

NAVIER AND STOKES MEET POINCARÉ AND DULAC

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Dedicated to Claude-Michel Brauner on the occasion of his 70th birthday.

Abstract This paper surveys various precise (long-time) asymptotic results for the solutions of the Navier-Stokes equations with potential forces in bounded domains. It turns out that the asymptotic expansion leads surprisingly to a kind of Poincaré-Dulac normal form of the Navier-Stokes equations. We will also discuss some related results and a few open issues.

Keywords Navier-Stokes, asymptotic expansion, Poincaré-Dulac, normal form, normalization map.

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1. Introduction and some historical tidbits

Claude-Louis Navier (1785–1836) and George B. Stokes (1819–1903), of course, never met Henri Poincaré (1854–1912) and Henri Dulac (1870–1955) as we will recall in the small historical section below. (For more details, and more generally for a fascinating account of the history of fluid dynamics, see [19]). However their mathematical theory of dynamical systems and physical theory of fluid mechanics have finally met more than a hundred years after their initial contributions.

We will not comment on Stokes and Poincaré who are well-known scientists but make a few (may be not so well known) remarks on Dulac, and mainly on Navier.

Claude Louis Marie Henri Navier was an “X-Ponts” engineer in the jargon of *Grandes Ecoles*, first trained at the Ecole Polytechnique, then at the Ecole des Ponts et Chaussées, one of the *Ecoles d’applications* such as the *Ecole des Mines* (Augustin Louis Cauchy was an “X-Ponts”, Henri Poincaré was an “X-Mines”). He was in the main stream of French theoretical continuum mechanics of this time.

A major figure of French Mechanics of this time, Adhémar Barré de Saint-Venant (1797–1886), also an X-Ponts, was a former student and successor of Navier. Among many other things, he derived the so-called *Saint-Venant* system (or *shallow water system*). He was the advisor and protector of Joseph Boussinesq (1842–1929) who made fundamental contributions in Fluid Mechanics, in particular on the theory of water waves.

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As noticed by Olivier Darrigol in his book [19],

“Navier and other Polytechnicians’ efforts to reconcile theoretical and applied mechanics had no clear effect on French engineering practice. Industry prospered much faster in Britain, despite the lesser mathematical training of its engineers. Some of Navier’s colleagues saw this and ridiculed the use of transcendental mathematics in concrete problems of construction. In the mid-1820s, a spectacular incident apparently justified their disdain. Navier’s chef-d’oeuvre, a magnificent suspended bridge at the Invalides, had to be dismantled in the final stage of its construction”.

Actually Navier was probably most famous in his time for the “disaster” of the *pont des Invalides*, the first suspended bridge over the Seine river. In fact Navier had mis-estimated the direction of the force exerted by the chain on the stone. This could have been corrected easily but the hostile municipal authorities decided the dismantlement of Navier bridge. Honoré de Balzac is alluding to this incident (rather ironically) in his novel *Le curé de village*:

“La France entière a vu le désastre, au coeur de Paris, du premier pont suspendu que voulut élever un ingénieur, membre de l’Académie des Sciences, triste chute qui fut causée par des fautes que ni le constructeur du canal de Briare, sous Henri IV, ni le moine qui a bâti le Pont-Royal, n’eussent faites, et que l’Administration consola en l’appelant au Conseil Général (des Ponts et Chaussées). Les Ecoles Spéciales seraient-elles donc des fabriques d’incapacités? Ce sujet exige de longues observations”.

(*Translation*) “All France knew of the disaster which happened in the heart of Paris to the first suspended bridge built by an engineer, a member of the Academy of Sciences, a melancholy collapse cause by blunders such as none of the ancient engineers, the man who cut the canal at Briare in Henry’s IV time, or the monk who built the Pont Royal-would have made; but our administration consoled its engineer for his blunder by making him a member of the Council general. (of the *Ponts et Chaussées*). Are our *Ecoles Spéciales* producers of incapacities? This topic deserves lengthy observations”.

According to Saint-Venant however, the dismantlement of the bridge was more than a local administrative deficiency:

“At that time there already was a surge of the spirit of denigration, not only of the “savants” but also of science, disparaged under the name of *theory* opposed to *practice*; one henceforth exalted practice in its most material aspects, and prevented that higher mathematics could not help, as if, when it comes to results, it made sense to distinguish between the more or less elementary or transcendent procedures that led to them in an equally logical manner. Some “savants” supported or echoed these unfounded criticisms”.

As an aside, we note that there are some estimates of velocity fields in infinite dimensional spaces which have useful consequences, e.g. [29].

Navier was nevertheless a great scientist and, coming back to our subject, he derived what are known as the Navier-Stokes equations in a 1823 Mémoire. Further (different) derivations are due to Poisson (1831), Saint-Venant (1834) and Stokes (1843).

The last member of the quartet in title is the least famous of them. Henri Dulac (1870-1955), a former student of Ecole Polytechnique, was a professor at the University of Lyon and a corresponding member of the French Academy of Sciences. He was a specialist of the geometric theory of ordinary differential equations and developed in particular, after Poincaré, the theory of *normal forms*.

1.1. The Navier-Stokes equations for viscous, incompressible fluid flows

We now recall briefly the derivation of Navier-Stokes equations (NSE), based on conservation laws and the choice of a constitutive equation. For complete background on NSE see e.g. [17, 28, 49, 70, 71].

We study fluid flows in Euclidean space of dimension $n = 2, 3$. Let ρ denote the density of the fluid, and u its velocity.

- Conservation of mass:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0.$$

We will consider only the case when the density ρ is constant, so that the conservation of mass reduces to

$$\operatorname{div} u = 0.$$

We refer to this as the incompressibility condition.

- Conservation of momentum (Newton's law) for a general fluid:

$$\rho(\partial_t u + (u \cdot \nabla)u) = \operatorname{div}(-\tilde{p}\mathbf{I} + \mathbf{T}) + \tilde{f},$$

where \tilde{p} is the (scalar) pressure, \mathbf{T} is the *extra-stress* tensor, and \tilde{f} represents body forces. Here, we use the standard notation $u \cdot \nabla = \sum_i u_i \partial_{x_i}$.

When $\mathbf{T} \equiv 0$, one obtains the Euler equations (1755).

- Constitutive law: For a *Newtonian viscous fluid*, \mathbf{T} at the present time t is just proportional to the rate of deformation tensor $\mathbf{D}(u) = (\nabla u + (\nabla u)^T)/2$ at time t , that is

$$\mathbf{T} = \mu \mathbf{D}(u),$$

where μ is the dynamic viscosity coefficient. (For a general *non-Newtonian fluid*, \mathbf{T} can be a complicated function of the past history of the deformations).

Finally, one obtains the Navier-Stokes equations (NSE)

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f, \\ \operatorname{div} u = 0, \end{cases} \quad (1.1)$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $p = \tilde{p}/\rho$, and $f = \tilde{f}/\rho$. For simplicity, we will just call ν viscosity, p pressure, and f body force.

The system (1.1) consists of $(n + 1)$ equations for $(n + 1)$ unknowns, namely, $u \in \mathbb{R}^n$ and $p \in \mathbb{R}$. It will be completed with initial and boundary conditions in our considerations.

1.2. Functional setting

We consider the following two cases of fluid flows.

The first scenario is when the fluid is confined a smooth, bounded domain Ω of \mathbb{R}^n , and the velocity satisfies the no-slip boundary condition, i.e., $u = 0$ on $\partial\Omega$. We set in this case

$$\mathcal{V} = \{v \in C_0^\infty(\Omega)^n : \operatorname{div} v = 0\}.$$

The second scenario is when (u, p) are defined in the whole space \mathbb{R}^n , but are L -periodic, for some $L > 0$, in all there Cartesian coordinates. Then u and p are considered as functions on the domain

$$\Omega = \mathbb{R}^n / [0, L]^n. \quad (1.2)$$

We usually refer to this Ω as a periodic domain, and say u and p satisfy the periodicity boundary condition on $[0, L]^n$. By a remarkable Galilean transformation, we assume, without loss of generality, that u has zero averages over Ω , i.e.,

$$\int_{\Omega} u(x, t) dx = 0.$$

We then define the space

$$\mathcal{V} = \left\{ \mathbb{R}^n\text{-valued } L\text{-periodic trigonometric polynomial } v : \operatorname{div} v = 0, \int_{\Omega} v dx = 0 \right\}.$$

In both cases, we will use the classical spaces:

$$\begin{aligned} H &= \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^n, \\ V &= \text{closure of } \mathcal{V} \text{ in } H^1(\Omega)^n, \end{aligned}$$

with norms

$$\|u\|_H = |u| = \left(\int_{\Omega} |u(x)|^2 dx \right)^{1/2}, \quad \|u\|_V = \|u\| = \left(\int_{\Omega} |\nabla v(x)|^2 dx \right)^{1/2}.$$

Note that notation $|\cdot|$ is used to denote the H -norm and the standard Euclidean norm on \mathbb{C}^n . However, its meaning will be clear in the context.

We denote the standard inner products of $L^2(\Omega)^k$, for $k \in \mathbb{N}$, by the same notation $\langle \cdot, \cdot \rangle$.

The norm in the Sobolev space $H^m(\Omega)$ is denoted by $\|\cdot\|_m$. We also denote

$$\mathcal{E}^m(\Omega) = H \cap H^m(\Omega) \text{ for } m \geq 0, \text{ and } \mathcal{E}^\infty(\Omega) = \bigcap_{m=0}^{\infty} \mathcal{E}^m(\Omega).$$

One has the Helmholtz-Leray decomposition for the case of no-slip boundary condition,

$$L^2(\Omega)^n = H \oplus \{\nabla\varphi : \varphi \in H^1(\Omega)\}. \quad (1.3)$$

and for the case of periodicity boundary condition,

$$\left\{ v \in L^2(\Omega)^n : \int_{\Omega} v \, dx = 0 \right\} = H \oplus \{ \nabla \varphi : \varphi \in H^1(\Omega) \}. \tag{1.4}$$

We define P to be the (Leray) orthogonal projection in $L^2(\Omega)^n$ onto H .

We assume at the moment that (u, p) are classical solutions of NSE. Thanks to (1.3) and (1.4), we have $P(\nabla p) = 0$. With this observation, we can reduce the unknowns of NSE from (u, p) to u only, by projecting the NSE to the space H . Having that in mind, we define the Stokes operator A by

$$Au = -P\Delta u$$

(with the ad hoc boundary conditions), and also define the bilinear form

$$B(v, w) = P[(v \cdot \nabla)w].$$

Assume f is a potential, i.e., $f = -\nabla\psi$, then, thanks to (1.3) and (1.4) again, $Pf = 0$.

Hence, applying the Leray projection P to the NSE, and using the decomposition (1.3) or (1.4), we rewrite the NSE (1.1) in the *functional form* as:

$$\begin{cases} \frac{du}{dt} + \nu Au + B(u, u) = 0, \\ u(0) = u_0, \end{cases} \tag{1.5}$$

where u_0 is a given initial data in H .

This functional form (1.5) will be the focus of our study in this paper.

1.3. Basic facts

The Stokes operator A is an unbounded, self-adjoint operator in H with domain

$$D(A) = V \cap H^2(\Omega)^n.$$

Its spectrum $\sigma(A)$ consists of an unbounded sequence of real eigenvalues

$$0 < \Lambda_1 < \Lambda_2 < \dots < \Lambda_k < \dots, \tag{1.6}$$

with corresponding multiplicities $m_1, m_2, \dots, m_k, \dots$ (See e.g. [13].)

The orthogonal projection in H on the eigenspace of A corresponding to Λ_j will be denoted by R_{Λ_j} .

We denote

$$\mathcal{S}(A) = \{0 < \mu_1 = \Lambda_1 < \mu_2 < \mu_3 < \dots\},$$

the additive semi-group generated by the Λ_k 's.

In the periodic case,

$$\sigma(A) = \{4\pi^2|\mathbf{k}|^2/L^2 : \mathbf{k} \in \mathbb{Z}^n, \mathbf{k} \neq 0\},$$

hence,

$$\Lambda_1 = 4\pi^2/L^2 \quad \text{and} \quad \sigma(A) \subset \{n\Lambda_1 : n \in \mathbb{N}\}. \tag{1.7}$$

By scaling the spatial and time variables, we can further assume, without loss of generality,

$$\nu = 1 \quad \text{and} \quad L = 2\pi, \quad (1.8)$$

thus,

$$\Lambda_1 = 1, \quad \sigma(A) \subset \mathbb{N}, \quad \text{and} \quad \mathcal{S}(A) = \mathbb{N}. \quad (1.9)$$

It has been known since Leray's fundamental papers ([50–52]) that

- (i) For every initial data $u_0 \in H$, problem (1.5) has a (Leray-Hopf) weak solution u (see e.g. [17, 28, 49, 53, 70, 71]), that is,

$$u \in C([0, \infty); H_w) \cap L^2_{\text{loc}}([0, \infty); V), \quad u' \in L^{4/3}_{\text{loc}}([0, \infty); V'),$$

satisfying (1.5) in the dual space V' of V , and the energy inequality

$$\frac{1}{2}|u(t)|^2 + \int_{t_0}^t \|u(\tau)\|^2 d\tau \leq \frac{1}{2}|u(t_0)|^2$$

holds for $t_0 = 0$ and almost all $t_0 \in (0, \infty)$, and all $t \geq t_0$.

Above, H_w denotes the space H endowed with the weak topology.

If I is a closed interval in $[0, \infty)$, then a weak solution u is *regular* on I if $u \in C(I; V)$.

- (ii) This weak solution becomes regular on $[T_0, \infty)$, for some $T_0 = T_0(\nu, u_0) \geq 0$.
 (iii) It is not known whether a (Leray-Hopf) weak solution is unique.
 (iv) If u is regular on $I = [t_0, t_1]$, then u is uniquely determined on I by $u(t_0)$.
 (v) It is known that any (Leray-Hopf) weak solution u satisfies

$$\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu \|u(t)\|^2 \leq 0 \text{ in the distribution sense on } (0, \infty).$$

- (vi) Any regular solution u on $[0, \infty)$ satisfies the equation

$$\frac{1}{2} \frac{d}{dt} |u(t)|^2 + \nu \|u(t)\|^2 = 0 \text{ on } (0, \infty). \quad (1.10)$$

Because of property (ii) above, and that our goal is to study long-time behavior of solutions to NSE, we will, without loss of generality, mainly consider regular solutions on $[0, \infty)$.

Let \mathcal{R} denote the set of initial data in V leading to global regular solutions. Then \mathcal{R} is an open subset of V , and, particularly, $\mathcal{R} = V$ when $d = 2$.

Obviously, $u = 0$ is a trivial regular solution on $[0, \infty)$. Hence, \mathcal{R} contains a neighborhood of 0. However, proving or disproving that $\mathcal{R} = V$ when $d = 3$ is still an outstanding open problem.

Here afterward, we will call a regular solution u on $[0, \infty)$, that is when $u(0) \in \mathcal{R}$, simply a *regular solution*.

For a regular solution u , one has from (1.10) and the Poincaré inequality, i.e.,

$$\Lambda_1 |u|^2 \leq \|u\|^2,$$

that

$$|u(t)|^2 \leq |u_0|^2 e^{-2\nu\Lambda_1 t}, \quad \forall t \geq 0.$$

That is $|u(t)|^2$ must decay exponentially as $t \rightarrow \infty$ at the rate at least $2\nu\Lambda_1$.

1.4. Aim and outline of the paper

A natural question (raised by P. Lax to C. Foias) is then to ask whether or not this decay rate is optimal. In an early work, Dyer and Edmunds [21] prove that any non-trivial, regular solution u has $|u(t)|^2$ also bounded below by an exponential function of t . However, this answer is far from being definitive in describing the exact asymptotic behavior of a non-trivial, regular solution. In the following sections, we present the mathematical developments of the problem which lead to the asymptotic expansion and normal form theory (for NSE).

The paper is organized as follows. In section 2, the Dirichlet quotient is proved to converge, as $t \rightarrow \infty$ to an eigenvalue of the Stokes operator. The asymptotic behavior of the regular solutions are studied. The set \mathcal{R} is decomposed into non-linear manifolds M_k 's, which characterize the rate of the decay for the solutions. In section 3, each regular solution is proved to admit an asymptotic expansion in terms of exponential decays and polynomials in time. The application to analysis of the helicity is also presented. In section 4, we review the classical Poincaré-Dulac theory of normal forms for ordinary differential equations (ODEs). In section 5, it is shown that the asymptotic expansion reduces to a normal form, which, originally, is in a Fréchet space with very weak topology. It is then studied in suitable Banach spaces. In such a weighted normed space, the normalization map is continuous and the normal form for NSE is a well-posed infinite ODE system. In section 6, the inverse of the normalization map is written as a formal power series in E^∞ , an appropriate topological vector subspace of C^∞ . It is then used to reduce the NSE to a Poincaré-Dulac normal form on E^∞ . In section 7, we review more related results and pose some open questions.

2. Limit of the Dirichlet quotients

By re-writing the energy equality (1.10) in the form

$$\frac{1}{2} \frac{d}{dt} |u|^2 + \nu \frac{\|u\|^2}{|u|^2} |u|^2 = 0,$$

it is natural to study the limit as $t \rightarrow \infty$ of the *Dirichlet quotients*

$$\lambda(t) = \frac{\|u(t)\|^2}{|u(t)|^2}.$$

This is the beginning of a long process leading eventually to a *normal form* of NSE. One has the following results ([30–32]).

Theorem 2.1 ([30–32]). *Let $u_0 \in \mathcal{R} \setminus \{0\}$.*

- (i) $\lim_{t \rightarrow \infty} \lambda(t) = \Lambda(u_0)$ exists and belongs to $\sigma(A)$.
- (ii) $\lim_{t \rightarrow \infty} e^{\nu \Lambda(u_0) t} u(t)$ exists and belongs to $R_{\Lambda(u_0)} H$.
- (iii) There exist analytic submanifolds M_k , $k = 1, 2, \dots$, of \mathcal{R} having codimension $m_1 + m_2 + \dots + m_k$ such that

$$\mathcal{R} = M_0 \supset M_1 \supset M_2 \supset \dots$$

- (iv) M_k is invariant by the nonlinear semi-group $S(t)$ generated by the Navier-Stokes equation, that is $S(t)M_k \subset M_k, \forall t \geq 0$.
- (v) $u_0 \in M_{k-1} \setminus M_k$ if and only if $\Lambda(u_0) = \Lambda_k$, for $k = 1, 2, \dots$. Consequently, $u_0 \in M_{k-1}$ if and only if $\Lambda(u_0) \geq \Lambda_k$.
- (vi) The tangent space of M_k at 0 is M_k^{lin} , for $k = 1, 2, \dots$, where

$$M_k^{\text{lin}} = \{u_0 \in V; R_{\Lambda_1} u_0 = \dots = R_{\Lambda_{k-1}} u_0 = 0\}.$$

Proof. (i) We recall elementary estimates for regular solutions and large t :

$$\int_t^\infty \|u(\tau)\|^2 \leq C|u(t)|^2, \quad \int_t^\infty \|Au(\tau)\|^2 \leq C\|u(t)\|^2. \quad (2.1)$$

Let $v(t) = u(t)/|u(t)|$. Note that $|v(t)| = 1$ and $\|v(t)\|^2 = \lambda(t)$. One can derive a differential equation for $\lambda(t)$:

$$\frac{1}{2} \frac{d\lambda}{dt} + \nu|(A - \lambda)v|^2 = -|u|\langle B(v, v), (A - \lambda)v \rangle. \quad (2.2)$$

It follows that

$$\frac{d\lambda}{dt} + \nu|(A - \lambda)v|^2 \leq C\|u\|\|Au\|\lambda.$$

Neglecting the second term on the left-hand side, and using Gronwall lemma together with (2.1), we obtain for sufficiently large $t' > t > 0$ that

$$\lambda(t') \leq \lambda(t)e^{C \int_t^\infty \|u(\tau)\|\|Au(\tau)\|d\tau} \leq \lambda(t)e^{C'|u(t)|\|u(t)\|}.$$

Using the fact that $|u(t)|$ and $\|u(t)\|$ go to zero as $t \rightarrow \infty$, and letting $t' \rightarrow \infty$, then $t \rightarrow \infty$, we obtain

$$0 < \limsup_{t' \rightarrow \infty} \lambda(t') \leq \liminf_{t \rightarrow \infty} \lambda(t) < \infty.$$

Thus

$$\lim_{t \rightarrow \infty} \lambda(t) = \Lambda \text{ exists and belongs to } (0, \infty).$$

By (2.2), $(A - \lambda)v \in L^2(t, \infty)$. Then there exist $t_j \rightarrow \infty$ such that $(A - \lambda(t_j))v(t_j) \rightarrow 0$ and $v(t_j) \rightarrow \bar{v}$ in H . Thus $Av(t_j) \rightarrow \Lambda\bar{v}$. Since A is a closed operator, this yields $\bar{v} \in \mathcal{D}(A)$ and $A\bar{v} = \Lambda\bar{v}$. Note also that $|\bar{v}| = 1$, hence, $\Lambda \in \sigma(A)$.

(ii) After this eigenvalue Λ is established, ones can prove in [31, Proposition 1 and Lemma 1] that

$$\|u(t)\| \leq Ce^{-\nu\Lambda t}, \quad (2.3)$$

$$\lim_{t \rightarrow \infty} \|(I - R_\Lambda)e^{\nu\Lambda t}u(t)\| = 0. \quad (2.4)$$

It remains to deal with $R_\Lambda e^{\nu\Lambda t}u(t)$. We have

$$\frac{d}{dt}(R_\Lambda e^{\nu\Lambda t}u(t)) + e^{\nu\Lambda t}R_\Lambda B(u(t), u(t)) = 0.$$

Hence, for $s > t > 0$:

$$e^{\nu\Lambda s}R_\Lambda u(s) - e^{\nu\Lambda t}R_\Lambda u(t) = - \int_t^s e^{\nu\Lambda\tau}R_\Lambda B(u(\tau), u(\tau))d\tau.$$

By the exponential decay (2.3) of $u(t)$, we see that the right-hand side goes to 0 as $s, t \rightarrow \infty$. Therefore, by Cauchy's criterion,

$$\lim_{t \rightarrow \infty} e^{\nu \Lambda t} R_{\Lambda} u(t) \text{ exists and belongs to } R_{\Lambda} H.$$

Together with (2.4), this yields (ii).

(iii) The sets M_k 's can be defined as level sets $M_k = \Phi_k^{-1}(0)$, where the functions Φ_k 's are as follows. For $k = 1$, $\Phi_1 : \mathcal{R} \rightarrow R_{\Lambda_1} V$ is given by

$$\Phi_1(v) = R_{\Lambda_1} v - \int_0^{\infty} e^{\Lambda_1 t} R_{\Lambda_1} B(S(t)v, S(t)v) dt.$$

For $k \geq 2$, the functions $\Phi_k : M_{k-1} \rightarrow R_{\Lambda_k} V$ is defined by

$$\Phi_k(v) = R_{\Lambda_k} v - \int_0^{\infty} e^{\Lambda_k t} R_{\Lambda_k} B(S(t)v, S(t)v) dt.$$

Then the analyticity of M_k 's results from the analyticity of the mapping $(t, v) \rightarrow S(t)v$ which is due to [22]. □

Remark 2.1. The following remarks are in order.

- (a) (L. Tartar) In the case where Ω is only bounded in one direction (so that Poincaré inequality holds), one has also that the $\lim_{t \rightarrow \infty} \lambda(t) = \Lambda((u_0))$ exists and belongs to the spectrum of A , which is not necessarily discrete.
- (b) One can prove that the rate of decay given by $\Lambda(u_0)$ gives also the decay rate of higher Sobolev norms, and also the convergence of $u(t)e^{t\Lambda(u_0)}$ in all H^s for any $s > 0$, see [34, 43]. We refer to [39] for various extensions of the convergence of the Dirichlet quotients to other situations, in particular Navier-Stokes and MHD equations on compact Riemannian manifolds.
- (c) We recall that for any $u_0 \in H$, the NSE possesses a weak solution u which becomes regular for t sufficiently large. In this case the Dirichlet quotient $\lambda(t)$ converges to an eigenvalue $\Lambda(u(\cdot))$ of $\sigma(A)$ that, by lack of uniqueness, depends *a priori* on the whole solution u .
- (d) The manifolds M_k 's are apparently the only known nonlinear manifolds invariant under the the Navier-Stokes flow.
- (e) The geometry of parts of the M_k 's that are far from the origin is unknown (see however below for a specific property in the periodic case).

The invariant manifolds M_k 's can also be characterized as in the next result.

Theorem 2.2 ([31, Corollary 2]). *The necessary and sufficient condition for $u_0 \in M_{k-1}$, or equivalently, $\Lambda(u_0) \geq \Lambda_k$, with $k \geq 2$, is*

$$\lim_{t \rightarrow \infty} e^{\nu \Lambda_j t} R_{\Lambda_j} S(t)u_0 = 0 \quad \forall j = 1, 2, \dots, k - 1. \tag{2.5}$$

Note in (2.5) that it only requires the projection R_{Λ_j} of $S(t)u_0$, not the whole $S(t)u_0$.

One has further results on properties of the manifolds M_k 's in the periodic case.

Theorem 2.3 ([31, Remark 7], [32, Theorem 2 and Proposition 4]). *In the periodic case, each M_k is a smooth analytic, truly nonlinear manifold in \mathcal{R} , and contains a linear submanifold L_k of infinite dimension. Consequently, M_k is unbounded in V .*

Proof. For the nonlinearity, we argue by contradiction. Suppose M_k is linear. Then it must coincide with its tangential linear manifold at 0, which is M_k^{lin} . Together with the invariance of M_k under the semigroup $S(t)$, it follows that

$$\sum_{j=1}^k R_{\Lambda_j} B(v, v) = 0 \text{ whenever } v \in \mathcal{D}(A) \text{ such that } \sum_{j=1}^k R_{\Lambda_j} v = 0.$$

By construction of an explicit counter example (see [32]), this fact is shown to be not true.

The construction of the invariant linear submanifolds L_k 's is based on special motions of the Navier-Stokes equations in the periodic case that we describe now.

For $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n \setminus \{0\}$ such that $k_1 + \dots + k_n = 0$, we consider

$$u(x, t) = (\varphi(\mathbf{k} \cdot x, t), \dots, \varphi(\mathbf{k} \cdot x, t)) \quad (n \text{ times}), \quad (2.6)$$

where $\varphi(y, t)$ is a scalar function with $y, t \in \mathbb{R}$. One can verify that

$$\operatorname{div} u = 0 \text{ and } (u \cdot \nabla)u = 0.$$

Then any (spatial) L -periodic solution φ of the linear heat equation

$$\frac{\partial \varphi}{\partial t} - \nu |\mathbf{k}|^2 \frac{\partial^2 \varphi}{\partial y^2} = 0$$

leads to a solution of the NSE of the form $u(x, t)$ in (2.6) and $p = \text{constant}$. Clearly, such a solution satisfies

$$\|u(t)\|_m \leq C_m e^{-\nu |\mathbf{k}|^2 t} \quad \forall m \geq 0.$$

Thus, $u(0) \in M_k$ if $|\mathbf{k}|^2 > \Lambda_k$.

Based on the above observation, we set

$$\mathcal{U} = \{u \in V : u(x) = (\varphi(\mathbf{k} \cdot x), \dots, \varphi(\mathbf{k} \cdot x)), \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n, \\ k_1 + \dots + k_n = 0, \varphi(y) \text{ is } L\text{-periodic on } \mathbb{R}\}.$$

Then $\mathcal{U} \cap M_k^{\text{lin}}$ is an infinite-dimensional submanifold of M_k . \square

Remark 2.2. The family of manifolds constructed in Theorem 2.3 is extended to the following more general ones, which are also used to analyze the decay of the helicity, see [23, 24] and subsection 3.4 below.

Consider the periodic case in \mathbb{R}^3 . Let \mathbf{a} be a vector in \mathbb{R}^3 such that its orthogonal plane has nontrivial intersection with \mathbb{Z}^3 , this means

$$\mathbf{a}^\perp := \{\mathbf{k} \in \mathbb{Z}^3, \mathbf{k} \cdot \mathbf{a} = 0\} \neq \{0\}.$$

Define the linear manifold $\mathcal{M}_{\mathbf{a}^\perp}$ in V by

$$\mathcal{M}_{\mathbf{a}^\perp} = \{u \in V : u = \sum_{\mathbf{k} \in \mathbf{a}^\perp} a_{\mathbf{k}} e^{i\mathbf{k} \cdot x}, a_{\mathbf{k}} \text{ is (complex) collinear to } \mathbf{a}, \text{ for all } \mathbf{k} \in \mathbf{a}^\perp\}. \quad (2.7)$$

For $u_0 \in \mathcal{M}_{\mathbf{a}^\perp}$, $u(t) = e^{-tA}u_0$ belongs to $\mathcal{M}_{\mathbf{a}^\perp}$ for all $t \in [0, \infty)$ and solves the linearized NSE

$$\begin{cases} \frac{du}{dt} + Au = 0, & t > 0, \\ u(0) = u_0 \in V. \end{cases}$$

as well as the NSE (1.5). The fact that the nonlinear term $B(u(t), u(t))$ vanishes in this situation can be easily verified. Therefore, $\mathcal{M}_{\mathbf{a}^\perp}$ is an invariant linear manifold in \mathcal{R} . Clearly, the cardinality of \mathbf{a}^\perp is infinite, and hence $\mathcal{M}_{\mathbf{a}^\perp}$ is infinite-dimensional.

Remark 2.3. Regarding the structure of the set \mathcal{R} , we have the following remarks.

- (a) Since \mathcal{R} is an open set, Theorem 2.3 implies that in the periodic case, it contains an *unbounded* open subset of V . This fact was apparently unknown before in the periodic case. Also, the construction of the linear manifolds L_k is explicit. (See [75] for another construction of arbitrary large solutions in the three-dimensional periodic case).
- (b) When Ω is a bounded, open set of \mathbb{R}^3 (with Dirichlet boundary conditions), Bondarevsky ([3]) has proved, by a totally different method, that \mathcal{R} contains a star-shaped, unbounded open subset.
- (c) In the case $\Omega = \mathbb{R}^3$, one can also obtain the global existence of solutions starting from initial data with small low frequencies but allowing large high frequencies (oscillations). See Cannone, Meyer and Planchon ([11]), and Chemin and Gallagher ([14, 15]).

Remark 2.4. The Dirichlet quotients (interpreted as the ratio of the *enstrophy* over the *energy*) have been used to study geophysical flows, in particular to give a precise mathematical sense (and justify) the physicists' *selective decay principle*:

After a long time, solutions of the quasi-geostrophic equations and/or the two-dimensional incompressible Navier-Stokes equations approach those states which minimize the enstrophy for a given energy.

For more details we refer to [55–57, 73, 74].

Remark 2.5. In relation with [30, 31], Kukavica [47], in the 2D periodic case, has provided upper bounds on the Hausdorff length \mathcal{H}^1 of the level sets of the vorticity and the stream function and expressed them for large t in term of the corresponding eigenvalue Λ_k .

Remark 2.6. In a totally different context, the Dirichlet quotients have been used in [18] to study the backward behavior of solutions to the periodic Navier-Stokes equations with (non-potential) time-independent body forces. More precisely it is proven there that the set of initial data for which the solution exists for all negative times and has exponential growth is rather rich, actually it is dense in the phase space of the NSE, answering positively a question of Bardos-Tartar [2].

Coming back to the study of NSE with potential forces, it has been proven in [33], by extending a result of Hartman ([44] chap. IX, th. 6.2) for ODEs, that the NSE have invariant manifolds with “slow” decay. More precisely,

Theorem 2.4 ([33]). *For any $k = 1, 2, \dots$ there exist an open neighborhood U_k of 0 in \mathcal{R} and a submanifold F_k without boundary of U_k such that*

- (i) F_k is C^1 and analytic outside the origin.
- (ii) $\dim F_k = m_1 + m_2 + \dots + m_k = N_k$.
- (iii) F_k is invariant, i.e. $S(t)F_k \subset F_k, \forall t \geq 0$.
- (iv) The tangent space to F_k at the origin is $(R_{\Lambda_1} + \dots + R_{\Lambda_k})H$.
- (v) $v \in F_k \setminus \{0\}$ implies $\Lambda(v) \leq \Lambda_k$.

Corollary 2.1. $M_k \cap F_{k+1}$ is an invariant submanifold of U_k , of dimension m_k that satisfies

- (i) $\Lambda(u) = \Lambda_k$ for all $u \in M_k \cap F_{k+1} \setminus \{0\}$.
- (ii) M_{k+1}, F_{k-1} and $M_k \cap F_{k+1}$ are transverse at 0.

Remark 2.7. (a) While the manifolds M_k 's are unique, this is not the case of the F_k 's.

- (b) In the $2D$ -periodic case, one can take $F_1 = R_{\Lambda_1}H$. This results from the fact that then the function $t \mapsto \|S(t)v\|^2/|S(t)v|^2$ is decreasing for any nonzero $v \in V$. It is a consequence of (2.2) and the orthogonal properties

$$\langle B(u, u), u \rangle = 0 \quad \text{and} \quad \langle B(u, u), Au \rangle = 0,$$

which make the right-hand side of (2.2) vanish.

- (c) For some specific examples (such as the viscous Burgers equation or a nonlocal version of it), one can prove that the manifolds F_k 's are global.

Remark 2.8. We refer to [10] for construction of invariant manifolds in a rather general setting, and, in particular, to its Appendix B5 for illuminating comments on slow manifolds.

Remark 2.9. The nonlinear spectral manifolds have been used in [54] to study asymptotic stability issues for the periodic two-dimensional Navier-Stokes equations.

The results in Theorem 2.1 suggest that one can go further and look for an asymptotic expansion of the solution. This will eventually lead to the normal form.

We first introduce a technical notion on the spectrum of A . More generally,

Definition 2.1. Let A be a, possibly unbounded, linear operator in a space X with spectrum $\sigma(A)$.

- (i) A resonance in $\sigma(A)$ is a relation of the type

$$a_1\Lambda_1 + a_2\Lambda_2 + \dots + a_k\Lambda_k = \Lambda, \tag{2.8}$$

for some $\Lambda, \Lambda_1, \Lambda_2, \dots, \Lambda_k \in \sigma(A)$, and some positive integers a_1, a_2, \dots, a_k with $a_1 + a_2 + \dots + a_k \geq 2$.

- (ii) If $\Lambda \in \sigma(A)$ satisfies (2.8), then we say Λ is resonant.
- (iii) If $\sigma(A)$ has a resonance then we say it is resonant, otherwise nonresonant.
- (iv) In case A is the Stokes operator with the spectrum described in (1.6), and $\Lambda = \Lambda_{k+1}$ for some $k \geq 1$, then (2.8) is equivalent to

$$a_1\Lambda_1 + a_2\Lambda_2 + \dots + a_k\Lambda_k = \Lambda_{k+1}, \quad \text{for some } a_1, a_2, \dots, a_k \in \mathbb{N} \cup \{0\}.$$

We note that the periodic boundary conditions on $[0, L]^n$ always lead to resonances because $\Lambda_2 = 2\Lambda_1$. On the other hand, for periodic boundary conditions on general cubes $[0, L_1] \times [0, L_2] \times [0, L_3]$, the spectrum of the Stokes operator A is non-resonant for a dense set of periods $(L_1, L_2, L_3) \in (0, \infty)^3$.

It has been recently proven ([16]) that in the case of Dirichlet boundary conditions, the spectrum of A is non-resonant generically with respect to the domain. More precisely let \mathcal{D}_l^3 be the set of bounded domains in \mathbb{R}^3 with C^l boundary equipped with a suitable topology. For any $\Omega \in \mathcal{D}_l^3$ we denote by $\mathcal{D}_l^3(\Omega)$ the Banach manifold obtained as the set of images $(Id + u)(\Omega)$ by $u \in W^{l+1, \infty}(\Omega, \mathbb{R}^3)$ which are diffeomorphic to Ω .

The main result in [16] is that generically with respect to $\Omega \in \mathcal{D}_5^3$ the spectrum of A is non-resonant, in the sense that the set of domains in $\mathcal{D}_5^3(\Omega)$ for which the non-resonance property holds contains an intersection of open and dense subsets of $\mathcal{D}_5^3(\Omega)$. This result is established as a consequence of the fact that generically with respect to $\Omega \in \mathcal{D}_4^3$, the eigenvalues of A are simple.

3. The asymptotic expansion

In this section we obtain asymptotic expansions, as time tends to infinity, for regular solutions of NSE. These expansions are of the following type.

Definition 3.1. Let X be a real vector space.

(a) An X -valued polynomial is a function $t \in \mathbb{R} \mapsto \sum_{n=1}^d a_n t^n$, for some $d \geq 0$, and a_n 's belonging to X .

(b) When $(X, \|\cdot\|)$ is a normed space, a function $g(t)$ from $(0, \infty)$ to X is said to have the asymptotic expansion

$$g(t) \sim \sum_{n=1}^{\infty} g_n(t) e^{-\alpha_n t} \text{ in } X,$$

where $(\alpha_n)_{n=1}^{\infty}$ is a strictly increasing sequence of positive numbers, $g_n(t)$'s are X -valued polynomials, if for all $N \geq 1$, there exists $\varepsilon_N > 0$ such that

$$\left\| g(t) - \sum_{n=1}^N g_n(t) e^{-\alpha_n t} \right\| = \mathcal{O}(e^{-(\alpha_N + \varepsilon_N)t}) \text{ as } t \rightarrow \infty.$$

Throughout, A is the Stokes operator.

3.1. The non-resonant case

Assume $\sigma(A)$ is non-resonant.

Theorem 3.1 ([34, Theorem 2]). *Let u be a regular solution. For each $N \in \mathbb{N}$, one has the expansion in H :*

$$u(t) = W_{\mu_1} e^{-\nu\mu_1 t} + W_{\mu_2} e^{-\nu\mu_2 t} + \dots + W_{\mu_N} e^{-\nu\mu_N t} + v_N(t), \quad \forall t > 0,$$

where $W_{\mu_j} = W_{\mu_j}(u_0) \in \mathcal{E}^\infty(\Omega) \cap V$ for $j = 1, \dots, N$, and

$$v_N \in C([0, \infty; V) \cap L_{loc}^2(0, \infty; D(A)) \cap C^\infty([t_0, \infty); \mathcal{E}^\infty(\Omega) \cap V), \quad \forall t_0 > 0. \quad (3.1)$$

Moreover,

- (i) $\|v_N(t)\|_m = O(e^{-\nu(\mu_N + \epsilon_N)t})$, for some $\epsilon_N > 0$, $m = 0, 1, 2, \dots$
- (ii) $R_{\Lambda_j} W_{\Lambda_j} = W_{\Lambda_j}$ for $\Lambda_j \leq \mu_N$.
- (iii) For $\mu_j = \alpha_1 \Lambda_1 + \dots + \alpha_{j-1} \Lambda_{j-1}$, with $\alpha_1 + \dots + \alpha_{j-1} \geq 2$, W_{μ_j} is a polynomial of $W_{\Lambda_1}, \dots, W_{\Lambda_{j-1}}$ which is homogeneous of degree $\leq \alpha_1$ in W_{Λ_1} , of degree $\leq \alpha_2$ in W_{Λ_2}, \dots , of degree $\leq \alpha_{j-1}$ in $W_{\Lambda_{j-1}}$. More precisely, one has in this case

$$\nu(A - \mu_j I)W_{\mu_j} + \sum_{\mu_i + \mu_k = \mu_j} B(W_{\mu_i}, W_{\mu_k}) = 0.$$

Remark 3.1. It is clear that the case $W_{\mu_1} = W_{\mu_2} = \dots = W_{\mu_{j-1}} = 0$ and $W_{\mu_j} \neq 0$ corresponds to $\mu_j = \Lambda(u_0)$.

The rather technical proof is by induction on N , see details in [34].

3.2. The resonant case

Assume $\sigma(A)$ is resonant.

Theorem 3.2 ([34, Theorem 4]). *Let u be a regular solution with initial data $u_0 \in \mathcal{R}$. For any $N \in \mathbb{N}$, one has the asymptotic expansion in H :*

$$u(t) = W_{\mu_1}(t)e^{-\nu\mu_1 t} + W_{\mu_2}(t)e^{-\nu\mu_2 t} + \dots + W_{\mu_N}(t)e^{-\nu\mu_N t} + v_N(t), \quad \forall t > 0,$$

where $W_{\mu_j}(t) = W_{\mu_j}(t, u_0)$ is a $V \cap \mathcal{E}^\infty(\Omega)$ -valued polynomial in t , and v_N satisfies (3.1). Moreover,

- (i) $\|v_N(t)\|_m = O(e^{-\nu(\mu_N + \epsilon_N)t})$, for some $\epsilon_N > 0$, $m = 0, 1, 2, \dots$
- (ii) $d_j^0 = \deg W_{\mu_j} \leq j - 1, \quad j = 1, \dots, N$.
- (iii) If $\Lambda_j \leq \mu_N$ is a non-resonant eigenvalue, then W_{Λ_j} is constant in t and $R_{\Lambda_j} W_{\Lambda_j} = W_{\Lambda_j}$.
- (iv) If $\mu_j \leq \mu_N$ is not a non-resonant eigenvalue, then $W_{\mu_j}(t)$ satisfies the equation

$$\frac{dW_{\mu_j}(t)}{dt} + \nu(A - \mu_j)W_{\mu_j}(t) + \sum_{\mu_l + \mu_k = \mu_j} B(W_{\mu_l}(t), W_{\mu_k}(t)) = 0, \quad \forall t \in \mathbb{R}.$$

- (v) If Λ_j is a resonant eigenvalue, one has

$$\deg W_{\Lambda_j} \leq \max_{\mu_l + \mu_k = \Lambda_j} (d_l^0 + d_k^0) + 1.$$

Moreover, $R_{\Lambda_k} W_{\Lambda_j}(t)$ for $k \neq j$, and the coefficients of order ≥ 1 in $R_{\Lambda_j} W_{\Lambda_j}(t)$ are obtained from $R_{\Lambda_1} W_{\Lambda_1}(0), \dots, R_{\Lambda_{j-1}} W_{\Lambda_{j-1}}(0)$, via successive integrations of explicit elementary functions.

- (vi) If $\mu_j \notin \sigma(A)$, $W_{\mu_j}(t)$ is obtained from $R_{\Lambda_1} W_{\Lambda_1}(0), \dots, R_{\Lambda_{k_j}} W_{\Lambda_{k_j}}(0)$, via successive integrations of elementary explicit functions, where

$$\Lambda_{k_j} = \max \{ \Lambda \in \sigma(A); \Lambda < \mu_j \}. \tag{3.2}$$

One has also $\deg W_{\mu_j} = d_j^0 \leq \sup_{\mu_l + \mu_k = \mu_j} (d_l^0 + d_k^0)$.

Proof. The proof is also technical and by induction on N . We merely sketch the main steps for (i) only.

- First step. We recall that

$$\|u(t)\| = O(e^{-\nu\Lambda_1 t}).$$

One can prove the limit in V :

$$\lim_{t \rightarrow \infty} e^{\nu\Lambda_1 t} R_{\Lambda_1} u(t) = \xi_1 \in R_{\Lambda_1} H,$$

and then establish

$$e^{\nu\Lambda_1 t} \|(I - R_{\Lambda_1})u(t)\|_m = O(e^{-\delta t}) \text{ for some } \delta > 0, \quad \forall m \geq 0.$$

- Induction step. Let $v_N(t) = u(t) - \sum_{j=1}^N W_{\mu_j}(t)e^{-\nu\mu_j t}$. Assume

$$\|v_N(t)\|_m = O(e^{-\delta t}) \text{ for some } \delta > 0, \quad \forall m \geq 0.$$

Write the equation for $w_N = e^{\nu\mu_{N+1}t} v_N(t)$ as

$$\frac{dw_N}{dt} + (A - \mu_{N+1})w_N + \sum_{\mu_\ell + \mu_j = \mu_{N+1}} B(W_{\mu_\ell}(t), W_{\mu_j}(t)) = h_N(t),$$

where

$$\|h_N(t)\|_m = O(e^{-\delta t}) \text{ for some } \delta > 0, \quad \forall m \geq 0.$$

We apply the projector R_{Λ_k} and obtain the equation for $w_{N,k} = R_{\Lambda_k} w_N$:

$$\frac{dw_{N,k}}{dt} + (\Lambda_k - \mu_{N+1})w_{N,k} + p_{N,k}(t) = R_{\Lambda_k} h_N(t), \tag{3.3}$$

where $p_{N,k}$ is a polynomial in t .

This equation is an ODE of the type:

$$\frac{dw}{dt} + \alpha w + p(t) = g(t) = O(e^{-\delta t}) \text{ in } R_{\Lambda_k} H,$$

where $p(t)$ is a polynomial. When either $\alpha \geq 0$, or $\alpha < 0$ with $\lim_{t \rightarrow \infty} (e^{\alpha t} w(t)) = 0$, there exists a polynomial solution $q(t)$ of

$$\frac{dq}{dt} + \alpha q + p(t) = 0$$

such that

$$|w(t) - q(t)| = O(e^{-\delta' t}) \text{ for some } \delta' \in (0, \delta).$$

Using this fact we approximate $w_{N,k}$ in (3.3) by a polynomial $q_{N+1,k} \in R_{\Lambda_k} H$. We then define $W_{\mu_{N+1}}(t) = \sum_{k=1}^\infty q_{N+1,k}(t)$. The function $W_{\mu_{N+1}}(t)$ is proved to be a polynomial and satisfies

$$\|w_N(t) - W_{\mu_{N+1}}(t)\|_m = O(e^{-\varepsilon t}) \text{ for some } \varepsilon > 0, \quad \forall m \geq 0.$$

This implies

$$\|v_N(t) - W_{\mu_{N+1}}(t)e^{-\nu\mu_{N+1}t}\|_m = O(e^{-(\nu\mu_{N+1} + \varepsilon)t}), \quad \forall m \geq 1,$$

which proves the induction step.

- Note that, in dealing with the higher norms $\|\cdot\|_m$, the proof in [34] estimates $\|d^{(j)}u/dt^j\|_m$ for all $j \geq 1$. □

Remark 3.2. The first coefficient $W_{\mu_j}(t)$ which is not identically zero in the expansion corresponds to $\mu_j = \Lambda(u_0)$. In this case it is constant in t and belongs to $R_{\Lambda(u_0)}H$.

Notation. Based on Theorems 3.1 and 3.2, and according to Definition 3.1, we have

$$u(t) \sim \sum_{j=1}^{\infty} W_{\mu_j}(t) e^{-\nu\mu_j t} \quad \text{in } H^m(\Omega)^n, \quad \forall m \in \mathbb{N},$$

which we will simply write

$$u(t) \sim \sum_{j=1}^{\infty} W_{\mu_j}(t) e^{-\nu\mu_j t}. \quad (3.4)$$

3.3. The asymptotic expansion in Gevrey spaces

Theorem 3.2 has been recently improved in [45] where it is proved in particular that the asymptotic expansion in the 3D-periodic case actually holds in all Gevrey classes.

Consider the periodic case (1.2) with $n = 3$ and assume (1.8). Recall that one has the properties (1.9). We first describe the relevant Gevrey classes.

For $\alpha, \sigma \in \mathbb{R}$, and $u = \sum_{\mathbf{k} \in \mathbb{Z}^3 \setminus \{0\}} \hat{u}(\mathbf{k}) e^{i\mathbf{k} \cdot x}$, we define

$$A^\alpha u = \sum_{\mathbf{k} \in \mathbb{Z}^3 \setminus \{0\}} |\mathbf{k}|^{2\alpha} \hat{u}(\mathbf{k}) e^{i\mathbf{k} \cdot x},$$

and the Gevrey class

$$G_{\alpha, \sigma} = D(A^\alpha e^{\sigma A^{1/2}}) = \{u \in H; |u|_{\alpha, \sigma} \stackrel{\text{def}}{=} |A^\alpha e^{\sigma A^{1/2}} u| < \infty\},$$

so that the domain of A^α is $D(A^\alpha) = G_{\alpha, 0}$. Also $D(A^0) = H$, $D(A^{1/2}) = V$.

The next theorem improves Theorem 3.2 for the periodic case for any weak solution.

Theorem 3.3 ([45, Theorem 1.1]). *The expansion in Theorem 3.2 holds in any Gevrey class $G_{\alpha, \sigma}$ with $\alpha, \sigma > 0$. More precisely, for any (Leray-Hopf) weak solution u of the NSE, there exist polynomials q_n 's in t valued in \mathcal{V} such that if $\alpha, \sigma > 0$ and $N \geq 1$ then*

$$|u(t) - \sum_{n=1}^N q_n(t) e^{-nt}|_{\alpha, \sigma} = O(e^{-(N+\epsilon)t}) \quad \text{as } t \rightarrow \infty, \quad \text{for any } \epsilon \in (0, 1).$$

By working with the Gevrey norms, the proof in [45] avoids the estimates of $\|d^{(j)}u/dt^j\|_m$ for all $j \geq 0$ and $m \geq 0$.

Before moving to the normal form theory for the NSE, we review other applications of the Dirichlet quotients techniques and asymptotic expansions.

3.4. Application: asymptotic behavior of the helicity

It turns out that the techniques developed to study the Dirichlet quotients can be used to obtain information on the asymptotic behavior of the helicity for Navier-Stokes equations with potential forces. This is the object of the papers [23, 24]. We will focus on the results of [23] where the (3D, periodic) deterministic case is considered. (Interested readers can read [24] which deals with the statistical case).

For a regular solution u of the NSE, the helicity is a scalar quantity defined by

$$\mathcal{H}(t) = \int_{\Omega} u(x, t) \cdot \omega(x, t) dx, \quad \text{where } \omega = \text{curl } u.$$

In the inviscid case ($\nu = 0$) the invariance of the helicity was noticed by Moreau [62]. The first thorough study of the helicity and of its density for inviscid incompressible flows was carried out by Moffatt [60], who gave in particular a connection of the helicity to the topological invariants and dynamics of the vortex tubes as well as the first examples of physically relevant fluid flows with non-zero helicity. There is a general agreement that helicity plays an important role in magneto-hydro dynamics, but not in the dynamics of neutral flows (that is, solutions of the Euler or the Navier-Stokes equations). However, theoretical, empirical and numerical evidence indicate that the helicity can provide insights into the nature of the fluid flows, at least in the case when the viscosity is small and this motivates the present study.

In the periodic case with our choice of Ω in (1.2) ($n = 3$), we recall from (1.7) that the first eigenvalue of the Stokes operator A is $\Lambda_1 = 4\pi^2/L^2$, and the other ones are among $n\Lambda_1$, with $n \in \mathbb{N}$. The previous results on the limit of the Dirichlet quotients together with Cauchy-Schwarz inequality imply that the helicity of a regular solution tends to zero as $t \rightarrow \infty$, at least with a rate $2\nu\Lambda_1 n_0$, where n_0 depends on the initial data. However, due to possible changes of sign and cancellations in $u \cdot \omega$, this does not imply that it has the same decay rate $2\nu\Lambda_1 n_0$, as that of the energy. In particular, it was not known whether the helicity could change sign or vanish infinitely times as $t \rightarrow \infty$. An answer to those questions is found in [23].

We further define related quantities

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \int_{\Omega} |u(x, t)|^2 dx \quad (\text{kinetic energy}), \\ \mathcal{F}(t) &= \int_{\Omega} |\omega(x, t)|^2 dx \quad (\text{rate of energy dissipation/viscosity}), \\ \mathcal{I}(t) &= \int_{\Omega} \omega(x, t) \cdot (\nabla \times \omega(x, t)) dx. \end{aligned}$$

These entities satisfy the following (balance) equations:

$$\frac{d}{dt} \mathcal{E}(t) + \nu \mathcal{F}(t) = 0,$$

$$\frac{1}{2} \frac{d}{dt} \mathcal{H}(t) + \nu \mathcal{I}(t) = 0.$$

We now assume (1.8), and recall that (1.9) holds true.

Theorem 3.4 ([23, Theorem 3.1]). *For any regular solution of the NSE,*

- (i) *Either the helicity becomes non-zero and decays as $t^d e^{-2hot}$, where $d \geq 0$ and h_0 are integers depending on u_0 with*

$$\lim_{t \rightarrow \infty} \frac{\mathcal{I}(t)}{\mathcal{H}(t)} = h_0.$$

- (ii) *Or it is identically zero.*

Proof. Using asymptotic expansion of u , we derive for the helicity that

$$\mathcal{H}(t) \sim \sum_{j=1}^{\infty} \phi_j(t) e^{-jt},$$

where the ϕ_j 's are polynomials in $t \in \mathbb{R}$.

If one of ϕ_j 's is not a zero polynomial, then we obtain case (i). Otherwise, $\mathcal{H}(t)$ decays to zero, as $t \rightarrow \infty$, faster than any exponential functions. This fact itself cannot yield conclusion for case (ii). More properties of the solution u and helicity \mathcal{H} are needed. For those, we complexify the NSE in time and denote the resulting solution and helicity by $u(\zeta)$ and $\mathcal{H}(\zeta)$, for the complex time $\zeta \in \mathbb{C}$. These functions are proved to be analytic and bounded in a domain E which, see [23, Propositions 8.3 and 8.4], contains $(0, \infty)$ and an open set

$$t_0 + D, \tag{3.5}$$

where t_0 is a certain positive time, and

$$D = \{\tau + i\sigma \in \mathbb{C} : \tau > 0, |\sigma| < \sqrt{2\tau} e^{\alpha\tau}\} \tag{3.6}$$

for some positive constant α . Then, see [25, Lemma B.2], the transformation

$$\varphi(\zeta) = \zeta - \frac{1}{\alpha} \log(1 + \alpha\zeta) \tag{3.7}$$

conformally maps D to a set containing the right half plane H_0 . Moreover,

$$\varphi([0, \infty)) = [0, \infty).$$

Define the function $\mathcal{H}_{t_0}(\zeta) = \mathcal{H}(t_0 + \zeta)$. We have $\mathcal{H}_{t_0} \circ \varphi^{-1}$ is analytic, bounded on H_0 and, see [23, Lemma 8.5], satisfies

$$\limsup_{0 < \eta \rightarrow \infty} e^{\beta\eta} |(\mathcal{H}_{t_0} \circ \varphi^{-1})(\eta)| \leq \limsup_{0 < \zeta \rightarrow \infty} e^{\beta\zeta} |\mathcal{H}_{t_0}(\zeta)| = 0 \quad \forall \beta > 0.$$

Then applying Phragmen-Lindeloff type estimates, see [23, Proposition C.1], we infer that $(\mathcal{H}_{t_0} \circ \varphi^{-1})(\eta) = 0$ for all $\eta \in H_0$. This implies $\mathcal{H}(\zeta) = 0$ on the open, non-empty set $D^* = t_0 + \varphi^{-1}(H_0)$, which, by analyticity, yields that $\mathcal{H}(t) = 0$ for all $t \in (0, \infty)$. \square

The next theorem shows that the case where the helicity is non-zero is generic.

Theorem 3.5 ([23, Theorem 3.2]). *Let \mathcal{R}_1 and \mathcal{R}_0 be the sets of initial data $u_0 \in \mathcal{R}$ corresponding to cases (i) and (ii) in Theorem 3.4, respectively. Then \mathcal{R}_1 is open and dense in \mathcal{R} while \mathcal{R}_0 is closed and contains an infinite union of linear, closed infinite dimensional manifolds.*

The asymptotic decay of helicity is precisely described in the next result.

Theorem 3.6 ([23, Theorem 3.4]). *Let $u_0 \in \mathcal{R} \setminus \{0\}$ and $n_0 = \Lambda(u_0)$. Then*

$$\lim_{t \rightarrow \infty} \frac{\mathcal{H}(t)}{|u(t)|^2} = \alpha_0, \quad \text{where } \alpha_0 \in [-n_0, n_0].$$

Moreover, for any $n \in \sigma(A)$ and $\alpha \in [-\sqrt{n}, \sqrt{n}]$, there exists $u_0 \in \mathcal{R}$ such that the corresponding solution u satisfies

$$\Lambda(u_0) = n \text{ and } \lim_{t \rightarrow \infty} \frac{\mathcal{H}(t)}{|u(t)|^2} = \alpha.$$

Note that one has the norm relation $\|u(t)\| = |\omega(t)|$, hence,

$$\Lambda(u_0) = \lim_{t \rightarrow \infty} \lambda(t) = \lim_{t \rightarrow \infty} \frac{\|u(t)\|^2}{|u(t)|^2} = \lim_{t \rightarrow \infty} \frac{|\omega(t)|^2}{|u(t)|^2}.$$

If $\alpha_0 \neq 0$ in the previous theorem, one is in case (i) of Theorem 3.4 (helicity decays) with $d = 0$ and $h_0 = n_0 \in \sigma(A)$. The situation where $\alpha_0 = 0$ is considered in the next theorem.

Theorem 3.7 ([23, Theorem 3.5]). *For any $n \in \sigma(A)$ and $M > 0$, there exists an initial data $u_0 \in \mathcal{R}$ such that one is in case (i) of Theorem 3.4 with $n_0 = n$ and $h_0 \geq n_0 + M$, and such that*

$$\frac{\mathcal{H}(t)}{|u(t)|^2} = O(e^{-2Mt}) \quad \text{when } t \rightarrow \infty.$$

Moreover, there exist solutions whose helicity satisfies the condition

$$\lim_{t \rightarrow \infty} \mathcal{H}(t)t^{-d}e^{2h_0t} \text{ exists and is not zero,}$$

where $d > 0$ or h_0 is not an eigenvalue of A .

Comments on the proofs of Theorems 3.5–3.7.

- The properties of the set \mathcal{R}_0 of initial data leading to an identically zero helicity result from a study of the spectrum of the curl operator and of a global stability result of NSE in 3D ([66]).
- Examples of linear submanifolds of \mathcal{R}_0 are, among others, the family $\mathcal{M}_{\mathbf{a}\perp}$ presented in Remark 2.2.

4. The Poincaré-Dulac theory of normal forms

This section reviews very briefly the Poincaré-Dulac theory of normal forms. This is, of course, a classical topic in dynamical systems, initiated by Poincaré and, later, Dulac (see [20, 65]) to analyze the dynamics of a nonlinear system of ODEs in the neighborhood of a singular point. The theory of normal forms for finite dimensional dynamical systems has tremendously grown since and we refer for instance to the books [1, 8, 9] for a modern treatment.

The next theorem is extracted from Poincaré’s thesis (1879).

Theorem 4.1 (Poincaré's thesis 1879 [64]). *If the eigenvalues of the matrix A are nonresonant, the equation*

$$\frac{dx}{dt} + Ax + \sum_{d=2}^{\infty} \phi^{[d]}(x) = 0, \quad (4.1)$$

where each $\phi^{[d]}$ is a homogeneous polynomial of degree d in \mathbb{R}^n , reduces to the linear equation

$$\frac{dy}{dt} + Ay = 0$$

by a formal change of variable

$$x = y + \sum_{d=2}^{\infty} \psi^{[d]}(y), \quad (4.2)$$

where each $\psi^{[d]}$ is a homogeneous polynomial of degree d .

Retrospectively the following extract of Bonnet and Darboux report on Poincaré's thesis (see [42] page 331) is somewhat amazing:

... Quelques lemmes de l'introduction ont aussi paru dignes d'intérêt. Le reste de la thèse est un peu confus et prouve que l'auteur n'a pu encore parvenir à exprimer ses idées d'une manière claire et simple. Néanmoins la Faculté tenant compte de la grande difficulté du sujet et du talent qu'a montré M. Poincaré lui a conféré avec trois boules blanches le grade de docteur.

(Translation) ... the remainder of the thesis is a little confused and shows that the author was still unable to express his ideas in a clear and simple manner. Nevertheless, considering the great difficulty of the subject and the talent demonstrated, the faculty recommends that M. Poincaré be granted the degree of Doctor with all privileges.

The resonant case was treated in Dulac's thesis (1903).

Definition 4.1. Suppose an $n \times n$ matrix A has eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors ξ_1, \dots, ξ_n . For each $x \in \mathbb{R}^n$, let x_i be its coordinate with respect to ξ_i . A monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \xi_k$ of degree two or higher is called resonant if

$$\lambda_k = \alpha_1 \lambda_1 + \alpha_2 \lambda_2 + \dots + \alpha_n \lambda_n.$$

Theorem 4.2 (Dulac [20]). *The equation (4.1) reduces, by a formal change of variable (4.2), to the canonical form*

$$\frac{dy}{dt} + Ay + \sum_{d=2}^{\infty} \Theta^{[d]}(y) = 0,$$

where all monomials in each $\Theta^{[d]}$ are resonant.

5. A normalization map and a normal form for NSE

There have been many important works concerning the use of normal forms for nonlinear partial differential equations, starting from the seminal paper of Shatah [68] and it is outside the scope of the present paper to review them. Let us just emphasize that they essentially aimed to reducing a quadratic nonlinear dispersive partial differential equation to a cubic one in order to use the dispersive estimates to prove the existence of global small solutions. On the other hand, our normal form for the Navier-Stokes equations is obtained as a by-product of the asymptotic expansion.

More precisely, one can use the *generating part* of the asymptotic expansion to construct a normalization of the NSE.

We first define the Fréchet space

$$\mathcal{S}_A = R_{\Lambda_1}H \oplus R_{\Lambda_2}H \oplus \dots$$

endowed with the topology of convergence of components. The Stokes operator A extends trivially to \mathcal{S}_A .

5.1. The non-resonant case

Assume $\sigma(A)$ is non-resonant.

Theorem 5.1 ([34, Theorem 3, Corollaries 1 and 2]). *Define the mapping $W : \mathcal{R} \rightarrow \mathcal{S}_A$ by*

$$W(u_0) = W_{\Lambda_1}(u_0) \oplus W_{\Lambda_2}(u_0) \oplus \dots \text{ for } u_0 \in \mathcal{R}.$$

Then:

- (i) W is analytic and one-to-one.
- (ii) W linearizes the NSE in the sense

$$W(u(t)) = e^{-\nu A t} W(u_0), \quad \forall u_0 \in \mathcal{R}, \quad \forall t \geq 0.$$

- (iii) $u_0 \in M_k$ if and only if the first k components of $W(u_0)$ vanish.

In this non-resonant case, $v(t) = W(u(t))$ is, thanks to (ii), a solution of the *linear* Navier-Stokes equations in the large space \mathcal{S}_A , i.e.,

$$\frac{dv}{dt} + Av = 0. \tag{5.1}$$

Thus, the mapping W transforms the (nonlinear) NSE (1.5) to the linear one (5.1), which is the case of Theorem 4.1. Therefore, we call W a normalization map, even though it is not a formal series.

5.2. The resonant case

Assume $\sigma(A)$ is resonant. In view of the structure of the asymptotic expansion in this case, considering the following mapping W is natural.

Theorem 5.2 ([34, Theorem 5]). *The mapping $W : \mathcal{R} \rightarrow \mathcal{S}_A$ given by*

$$W(u_0) = R_{\Lambda_1}W_{\Lambda_1}(0, u_0) \oplus R_{\Lambda_2}W_{\Lambda_2}(0, u_0) \oplus R_{\Lambda_3}W_{\Lambda_3}(0, u_0) \oplus \dots \text{ for } u_0 \in \mathcal{R},$$

is analytic and one-to-one.

We will see that this mapping W also plays the role of a normalization map. First, we find important polynomials that are essential in transforming the NSE into a normal form.

Let $W_j(u_0)$ denote the j -th component of $W(u_0)$, that is, $W_j(u_0) = R_{\Lambda_j} W_{\Lambda_j}(0, u_0)$.

For $j \in \mathbb{N}$, let $k_j \in \mathbb{N}$ be determined by $\Lambda_{k_j} = \mu_j$ if $\mu_j \in \sigma(A)$, or by (3.2) otherwise.

Lemma 5.1 ([34, Lemma 7]). *For every $j = 1, 2, 3, \dots$, there exists a multilinear function \mathcal{P}_j , defined on $R_{\Lambda_1}H \oplus \dots \oplus R_{\Lambda_{k_j}}H$, depending on $\sigma(A)$, B , ν , such that*

$$W_{\mu_j}(u_0) = \mathcal{P}_j(W_1(u_0), W_2(u_0), \dots, W_{k_j}(u_0)).$$

According to Lemma 5.1, each function \mathcal{P}_j , for $j = 1, 2, 3, \dots$, is a polynomial with values in $E^\infty \cap V$ (that can be explicitly constructed by induction), defined on $R_{\Lambda_1}H \oplus R_{\Lambda_2}H \oplus \dots \oplus R_{\Lambda_{k_j}}H$.

These polynomials have the following supplementary properties [34, Lemma 7]:

- If $\mathcal{M}(x_1, x_2, \dots, x_{k_j})$ is a monomial in $\mathcal{P}_j(x_1, \dots, x_{k_j})$ of degree m_1, \dots, m_{k_j} in x_1, \dots, x_{k_j} respectively, then

$$m_1\Lambda_1 + \dots + m_{k_j}\Lambda_{k_j} = \mu_j. \tag{5.2}$$

- Furthermore, if μ_j is an eigenvalue Λ_k , then $k_j = k$ and

$$\mathcal{P}_j(x_1, \dots, x_k) = x_k + \text{higher order terms in } x_1, \dots, x_{k-1}.$$

With these polynomials \mathcal{P}_j 's, we are ready to rewrite the NSE under the transformation $W(u(t))$.

Theorem 5.3 ([34, Theorem 6]). *Let $u_0 \in \mathcal{R}$ and $u(t) = S(t)u_0$ be the corresponding global solution of the NSE. The (\mathcal{S}_A -valued) function $v(t) = W(u(t))$ satisfies the equation*

$$\frac{dv(t)}{dt} + \nu Av(t) + \mathcal{B}(v(t)) = 0 \text{ in } \mathcal{S}_A, \tag{5.3}$$

where, for $v = v_1 \oplus v_2 \oplus \dots \in \mathcal{S}_A$, $\mathcal{B}(v) = (\mathcal{B}_k(v))_{k=1}^\infty \in \mathcal{S}_A$ with

$$\mathcal{B}_k(v) = \sum_{\mu_l + \mu_j = \Lambda_k} R_{\Lambda_k} B(\mathcal{P}_l(v_1, \dots, v_{k_l}), \mathcal{P}_j(v_1, \dots, v_{k_j})),$$

where the polynomials \mathcal{P}_l and \mathcal{P}_j are defined as in Lemma 5.1.

From the relation (5.2), one can prove also that each monomial in $\mathcal{B}(v)$ is resonant, i.e., if $\mathcal{M}(v_1, v_2, \dots, v_k)$, for some $k \in \mathbb{N}$, is a (nonzero) monomial in \mathcal{B} of degree m_1, \dots, m_k in v_1, \dots, v_k respectively, and $\mathcal{M} \in R_\Lambda H$ for some $\Lambda \in \sigma(A)$, then

$$m_1\Lambda_1 + \dots + m_k\Lambda_k = \Lambda. \tag{5.4}$$

Thus, W transforms NSE (1.5) to (5.3), which satisfies the resonance condition as in Theorem 4.2. Therefore, we, again, call W a normalization map, and equation (5.3) a normal form of NSE.

Although the normal form (5.3) is nonlinear in \mathcal{S}_A , it can be solved by successive integration of an infinite set of non-homogeneous linear differential equations in $R_{\Lambda_k}H$, $k = 1, 2, \dots$, each one having an already known non-homogeneous part.

Remark 5.1. Minea ([59]) shows that this type of normalization, when applied to ODEs, is a normalization in the sense of Bruno ([6, 7]).

5.3. Further results in the 3D periodic case

The papers [25, 26] aim to answer the following questions.

- When does the asymptotic expansion actually converge?
- In what natural normed spaces is the normal form a well-behaved infinite-dimensional system of ODEs?
- What is the range of the normalization map?

Partial answers to those questions are established in the 3D periodic case, namely:

- In paper ([25]): Construction of a suitable Banach space $\mathcal{S}_A^* \subset \mathcal{S}_A$ on which the normal form is a well-posed system *near the origin*. The norm $\|\bar{u}\|_*$ of $\bar{u} = (u_n)_{n=1}^\infty \in \mathcal{S}_A^*$ is

$$\|u\|_* = \sum_{n=1}^\infty \rho_n \|\nabla u_n\|_{L^2(\Omega)}, \tag{5.5}$$

where $(\rho_n)_{n=1}^\infty$ is a sequence of positive weights.

- In paper ([26]): choice of a suitable set of weights ρ_n such that the normalization map $W : \mathcal{R} \rightarrow \mathcal{S}_A^*$ is continuous and such that the normal form of the NSE is well-posed in *the entire space* \mathcal{S}_A^* .

We consider thus the periodic case in \mathbb{R}^3 with the standard setting (1.2), (1.8), (1.9) for dimension $n = 3$.

For $N \in \mathbb{N}$, we denote by R_N the projection from H onto the eigenspace of A corresponding to N in case $N \in \sigma(A)$, and set $R_N = 0$ in case $N \notin \sigma(A)$.

We note that this definition of R_N is only different from that in subsection 1.3 by the change of the index. It aims to unify calculations and make them more efficient in lengthy proofs.

The definitions of polynomials \mathcal{P}_j 's and the normal form (5.3) can be expressed more explicitly as follows. We recall that the asymptotic expansion (3.4) for a regular solution u of the NSE with initial data $u_0 \in \mathcal{R}$ is

$$u(t) \sim \sum_{j=1}^\infty q_j(t)e^{-jt},$$

where $q_j(t)$'s are polynomials in t with values in \mathcal{V} , and are unique polynomial solutions of the following ODEs

$$q'_j(t) + (A - j)q_j(t) + \beta_j(t) = 0, \quad t \in \mathbb{R}, \quad R_j q_j(0) = W_j(u_0),$$

with

$$\beta_1(t) = 0, \quad \beta_j(t) = \sum_{k+l=j} B(q_k(t), q_l(t)), \quad \text{for } j > 1.$$

For $\xi = (\xi_n)_{n=1}^\infty \in \mathcal{S}_A$ arbitrary, the polynomial solutions of the preceding system with initial conditions $R_j q_j(0) = \xi_j$ are explicitly given by the recursive formula:

$$q_j(t, \xi) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau + \sum_{n \geq 0} (-1)^{n+1} (A - j)^{-n-1} \frac{d^n}{dt^n} (I - R_j) \beta_j, \quad j \in \mathbb{N},$$

where

$$(A - j)^{-n-1}u = \sum_{|\mathbf{k}|^2 \neq j} \frac{a_{\mathbf{k}}}{(|\mathbf{k}|^2 - j)^{n+1}} e^{i\mathbf{k} \cdot x},$$

for $u = \sum_{|\mathbf{k}|^2 \neq j} a_{\mathbf{k}} e^{i\mathbf{k} \cdot x} \in (I - R_j)H$.

The \mathcal{S}_A -valued function $\xi(t) = (\xi_j(t))_{j=1}^\infty = (W_j(u(t)))_{j=1}^\infty = W(u(t))$ satisfies the system of ODEs :

$$\begin{cases} \frac{d\xi_1(t)}{dt} + A\xi_1(t) = 0, \\ \frac{d\xi_j(t)}{dt} + A\xi_j(t) + \sum_{k+l=j} R_j B(\mathcal{P}_k(\xi(t)), \mathcal{P}_l(\xi(t))) = 0, \quad j > 1. \end{cases} \tag{5.6}$$

Above, $\mathcal{P}_j(\xi) = q_j(0, \xi)$ for $\xi \in \mathcal{S}_A$ and $j \geq 1$. Then each function $\mathcal{P}_j(\xi)$, for $\xi = (\xi_n)_{n=1}^\infty$, is a \mathcal{V} -valued polynomial in the variables $\xi_1, \xi_2, \dots, \xi_j$, each of which belongs to a finite dimensional space. For instance,

$$\mathcal{P}_1(\xi) = \xi_1, \quad \mathcal{P}_2(\xi) = \xi_2 - (A - 2)^{-1}(I - R_2)B(\xi_1, \xi_1).$$

We define $\mathcal{B} = (\mathcal{B}_j)_{j=1}^\infty$ where

$$\mathcal{B}_1(\xi) = 0, \quad \text{and } \mathcal{B}_j(\xi) = \sum_{k+l=j} R_j B(\mathcal{P}_k(\xi), \mathcal{P}_l(\xi)) \quad \text{for } j > 1.$$

We rewrite the system (5.6) in a vectorial form in \mathcal{S}_A as

$$\frac{d\xi}{dt} + A\xi + \mathcal{B}(\xi) = 0. \tag{5.7}$$

One can verify that each monomial in \mathcal{B} satisfies the resonance condition, see (5.4). Therefore, (5.7) is a normal form of NSE in \mathcal{S}_A under the transformation $\xi = W(u)$.

The solution of the normal form (5.7) with initial data $\xi^0 = (\xi_n^0)_{n=1}^\infty \in \mathcal{S}_A$ is precisely

$$(R_n q_n(t, \xi^0) e^{-nt})_{n=1}^\infty.$$

We denote this solution by $S_{\text{normal}}(t)\xi^0$.

Next, we investigate the convergence of the asymptotic expansion. We introduce and make use of the following construction of regular solutions. It is motivated by the asymptotic expansion itself.

We decompose the initial data u^0 in V as

$$u^0 = \sum_{n=1}^\infty u_n^0. \tag{5.8}$$

We find the solution $u(t)$ of the form

$$u(t) = \sum_{n=1}^\infty u_n(t), \tag{5.9}$$

where for each n ,

$$\frac{du_n(t)}{dt} + Au_n(t) + B_n(t) = 0, \quad t > 0, \tag{5.10}$$

with initial condition

$$u_n(0) = u_n^0, \tag{5.11}$$

where

$$B_1(t) \equiv 0, \quad B_n(t) = \sum_{j+k=n} B(u_j(t), u_k(t)) \text{ for } n > 1.$$

System (5.10) will be called the extended NSE. We denote by $S_{\text{ext}}(t)$ the semi-group generated by solutions of this system.

It turns out that such a construction (5.8)–(5.11), indeed, produces regular solutions of the form (5.9) for NSE with the initial condition (5.8). The following existence theorem is a special case of Corollary 3.5 [25] with specific parameter $\rho_0 > 1 = \rho$.

Theorem 5.4 ([25, Corollary 3.5]). *Let $(u_n^0)_{n=1}^\infty$ be a sequence in V such that*

$$\limsup_{n \rightarrow \infty} \|u_n^0\|^{1/n} < 1.$$

Let $(u_n(t))_{n=1}^\infty$ be the solutions to (5.10) and (5.11). Then there exists $T > 0$ such that $u(t) \stackrel{\text{def}}{=} \sum_{n=1}^\infty u_n(t)$ is the regular solution of the NSE on $[0, T]$ with initial data $u^0 = \sum_{n=1}^\infty u_n^0$.

We now make the connections between the solutions of the extended NSE with the asymptotic expansions of solutions of NSE.

If a regular solution $u(t)$ has the expansion $\sum_{n=1}^\infty W_n(t, u^0)e^{-nt}$, then formally we wish for

$$u^0 = \sum_{n=1}^\infty W_n(0, u^0).$$

Therefore, we set $u_n^0 = W_n(0, u^0)$ in the extended NSE. Then solutions $u_n(t)$ of the extended NSE are exactly $W_n(t, u^0)e^{-nt}$. Hence, conclusion in Theorem 5.4 on $u_n(t)$, helps us make conclusion on $\sum_{n=1}^\infty W_n(t, u^0)e^{-nt}$.

Notation. For $u^0 \in \mathcal{R}$, we denote by $u(t, u^0)$ the regular solution of NSE on $[0, \infty)$ with initial data $u(0) = u^0$.

First, we have a small initial data result.

Theorem 5.5 ([25, Theorem 5.3]). *There exists $\varepsilon_0 > 0$ such that if $u^0 \in \mathcal{R}$ with*

$$\sum_{n=1}^\infty \|W_n(0, u^0)\| < \varepsilon_0, \tag{5.12}$$

then $\sum_{n=1}^\infty W_n(t, u^0)e^{-nt}$ converges in V for all $t > 0$ to the regular solution $u(t, u^0)$.

Second, we have a large time result for large initial data.

Theorem 5.6 ([25, Theorem 5.10]). *Suppose $u^0 \in \mathcal{R}$ and*

$$\limsup_{n \rightarrow \infty} \|W_n(0, u^0)\|^{1/n} < \infty. \tag{5.13}$$

Then there exists $T > 0$ such that the function $v(t) \stackrel{\text{def}}{=} \sum_{n=1}^\infty W_n(t, u^0)e^{-nt}$ is absolutely convergent in V , uniformly in $t \in [T, \infty)$, the series $\sum_{n=1}^\infty W_n(t, u^0)e^{-nt}$ is the asymptotic expansion of $v(t)$, and $v(t) = u(t, u^0)$ for all $t \in [T, \infty)$.

Note in Theorem 5.6 that it is not known whether the sum $\sum_{n=1}^\infty W_n(t, u^0)e^{-nt}$ converges to a solution in short time.

Although the conclusions in Theorems 5.5 and 5.6 are satisfactory, it is not known whether the condition (5.12) or (5.13) holds true for a non-zero u^0 . At the moment, we do not know whether $\sum_{n=1}^\infty W_n(0, u^0)$ and $\sum_{n=1}^\infty W_n(t, u^0)$ converge in V , i.e., with respect to the norm $\|\cdot\|$. However, one may hope to obtain some convergence in weaker norms. Therefore, we study, in the following, the asymptotic expansions with a different approach, which uses suitable weighted normed spaces.

Let $V^\infty = V \oplus V \oplus V \oplus \dots$. Define

$$W(t, \cdot) : u \in \mathcal{R} \mapsto (W_n(t, u)e^{-nt})_{n=1}^\infty \in V^\infty,$$

$$Q(t, \cdot) : \bar{\xi} \in \mathcal{S}_A \mapsto (q_n(t, \bar{\xi})e^{-nt})_{n=1}^\infty \in V^\infty.$$

We now proceed to the construction of the normed spaces.

Definition 5.1 (Fast decaying weights). Let $(\tilde{\kappa}_n)_{n=2}^\infty$ be a fixed sequence of real numbers in the interval $(0, 1]$ satisfying

$$\lim_{n \rightarrow \infty} (\tilde{\kappa}_n)^{1/2^n} = 0.$$

We define the sequence of positive weights $(\rho_n)_{n=1}^\infty$ by

$$\rho_1 = 1, \quad \rho_n = \tilde{\kappa}_n \gamma_n \rho_{n-1}^2, \quad n > 1,$$

where $\gamma_n \in (0, 1]$ are known and decrease to zero faster than n^{-n} .

For $\bar{u} = (u_n)_{n=1}^\infty \in V^\infty$, let $\|\bar{u}\|_\star$ be defined by the formula (5.5). Define the spaces

$$V^\star = \{ \bar{u} \in V^\infty : \|\bar{u}\|_\star < \infty \} \quad \text{and} \quad \mathcal{S}_A^\star = \mathcal{S}_A \cap V^\star.$$

Clearly $(V^\star, \|\cdot\|_\star)$ and $(\mathcal{S}_A^\star, \|\cdot\|_\star)$ are Banach spaces.

It turns out that the extended NSE is well-posed in the space V^\star .

Theorem 5.7 ([25, Theorem 4.3]). *If $\bar{u}^0 \in V^\star$, then $S_{\text{ext}}(t)\bar{u}^0 \in V^\star$ for all $t > 0$. More precisely,*

$$\|S_{\text{ext}}(t)\bar{u}^0\|_\star \leq M e^{-t}, \quad t > 0,$$

where $M > 0$ depends on ρ_n, κ_n and $\|\bar{u}^0\|_\star$.

Theorem 5.8 ([26, Theorem 3.8]). *For each $t \in [0, \infty)$, $S_{\text{ext}}(t)$ is continuous from V^\star to V^\star . More precisely, for any $\bar{u}^0 \in V^\star$ and $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\|S_{\text{ext}}(t)\bar{v}^0 - S_{\text{ext}}(t)\bar{u}^0\|_\star < \varepsilon e^{-t},$$

for all $\bar{v}^0 \in V^\star$ satisfying $\|\bar{v}^0 - \bar{u}^0\|_\star < \delta$ and for all $t \geq 0$.

As for the normal form and normalization map, one obtains the following well-posedness and continuity results.

Theorem 5.9 ([26, Theorem 4.1]). *Let $\bar{\xi} = (\xi_n)_{n=1}^\infty \in \mathcal{S}_A^\star$. Then $S_{\text{normal}}(t)\bar{\xi} \in \mathcal{S}_A^\star$ for all $t \geq 0$. Moreover,*

$$\|S_{\text{normal}}(t)\bar{\xi}\|_\star \leq M e^{-t}, \quad t > 0,$$

where M is a positive number depending on $\|\bar{\xi}\|_\star$ and the sequence $(\rho_n)_{n=1}^\infty$.

Theorem 5.9 shows that the semigroup $S_{\text{normal}}(t), t \geq 0$, generated by the solutions of the normal form (5.7) leaves invariant the whole space \mathcal{S}_A^* . Furthermore, one can establish the continuity (but not necessarily Lipschitz continuity) of each $S_{\text{normal}}(t)$ as a map from \mathcal{S}_A^* to \mathcal{S}_A^* .

Theorem 5.10 ([26, Theorem 4.2] and [25, Theorem 7.4]). *For each $t \geq 0$, the map*

$$\bar{\xi} \in \mathcal{S}_A^* \mapsto S_{\text{normal}}(t)\bar{\xi} \in \mathcal{S}_A^* \text{ is continuous.}$$

In particular, there exists $\varepsilon_0 > 0$ such that if $\bar{\xi}, \bar{\chi} \in \mathcal{S}_A^$ and $\|\bar{\xi}\|_*, \|\bar{\chi}\|_* < \varepsilon_0$, then*

$$\|S_{\text{normal}}(t)\bar{\xi} - S_{\text{normal}}(t)\bar{\chi}\|_* \leq 4e^{1/8}e^{-t} \|\bar{\xi} - \bar{\chi}\|_* \quad \forall t \geq 0.$$

According to Theorem 5.10, the normal form (5.7) is a well-posed system of ODEs in the infinite dimensional Banach space \mathcal{S}_A^* .

Theorem 5.11 ([26, Theorems 5.9 and 5.21]). *The normalization map W is a continuous function from \mathcal{R} to \mathcal{S}_A^* .*

We summarize our results stated above in the commutative diagram (Figure 1) in which all mappings are continuous.

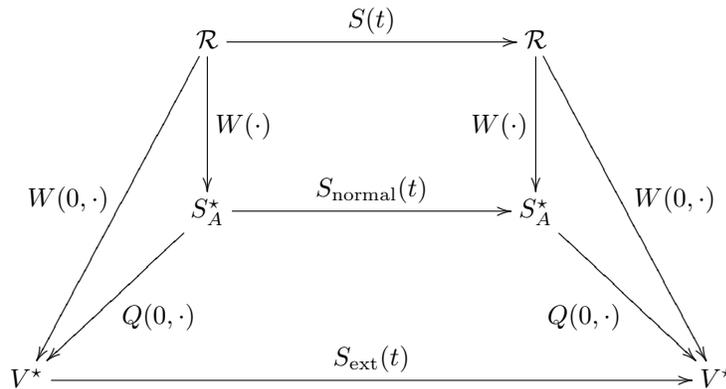


Figure 1. Commutative diagram for mappings and spaces ([26]).

The complete proofs of Theorems 5.9, 5.10 and 5.11 are lengthy and technical and giving them exceeds the scope of this survey. We refer the reader to the papers [25, 26] for details. They involve the complexification of NSE and extended NSE, for which the solutions are analytic in the complex time in a large domain of the form (3.5) and (3.6). Using an appropriate transformation, see (3.7), to map this domain to the right half plane, and utilizing some Phragmen-Lindelöff estimates, we can obtain recursive estimates for each step $W_n(u^0)$, $q_n(0, W(u^0))$, $q_n(\zeta, W(u^0))$, and $u(\zeta) - \sum_{j=1}^n q_j(\zeta, W(u^0))e^{-j\zeta}$ for complex time ζ . They, of course, depend on the weights ρ_n 's. Then the sum, say, $\sum_{n=1}^\infty \rho_n \|q_n(0, W(u^0))\|$ is convergent when ρ_n 's are chosen specifically and decay to zero extremely fast.

6. Navier and Stokes meet Poincaré and Dulac

It was not totally clear that the normal form theory for the NSE derived in section 5 could be related to the Poincaré-Dulac theory presented in section 4. It turns out to be the case, at least in the periodic case.

We consider thus the periodic case and use the same notation as in subsection 5.3.

In that subsection, (5.7) is a normal form of NSE in a suitable Banach space \mathcal{S}_A^* . This space, however, is too big to make a link with the *concrete* (formal series) approach of the Poincaré-Dulac theory. Such link was finally established in [27]. In short,

- The system (5.7), indeed, provides a Poincaré-Dulac normal form of the NSE, and is obtained by a (formal) explicit change of variables. The change of variables is a formal series expansion of the inverse of the normalization map W .
- Each homogeneous term in the formal series is well-defined in suitable Sobolev spaces.

We present below the precise results and provide main ideas and techniques in their proofs.

The following topological vector space will be essential in our study

$$E^\infty = C^\infty(\mathbb{R}^3, \mathbb{R}^3) \cap V \subset \mathcal{S}_A. \quad (6.1)$$

It is endowed with the topology generated by the family of norms $|A^\alpha \cdot|$ for all $\alpha \geq 0$.

First, we give an explicit definition of a normal form for the NSE, which is an analogue to classical ones by Poincaré and Dulac reviewed in section 4. We start with homogeneous polynomials and resonant monomials in infinite-dimensional spaces.

Definition 6.1. Let $Q \in \mathcal{H}^{[d]}(E^\infty)$, the space of homogeneous polynomials in $\xi \in E^\infty$ of order d . Then $Q(\xi)$ ($\xi \in E^\infty$ and $\xi_j = R_j \xi$, $j \in \mathbb{N}$), is a monomial of degree $\alpha_{k_i} > 0$ in ξ_{k_i} where $i = 1, 2, \dots, m$, $\alpha_{k_1} + \dots + \alpha_{k_m} = d$ and $k_1 < k_2 < \dots < k_m$, if it can be represented as

$$Q(\xi) = \tilde{Q}(\underbrace{\xi_{k_1}, \dots, \xi_{k_1}}_{\alpha_{k_1}}, \underbrace{\xi_{k_2}, \dots, \xi_{k_2}}_{\alpha_{k_2}}, \dots, \underbrace{\xi_{k_m}, \dots, \xi_{k_m}}_{\alpha_{k_m}}), \quad (6.2)$$

where $\tilde{Q}(\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(d)})$ is a continuous d -linear map from $(E^\infty)^d$ to E^∞ .

The monomial $Q(\xi)$ defined by (6.2), with degree $d \geq 2$, is called *resonant* if

$$\sum_{i=1}^m \alpha_{k_i} k_i = j \text{ and } Q = R_j Q \neq 0.$$

Although the definition of resonant monomials in Definition 6.1 is more abstract than that in Definition 4.1, they are essentially the same, see details in [27, Lemmas 4.4 and 4.6].

Definition 6.2. A differential equation in an infinite dimensional space E

$$\frac{d\xi}{dt} + A\xi + \sum_{d=2}^{\infty} \Phi^{[d]}(\xi) = 0 \tag{6.3}$$

is a Poincaré–Dulac normal form for the NSE if

- (i) Each $\Phi^{[d]}$ belongs to $\mathcal{H}^{[d]}(E)$, the space of homogeneous polynomials of order d , and $\Phi^{[d]}(\xi) = \sum_{k=1}^{\infty} \Phi_k^{[d]}(\xi)$, where all $\Phi_k^{[d]} \in \mathcal{H}^{[d]}(E)$ are resonant monomials,
- (ii) Equation (6.3) is obtained from NSE by a formal change of variable

$$u = \sum_{d=1}^{\infty} \Psi^{[d]}(\xi), \quad \text{where } \Psi^{[d]} \in \mathcal{H}^{[d]}(E). \tag{6.4}$$

To establish a normal form theory for the NSE, according to Definition 6.2, we need to identify the framework E , the normal form (6.3), and the formal change of variable (6.4).

The framework. We will use the space E^∞ defined by (6.1).

The normal form. The natural candidate for the normal form is (5.7). However, we must rewrite it in the power series form.

Let $\mathcal{P}_j^{[d]}(\xi)$ and $\mathcal{B}_j^{[d]}(\xi)$ denote the sum of all homogeneous monomials of degree d of $\mathcal{P}_j(\xi)$ and $\mathcal{B}_j(\xi)$, respectively. Then the series $\sum_j \mathcal{P}_j^{[d]}(\xi)$ and $\sum_j \mathcal{B}_j^{[d]}(\xi)$ converge in E^∞ to continuous polynomials $\mathcal{P}^{[d]}(\xi)$ and $\mathcal{B}^{[d]}(\xi)$, respectively, see Theorem 6.2 below.

The system (5.7) is rewritten in the formal power series as

$$\frac{d\xi}{dt} + A\xi + \sum_{d=2}^{\infty} \mathcal{B}^{[d]}(\xi) = 0. \tag{6.5}$$

Inheriting the spectral property (5.4), each polynomial $\mathcal{B}^{[d]}(\xi)$ in (6.5) can be verified to be resonant. Hence, system (6.5) is a potential Poincaré–Dulac normal form, except that it is missing a power series change of variable.

The formal change of variable. We already know that NSE reduces to (6.5) by the transformation $\xi = W(u)$. Therefore, u should be $W^{-1}(\xi)$. Of course, W^{-1} is not rigorously defined on E^∞ and, additionally, not expressed in the power series form. To resolve these, we heuristically argue that

$$u = \sum_{j=1}^{\infty} q_j(0, \xi) = \sum_{j=1}^{\infty} \sum_{d=1}^j q_j^{[d]}(0, \xi) = \sum_{d=1}^{\infty} \sum_{j=d}^{\infty} q_j^{[d]}(0, \xi) = \sum_{d=1}^{\infty} \mathcal{P}^{[d]}(\xi).$$

Note that $\mathcal{P}^{[1]}(\xi) = \xi$. Thus, the formal change of variable would be

$$u = \xi + \sum_{d=2}^{\infty} \mathcal{P}^{[d]}(\xi). \tag{6.6}$$

The change of variable (6.6) is considered as the formal inverse of the normalization map W .

It turns out that, these arguments can be made rigorous and we obtain the following result.

Theorem 6.1 ([27, Theorem 4.9]). *The system (6.5) is a Poincaré-Dulac normal form in E^∞ for the NSE (1.5), and is obtained by the formal change of variable (6.6).*

The proof of Theorem 6.1 relies on recursive formulas giving the homogeneous terms of the normal form. The main tool to estimate their Sobolev norms is the following family of homogeneous gauges $[[\xi]]_{d,n}$.

We introduce the set of general multi-indices $GI = \bigcup_{n=1}^\infty GI(n)$ where for $n \geq 1$,

$$GI(n) = \{ \bar{\alpha} = (\alpha_k)_{k=1}^\infty, \alpha_k \in \{0, 1, 2, \dots\}, \alpha_k = 0 \text{ for } k > n \text{ or } k \notin \sigma(A) \}.$$

For $\bar{\alpha} \in GI$, define

$$|\bar{\alpha}| = \sum_{k=1}^\infty \alpha_k \quad \text{and} \quad \|\bar{\alpha}\| = \sum_{k=1}^\infty k\alpha_k.$$

Definition 6.3 (Homogeneous gauges). Let $\xi = (\xi_k)_{k=1}^\infty \in S_A$ and $\bar{\alpha} = (\alpha_k)_{k=1}^\infty \in GI$, define

$$[\xi]^{\bar{\alpha}} = \prod_{\alpha_k > 0} |\xi_k|^{\alpha_k}.$$

For $n \geq d \geq 1$, define

$$[[\xi]]_{d,n} = \left(\sum_{|\bar{\alpha}|=d, \|\bar{\alpha}\|=n} [\xi]^{2\bar{\alpha}} \right)^{1/2}.$$

One can easily compare $[[\xi]]_{d,n}$ with the usual norm $|\xi|$ by

$$[[\xi]]_{d,n} \leq \left(\sum_{\bar{\alpha} \in GI(n), |\bar{\alpha}|=d} [\xi]^{2\bar{\alpha}} \right)^{1/2} \leq |P_n \xi|^d.$$

Moreover, one has the following multiplicative and Poincaré inequalities.

Lemma 6.1 ([27, Lemma 2.1]). *Let $\xi \in S_A$, $n \geq d \geq 1$ and $n' \geq d' \geq 1$. Then*

$$[[\xi]]_{d,n} \cdot [[\xi]]_{d',n'} \leq e^{d+d'} [[\xi]]_{d+d',n+n'}. \tag{6.7}$$

Note that the constant on the right-hand side of (6.7) is independent of n, n' .

Lemma 6.2 ([27, Lemma 2.2]). *For any $\xi \in S_A$, any numbers $\alpha, s \geq 0$ and $n \geq d \geq 1$, one has*

$$[[A^\alpha \xi]]_{d,n} \leq \left(\frac{d}{n}\right)^s [[A^{\alpha+s} \xi]]_{d,n} \leq \left(\frac{d}{n}\right)^s |P_n A^{\alpha+s} \xi|^d.$$

The main advantage of the gauges $[[\cdot]]_{d,n}$ is that they efficiently track the norm contribution of each variable in estimates of homogeneous polynomials. It leads to simple and convenient bounds, see (6.12) below, for recursively defined, complicated $\mathcal{P}_j^{[d]}(\xi)$ and $\mathcal{B}_j^{[d]}(\xi)$.

Convergence of homogeneous polynomials. For $d \geq 1$, let

$$\mathcal{P}^{[d]}(\xi) = \sum_{j=d}^\infty \mathcal{P}_j^{[d]}(\xi) = \sum_{j=d}^\infty q_j^{[d]}(0, \xi), \tag{6.8}$$

and $d \geq 2$, let

$$\mathcal{B}^{[d]}(\xi) = \sum_{j=1}^{\infty} \mathcal{B}_j^{[d]}(\xi) = \sum_{j=1}^{\infty} \sum_{k+l=j} \sum_{m+n=d} R_j B(\mathcal{P}_k^{[m]}(\xi), \mathcal{P}_l^{[n]}(\xi)). \tag{6.9}$$

Theorem 6.2 ([27, Theorems 3.4 and 3.5, Lemma 4.1]). *Let $\alpha \geq 1/2$. Then $\mathcal{P}^{[d]}(\xi)$, defined in (6.8) for $d \geq 1$, and $\mathcal{B}^{[d]}(\xi)$, defined in (6.9) for $d \geq 2$, are continuous homogeneous polynomials from $\mathcal{D}(A^{\alpha+3d/2})$ to $\mathcal{D}(A^\alpha)$, and satisfy*

$$|A^\alpha \mathcal{P}^{[d]}(\xi)| \leq \sum_{j=d}^{\infty} |A^\alpha \mathcal{P}_j^{[d]}(\xi)| \leq M(\alpha, d) |A^{\alpha+3d/2} \xi|^d, \tag{6.10}$$

$$|A^\alpha \mathcal{B}^{[d]}(\xi)| \leq \sum_{n=1}^{\infty} |A^\alpha \mathcal{B}_n^{[d]}(\xi)| \leq C(\alpha, d) |A^{\alpha+3d/2} \xi|^d, \tag{6.11}$$

for some positive constants $M(\alpha, d)$ and $C(\alpha, d)$.

Consequently, the series $\mathcal{P}^{[d]}(\xi) = \sum_{j=d}^{\infty} \mathcal{P}_j^{[d]}(\xi)$ and $\mathcal{B}^{[d]}(\xi) = \sum_{j=d}^{\infty} \mathcal{B}_j^{[d]}(\xi)$, converge in E^∞ , and are continuous homogeneous polynomials of degree d from E^∞ to E^∞ .

Proof. By induction, ones first prove that

$$|A^\alpha \mathcal{P}_j^{[d]}(\xi)| \leq c(\alpha, d) \left[\left[A^{\alpha+\frac{3}{2}(d-1)} \xi \right] \right]_{d,j}. \tag{6.12}$$

Then using inequality (6.2),

$$\begin{aligned} \sum_{j=1}^{\infty} |A^\alpha \mathcal{P}_j^{[d]}(\xi)| &\leq \sum_{j=d}^{\infty} c(\alpha, d) \left[\left[A^{\alpha+(3/2)(d-1)} \xi \right] \right]_{d,j} \\ &\leq \sum_{j=d}^{\infty} c(\alpha, d) \left(\frac{d}{j}\right)^{3/2} |A^{\alpha+(3/2)(d-1)+3/2} \xi|^d = M(\alpha, d) |A^{\alpha+3d/2} \xi|^d, \end{aligned}$$

which proves (6.10). Inequality (6.11) for $\mathcal{B}^{[d]}(\xi)$ is proved similarly. □

Proof of Theorem 6.1. By virtue of Theorem 6.2, the ODE (6.5) and change of variable (6.6) are meaningful in E^∞ now. It still remains to prove that (6.5), indeed, comes from NSE and (6.6). One can derive from NSE (1.5) the formal ODE for $\xi(t)$ under the transformation (6.6) as

$$\frac{d\xi}{dt} + \sum_{d=1}^{\infty} Q^{[d]}(\xi) = 0, \tag{6.13}$$

where $Q^{[1]}(\xi) = A\xi$ and, for $d \geq 2$,

$$Q^{[d]}(\xi) = \sum_{k+l=d} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)) - \sum_{\substack{2 \leq k, l \leq d-1 \\ k+l=d+1}} D\mathcal{P}^{[k]}(\xi)(Q^{[l]}(\xi)) + H_A^{(d)} \mathcal{P}^{[d]}(\xi),$$

with

$$H_A^{(d)} \mathcal{P}^{[d]}(\xi) = A\mathcal{P}^{[d]}(\xi) - D\mathcal{P}^{[d]}(\xi)A\xi$$

being the Poincaré homology operator.

It turns out that, see [27, Proposition 4.7],

$$Q^{[d]}(\xi) = \mathcal{B}^{[d]}(\xi) \text{ for all } \xi \in E^\infty, d \geq 2.$$

Therefore, the transformed system (6.13) is the same as (6.5) whose resonance conditions are already met. This implies that (6.5) is a Poincaré-Dulac normal form of the NSE (1.5) by the change of variable (6.6). \square

7. Final comments

7.1. Other related results

1. The recent paper [46] obtains the asymptotic expansion of the same type for weak solutions of NSE in periodic domains with exponentially decaying (non-potential) outer body forces satisfying an asymptotic expansion of the type

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-nt}$$

in appropriate (Sobolev-Gevrey type) functional spaces. See also [12] for a new result for power-decaying forces.

2. When $\Omega = \mathbb{R}^n$, because of lack of the Poincaré inequality, the situation is drastically different and the decay rate is only algebraic. The techniques and their proofs are quite different than those used for the bounded domains. We refer to [4, 5, 38, 58, 67, 72] and the references therein. In particular, Kukavica and Reis ([48]) obtain a precise space-time asymptotic of smooth solutions in a weighted space.
3. Section 7 of [34] focuses on the viscous Burgers equation and the Minea system. In the case of the viscous Burgers equation, the normalizing mapping W can be explicitly computed in terms of the Cole-Hopf transform
4. The asymptotic expansion is also established for dissipative wave equations by Shi in [69].

7.2. Some open issues

We indicate below a few open questions related to the topics considered in the present paper.

1. The classical Poincaré-Dulac theory for ODEs has a second part concerning the convergence of the formal series which give the change of variables, and the convergence of the series in the normal form (see [1]). The extension of these convergence results to the NSE seems to be an open problem.
2. Complete our knowledge of the normalization map and of the normal form.
3. What happens in 3D when the initial data $u_0 \in \mathcal{R}$ is near the boundary $\partial\mathcal{R}$ in case $\mathcal{R} \neq V$?
4. It is very likely that the normal form theory studied here extends to the NSE posed on a compact Riemann manifold (*e.g.* the Euclidean sphere in \mathbb{R}^3) (see [39] for results on the asymptotic decay). In particular, all specific results obtained in the periodic case should have a counterpart in this framework.

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