists a function $\Phi : \mathbb{R}^d \to \mathbb{R}$ such that

(i) $Q^{-1}(\mathrm{div}Q - \mathbf{b}) = \nabla\Phi$ where $(\mathrm{div}Q)_j := \sum_{i=1}^d D_i q_{ij}$ for any $j = 1, \ldots, d$;

(ii) $e^{-\Phi} \in L^1(\mathbb{R}^d)$;

(iii) there exists two positive constants k_1 and $k_2 \in (0, 1)$ such that

$$\langle Q(x)(J\mathbf{b}(x))^*\xi, \xi\rangle + \langle Q(x)\xi, \nabla\mathrm{Tr}(Q(x)S)\rangle - \mathrm{Tr}((\nabla(Q(x)\xi))Q(x)S)$$
$$\leq k_1|\sqrt{Q}(x)\xi|^2 + k_2|\sqrt{Q}(x)S\sqrt{Q}(x)|^2 \tag{3.23}$$

for any $x, \xi \in \mathbb{R}^d$ and any $d \times d$ symmetric matrix S.

Let $\{\mu_j : j = 1, \ldots, m\}$ be a system of invariant measures for $\mathbf{T}(t)$ which consists of measures absolutely continuous with respect to the Lebesgue measure. Since μ is equivalent to the Lebesgue measure, there exists a vector valued function $\boldsymbol{\rho}$ such that each ρ_j belongs to $L^1(\mathbb{R}^d, \mu)$ and $d\mu_j = \rho_j d\mu$ $(j = 1, \ldots, m)$. For $\boldsymbol{\rho}$ smooth enough, next theorem relates the invariance property of the family $\{\mu_j : j = 1, \ldots, m\}$ to a first-order differential equation that $\boldsymbol{\rho}$ has to satisfy.

Theorem 3.4. *Under Hypothesis* 2.1, *assume that* $|Q(x)| \leq c(1 + |x|^2)$ *for any* $x \in \mathbb{R}^d$, *that the map* $x \mapsto |Q(x)|$ *belongs to* $L^1(\mathbb{R}^d, \mu)$ *and that* μ *is symmetrizing for the scalar semigroup* $T(t)$. *Further, let* $\{\mu_i : i = 1, \ldots, m\}$ *be a family of Borel finite measures, absolutely continuous with respect to the Lebesgue measure. Suppose that* $\boldsymbol{\rho}$ *solves the first-order differential equations* $(Q\nabla\rho_i)_j - (B_j^*\boldsymbol{\rho})_i = 0$ *in* $\mathbb{R}^d$ *for any* $i, j = 1, \ldots, m$, *with* $\rho_i \in L^2(\mathbb{R}^d, \mu) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ *for some* $p < +\infty$ *and any* i *as above. Then,* $\{\mu_i : i = 1, \ldots, m\}$ *is a system of invariant measures for* $\mathbf{T}(t)$.

Proof. We split the proof into two steps.

Step 1. Here, we prove that the set

$$D_\mu(\boldsymbol{\mathcal{A}}) := \{\mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}}) : |\sqrt{Q}\nabla f_i| \in L^2(\mathbb{R}^d, \mu), \ i = 1, \ldots, m\}$$

is a core for $(\boldsymbol{\mathcal{A}}, D_{\max}(\boldsymbol{\mathcal{A}}))$ with respect to the mixed topology** of $C_b(\mathbb{R}^d; \mathbb{R}^m)$. To begin with, we prove that $\mathbf{R}(n)$ (see (2.8)) preserves $D_\mu(\boldsymbol{\mathcal{A}})$ for any $n \in \mathbb{N}$. For this purpose, we observe that, in view of Proposition 2.4, we can estimate

$$
|\sqrt{Q}\nabla(\mathbf{R}(n)\mathbf{f})_i|^2 = \left\langle Q \int_0^{+\infty} e^{-nt}\nabla(\mathbf{T}(t)\mathbf{f})_i(\cdot)dt, \int_0^{+\infty} e^{-nt}\nabla(\mathbf{T}(t)\mathbf{f})_i(\cdot)dt \right\rangle
$$

$$
\leq |Q| \left| \int_0^{+\infty} e^{-nt}\nabla(\mathbf{T}(t)\mathbf{f})_i(\cdot)dt \right|^2
$$

$$
\leq c|Q|\|\mathbf{f}\|_\infty \left| \int_0^{+\infty} e^{-nt}(1 \vee t^{-\frac{1}{2}})dt \right|^2 \leq c|Q|\|\mathbf{f}\|_\infty.
$$

Since $|Q| \in L^1(\mathbb{R}^d, \mu)$ we deduce that $\mathbf{R}(n)\mathbf{f} \in D_\mu(\boldsymbol{\mathcal{A}})$ for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$.

Now, for any $\mathbf{g} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and $n \in \mathbb{N}$ such that[††] $n > [\beta]$ (see (2.6)), we consider the function $n\mathbf{R}(n)\mathbf{g}$. Note that $\|\mathbf{R}(n)\mathbf{g}\|_{C_b(\mathbb{R}^d; \mathbb{R}^m)} \leq (n - \beta)^{-1}$ for any n as above. Moreover, $n\mathbf{R}(n)\mathbf{g}$ converges to $\mathbf{g}$ locally uniformly in $\mathbb{R}$ as n tends to $+\infty$. This is clear if $\mathbf{g}$ belongs to $D_{\max}(\boldsymbol{\mathcal{A}})$. Indeed, we can split $n\mathbf{R}(n)\mathbf{g} = \mathbf{g} - \mathbf{R}(n)\boldsymbol{\mathcal{A}}\mathbf{g}$ for any n and, by the above estimate, $\mathbf{R}(n)\boldsymbol{\mathcal{A}}\mathbf{g}$ vanishes uniformly in $\mathbb{R}^d$ as n tends to $+\infty$. Suppose that $\mathbf{g} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$. Since $C_c^\infty(\mathbb{R}^d; \mathbb{R}^m) \subset D_{\max}(\boldsymbol{\mathcal{A}})$, we can determine a sequence $(\mathbf{g}_m) \subset D_{\max}(\boldsymbol{\mathcal{A}})$, bounded with respect to the sup-norm, which converges to $\mathbf{g}$ locally uniformly in $\mathbb{R}^d$. We split

$$
\|n\mathbf{R}(n)\mathbf{g} - \mathbf{g}\|_{C_b(B(0,r);\mathbb{R}^m)} \leq \|n\mathbf{R}(n)(\mathbf{g} - \mathbf{g}_m)\|_{C_b(B(0,r);\mathbb{R}^m)} + \|\mathbf{g}_m - \mathbf{g}\|_{C_b(B(0,r);\mathbb{R}^m)}
$$

$$
+ \|n\mathbf{R}(n)\mathbf{g}_m - \mathbf{g}_m\|_{C_b(B(0,r);\mathbb{R}^m)}
$$

$$
\leq \|\mathbf{T}(\cdot)(\mathbf{g} - \mathbf{g}_m)\|_{C_b((0,+\infty) \times B(0,r);\mathbb{R}^m)}
$$

$$
+ \|\mathbf{g}_m - \mathbf{g}\|_{C_b(B(0,r);\mathbb{R}^m)}
$$

$$
+ \|n\mathbf{R}(n)\mathbf{g}_m - \mathbf{g}_m\|_{C_b(B(0,r);\mathbb{R}^m)}
$$

for any $m \in \mathbb{N}$, $n > [\beta]$ and $r > 0$. Taking Proposition 2.5 into account, we can let first n and then m tend to $+\infty$ in the first and last side of the previous chain of inequalities and conclude that $n\mathbf{R}(n)\mathbf{g}$ converges to $\mathbf{g}$ locally uniformly in $\mathbb{R}^d$.

Given $\mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}})$, the sequence $(\mathbf{f}_n)$ we are looking for can be defined by setting $\mathbf{f}_n = n\mathbf{R}(n)\mathbf{f}$ for any $n > [\beta]$.

Step 2. In view of Proposition 3.1, to prove that the system $\{\mu_i : i = 1, \ldots, m\}$ is invariant for $\mathbf{T}(t)$ we need to show that $\sum_{i=1}^m \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{f})_i d\mu_i = 0$ for any $\mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}})$. By Step 1, we can limit ourselves to proving that the previous formula holds true for any $\mathbf{f} \in D_\mu(\boldsymbol{\mathcal{A}})$. So, let us fix one such function $\mathbf{f}$ and let $(\vartheta_n) \in C_c^\infty(\mathbb{R}^d)$ be a sequence of cut-off functions such that $\chi_{B(0,n)} \leq \vartheta_n \leq \chi_{B(0,n+1)}$ and $|\nabla\vartheta_n| \leq cn^{-1}$ for any $n \in \mathbb{N}$. We set $\mathbf{f}_n := \vartheta_n\mathbf{f}$ and using (2.3) and (3.22) we obtain that

$$
\sum_{i=1}^m \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{f}_n)_i d\mu_i = \sum_{i=1}^m \left[\int_{\mathbb{R}^d} (\mathcal{A}f_{n,i})\rho_i d\mu + \sum_{j=1}^d \int_{\mathbb{R}^d} (B_j D_j \mathbf{f}_n)_i \rho_i d\mu \right]
$$

$$
= \sum_{i=1}^m \left[-\int_{\mathbb{R}^d} \langle Q\nabla f_{n,i}, \nabla\rho_i \rangle d\mu + \sum_{j=1}^d \int_{\mathbb{R}^d} (B_j D_j \mathbf{f}_n)_i \rho_i d\mu \right]
$$

** i.e., for any $\mathbf{f} \in D_{\max}(\boldsymbol{\mathcal{A}})$ there exists a sequence $(\mathbf{f}_n) \subset D_\mu(\boldsymbol{\mathcal{A}})$ such that $\sup_{n \in \mathbb{N}}(\|\mathbf{f}_n\|_\infty + \|\boldsymbol{\mathcal{A}}\mathbf{f}_n\|_\infty < +\infty)$, $\mathbf{f}_n$ and $\boldsymbol{\mathcal{A}}\mathbf{f}_n$ converge to $\mathbf{f}$ and $\boldsymbol{\mathcal{A}}\mathbf{f}$, respectively, locally uniformly in $\mathbb{R}^d$ as n tends to $+\infty$.

†† Here, $[\beta]$ denotes the integer part of β.

$$= \sum_{i=1}^{m} \sum_{j=1}^{d} \int_{\mathbb{R}^d} D_j f_{n,i}[(B_j^* \boldsymbol{\rho})_i - (Q\nabla\rho_i)_j] d\mu = 0 \qquad (3.24)$$

for $n \in \mathbb{N}$. Now we show that the first side of (3.24) converges to $\sum_{i=1}^{m} \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{f})_i d\mu_i$ as n tends to $+\infty$. For this purpose, we observe that for any $n \in \mathbb{N}$ it holds that

$$(\boldsymbol{\mathcal{A}}\mathbf{f}_n)_i = \vartheta_n (\boldsymbol{\mathcal{A}}\mathbf{f})_i + f_i \mathcal{A}\vartheta_n + \sum_{k=1}^{m} \sum_{j=1}^{d} f_k (B_j)_{ik} D_j \vartheta_n + 2\langle Q\nabla\vartheta_n, \nabla f_i\rangle.$$

Integrating this formula over $\mathbb{R}^d$ with respect to μ_i, summing up over i from 1 to m, using again (3.22), to write $\int_{\mathbb{R}^d} (\mathcal{A}\vartheta_n) f_i \rho_i d\mu = -\int_{\mathbb{R}^d} \langle Q\nabla\vartheta_n, \nabla(f_i\rho_i)\rangle d\mu$, and the assumption on $\boldsymbol{\rho}$, we get

$$\sum_{i=1}^{m} \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{f}_n)_i d\mu_i = \sum_{i=1}^{m} \int_{\mathbb{R}^d} \vartheta_n (\boldsymbol{\mathcal{A}}\mathbf{f})_i d\mu_i + \sum_{i=1}^{m} \int_{\mathbb{R}^d} \langle Q\nabla\vartheta_n, \nabla f_i\rangle \rho_i d\mu$$

$$+ \int_{\mathbb{R}^d} \sum_{k=1}^{m} \sum_{j=1}^{d} f_k (B_j)_{ik} D_j \vartheta_n d\mu_i - \sum_{k=1}^{m} \int_{\mathbb{R}^d} f_k \langle Q\nabla\vartheta_n, \nabla\rho_k\rangle d\mu$$

$$= \sum_{i=1}^{m} \int_{\mathbb{R}^d} \vartheta_n (\boldsymbol{\mathcal{A}}\mathbf{f})_i d\mu_i + \sum_{k=1}^{m} \int_{\mathbb{R}^d} \langle \sqrt{Q}\nabla\vartheta_n, \sqrt{Q}\nabla f_k\rangle \rho_k d\mu. \quad (3.25)$$

By the dominated convergence theorem, the first term in the last side of (3.25) converges to $\sum_{i=1}^{m} \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{f})_i d\mu_i$ as n tends to $+\infty$. In addition

$$\int_{\mathbb{R}^d} |\langle \sqrt{Q}\nabla\vartheta_n, \sqrt{Q}\nabla f_i\rangle \rho_i| d\mu \leq c \int_{\mathbb{R}^d \setminus B(0,n)} \frac{|\sqrt{Q}|}{n} |\sqrt{Q}\nabla f_i| |\rho_i| d\mu$$

$$\leq c\|\sqrt{Q}\nabla f_i\|_{L^2(\mathbb{R}^d,\mu)} \int_{\mathbb{R}^d \setminus B(0,n)} \rho_i^2 d\mu$$

which vanishes as n tends to $+\infty$, since $\rho_i \in L^2(\mathbb{R}^d, \mu)$. As a byproduct, we conclude that $\sum_{i=1}^{m} \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{f}_n)_i d\mu_i$ converges to $\sum_{i=1}^{m} \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{f})_i d\mu_i$ as n tends to $+\infty$. We have so proved that $\sum_{i=1}^{m} \int_{\mathbb{R}^d} (\boldsymbol{\mathcal{A}}\mathbf{f})_i d\mu_i = 0$. This completes the proof. $\qquad \square$

Example 3.1. Here we assume $d = 1$ and $m = 2$. In this case $(\boldsymbol{\mathcal{A}}\boldsymbol{\zeta})(x) = q(x)\boldsymbol{\zeta}''(x) + b(x)\boldsymbol{\zeta}'(x) + B\boldsymbol{\zeta}'(x)$ for any $x \in \mathbb{R}$, on smooth functions $\boldsymbol{\zeta} : \mathbb{R} \to \mathbb{R}^2$. We suppose that q, b and the entries of the matrix-valued function B satisfy Hypotheses 2.1 and the function $x \mapsto -\log(q(x)) + \int_0^x (q(s))^{-1}b(s)ds$ belongs to $L^1(\mathbb{R})$. In this case, Remark 3.2 is satisfied and

$$\mu(dx) = \frac{c}{q(x)} \exp\Big(\int_0^x \frac{b(s)}{q(s)} ds\Big) dx$$

for a suitable positive constant c. Note that condition (iii) in Remark 3.2 reduces to $q(x)b'(x)\xi^2 \leq k_1 q(x)\xi^2 + k_2(q(x))^2 s^2$ for any $s, \xi, x \in \mathbb{R}$ and some constants $k_1 > 0$ and $k_2 \in (0,1)$, which is trivially satisfied since $b' < 0$ in $\mathbb{R}$ due to Hypothesis 2.1(iv).

In order to compute a system of invariant measures associated to the vector-valued semigroup associated to $\boldsymbol{\mathcal{A}}$ we further assume that $|q(x)| \leq c(1 + x^2)$ for any $x \in \mathbb{R}$, and $B_{11}(x) + B_{12}(x) = B_{21}(x) + B_{22}(x) =: \beta(x)$ for any $x \in \mathbb{R}$. From

Hypothesis 2.1(ii) it follows that the functions B_{ij} $(i, j = 1, 2)$ grow at most linearly as $|x|$ tends to $+\infty$. We solve the system $q\boldsymbol{\rho}' = B^*\boldsymbol{\rho}$. Due to the above condition on the sum of the rows of B, we easily see that $(\rho_1 + \rho_2)' = q^{-1}(B_{11} + B_{12})(\rho_1 + \rho_2)$. Hence,

$$\rho_1(x) + \rho_2(x) = c_1 \exp\left(\int_0^x \frac{\beta(t)}{q(t)} dt\right), \qquad x \in \mathbb{R},$$

for some positive constant c_1. Using this equation to write ρ_2 in terms of ρ_1 and replacing in the first equation of the above system, we easily see that

$$\rho_1(x) = \exp\left(\int_0^x \frac{\gamma(t)}{q(t)} dt\right)\left[c_2 + c_1 \int_0^x \exp\left(\int_0^t \frac{\beta(s) - \gamma(s)}{q(s)} ds\right)\frac{B_{21}(t)}{q(t)} dt\right],$$

and

$$\rho_2(x) = \exp\left(\int_0^x \frac{\gamma(t)}{q(t)} dt\right)\left[-c_2 + c_1 \exp\left(\int_0^x \frac{\beta(t) - \gamma(t)}{q(t)} dt\right)\right.$$
$$\left. - c_1 \int_0^x \exp\left(\int_0^t \frac{\beta(s) - \gamma(s)}{q(s)} ds\right)\frac{B_{21}(t)}{q(t)} dt\right]$$
$$= \exp\left(\int_0^x \frac{\gamma(t)}{q(t)} dt\right)\left[-c_2 + c_1 \exp\left(\int_0^x \frac{\beta(t) - \gamma(t)}{q(t)} dt\right)\right.$$
$$- c_1 \int_0^x \exp\left(\int_0^t \frac{\beta(s) - \gamma(s)}{q(s)} ds\right)\frac{\beta(t) - \gamma(t)}{q(t)} dt$$
$$\left. + c_1 \int_0^x \exp\left(\int_0^t \frac{\beta(s) - \gamma(s)}{q(s)} ds\right)\frac{B_{12}(t)}{q(t)} dt\right]$$
$$= \exp\left(\int_0^x \frac{\gamma(t)}{q(t)} dt\right)\left[-c_2 + c_1 + c_1 \int_0^x \exp\left(\int_0^t \frac{\beta(s) - \gamma(s)}{q(s)} ds\right)\frac{B_{12}(t)}{q(t)} dt\right]$$

for some positive constant $c_2 \in \mathbb{R}$, where $\gamma = B_{11} - B_{21}$.

Now, we consider two concrete cases.

Case 1. Here, we assume $q(x) = 1$, $b(x) = -x$ for any $x \in \mathbb{R}$ and

$$B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

This means that the scalar operator is the Ornstein-Uhlenbeck operator and the invariant measure of the associated Ornstein-Uhlenbeck operator $T(t)$ is the Gaussian measure $\mu(dx) = (2\pi)^{-1/2}e^{-x^2/2}dx$. Hence, condition (ii) in Remark 3.2 is clearly satisfied.

From the above formulas for ρ_1 and ρ_2, we get

$$\rho_1(x) = a_1 e^x + a_2 e^{-x}, \qquad \rho_2(x) = -a_1 e^x + a_2 e^{-x}, \qquad x \in \mathbb{R},$$

for any $a_1, a_2 \in \mathbb{R}$.

Case 2. If $q(x) = 1 + x^2$, $b(x) = -b_0 x(1 + x^2)$ and $B_{ij}(x) = b_{ij}x$ for any $x \in \mathbb{R}$ and some positive constants b_0 and b_{ij} $(i, j = 1, 2)$ such that

$$\left(\sum_{i,j=1}^{2} b_{ij}^2 \right)^{\frac{1}{2}} + \left(\max_{1 \leq i,j \leq 2} |b_{ij}| + 1 \right)^2 < b_0,$$

then Hypotheses 2.1 are clearly satisfied. In particular, for any $h \in \mathbb{N}$, the function φ, defined by $\varphi(x) = (1 + x^2)^h$ satisfies Hypothesis 2.1(iii). Moreover it is quite easy to show that the density of the invariant measure μ associated to the scalar semigroup $T(t)$ is the function $x \mapsto (\pi e^{b/2}\text{erfc}(2^{-1/2}b^{1/2}))(1+x^2)^{-1}\exp(-bx^2/2)dx$ for any $x \in \mathbb{R}$. Again, this implies that condition (ii) in Remark 3.2 holds true. It turns out that

$$\begin{cases} \rho_1(x) = (1 + x^2)^{\frac{b_{11}-b_{21}}{2}} \left(c_2 + \frac{c_1}{2} b_{21} \log(1 + x^2) \right), \\ \rho_2(x) = (1 + x^2)^{\frac{b_{11}-b_{21}}{2}} \left(c_1 - c_2 + \frac{c_1}{2} b_{12} \log(1 + x^2) \right), \end{cases} \quad (3.26)$$

for any $x \in \mathbb{R}$ and $c_1, c_2 \in \mathbb{R}$, if $b_{12} = -b_{21}$, and

$$\begin{cases} \rho_1(x) = (1 + x^2)^{\frac{b_{11}-b_{21}}{2}} \left(c_2 + c_1 \frac{b_{21}}{b_{12}+b_{21}} (1 + x^2)^{\frac{b_{12}+b_{21}}{2}} - c_1 \frac{b_{21}}{b_{12}+b_{21}} \right), \\ \rho_2(x) = (1 + x^2)^{\frac{b_{11}-b_{21}}{2}} \left(-c_2 + c_1 \frac{b_{12}}{b_{12}+b_{21}} (1 + x^2)^{\frac{b_{12}+b_{21}}{2}} + c_1 \frac{b_{21}}{b_{12}+b_{21}} \right), \end{cases}$$

for any $x \in \mathbb{R}$ and $c_1, c_2 \in \mathbb{R}$, otherwise. Note that in both cases, the functions ρ_1 and ρ_2 belongs to $H^1_{\text{loc}}(\mathbb{R}) \cap L^q(\mathbb{R}, \mu)$ for any $q < +\infty$.

Remark 3.3. We stress that, if ρ_1 and ρ_2 are given by (3.26) and B is not diagonal, then, for any choice of the constants c_1 and c_2, at least one between $\mu_1 = \rho_1 d\mu$ and $\mu_2 = \rho_2 d\mu$ is not a positive measure. Indeed, suppose to fix the ideas that $b_{12} < 0$. Then, ρ_2 is positive in a neighborhood of $+\infty$ if and only if $c_1 \leq 0$. Since $\rho_2(0) = c_1 - c_2$, also c_2 should be non positive and c_1, c_2 can not be both zero. If $c_2 < 0$, then $\rho_1(0) < 0$, otherwise, if $c_2 = 0$, then $c_1 < 0$ and $\rho_2(0) = c_1 < 0$.

3.2. Asymptotic behaviour of the semigroup in $C_b(\mathbb{R}^d; \mathbb{R}^m)$

As in the scalar case, the systems of invariant measures $\{\mu_j^i : j = 1, \ldots, m\}$ $(i = 1, \ldots, m)$ provided by Theorem 3.1 allow us to characterize the asymptotic behaviour of $\mathbf{T}(t)$ in $C_b(\mathbb{R}^d; \mathbb{R}^m)$ as t tends to $+\infty$.

Theorem 3.5. *For any* $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ *and* $i = 1, \ldots, m$, *it holds that*

$$\lim_{t \to +\infty} (\mathbf{T}(t)\mathbf{f})_i = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\mu_j^i,$$

locally uniformly in $\mathbb{R}^d$.

Proof. Fix $t > 0$, $x \in \mathbb{R}^d$ and $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$. Using the invariance property (3.1) and taking into account that $\mu_j^i(\mathbb{R}^d) = \delta_{ij}$ for any $i, j = 1, \ldots, m$ (see Subsection

3.1), we can write

$$\left| (\mathbf{T}(t)\mathbf{f})_i(x) - \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j(y) d\mu_j^i(y) \right| = \left| \sum_{j=1}^{m} \int_{\mathbb{R}^d} \left((\mathbf{T}(t)\mathbf{f})_j(x) - (\mathbf{T}(t)\mathbf{f})_j(y) \right) d\mu_j^i(y) \right|$$

for any $i = 1, \dots, m$. Now, for any $t \geq 1$ set $B^t := B(0, e^{-\sigma_2 t/2})$. Thus, using estimate (2.18), recalling that $\gamma^{-1} \leq \gamma_0$, $\varphi \geq 1$ in $\mathbb{R}^d$ and taking Proposition 3.2 into account, we deduce that

$$\left| \int_{\mathbb{R}^d} \left((\mathbf{T}(t)\mathbf{f})_j(x) - (\mathbf{T}(t)\mathbf{f})_j(y) \right) d\mu_j^i(y) \right|$$

$$\leq \int_{\mathbb{R}^d \setminus B^t} \left| \left((\mathbf{T}(t)\mathbf{f})_j(x) - (\mathbf{T}(t)\mathbf{f})_j(y) \right) \right| d\mu_j^i(y)$$

$$+ \int_{B^t} \left| \left((\mathbf{T}(t)\mathbf{f})_j(x) - (\mathbf{T}(t)\mathbf{f})_j(y) \right) \right| d\mu_j^i(y)$$

$$\leq c \|\mathbf{f}\|_\infty \int_{\mathbb{R}^d \setminus B^t} \varphi^{\frac{1}{\gamma}} d|\mu_j^i| + c e^{\sigma_2 t} \int_{B^t} |x - y| d|\mu_j^i|(y)$$

$$\leq c \|\mathbf{f}\|_\infty \int_{\mathbb{R}^d \setminus B^t} \varphi^{\gamma_0} d|\mu_j^i| + c \left(e^{\sigma_2 t} |x| + e^{\frac{1}{2}\sigma_2 t} \right)$$

for any $i, j = 1, \dots, m$. Letting t tend to $+\infty$ we obtain that the right-hand side tends to 0 locally uniformly with respect to $x \in \mathbb{R}^d$ and this yields the assertion. $\qquad \square$

A. A priori estimates

Theorem A.1. *Let $\mathbf{u}$ belong to $W_{\mathrm{loc}}^{2,p}(\mathbb{R}^d; \mathbb{R}^m)$ for some $1 < p < +\infty$. Then, for any pair of open bounded sets Ω_1 and Ω_2, Ω_1 being compactly contained in Ω_2, there exists a positive constant c, depending on d, p, Ω_1, Ω_2 the ellipticity constant of the operator $\boldsymbol{\mathcal{A}}$ and the Hölder norm of its coefficients over Ω_2, but independent of $\mathbf{u}$, such that*

$$\|\mathbf{u}\|_{W^{2,p}(\Omega_1; \mathbb{R}^m)} \leq c(\|\mathbf{u}\|_{L^p(\Omega_2; \mathbb{R}^m)} + \|\boldsymbol{\mathcal{A}}\mathbf{u}\|_{L^p(\Omega_2; \mathbb{R}^m)}). \qquad (A.1)$$

Proof. We divide the proof into two steps. In the first step we prove the claim when $\Omega_1 = B(0, r)$ and $\Omega_2 = B(0, 2r)$, $r > 0$. In the second one, we complete the proof.

Step 1. For any $n \in \mathbb{N}$, we set $r_n = (2 - 2^{-n})r$. Clearly, $r_0 = r$ and $r_\infty = 2r$. We also set

$$\vartheta_n(x) = \vartheta \left(1 + \frac{|x| - r_n}{r_{n+1} - r_n} \right), \qquad x \in \mathbb{R}^d, \ n \in \mathbb{N},$$

where $\vartheta \in C^\infty(\mathbb{R})$ satisfies $\chi_{(-\infty, 1]} \leq \vartheta \leq \chi_{(-\infty, 2]}$. Clearly, each function ϑ_n belongs to $C_c^\infty(\mathbb{R}^d)$, is such that $0 \leq \vartheta_n \leq 1$, $\vartheta_n = 1$ in $B(0, r_n)$ and $\mathrm{supp}(\vartheta_n) \subset B(0, r_{n+1})$. Moreover $\|\vartheta_n\|_{C_b^h(\mathbb{R}^d)} \leq 2^{hn} c_r$ for $h = 1, 2$.

Applying classical global L^p-estimates to the functions $\mathbf{v}_n := \vartheta_n \mathbf{u}$, which belong to $W^{2,p}(\mathbb{R}^d; \mathbb{R}^m)$ as well as the interpolative estimate $\| |J_x \mathbf{v}_{n+1}| \|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \leq$

$c(\varepsilon\|D^2\mathbf{v}_{n+1}\|_{L^p(\mathbb{R}^d;\mathbb{R}^m)} + \varepsilon^{-1}\|\mathbf{v}_{n+1}\|_{L^p(\mathbb{R}^d;\mathbb{R}^m)})$, which holds true for any $\varepsilon > 0$, we deduce that

$$
\begin{aligned}
\|\mathbf{v}_n\|_{W^{2,p}(\mathbb{R}^d;\mathbb{R}^m)} &\leq c_r(\|\mathbf{v}_n\|_{L^p(\mathbb{R}^d;\mathbb{R}^m)} + \|\mathcal{A}\mathbf{v}_n\|_{L^p(\mathbb{R}^d;\mathbb{R}^m)}) \\
&\leq c_r(\|\mathcal{A}\mathbf{u}\|_{L^p(B(0,2r);\mathbb{R}^m)} + 4^n\|\mathbf{u}\|_{L^p(B(0,2r);\mathbb{R}^m)} \\
&\quad + 2^n\||J_x\mathbf{v}_{n+1}|\|_{L^p(\mathbb{R}^d;\mathbb{R}^m)}) \\
&\leq c_r(\|\mathcal{A}\mathbf{u}\|_{L^p(B(0,2r);\mathbb{R}^m)} + (2^n\varepsilon^{-1} + 4^n)\|\mathbf{u}\|_{L^p(B(0,2r);\mathbb{R}^m)} \\
&\quad + 2^n\varepsilon\|\mathbf{v}_{n+1}\|_{W^{2,p}(\mathbb{R}^d;\mathbb{R}^m)})
\end{aligned}
$$

for any $\varepsilon \in (0,1)$, where the constant c depends also on d, m, p, the ellipticity constant of the operator $\mathcal{A}$ and the Hölder norm of its coefficients over $B(0,2r)$. Choosing $\varepsilon = c^{-1}2^{-n-4}$ the previous inequality becomes

$$
\begin{aligned}
&\|\mathbf{v}_n\|_{W^{2,p}(\mathbb{R}^d;\mathbb{R}^m)} - 2^{-4}\|\mathbf{v}_{n+1}\|\|_{W^{2,p}(\mathbb{R}^d;\mathbb{R}^m)} \\
&\leq c_r\|\mathcal{A}\mathbf{u}\|_{L^p(B(0,2r);\mathbb{R}^m)} + 4^{n+2}c_r\|\mathbf{u}\|_{L^p(B(0,2r);\mathbb{R}^m)}.
\end{aligned}
$$

Multiplying both the terms by 2^{4n} and summing over n from 0 to $k \in \mathbb{N}$ we get

$$
\begin{aligned}
&\|\mathbf{v}_0\|_{W^{2,p}(\mathbb{R}^d;\mathbb{R}^m)} - 2^{-4k-4}\|\mathbf{v}_{k+1}\|_{W^{2,p}(\mathbb{R}^d;\mathbb{R}^m)} \\
&\leq c_r(\|\mathcal{A}\mathbf{u}\|_{L^p(B(0,2r);\mathbb{R}^m)} + \|\mathbf{u}\|_{L^p(B(0,2r);\mathbb{R}^m)}).
\end{aligned}
$$

Since $\|\vartheta_{k+1}\|_{C_b^2(\mathbb{R}^d)} \leq 4^k c_r$, for any $k \in \mathbb{N}$, the second term in the left-hand side of the previous inequality vanishes as k tends to $+\infty$ and this allows us to conclude the proof in this particular case, recalling that $\mathbf{v}_0 = \mathbf{u}$ on $B(0,r)$.

  Step 2. Here, we complete the proof using a covering argument. Let Ω_1 and Ω_2 be as in the statement of the theorem. Further, fix $0 < r < \text{dist}(\Omega_1, \partial\Omega_2)$. By compactness we can cover Ω_1 by a finite number of balls of radius r, i.e., there exist $x_1, \ldots, x_k$ in Ω_1 such that $\overline{\Omega_1} \subset \bigcup_{i=1}^k (B(x_i, r))$. Due to the choice of r, $\bigcup_{i=1}^k (B(x_i, r)) \subset \Omega_2$. By a translation, we can easily extend estimate (A.1) to balls centered at any point $x_0 \in \mathbb{R}^d$. Hence, we can write

$$
\begin{aligned}
\|\mathbf{u}\|_{W^{2,p}(\Omega_1;\mathbb{R}^m)} &\leq \sum_{i=1}^k \|\mathbf{u}\|_{W^{2,p}(B(x_i,r);\mathbb{R}^m)} \\
&\leq c\sum_{i=1}^k (\|\mathbf{u}\|_{L^p(B(x_i,2r);\mathbb{R}^m)} + \|\mathcal{A}\mathbf{u}\|_{L^p(B(x_i,2r);\mathbb{R}^m)}) \\
&\leq c(\|\mathbf{u}\|_{L^p(\Omega_2;\mathbb{R}^m)} + \|\mathcal{A}\mathbf{u}\|_{L^p(\Omega_2;\mathbb{R}^m)})
\end{aligned}
$$

and the claim is so proved.   □

Theorem A.2 (Theorem A.2, Addona etc [1]). *Let $\mathbf{u} \in C_{\mathrm{loc}}^{1+\alpha/2,2+\alpha}((0,T]\times\mathbb{R}^d;\mathbb{R}^m)$ satisfy the differential equation $D_t\mathbf{u} = \mathcal{A}\mathbf{u} + \mathbf{g}$ in $(0,T] \times \mathbb{R}^d$, for some $\mathbf{g} \in C_{\mathrm{loc}}^{\alpha/2,\alpha}((0,T] \times \mathbb{R}^d;\mathbb{R}^m)$ and $T > 0$. Then, for any $\tau \in (0,T)$ and any pair of bounded open sets Ω_1 and Ω_2, with Ω_1 being compactly contained in Ω_2, there exists a positive constant c, independent of $\mathbf{u}$, such that*

$$
\begin{aligned}
&\|\mathbf{u}\|_{C^{1+\alpha/2,2+\alpha}((\tau,T)\times\Omega_1;\mathbb{R}^m)} \\
&\leq c(\|\mathbf{u}\|_{C_b((\tau/2,T)\times\Omega_2;\mathbb{R}^m)} + \|\mathbf{g}\|_{C^{\alpha/2,\alpha}((\tau/2,T)\times\Omega_2;\mathbb{R}^m)}).
\end{aligned}
$$

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