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GLOBAL CONVERGENCE OF AN ISENTROPIC EULER-POISSON SYSTEM IN $\mathbb{R}^+ \times \mathbb{R}^d$ *

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Abstract We prove the global-in-time convergence of an Euler-Poisson system near a constant equilibrium state in the whole space $\mathbb{R}^d$, as physical parameters tend to zero. The result follows from the uniform global existence of smooth solutions by means of energy estimates together with compactness arguments. For this purpose, we establish uniform estimates for $\operatorname{div} u$ and $\operatorname{curl} u$ instead of ∇u.

Keywords Euler-Poisson system, uniform global smooth solution, energy estimate, compactness and convergence.

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1. Introduction

The Euler-Poisson system plays a vital role in modelling of unmagnetized plasmas and semiconductor devices. Let n, u and ϕ be the density, the velocity of the electrons and the electric potential, respectively. These variables depend on the time $t \geq 0$ and the position $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $d = 1, 2, 3$. The scaled isentropic Euler-Poisson system for electrons is written as (see [4, 16]) :

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ m\big[\partial_t(nu) + \operatorname{div}(nu \otimes u)\big] + \nabla p(n) = n\nabla\phi - mnu, \\ -\lambda^2 \Delta\phi = 1 - n, \quad \text{in } (0, +\infty) \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

for $t > 0$ and $x \in \mathbb{R}^d$. Here, p is the pressure function, assumed to be strictly increasing on $(0, +\infty)$. The parameters m, $\lambda \in (0, 1]$ stand for the ratio of electron mass to ion mass and the Debye length, respectively. For convenience, we use a new parameter $\varepsilon \in (0, 1]$ defined by $m = \varepsilon^2$. We consider the Cauchy problem for (1.1) with initial conditions depending on the parameters :

$$t = 0 \; : \quad (n, u) = (n_0^\nu, u_0^\nu), \quad \text{in } \mathbb{R}^d, \tag{1.2}$$

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where $\nu = (\varepsilon, \lambda)$. The aim of the paper is to study the global-in-time convergence of the system near the constant equilibrium state $(n, u, \phi) = (1, 0, 0)$ in the whole space $\mathbb{R}^d$ as $\varepsilon \to 0$ and $\lambda \to 0$, which are referred to as the zero-electron mass limit and the quasi-neutral limit, respectively.

For smooth solutions in a non-vacuum region, the momentum equation in (1.1) can be written as

$$\varepsilon^2 \partial_t u + \varepsilon^2 (u \cdot \nabla) u + \nabla h(n) = \nabla \phi - \varepsilon^2 u, \tag{1.3}$$

where h is the enthalpy function defined by $h'(n) = p'(n)/n$. It is clear that h is strictly increasing on $(0, +\infty)$. In order that ϕ is uniquely determined, we assume that

$$\lim_{|x| \to +\infty} \phi(t, x) = 0, \quad \forall\, t \geq 0, \tag{1.4}$$

with

$$|x| = \sqrt{x_1^2 + \cdots x_d^2}.$$

From the Poisson equation in (1.1), ϕ can be expressed as

$$\phi = \frac{1}{\lambda^2} \Gamma * (1 - n).$$

where $*$ is the convolution operator and

$$\Gamma(x) = \begin{cases} -\dfrac{1}{2}|x|, & \text{if } d = 1, \\ -\dfrac{1}{2\pi} \ln |x|, & \text{if } d = 2, \\ -\dfrac{1}{4\pi |x|^2}, & \text{if } d = 3. \end{cases}$$

In particular, let ϕ_0^ν be the unique solution of the following problem

$$-\lambda^2 \Delta \phi_0^\nu = 1 - n_0^\nu, \quad \lim_{|x| \to +\infty} \phi_0^\nu(x) = 0.$$

Then we have

$$\phi_0^\nu = \frac{1}{\lambda^2} \Gamma * (1 - n_0^\nu).$$

From the result of Lax [13] and Kato [12] (see also [15]), we obtain the local existence of smooth solutions as follows. Let $s > \frac{d}{2} + 1$ be an integer and $(n_0^\nu - 1, u_0^\nu) \in H^s(\mathbb{R}^d)$ with $n_0^\nu \geq$ const. > 0. There exists a positive constant T_* (depending on ν) such that the Cauchy problem (1.1)-(1.2) with (1.4) admits a unique local smooth solution (n, u, ϕ) defined on $[0, T_*]$. Moreover, the solution satisfies

$$(n - 1, u, \nabla \phi) \in C\big([0, T_*]; H^s(\mathbb{R}^d)\big) \cap C^1\big([0, T_*]; H^{s-1}(\mathbb{R}^d)\big), \quad n \geq \text{const.} > 0.$$

The zero-electron mass limit and the quasi-neutral limit have attracted a lot of attention for system (1.1). See for instance [2, 5, 6, 20, 22, 23]. Both limits lead to incompressible Euler equations. For periodic smooth solutions with large initial data, they were justified in [2] and in [5, 22], respectively. Remark that in the Poisson equation in (1.1), the ion density is equal to 1. When the ion density is not a constant, the quasi-neutral limit was studied in [20]. In work [6], the quasi-neutral limit was justified for an Euler-Poisson for ions in a bounded domain with one-dimensional boundary layer formation, where the semilinear Poisson equation is derived from the Boltzmann relation. All these results are only valid on a uniform finite time interval.

Recently, the second author of the present paper considered the limits of periodic smooth solutions with initial data close to the constant equilibrium state $(n, u, \phi) = (1, 0, 0)$ and he proved the global-in-time convergence of the system [19]. The proof of the results in both limits relies on classical energy estimates for Euler equations together with compactness arguments. It depends strongly on the periodic property of the solutions for which the Poincaré inequality can be applied. As a consequence, an important estimate is obtained in the periodic case, namely (see Lemma 3.2 in [19]),

$$C^{-1}\|n-1\|_{s-1} \leq \lambda^2\|\nabla\phi\|_s \leq C\|n-1\|_{s-1},$$

where $C > 0$ is a constant independent of ν.

In this paper, we consider the global-in-time convergence of the system in the whole space $\mathbb{R}^d$ where the Poincaré inequality is not available. For this purpose, we have to avoid the above estimate in the analysis. Our proof is based on a single uniform estimate with respect to the time and the parameters. The main technique is to estimate $\operatorname{div} u$ and $\operatorname{curl} u$ instead of ∇u. This leads to uniform estimates which allow to apply compactness arguments.

We remark that, when the parameters are fixed constants, the global existence of smooth solutions near the constant equilibrium state for (1.1) was obtained in [1,10]. In (1.3), the last term $-\varepsilon^2 u$ stands for a dissipation property for the velocity. It plays an essential role in the proof of the global existence. Without this term, the global existence of smooth solutions can be still proved under an additional assumption that the flow is irrotational, namely, $\operatorname{curl} u = 0$. See for instance [7–9, 11]). Finally, we also mention results on the global-in-time convergence of a non-isentropic Euler-Poisson system [14, 23]. However, (1.1) cannot be derived from the system studied in [14, 23], even at a formal level.

The rest of the paper is organized as follows. Section 2 includes the main results of the paper and two preliminary lemmas that will be used later. The results consist of the uniform global existence of smooth solutions and their convergence as each of the parameters tends to zero. Section 3 is devoted to uniform energy estimates and a time dissipation estimate for the density. In the last section we give the proof of the main results.

2. Statement of main results

Let us first introduce some notations. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ with $d \geq 1$, we denote

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}} \quad \text{with} \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

For simplicity, the norms of $H^s(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ are denoted by $\|\cdot\|_s$, $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively.

The main results of the paper are the three theorems stated below. Theorem 2.1 concerns a result on the uniform global existence of smooth solutions with respect to the parameters. Theorems 2.2 and 2.3 concern the convergence of the system in the zero-electron mass limit $\varepsilon \to 0$ and the quasi-neutral limit $\lambda \to 0$, respectively. Both limit systems are incompressible Euler equations.

Theorem 2.1. *Let $s > \frac{d}{2} + 1$ be an integer with $d = 1, 2, 3$. There exist two constants $\delta > 0$ and $C > 0$, such that for all $\varepsilon,\ \lambda \in (0, 1]$ and for all initial data (n_0^ν, u_0^ν) verifying*

$$\frac{1}{\varepsilon}\|n_0^\nu - 1\|_s + \|u_0^\nu\|_s + \frac{\lambda}{\varepsilon}\|\nabla\phi_0^\nu\|_s \le \delta, \tag{2.1}$$

where $\nu = (\varepsilon, \lambda)$, then there is a unique global solution (n, u, ϕ) to (1.1)-(1.2) such that

$$(n - 1, u, \nabla\phi) \in C\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big) \cap C^1\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big). \tag{2.2}$$

Moreover, for all $t > 0$, this solution satisfies a uniform estimate

$$\begin{aligned}&\frac{1}{\varepsilon^2}\|n(t) - 1\|_s^2 + \|u(t)\|_s^2 + \frac{\lambda^2}{\varepsilon^2}\|\nabla\phi(t)\|_s^2\\ &+ \int_0^t \Big(\frac{1}{\varepsilon^2}\|\nabla n(t')\|_{s-1}^2 + \frac{1}{\varepsilon^2\lambda^2}\|n(t') - 1\|_{s-1}^2 + \|u(t')\|_s^2\Big)dt'\\ \le\ & C\Big(\frac{1}{\varepsilon^2}\|n_0^\nu - 1\|_s^2 + \|u_0^\nu\|_s^2 + \frac{\lambda^2}{\varepsilon^2}\|\nabla\phi_0^\nu\|_s^2\Big).\end{aligned} \tag{2.3}$$

Theorem 2.2. *Let $\lambda = 1$, $s \ge 3$ and $d = 2, 3$. Let $\big(n^\varepsilon, u^\varepsilon, \phi^\varepsilon\big)_{\varepsilon>0}$ be the sequence of the corresponding solutions defined by Theorem 2.1. Then there exist functions $\big(\bar u, \bar\phi\big)$ with*

$$\bar u,\ \bar\phi \in L^2\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big), \quad \partial_t \bar u \in L^2\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big),$$

such that, as $\varepsilon \to 0$ and up to subsequences, we have

$$u^\varepsilon \rightharpoonup \bar u, \quad \text{weakly in } L^2\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big). \tag{2.4}$$

Moreover, $\big(\bar u, \bar\phi\big)$ is a global solution of incompressible Euler equations

$$\partial_t \bar u + (\bar u \cdot \nabla)\bar u + \bar u + \nabla\bar\phi = 0, \quad \operatorname{div} \bar u = 0. \tag{2.5}$$

In addition, if

$$\frac{1}{\varepsilon}\|n_0^\varepsilon - 1\|_1 + \|u_0^\varepsilon - u_0\|_1 + \frac{1}{\varepsilon}\|\nabla\phi_0^\varepsilon\| \le C\varepsilon, \tag{2.6}$$

with

$$u_0 \in H^s(\mathbb{R}^d), \quad \operatorname{div} u_0 = 0, \tag{2.7}$$

then the whole sequence $(u^\varepsilon)_{\varepsilon>0}$ converges and $\big(\bar u, \bar\phi\big)$ with $\lim\limits_{|x|\to+\infty} \bar\phi(t, x) = 0$ is a unique global smooth solution of (2.5) with an initial condition :

$$t = 0\ :\quad \bar u = u_0, \quad \text{in } \mathbb{R}^d. \tag{2.8}$$

Theorem 2.3. *Let $\varepsilon = 1$, $s \ge 3$ and $d = 2, 3$. Let $\big(n^\lambda, u^\lambda, \phi^\lambda\big)_{\lambda>0}$ be the sequence of the corresponding solutions given by Theorem 2.1. Then there exist functions $\big(\bar u, \bar\phi\big)$ with*

$$\bar u,\ \bar\phi \in L^2\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big), \quad \partial_t \bar u \in L^2\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big),$$

such that, as $\lambda \to 0$ and up to subsequences, we have

$$u^\lambda \rightharpoonup \bar{u}, \quad \textit{weakly in } L^2\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big). \tag{2.9}$$

Moreover, $(\bar{u}, \bar{\phi})$ is a global solution of incompressible Euler equations (2.5). In addition, if

$$\|n_0^\lambda - 1\|_1 + \|u_0^\lambda - u_0\|_1 + \lambda\|\nabla\phi_0^\lambda\| \le C\lambda, \tag{2.10}$$

with

$$u_0 \in H^s(\mathbb{R}^d), \quad \operatorname{div} u_0 = 0, \tag{2.11}$$

then the whole sequence $(u^\lambda)_{\lambda>0}$ converges and $(\bar{u}, \bar{\phi})$ with $\lim\limits_{|x|\to+\infty} \bar{\phi}(t,x) = 0$ is a unique global smooth solution of (2.5) and (2.8).

3. Uniform energy estimates

In this section, we suppose that $s > \frac{d}{2} + 1$ with $d = 1, 2, 3$. According to [17], in order to prove the uniform global existence of smooth solutions, it suffices to establish uniform estimates of solutions with respect to the time, ε and λ. Set

$$N = n - 1, \quad W = \begin{pmatrix} N \\ u \end{pmatrix}.$$

Let $T > 0$ and W be the smooth solution of (1.1) defined on time interval $[0, T]$. We denote

$$W_T = \sup_{0 \le t \le T} \|W(t)\|_s.$$

We want to show (2.3) for all $t \in [0, T]$.

3.1. Two useful lemmas

In the proof of Theorems 2.1-2.3, we need two lemmas below. Lemma 3.1 is a refined version of Moser-type calculus inequalities [15]. It is useful to establish the uniform energy estimates. The proof of Lemma 3.1 can be found in [18]. Lemma 3.2 is useful to study the convergence of the system to incompressible Euler equations. It only concerns $d = 2, 3$. For this, we define

$$\omega^\nu = \operatorname{curl} u^\nu, \quad \bar{\omega} = \operatorname{curl} \bar{u} \overset{def}{=} \begin{cases} \dfrac{\partial \bar{u}_2}{\partial x_1} - \dfrac{\partial \bar{u}_1}{\partial x_2}, & \text{if } d = 2, \\ \nabla \times \bar{u}, & \text{if } d = 3. \end{cases}$$

Lemma 3.1. *Let $s > \frac{d}{2} + 1$ be an integer and $d = 1, 2, 3$. Let $u \in H^s(\mathbb{R}^d)$. For all $\alpha \in \mathbb{N}^d$ with $1 \le |\alpha| \le s$, if $v \in H^{|\alpha|}(\mathbb{R}^d)$, then*

$$\|\partial_x^\alpha(uv) - u\partial_x^\alpha v\| \le C\|\nabla u\|_{s-1}\|v\|_{|\alpha|-1},$$

and

$$\|\partial_x^\alpha(uv)\| \le C\|u\|_s\|v\|_{|\alpha|}.$$

Lemma 3.2. *Let $s \geq 3$ and $s_0 \geq 0$ be integers. Let $u_0 \in H^s(\mathbb{R}^d)$ with $d = 2, 3$. Assume $(n^\nu, u^\nu)_{\nu>0}$ is a sequence of functions satisfying*

$$(n-1, u) \in C\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big) \cap C^1\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big), \tag{3.1}$$

$$\begin{cases} \partial_t n^\nu + \mathrm{div}(n^\nu u^\nu) = 0, \\ \mathrm{curl}\big(\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + u^\nu\big) = 0, \quad \text{in } (0, +\infty) \times \mathbb{R}^d, \end{cases} \tag{3.2}$$

and as $\nu \to 0$,

$$n^\nu \longrightarrow 1, \quad \text{strongly in } C\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big), \tag{3.3}$$

$$u^\nu \rightharpoonup \bar{u}, \quad \text{weakly-}* \text{ in } L^\infty\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big), \tag{3.4}$$

where $\bar{u} \in L^\infty\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big)$. Then there exists a function $\bar{\phi}$ such that $\nabla\bar{\phi} \in L^\infty\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big)$ and $(\bar{u}, \bar{\phi})$ satisfies incompressible Euler equations (2.5). Further, $\partial_t \bar{u} \in L^\infty\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big)$.

Proof. From (3.3)-(3.4), it is clear that

$$n^\nu u^\nu \longrightarrow \bar{u}, \quad \text{in the sense of distributions } \mathcal{D}'\big(\mathbb{R}^+ \times \mathbb{R}^d\big).$$

Therefore, the first equation of (3.2) implies that $\mathrm{div}\bar{u} = 0$. In addition, we also have

$$\nabla u^\nu \rightharpoonup \nabla \bar{u}, \ \ \mathrm{div}\, u^\nu \rightharpoonup \mathrm{div}\, \bar{u} = 0, \ \ \omega^\nu \rightharpoonup \bar{\omega}, \quad \text{weakly-}* \text{ in } L^\infty\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big).$$

Now we prove the strong compactness of the sequence $(\omega^\nu)_{\nu>0}$. Indeed, the second equation in (3.2) is equivalent to

$$\begin{cases} \partial_t \omega^\nu + (u^\nu \cdot \nabla)\omega^\nu + \omega^\nu = -\omega^\nu \mathrm{div}\, u^\nu, & \text{if } d = 2, \\ \partial_t \omega^\nu + (u^\nu \cdot \nabla)\omega^\nu - (\omega^\nu \cdot \nabla) u^\nu + \omega^\nu = -\omega^\nu \mathrm{div}\, u^\nu, & \text{if } d = 3. \end{cases} \tag{3.5}$$

Since $(u^\nu)_{\nu>0}$ is bounded in $L^\infty\big(\mathbb{R}^+; H^s(\mathbb{R}^d)\big)$ with $s \geq 3$ and $d = 2, 3$, $(\omega^\nu)_{\nu>0}$ is also bounded in $L^\infty\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big)$ and $\big(\partial_t \omega^\nu\big)_{\nu>0}$ is bounded in $L^\infty\big(\mathbb{R}^+; H^{s-2}(\mathbb{R}^d)\big)$. Let $T > 0$. By a classical compactness theorem (see [21]), for all $s_1 \in [0, s-1)$, $\big(\omega^\nu\big)_{\nu>0}$ is relatively compact in $C\big([0,T]; H^{s_1}_{loc}(\mathbb{R}^d)\big)$. As a consequence, up to a subsequence, we have

$$\omega^\nu \longrightarrow \bar{\omega}, \quad \text{strongly in } C\big([0,T]; H^{s_1}_{loc}(\mathbb{R}^d)\big), \tag{3.6}$$

which implies that

$$\nabla\omega^\nu \longrightarrow \nabla\bar{\omega}, \quad \text{strongly in } C\big([0,T]; H^{s_1-1}_{loc}(\mathbb{R}^d)\big), \quad \forall\, s_1 \in [1, s-1),$$

where

$$\bar{\omega} \in L^\infty\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big) \cap C\big([0,T]; H^{s_1}_{loc}(\mathbb{R}^d)\big).$$

Consequently, in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$, we have

$$(\omega^\nu \cdot \nabla) u^\nu \longrightarrow (\bar{\omega} \cdot \nabla)\bar{u}, \ \ \omega^\nu \mathrm{div}\, u^\nu \longrightarrow \bar{\omega}\mathrm{div}\, \bar{u} = 0, \ \ (u^\nu \cdot \nabla)\omega^\nu \longrightarrow (\bar{u} \cdot \nabla)\bar{\omega}.$$

This allows us to pass to the limit in (3.5) to obtain

$$\begin{cases} \partial_t \bar{\omega} + (\bar{u} \cdot \nabla)\bar{\omega} + \bar{\omega} = 0, & \text{if } d = 2, \\ \partial_t \bar{\omega} + (\bar{u} \cdot \nabla)\bar{\omega} - (\bar{\omega} \cdot \nabla)\bar{u} + \bar{\omega} = 0, & \text{if } d = 3, \end{cases}$$

namely,

$$\operatorname{curl}\big(\partial_t \bar{u} + (\bar{u}\cdot\nabla)\bar{u} + \bar{u}\big) = 0.$$

Hence, there exists $\bar{\phi} \in \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ such that (see [3])

$$\partial_t \bar{u} + (\bar{u}\cdot\nabla)\bar{u} + \bar{u} = -\nabla\bar{\phi},$$

which proves (2.5).

In order to recover the regularity of $(\bar{u}, \bar{\phi})$, we apply div to the last equation. Since $\operatorname{div}\bar{u} = 0$, we have

$$-\Delta\bar{\phi} = \operatorname{div}\big((\bar{u}\cdot\nabla)\bar{u}\big),$$

which implies that $\nabla\bar{\phi} \in L^\infty\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big)$. Finally, from (2.5), we also have $\partial_t \bar{u} \in L^\infty\big(\mathbb{R}^+; H^{s-1}(\mathbb{R}^d)\big)$. □

3.2. Energy estimates in L^2

Lemma 3.3. *For all $t \in [0,T]$, it holds that*

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^d}\Big(n|u|^2 + \frac{1}{\varepsilon^2}h'(\xi)N^2 + \frac{\lambda^2}{\varepsilon^2}\|\nabla\phi\|^2\Big)\mathrm{d}x + 2\int_{\mathbb{R}^d} n|u|^2\mathrm{d}x = 0. \tag{3.7}$$

Proof. The Euler equations in (1.1) admits the following entropy:

$$E(n,u) = \frac{1}{2}n|u|^2 + \frac{1}{\varepsilon^2}H(n) \quad \text{with } H'(n) = h(n).$$

The associated entropy flux is

$$F(n,u) = \frac{1}{2}n|u|^2 u + \frac{1}{\varepsilon^2}R(n)u \quad \text{with } R(n) = nh(n).$$

Then for smooth solutions of (1.1), the following entropy equality holds:

$$\partial_t E + \operatorname{div} F = \frac{1}{\varepsilon^2}nu\cdot\nabla\phi - n|u|^2.$$

The Taylor expansion for H around $n = 1$ yields

$$\begin{aligned} H(n) &= H(1) + H'(1)(n-1) + \frac{1}{2}H''(\xi)(n-1)^2 \\ &= H(1) + h(1)N + \frac{1}{2}h'(\xi)N^2, \end{aligned}$$

where ξ is between 1 and n. From the density equation

$$\partial_t N = -\operatorname{div}(nu),$$

we obtain

$$\partial_t E = \partial_t\Big(\frac{1}{2}n|u|^2 + \frac{1}{2\varepsilon^2}h'(\xi)N^2\Big) - \frac{1}{\varepsilon^2}h(1)\operatorname{div}(nu).$$

It follows that

$$\partial_t\Big(n|u|^2 + \frac{1}{\varepsilon^2}h'(\xi)N^2\Big) + 2\operatorname{div} F - \frac{2}{\varepsilon^2}h(1)\operatorname{div}(nu) = \frac{2}{\varepsilon^2}nu\cdot\nabla\phi - 2n|u|^2.$$

Integrating the above equation with respect to x yields

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^d}\Big(n|u|^2+\frac{1}{\varepsilon^2}h'(\xi)N^2\Big)\mathrm{dx}+2\int_{\mathbb{R}^d}n|u|^2\mathrm{dx}-\frac{2}{\varepsilon^2}\int_{\mathbb{R}^d}nu\cdot\nabla\phi\mathrm{dx}=0.$$

Using the Poisson equation and density equation again and integrating by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^d}nu\cdot\nabla\phi\mathrm{d}x&=-\int_{\mathbb{R}^d}\mathrm{div}(nu)\phi\mathrm{d}x\\&=\int_{\mathbb{R}^d}\partial_t n\phi\mathrm{d}x\\&=\lambda^2\int_{\mathbb{R}^d}\partial_t(\triangle\phi)\phi\mathrm{d}x\\&=-\lambda^2\int_{\mathbb{R}^d}\partial_t(\nabla\phi)\nabla\phi\mathrm{d}x\\&=-\frac{\lambda^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|^2.\end{aligned}$$

These last two relations imply (3.7). □

3.3. Higher order energy estimates

From now on, we denote by $C>0$ a generic constant independent of any time and the parameters. We want to estimate $\mathrm{div}\,u$ and $\mathrm{curl}\,u$ in $H^{s-1}(\mathbb{R}^d)$ by using the momentum equation. Then the estimate for ∇u follows from the equality

$$\|\nabla u\|^2=\|\mathrm{div}\,u\|^2+\|\mathrm{curl}\,u\|^2,\quad d=2,3.$$

Let $\beta\in\mathbb{N}^d$ with $|\beta|\le s-1$. We first consider the L^2 estimate of $\partial^\beta(\mathrm{div}\,u)$. Applying $\partial^\beta\mathrm{div}$ to (1.3) and taking the inner product with $2\partial^\beta(\mathrm{div}\,u)$ in L^2, we obtain the classical energy equality :

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\|\partial^\beta(\mathrm{div}\,u)\|^2+2\|\partial^\beta(\mathrm{div}\,u)\|^2=&-\frac{2}{\varepsilon^2}\langle\partial^\beta(\mathrm{div}(\nabla h(n)),\partial^\beta(\mathrm{div}u)\rangle\\&+\frac{2}{\varepsilon^2}\langle\partial^\beta(\triangle\phi),\partial^\beta(\mathrm{div}\,u)\rangle\\&-2\langle\partial^\beta(\mathrm{div}((u\cdot\nabla)u),\partial^\beta(\mathrm{div}\,u)\rangle,\end{aligned}\tag{3.8}$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product of $L^2(\mathbb{R}^d)$. We need to deal with each term on the right-hand side of (3.8), which are achieved in the following lemmas.

Lemma 3.4. *For all $t\in[0,T]$ and all $\beta\in\mathbb{N}^d$ with $|\beta|\le s-1$, it holds that*

$$-2\langle\partial^\beta(\mathrm{div}(\nabla h(n))),\partial^\beta(\mathrm{div}u)\rangle\le-\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle\frac{h'(n)}{n}\nabla(\partial^\beta N),\nabla(\partial^\beta N)\Big\rangle+C\|N\|_s^2\,\|u\|_s.\tag{3.9}$$

Proof. From the density equation, it is easy to obtain

$$\mathrm{div}\,u=-\frac{\partial_t N+u\cdot\nabla N}{n}.$$

Substituting this relation into the term on the left-hand side of (3.5), we get

$$\begin{aligned}&-\langle\partial^\beta(\mathrm{div}(\nabla h(n))),\partial^\beta(\mathrm{div}u)\rangle\\&=\Big\langle\mathrm{div}(\partial^\beta(h'(n)\nabla N)),\partial^\beta\Big(\frac{\partial_tN}{n}\Big)\Big\rangle+\Big\langle\mathrm{div}(\partial^\beta(h'(n)\nabla N)),\partial^\beta\Big(\frac{u\cdot\nabla N}{n}\Big)\Big\rangle.\end{aligned}\tag{3.10}$$

An integration by parts and a straightforward calculation yield

$$\begin{aligned}\Big\langle\mathrm{div}(\partial^\beta(h'(n)\nabla N)),\partial^\beta\Big(\frac{\partial_tN}{n}\Big)\Big\rangle&=-\Big\langle\partial^\beta(h'(n)\nabla N),\partial^\beta(\nabla\frac{\partial_tN}{n})\Big\rangle\\&=\Big\langle h'(n)\partial^\beta(\nabla N)-\partial^\beta(h'(n)\nabla N),\partial^\beta\Big(\nabla\frac{\partial_tN}{n}\Big)\Big\rangle\\&\quad-\Big\langle h'(n)\partial^\beta(\nabla N),\partial^\beta\Big(\nabla\frac{\partial_tN}{n}\Big)\Big\rangle\\&=\Big\langle h'(n)\partial^\beta(\nabla N)-\partial^\beta(h'(n)\nabla N),\partial^\beta\Big(\nabla\frac{\partial_tN}{n}\Big)\Big\rangle\\&\quad-\Big\langle h'(n)\partial^\beta(\nabla N),\partial^\beta\Big(\nabla\frac{\partial_tN}{n}\Big)-\frac{1}{n}\partial^\beta(\nabla(\partial_tN))\Big\rangle\\&\quad-\Big\langle h'(n)\partial^\beta(\nabla N),\frac{1}{n}\partial^\beta(\nabla(\partial_tN))\Big\rangle.\end{aligned}\tag{3.11}$$

For the first term on the right-hang side of (3.11), we use the density equation and an integration by parts to obtain

$$\begin{aligned}&\Big\langle h'(n)\partial^\beta(\nabla N)-\partial^\beta(h'(n)\nabla N),\partial^\beta\Big(\nabla\frac{\partial_tN}{n}\Big)\Big\rangle\\&=\Big\langle\partial^\beta(h'(n)\nabla N)-h'(n)\partial^\beta(\nabla N),\partial^\beta\Big(\nabla\frac{\mathrm{div}(nu)}{n}\Big)\Big\rangle\\&=-\Big\langle\mathrm{div}[\partial^\beta(h'(n)\nabla N)-h'(n)\partial^\beta(\nabla N)],\partial^\beta\Big(\frac{\mathrm{div}(nu)}{n}\Big)\Big\rangle\\&=-\Big\langle\mathrm{div}(\partial^\beta(h'(n)\nabla N))-h'(n)\mathrm{div}(\partial^\beta(\nabla N)),\partial^\beta\Big(\frac{\mathrm{div}(nu)}{n}\Big)\Big\rangle\\&\quad+\Big\langle\nabla h'(n)\partial^\beta(\nabla N),\partial^\beta\Big(\frac{\mathrm{div}(nu)}{n}\Big)\Big\rangle.\end{aligned}$$

Since $\nabla h'(n)-=h''(n)\nabla N$, it is clear that

$$\Big\langle\nabla h'(n)\partial^\beta(\nabla N),\partial^\beta\Big(\frac{\mathrm{div}(nu)}{n}\Big)\Big\rangle\le C\|N\|_s^2\|u\|_s.$$

Applying Lemma 3.1, we have

$$-\Big\langle\mathrm{div}(\partial^\beta(h'(n)\nabla N))-h'(n)\mathrm{div}(\partial^\beta(\nabla N)),\partial^\beta\Big(\frac{\mathrm{div}(nu)}{n}\Big)\Big\rangle\le C\|N\|_s^2\|u\|_s.$$

Therefore,

$$\Big\langle h'(n)\partial^\beta(\nabla N)-\partial^\beta(h'(n)\nabla N),\partial^\beta\Big(\nabla\frac{\partial_tN}{n}\Big)\Big\rangle\le C\|N\|_s^2\|u\|_s.\tag{3.12}$$

Similarly, we also have

$$-\Big\langle h'(n)\partial^\beta(\nabla N),\partial^\beta\Big(\nabla\frac{\partial_tN}{n}\Big)-\frac{1}{n}\partial^\beta(\nabla(\partial_tN))\Big\rangle\le C\|N\|_s^2\|u\|_s.\tag{3.13}$$

For the last term on the right-hand side of (3.11), we have

$$\begin{aligned}
&-\Big\langle h'(n)\partial^\beta(\nabla N), \frac{1}{n}\partial_t(\partial^\beta(\nabla N))\Big\rangle \\
&= -\frac{1}{2}\Big\langle \frac{h'(n)}{n}, \partial_t|\nabla(\partial^\beta N)|^2\Big\rangle \\
&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle \frac{h'(n)}{n}, |\nabla(\partial^\beta N)|^2\Big\rangle + \frac{1}{2}\Big\langle |\nabla(\partial^\beta N)|^2, \partial_t\frac{h'(n)}{n}\Big\rangle,
\end{aligned}$$

which implies that

$$-\Big\langle h'(n)\partial^\beta(\nabla N), \frac{1}{n}\partial_t(\partial^\beta(\nabla N))\Big\rangle \le -\frac{1}{2}\Big\langle \frac{h'(n)}{n}, \partial_t|\nabla(\partial^\beta N)|^2\Big\rangle + C\|N\|_s^2\,\|u\|_s. \tag{3.14}$$

Hence, from (3.11)-(3.14) we obtain

$$\Big\langle \mathrm{div}(\partial^\beta(h'(n)\nabla N)), \partial^\beta\Big(\frac{\partial_t N}{n}\Big)\Big\rangle \le -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle \frac{h'(n)}{n}\nabla(\partial^\beta N), \nabla(\partial^\beta N)\Big\rangle + C\|N\|_s^2\|u\|_s. \tag{3.15}$$

Next, for the last term on the right-hand side of (3.10), we have

$$\begin{aligned}
&\Big\langle \mathrm{div}(\partial^\beta(h'(n)\nabla N)), \partial^\beta\Big(\frac{u\cdot\nabla N}{n}\Big)\Big\rangle \\
&= \Big\langle \mathrm{div}[\partial^\beta(h'(n)\nabla N) - h'(n)\partial^\beta(\nabla N)], \partial^\beta\Big(\frac{u\cdot\nabla N}{n}\Big)\Big\rangle \\
&\quad -\Big\langle h'(n)\partial^\beta(\nabla N), \partial^\beta\nabla\Big(\frac{u\cdot\nabla N}{n}\Big) - \frac{\partial^\beta(\nabla(\nabla N))\cdot u}{n}\Big\rangle \\
&\quad -\Big\langle h'(n)\partial^\beta(\nabla N), \frac{\partial^\beta(\nabla(\nabla N))\cdot u}{n}\Big\rangle.
\end{aligned} \tag{3.16}$$

The first term on the right-hand side of (3.16) can be written as

$$\begin{aligned}
&\Big\langle \mathrm{div}[\partial^\beta(h'(n)\nabla N) - h'(n)\partial^\beta(\nabla N)], \partial^\beta\Big(\frac{u\cdot\nabla N}{n}\Big)\Big\rangle \\
&= \Big\langle \mathrm{div}(\partial^\beta(h'(n)\nabla N)) - h'(n)\mathrm{div}(\partial^\beta(\nabla N)), \partial^\beta\Big(\frac{u\cdot\nabla N}{n}\Big)\Big\rangle \\
&\quad -\Big\langle \nabla h'(n)\partial^\beta(\nabla N), \partial^\beta\Big(\frac{u\cdot\nabla N}{n}\Big)\Big\rangle.
\end{aligned}$$

Then using Lemma 3.1, it is easy to see that

$$\Big\langle \mathrm{div}[\partial^\beta(h'(n)\nabla N) - h'(n)\partial^\beta(\nabla N)], \partial^\beta\Big(\frac{u\cdot\nabla N}{n}\Big)\Big\rangle \le C\|N\|_s^2\|u\|_s.$$

Similarly, we have

$$-\Big\langle h'(n)\partial^\beta(\nabla N), \partial^\beta\nabla\Big(\frac{u\cdot\nabla N}{n}\Big) - \frac{u\cdot\partial^\beta(\nabla(\nabla N))}{n}\Big\rangle \le C\|N\|_s^2\|u\|_s.$$

For the last term, an integration by parts gives

$$\begin{aligned}
-\Big\langle h'(n)\partial^\beta(\nabla N), \frac{u\cdot\partial^\beta(\nabla(\nabla N))}{n}\Big\rangle &= -\Big\langle \partial^\beta(\nabla N), \frac{h'(n)u\cdot\nabla(\partial^\beta(\nabla N))}{n}\Big\rangle \\
&= -\frac{1}{2}\Big\langle \frac{h'(n)u}{n}, \nabla|\partial^\beta(\nabla N)|^2\Big\rangle \\
&= \frac{1}{2}\Big\langle \mathrm{div}\Big(\frac{h'(n)u}{n}\Big), |\partial^\beta(\nabla N)|^2\Big\rangle \\
&\le C\|N\|_s^2\|u\|_s.
\end{aligned}$$

It follows from (3.16) and the last three inequalities that

$$\Big\langle \operatorname{div}(\partial^\beta(h'(n)\nabla N)), \partial^\beta\Big(\frac{u\cdot\nabla N}{n}\Big)\Big\rangle \le C\|N\|_s^2\|u\|_s, \tag{3.17}$$

which combines (3.10) and (3.15) yields (3.9). □

Lemma 3.5. *For all $t\in[0,T]$ and all $\beta\in\mathbb{N}^d$ with $|\beta|\le s-1$, it holds that*

$$2\langle\partial^\beta(\triangle\phi), \partial^\beta(\operatorname{div} u)\rangle \le -\lambda^2\frac{\mathrm{d}}{\mathrm{dt}}\|\partial^\beta(\Delta\phi)\|^2 + \frac{C}{\lambda^2}\|N\|_{s-1}^2\|u\|_s, \tag{3.18}$$

$$-2\langle\partial^\beta(\operatorname{div}((u\cdot\nabla)u), \partial^\beta(\operatorname{div} u)\rangle \le C\|u\|_s^3. \tag{3.19}$$

Proof. We first prove (3.18). Applying $\partial_t\partial^\beta$ to the Poisson equation together with the density equation in (1.1) yields

$$\lambda^2\partial_t\partial^\beta(\Delta\phi) = \partial^\beta\partial_t N = -\partial^\beta(\operatorname{div}(nu)).$$

It follows that

$$\lambda^2\frac{\mathrm{d}}{\mathrm{dt}}\|\partial^\beta(\Delta\phi)\|^2 = -2\langle\partial^\beta\operatorname{div}(nu), \partial^\beta(\Delta\phi)\rangle.$$

From the above equality and $n=N+1$, we obtain

$$2\langle\partial^\beta\operatorname{div} u, \partial^\beta(\Delta\phi)\rangle = -\lambda^2\frac{\mathrm{d}}{\mathrm{dt}}\|\partial^\beta(\Delta\phi)\|^2 - 2\langle\partial^\beta(\Delta\phi), \partial^\beta(\operatorname{div}(Nu))\rangle.$$

By the Poisson equation, the last term above can be estimates as follows:

$$\begin{aligned}
-2\langle\partial^\beta(\Delta\phi), \partial^\beta(\operatorname{div}(Nu))\rangle &= -\frac{2}{\lambda^2}\langle\partial^\beta N, \partial^\beta(\operatorname{div}(Nu))\rangle\\
&= -\frac{2}{\lambda^2}\langle\partial^\beta N, \partial^\beta(N\operatorname{div} u)\rangle - \frac{2}{\lambda^2}\langle\partial^\beta N, \partial^\beta(\nabla N\cdot u)\rangle\\
&= -\frac{2}{\lambda^2}\langle\partial^\beta N, \partial^\beta(N\operatorname{div} u)\rangle - \frac{2}{\lambda^2}\langle\partial^\beta N, u\cdot\nabla(\partial^\beta N)\rangle\\
&\quad -\frac{2}{\lambda^2}\langle\partial^\beta N, \partial^\beta(\nabla N\cdot u) - u\cdot\nabla(\partial^\beta N)\rangle,
\end{aligned}$$

with

$$-\frac{2}{\lambda^2}\langle\partial^\beta N, u\cdot\nabla(\partial^\beta N)\rangle = -\frac{1}{\lambda^2}\langle u, \nabla|\partial^\beta N|^2\rangle.$$

Obviously, applying again Lemma 3.1 together these inequalities gives (3.18).

Now we prove (3.19) of which the left-hand side can be written as

$$\begin{aligned}
&-2\langle\partial^\beta(\operatorname{div}((u\cdot\nabla)u), \partial^\beta(\operatorname{div} u)\rangle\\
&= -2\sum_{j=1}^d\langle\partial^\beta[\nabla u_j\cdot\partial_{x_j}u + u_j\partial_{x_j}(\operatorname{div} u)], \partial^\beta(\operatorname{div} u)\rangle\\
&= -2\sum_{j=1}^d\langle\partial^\beta(\nabla u_j\cdot\partial_{x_j}u), \partial^\beta(\operatorname{div} u)\rangle\\
&\quad -2\sum_{j=1}^d\langle u_j\partial^\beta(\partial_{x_j}(\operatorname{div} u)), \partial^\beta(\operatorname{div} u)\rangle\\
&\quad -2\sum_{j=1}^d\langle\partial^\beta(u_j\partial_{x_j}(\operatorname{div} u)) - u_j\partial^\beta(\partial_{x_j}(\operatorname{div} u)), \partial^\beta(\operatorname{div} u)\rangle,
\end{aligned} \tag{3.20}$$

from which we obtain easily (3.19). □

Lemma 3.6. *For all $t \in [0,T]$ and all $\beta \in \mathbb{N}^d$ with $|\beta| \le s-1$, it holds that*

$$\frac{d}{dt}\Big(\|\partial^\beta(\operatorname{div} u)\|^2 + \frac{1}{\varepsilon^2}\Big\langle \frac{h'(n)}{n}\partial^\beta(\nabla N), \partial^\beta(\nabla N)\Big\rangle + \frac{\lambda^2}{\varepsilon^2}\|\partial^\beta \Delta\phi\|^2\Big) + 2\|\partial^\beta(\operatorname{div} u)\|^2$$
$$\le \frac{C}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2\|u\|_s + \frac{C}{\varepsilon^2}\|N\|_s^2\|u\|_s + C\|u\|_s^3. \tag{3.21}$$

Lemma 3.6 gives a conclusion of Lemmas 3.4-3.5 together with (3.8). When $d=1$, (3.21) is a usual classical energy estimate. When $d=2,3$, we have to consider the energy estimate for $\omega = \operatorname{curl} u$, which is given as follows.

Lemma 3.7. *For all $t \in [0,T]$ and all $\beta \in \mathbb{N}^d$ with $|\beta| \le s-1$, it holds that*

$$\frac{d}{dt}\|\partial^\beta\omega\|^2 + 2\|\partial^\beta\omega\|^2 \le C\|u\|_s^3. \tag{3.22}$$

Proof. Applying ∂^βcurl to (1.3) and taking the inner product with $\partial^\beta\omega$ in $L^2(\mathbb{R}^d)$, we have

$$\frac{d}{dt}\|\partial^\beta\omega\|^2 + 2\|\partial^\beta\omega\|^2 = -2\langle \partial^\beta((u\cdot\nabla)u), \partial^\beta\omega\rangle.$$

The term on the right-hand side of the above equality equals

$$\begin{cases} -2\langle \partial^\beta((u\cdot\nabla)\omega), \partial^\beta\omega\rangle - 2\langle \omega \operatorname{div} u, \partial^\beta\omega\rangle, & \text{if } d=2,\\ -2\langle \partial^\beta((u\cdot\nabla)\omega), \partial^\beta\omega\rangle - 2\langle \omega \operatorname{div} u, \partial^\beta\omega\rangle + 2\langle \partial^\beta(\omega\cdot\nabla)u, \partial^\beta\omega\rangle, & \text{if } d=3. \end{cases}$$

It is obvious that

$$2\langle \partial^\beta(\omega\cdot\nabla)u, \partial^\beta\omega\rangle \le C\|u\|_s^3,$$
$$-2\langle \omega\operatorname{div} u, \partial^\beta\omega\rangle \le C\|u\|_s^3,$$

and

$$-2\langle \partial^\beta((u\cdot\nabla)\omega), \partial^\beta\omega\rangle = -2\sum_{j=1}^d \big[\langle \partial^\beta(u_j\partial_{x_j}\omega) - u_j\partial^\beta(\partial_{x_j}\omega), \partial^\beta\omega\rangle + \langle u_j\partial^\beta(\partial_{x_j}\omega), \partial^\beta\omega\rangle\big].$$

By Lemma 3.1, we have

$$-2\sum_{j=1}^d \langle \partial^\beta(u_j\partial_{x_j}\omega) - u_j\partial^\beta(\partial_{x_j}\omega), \partial^\beta\omega\rangle \le C\|u\|_s^3.$$

We also have

$$-2\sum_{j=1}^d \langle u_j\partial^\beta(\partial_{x_j}\omega), \partial^\beta\omega\rangle = \sum_{j=1}^d \langle \partial_{x_j}u_j, |\partial^\beta\omega|^2\rangle \le C\|u\|_s^3.$$

Therefore,

$$-2\langle \partial^\beta((u\cdot\nabla)\omega), \partial^\beta\omega\rangle \le C\|u\|_s^3.$$

This proves

$$-2\langle \partial^\beta((u\cdot\nabla)u), \partial^\beta\omega\rangle \le C\|u\|_s^3,$$

and (3.22) follows. □

From Lemmas 3.3 and 3.6–3.7, we obtain the following energy estimate.

Lemma 3.8. *For all $t \in [0,T]$, it holds that*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\Big(\int_{\mathbb{R}^d} n|u|^2\mathrm{d}x + \|\nabla u\|_{s-1}^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} h'(\xi)N^2\mathrm{d}x + \frac{\lambda^2}{\varepsilon^2}\|\Delta\phi\|_{s-1}^2 + \frac{\lambda^2}{\varepsilon^2}\|\nabla\phi\|^2 \\
&+\frac{1}{\varepsilon^2}\sum_{|\beta|\le s-1}\Big\langle\frac{h'(n)}{n}\partial^\beta(\nabla N),\partial^\beta(\nabla N)\Big\rangle\Big) + 2\|u\|_s^2 \\
&\le \frac{C}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2\|u\|_s + \frac{C}{\varepsilon^2}\|N\|_s^2\,\|u\|_s + C\|u\|_s^3. \qquad (3.23)
\end{aligned}$$

Proof. Summing up (3.7) and (3.21)-(3.22) for all $|\beta| \le s-1$, we obtain (3.23). □

The next lemma gives a time dissipation estimate of N. The proof is omitted since it is similar to that of Lemma 3.5 in [19].

Lemma 3.9. *For all $t \in [0,T]$ and all $\beta \in \mathbb{N}^d$ with $|\beta| \le s-1$, it holds that*

$$\begin{aligned}
&2\sum_{|\beta|\le s-1}\frac{d}{dt}\big\langle\partial_x^\beta u, \partial_x^\beta(\nabla N)\big\rangle + \frac{C^{-1}}{\varepsilon^2}\|\nabla N\|_{s-1}^2 + \frac{1}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2 \\
&\le C\|u\|_s^2 + C\Big(\frac{1}{\varepsilon^2}\|N\|_s^3 + \|N\|_s\,\|u\|_s^2\Big). \qquad (3.24)
\end{aligned}$$

From Lemmas 3.8-3.9, it is easy to get the energy estimate in the following result.

Lemma 3.10. *For all $t \in [0,T]$, it holds that*

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^d} n|u|^2\mathrm{d}x + \|\nabla u\|_{s-1}^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} h'(\xi)N^2\mathrm{d}x + \frac{\lambda^2}{\varepsilon^2}\|\Delta\phi\|_{s-1}^2 + \frac{\lambda^2}{\varepsilon^2}\|\nabla\phi\|^2 \\
&+\frac{1}{\varepsilon^2}\sum_{|\beta|\le s-1}\Big\langle\frac{h'(n)}{n}\partial^\beta(\nabla N),\partial^\beta(\nabla N)\Big\rangle + 2\kappa\sum_{|\beta|\le s-1}\langle\partial^\beta u, \partial^\beta(\nabla N)\rangle\Big) \\
&+ 2\|u\|_s^2 + \frac{C^{-1}\kappa}{\varepsilon^2}\|\nabla N\|_{s-1}^2 + \frac{\kappa}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2 \qquad (3.25) \\
&\le \frac{C}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2\|u\|_s + \frac{C}{\varepsilon^2}\|N\|_s^2\,\|u\|_s + C\|u\|_s^3 + C\kappa\|u\|_s^2 + C\kappa\Big(\frac{1}{\varepsilon^2}\|N\|_s^3 + \|N\|_s\,\|u\|_s^2\Big).
\end{aligned}$$

where $\kappa > 0$ is a small constant to be chosen.

4. Proof of Theorems 2.1–2.3

The convergence of the system follows from uniform estimates with respect to the parameters ε and λ.

4.1. Proof of Theorem 2.1

Using (3.25) together with (2.1), we want to show (2.3), which implies the global existence of solutions.

We consider the terms on the right-hand side in (3.25). For any fixed $\kappa > 0$ and $\varepsilon, \lambda \in (0,1]$, when W_T is sufficiently small, we have

$$\begin{aligned}
\frac{C}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2\|u\|_s &\le \frac{\kappa}{4\varepsilon^2\lambda^2}\|N\|_{s-1}^2,\\
\frac{C}{\varepsilon^2}\|N\|_s^2\|u\|_s &\le \frac{1}{4}\Big(\frac{C^{-1}\kappa}{\varepsilon^2}\|\nabla N\|_{s-1}^2 + \frac{\kappa}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2\Big),\\
\frac{C\kappa}{\varepsilon^2}\|N\|_s^3 &\le \frac{1}{4}\Big(\frac{C^{-1}\kappa}{\varepsilon^2}\|\nabla N\|_{s-1}^2 + \frac{\kappa}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2\Big),\\
C\|u\|_s^3 &\le \frac{1}{2}\|u\|_s^2
\end{aligned}$$

and

$$C\kappa\|N\|_s\|u\|_s^2 \le \frac{1}{2}\|u\|_s^2.$$

Next, we choose κ sufficiently small such that

$$C\kappa\,\|u\|_s^2 \le \frac{1}{2}\|u\|_s^2.$$

It follows from (3.25) that

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^d} n|u|^2\mathrm{d}x + \|\nabla u\|_{s-1}^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} h'(\xi)N^2\mathrm{d}x + \frac{\lambda^2}{\varepsilon^2}\|\Delta\phi|_{s-1}^2 + \frac{\lambda^2}{\varepsilon^2}\|\nabla\phi\|^2\\
&+ \frac{1}{\varepsilon^2}\sum_{|\beta|\le s-1}\Big\langle\frac{h'(n)}{n}\partial^\beta(\nabla N), \partial^\beta(\nabla N)\Big\rangle + 2\kappa\sum_{|\beta|\le s-1}\langle\partial^\beta u, \partial^\beta(\nabla N)\rangle\Big)\\
&+ \frac{1}{4}\Big(2\|u\|_s^2 + \frac{C^{-1}\kappa}{\varepsilon^2}\|\nabla N\|_{s-1}^2 + \frac{\kappa}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2\Big) \le 0. \qquad (4.1)
\end{aligned}$$

On the other hand, when W_T is sufficiently small, we have $\frac{1}{2} \le n \le \frac{3}{2}$ and

$$2\kappa\sum_{|\beta|\le s-1}\langle\partial^\beta u, \partial^\beta(\nabla N)\rangle \le C\kappa\Big(\|u\|_s^2 + \frac{1}{\varepsilon^2}\|N\|_s^2\Big).$$

Since ξ is between 1 and n and h is strictly increasing, we also have

$$\begin{aligned}
&\int_{\mathbb{R}^d} n|u|^2\mathrm{d}x + \|\nabla u\|_{s-1}^2 \sim \|u\|_s^2,\\
&\int_{\mathbb{R}^d} h'(\xi)N^2\mathrm{d}x + \sum_{|\beta|\le s-1}\Big\langle\frac{h'(n)}{n}\partial^\beta(\nabla N), \partial^\beta(\nabla N)\Big\rangle \sim \|N\|_s^2,
\end{aligned}$$

where $\sim$ means the uniform equivalence with respect to the time and the parameters. Moreover, $\mathrm{curl}(\nabla\phi) = 0$ implies that

$$\|\Delta\phi\|_{s-1}^2 + \|\nabla\phi\|^2 \sim \|\nabla\phi\|_s^2.$$

Hence, taking κ small enough, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} n|u|^2\mathrm{d}x + \|\nabla u\|_{s-1}^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} h'(\xi)N^2\mathrm{d}x + \frac{\lambda^2}{\varepsilon^2}\|\Delta\phi\|_{s-1}^2 + \frac{\lambda^2}{\varepsilon^2}\|\nabla\phi\|^2\\
&+ \frac{1}{\varepsilon^2}\sum_{|\beta|\le s-1}\Big\langle\frac{h'(n)}{n}\partial^\beta(\nabla N), \partial^\beta(\nabla N)\Big\rangle + 2\kappa\sum_{|\beta|\le s-1}\langle\partial^\beta u, \partial^\beta(\nabla N)\rangle\\
\sim&\frac{1}{\varepsilon^2}\|N\|_s^2 + \|u\|_s^2 + \frac{\lambda^2}{\varepsilon^2}\|\nabla\phi\|_s^2 \overset{def}{=} Y_\nu(t).
\end{aligned}$$

From (2.1), $Y_\nu(0)$ is uniformly bounded with respect to ε and λ. Integrating (4.1) over $[0,t]$ yields

$$Y_\nu(t)+\int_0^t\Big(\frac{1}{\varepsilon^2}\|\nabla N\|_{s-1}^2+\frac{1}{\varepsilon^2\lambda^2}\|N\|_{s-1}^2+\|u\|_s^2\Big)\le CY_\nu(0),\quad \forall\, t\in[0,T].$$

This inequality implies (2.3) and the global existence of solutions. □

4.2. Proof of Theorems 2.2–2.3

We only prove Theorem 2.2, since the proof of Theorem 2.3 is similar. Let $\lambda=1$, from (2.1) and (2.3), it is evident that

$$n^\varepsilon\longrightarrow 1,\quad \text{strongly in }\ C\big(\mathbb{R}^+;H^s(\mathbb{R}^d)\big).$$

Moreover, the sequence $(u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2\big(\mathbb{R}^+;H^s(\mathbb{R}^d)\big)$. Then there is a function $\bar u\in L^2\big(\mathbb{R}^+;H^s(\mathbb{R}^d)\big)$ such that, up to a subsequence,

$$u^\varepsilon\rightharpoonup \bar u,\quad \text{weakly-}*\ \text{ in }\ L^2\big(\mathbb{R}^+;H^s(\mathbb{R}^d)\big).$$

Then, $(n^\varepsilon,u^\varepsilon)$ satisfies (3.2)-(3.4). Applying Lemma 3.2, there is a function $\bar\phi\in L^2\big(\mathbb{R}^+;H^s(\mathbb{R}^d)\big)$ such that $(\bar u,\bar\phi)$ is a global solution of the incompressible Euler equations.

Next, we consider the initial condition of $\bar u$. We define

$$Z_\varepsilon=\big\langle A_0(n)\partial_t W^\varepsilon,\partial_t W^\varepsilon\big\rangle+\frac{1}{\varepsilon^2}\|\nabla(\partial_t\phi^\varepsilon)\|^2,$$

where

$$A_0(n)=\begin{pmatrix}\frac{1}{\varepsilon^2}h'(n) & 0\\ 0 & n\mathbf{I}_d\end{pmatrix}.$$

From the definition of A_0, we have

$$Z_\varepsilon\sim\frac{1}{\varepsilon^2}\|\partial_t n^\varepsilon\|^2+\|\partial_t u^\varepsilon\|^2+\frac{1}{\varepsilon^2}\|\nabla(\partial_t\phi^\varepsilon)\|^2.$$

A similar energy estimate in L^2 of $\partial_t W^\varepsilon$ together with (2.3) gives (see also Lemma 3.4 in [19])

$$Z_\varepsilon'(t)+C_0\|\partial_t u^\varepsilon\|^2\le CZ_\varepsilon(t),\quad \forall\, t>0,$$

which implies that

$$Z_\varepsilon(t)\le Z_\varepsilon(0)e^{ct},\quad \forall\, t>0.$$

If (2.6)–(2.7) hold, from the system (1.1), it is easy to see that $\Big(\frac{1}{\varepsilon}\partial_t n^\varepsilon(0),\partial_t u^\varepsilon(0)\Big)_{\varepsilon>0}$ is bounded in $L^2(\mathbb{R}^d)$. Moreover, from the Poisson equation, we have

$$\Delta(\partial_t\phi^\varepsilon(0))=-\mathrm{div}(n_0^\varepsilon u_0^\varepsilon),$$

or equivalently,

$$\Delta(\partial_t\phi^\varepsilon(0))=-\mathrm{div}((n_0^\varepsilon-1)u_0^\varepsilon)-\mathrm{div}(u_0^\varepsilon-u_0).$$

It follows that
$$\frac{1}{\varepsilon}\|\nabla(\partial_t\phi)\| \le \frac{1}{\varepsilon}\|(n_0^\varepsilon - 1)u_0^\varepsilon\| + \frac{1}{\varepsilon}\|u_0^\varepsilon - u_0\|,$$
which is also a bounded number. We conclude that $\big(Z_\varepsilon(0)\big)_{\varepsilon>0}$ is a bounded sequence. Then $\big(Z_\varepsilon(t)\big)_{\varepsilon>0}$ is bounded for all $t>0$, and the sequence $\big(\partial_t u^\varepsilon\big)_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\mathbb{R}^d))$ for all fixed $T>0$. Consequently, by a classical compactness theorem (see [21]), for all $s_1 \in [0,s)$, the sequence $\big(u^\varepsilon\big)_{\varepsilon>0}$ is relatively compact in $C\big([0,T];H^{s_1}_{loc}(\mathbb{R}^d)\big)$. This allows us to define the initial condition of $\bar u$ in (2.8). Finally, the uniqueness of solutions to (2.5) and (2.8) implies the convergence of the whole sequence $(u^\varepsilon)_{\varepsilon>0}$. □

References

[1] G. Alì, *Global existence of smooth solutions of the N-dimensional Euler-Poisson model*, SIAM J. Appl. Math., 2003, 35(2), 389–422.

[2] G. Alì, L. Chen, A. Jüngel and Y. J. Peng, *The zero-electron-mass limit in the hydrodynamic model for plasmas*, Nonlinear Anal. TMA, 2010, 72(12), 4415–4427.

[3] J.Y. Chemin, *Fluides Parfaits Incompressibles*, Astérisque, 1995, 230.

[4] F. Chen, *Introduction to Plasma Physics and Controlled Fusion*, Plenum Press, New York, 1984, 1.

[5] S. Cordier and E. Grenier, *Quasineutral limit of an Euler-Poisson system arising from plasma physics*, Comm. Part. Diff. Equations, 2000, 25(5–6), 1099–1113.

[6] D. Gérard-Varet, D. Han-Kwan and F. Rousset, *Quasineutral limit of the Euler-Poisson system for ions in a domain with boundaries*, Indiana Univ. Math. J., 2013, 62(2), 359–402.

[7] P. Germain, N. Masmoudi and B. Pausader, *Nonneutral global solutions for the electron Euler-Poisson system in three dimensions*, SIAM J. Math. Anal., 2013, 45(1), 267–278.

[8] Y. Guo, *Smooth irrotational flows in the large to the Euler-Poisson system in* $\mathbb{R}^{3+1}$, Comm. Math. Phys., 1998, 195(2), 249–265.

[9] Y. Guo and B. Pausader, *Global smooth ion dynamics in the Euler-Poisson system*, Comm. Math. Phys., 2011, 303(1), 89–125.

[10] L. Hsiao, P. A. Markowich and S. Wang, *The asymptotic behavior of globally smooth solutions of the multidimensional isentropic hydrodynamic model for semiconductors*, J. Differential Equations, 2003, 192(1), 111–133.

[11] A. D. Ionescu and B. Pausader, *The Euler-Poisson system in 2D: global stability of the constant equilibrium solution*, Int. Math. Res. Not., 2013, 2013(4), 761–826.

[12] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Ration. Mech. Anal., 1975, 58(3), 181–205.

[13] P. D. Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, SIAM Regional Conf. Lecture, Philadelphia, 1973, 11.

[14] C. M. Liu and Y. J. Peng, *Convergence of non-isentropic Euler-Poisson systems for all time*, J. Math. Pures Appl., in press. DOI : 10.1016/j.matpur.2017.07.017.

[15] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New York, 1984.

[16] P. A. Markowich, C. A. Ringhofer and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, New York, 1990.

[17] T. Nishida, *Nonlinear Hyperbolic Equations and Related Topics in Fluids Dynamics*, Publications Mathématiques d'Orsay, 78-02, Université Paris-Sud, Orsay, 1978.

[18] Y. J. Peng, *Stability of non-constant equilibrium solutions for Euler-Maxwell equations*, J. Math. Pures Appl., 2015, 103(1), 39–67.

[19] Y. J. Peng, *Uniformly global smooth solutions and convergence of Euler-Poisson systems with small parameters*, SIAM J. Appl. Math., 2015, 27, 1355–1376.

[20] Y. J. Peng and Y. G. Wang, *Convergence of compressible Euler-Poisson equations to incompressible type Euler equations*, Asymptotic Anal., 2005, 41(2), 141–160.

[21] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl., 1987, 146(1), 65–96.

[22] S. Wang, *Quasineutral limit of Euler-Poisson system with and without viscosity*, Comm. Part. Diff. Equations, 2004, 29(3–4), 419–456.

[23] J. Xu and T. Zhang, *Zero-electron-mass limit of Euler-Poisson equations*, Discrete Contin. Dyn. Syst., 2013, 33(10), 4743–4768.