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ON ANISOTROPIC CAGINALP PHASE-FIELD TYPE MODELS WITH SINGULAR NONLINEAR TERMS

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Abstract Our aim in this paper is to study the well-posedness and the existence of the global attractor of anisotropic Caginalp phase-field type models with singular nonlinear terms. The main difficulty is to prove, in one and two space dimensions, that the order parameter remains in the physically relevant range and this is achieved by deriving proper a priori estimates.

Keywords Caginalp phase-field type models, anisotropy, singular nonlinear terms, well-posedness, global attractor.

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1. Introduction

The Caginalp phase-field system,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = T, \tag{1.1}$$

$$\frac{\partial T}{\partial t} - \Delta T = -\frac{\partial u}{\partial t}, \tag{1.2}$$

was proposed in [6] to model phase transition phenomena, such as melting-solidification phenomena. Here, u is the order parameter, T is the relative temperature (defined as $T = \tilde{T} - T_E$, where $\tilde{T}$ is the absolute temperature and T_E is the equilibrium melting temperature) and f is the derivative of a double-well potential F (a typical choice of potential is $F(s) = \frac{1}{4}(s^2-1)^2$, hence the usual cubic nonlinear term $f(s) = s^3 - s$). Furthermore, here and below, we set all physical parameters equal to one. This system has been much studied; we refer the reader to, e.g., [3–5, 8, 10, 12, 13, 15, 16, 18–20, 25–27, 31].

These equations can be derived as follows. One introduces the (total Ginzburg-Landau) free energy

$$\Psi_{GL} = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u) - uT - \frac{1}{2}T^2)\mathrm{d}x, \tag{1.3}$$

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where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^n$, $n = 1$ or 2, with boundary Γ), and the enthalpy

$$H = u + T. \tag{1.4}$$

As far as the evolution equation for the order parameter is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial t} = -\frac{D\Psi_{GL}}{Du}, \tag{1.5}$$

where $\frac{D}{Du}$ denotes a variational derivative with respect to u, which yields (1.1). Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -\mathrm{divq}, \tag{1.6}$$

where q is the heat flux. Assuming finally the usual Fourier law for heat conduction,

$$q = -\nabla T, \tag{1.7}$$

we obtain (1.2).

Now, one essential drawback of the Fourier law is that it predicts that thermal signals propagate at an infinite speed, which violates causality (the so-called paradox of heat conduction). To overcome this drawback, or at least to account for more realistic features, several alternatives to the Fourier law, based, e.g., on the Maxwell-Cattaneo law or recent laws from thermomechanics, have been proposed and studied, in the context of the Caginalp phase-field system, in [20, 21].

In the late 1960's, several authors proposed a heat conduction theory based on two temperatures (see [33–35]). More precisely, one now considers the conductive temperature T and the thermodynamic temperature θ. In particular, for simple materials, these two temperatures are shown to coincide. However, for non-simple materials, they differ and are related as follows:

$$\theta = T - \Delta T. \tag{1.8}$$

The Caginalp system, based on this two temperatures theory and the usual Fourier law, was studied in [3].

Our aim in this paper is to study a variant of the Caginalp phase-field system based on the type III thermomechanics theory with two temperatures recently proposed in [36].

In that case, the free energy reads, in terms of the (relative) thermodynamic temperature θ,

$$\Psi_{GL} = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2)\mathrm{d}x \tag{1.9}$$

and (1.5) yields, in view of (1.8), the following evolution equation for the order parameter:

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = T - \Delta T. \tag{1.10}$$

Furthermore, the enthalpy now reads

$$H = u + \theta = u + T - \Delta T, \tag{1.11}$$

which yields, owing to (1.6), the energy equation

$$\frac{\partial T}{\partial t} - \Delta \frac{\partial T}{\partial t} + \mathrm{divq} = -\frac{\partial u}{\partial t}. \tag{1.12}$$

Finally, the heat flux is given, in the type III theory with two temperatures, by (see [36])

$$q = -\nabla\alpha - \nabla T, \tag{1.13}$$

where

$$\alpha(t,x) = \int_0^t T(\tau,x)\mathrm{d}\tau + \alpha_0(x) \tag{1.14}$$

is the conductive thermal displacement.

In (1.3) (or (1.9)), the term $|\nabla u|^2$ models short-ranged interactions. It is however interesting to note that such a term is obtained by truncation of higher-order ones (see [30]); it can also be seen as a first-order approximation of a nonlocal term accounting for long-ranged interactions (see [11]).

G. Caginalp and E. Esenturk recently proposed in [7] higher-order phase-field models in order to account for anisotropic interfaces (see also [29] for other approaches which, however, do not provide an explicit way to compute the anisotropy). More precisely, these authors proposed the following modified (total) free energy

$$\Psi_{HOGL} = \int_\Omega \left(\frac{1}{2}\sum_{i=1}^{k}\sum_{|\beta|=i} a_\beta |\mathcal{D}^\beta u|^2 + F(u) - uT - \frac{1}{2}T^2\right)\mathrm{d}x, \tag{1.15}$$

where, for $\beta = (k_1, k_2, k_3) \in (\mathbb{N}\cup\{0\})^3$,

$$|\beta| = k_1 + k_2 + k_3$$

and, for $\beta \neq (0,0,0)$,

$$\mathcal{D}^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}}$$

(we agree that $\mathcal{D}^{(0,0,0)}v = v$). Noting that $T = \frac{\partial\alpha}{\partial t}$, this then yields the following evolution equation for the order parameter u:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{k}(-1)^i \sum_{|\beta|=i} a_\beta \mathcal{D}^{2\beta}u + f(u) = \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}. \tag{1.16}$$

In particular, for $k = 1$ (anisotropic Caginalp phase-field system), we have an equation of the form

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} + f(u) = \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t} \tag{1.17}$$

and, for $k = 2$ (fourth-order anisotropic Caginalp phase-field system), we have an equation of the form

$$\frac{\partial u}{\partial t} + \sum_{i,j=1}^{3} a_{ij}\frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} - \sum_{i=1}^{3} b_i \frac{\partial^2 u}{\partial x_i^2} + f(u) = \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}. \tag{1.18}$$

It is however important to note that, in phase transition, regular nonlinear terms actually are approximations of thermodynamically relevant logarithmic ones of the form $f(s) = -\lambda_1 s + \frac{\lambda_2}{2} \ln \frac{1+s}{1-s}$, $s \in (-1,1)$, $0 < \lambda_2 < \lambda_1$, which follow from a mean-field model (see [12]; in particular, the logarithmic terms correspond to the entropy of mixing).

Our aim in this paper is to consider the second-order anisotropic equation (1.17) and the energy equation

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{1.19}$$

with general singular nonlinear terms (containing the physically relevant logarithmic ones). In particular, we prove, in one and two space dimensions, the existence and uniqueness of classical solutions, as well as the existence of the global attractor of the associated dynamical system.

Here, we do not address the higher-order models. Indeed, when $k \geqslant 2$, we are not able to prove the existence of classical solutions and have to deal with a different notion of a solution, based on a variational inequality (see [9, 22]; see also [18]).

2. Setting of the problem

We consider the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} + f(u) = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \tag{2.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{2.2}$$

$$u = \alpha = 0 \quad \text{on} \quad \Gamma, \tag{2.3}$$

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{2.4}$$

We assume that

$$a_i > 0, \quad i \in \{1, 2, 3\}, \tag{2.5}$$

and we introduce the elliptic operator A defined by

$$\langle Av, w \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \sum_{i=1}^{3} a_i ((\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i})), \tag{2.6}$$

where $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$. Furthermore, $((.,.))$ denotes the usual L^2-scalar product, with associated norm $\|.\|$; more generally, we denote by $\|.\|_X$ the norm on the Banach space X. We can note that

$$(v, w) \in H_0^1(\Omega)^2 \mapsto \sum_{i=1}^{3} a_i ((\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i}))$$

is bilinear, symmetric, continuous and coercive, so that

$$A : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is indeed well defined. It then follows from elliptic regularity results for linear elliptic operators of order 2 (see [1,2]) that A is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

where, for $v \in D(A)$,

$$Av = -\sum_{i=1}^{3} a_i \frac{\partial^2 v}{\partial x_i^2}.$$

We further note that $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$ and, for $v \in D(A^{\frac{1}{2}})$,

$$((A^{\frac{1}{2}}v, A^{\frac{1}{2}}v)) = \sum_{i=1}^{3} a_i \|\frac{\partial v}{\partial x_i}\|^2.$$

We finally note that (see, e.g., [28]) $\|A.\|$ (resp., $\|A^{\frac{1}{2}}.\|$) is equivalent to the usual H^2-norm (resp., H^1-norm) on $D(A)$ (resp., $D(A^{\frac{1}{2}})$).

Remark 2.1. Note that similar properties hold for the operator $-\Delta$, with obvious changes.

Having this, we rewrite (2.1) as

$$\frac{\partial u}{\partial t} + Au + f(u) = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}. \tag{2.7}$$

As far as the nonlinear term f is concerned, we assume that

$$f \in C^1(-1,1), \quad f(0) = 0, \tag{2.8}$$

$$\lim_{s \to \pm 1} f(s) = \pm\infty, \quad \lim_{s \to \pm 1} f'(s) = +\infty, \tag{2.9}$$

$$f' \geqslant -c_0, \quad c_0 \geqslant 0. \tag{2.10}$$

In particular, it follows from (2.9)–(2.10) that

$$-c_1 \leqslant F(s) \leqslant f(s)s + c_2, \quad c_1, c_2 \geqslant 0, \quad s \in (-1,1), \tag{2.11}$$

where $F(s) = \int_0^s f(\tau)\mathrm{d}\tau$. Here, the only difficulty is to prove that $F(s) \leqslant f(s)s + c, c \geqslant 0, s \in (-1,1)$. To do so, it suffices to study the variations of the function $s \mapsto f(s)s - F(s) + \frac{c_0}{2}s^2$, whose derivate has, owing to (2.10), the sign of s.

We further assume that

$$(u_0, \alpha_0, \alpha_1) \in (H_0^1(\Omega) \cap H^3(\Omega))^3, \tag{2.12}$$

with

$$\|u_0\|_{L^\infty(\Omega)} < 1, \tag{2.13}$$

and that the following compatibility conditions hold:

$$\Delta u_0 = \Delta \alpha_0 = \Delta \alpha_1 = 0 \quad \text{on} \quad \Gamma. \tag{2.14}$$

Throughout the paper, the same letters c, c' and c'' denote (generally positive) constants which may vary from line to line. Similary, the same letter Q denotes (positive) monotone increasing (with respect to each argument) and continuous functions which may vary from line to line.

3. A priori estimates

The estimates derived in this section are formal, but they can easily be justified within a Galerkin scheme.

We first assume that u is regular and a priori satisfies

$$\|u\|_{L^\infty((0,T)\times\Omega)} < 1, \tag{3.1}$$

where $T > 0$ is an arbitrary final time.

We multiply (2.7) by $\frac{\partial u}{\partial t}$ and (2.2) by $\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}$, sum the two resulting equalities and integrate over Ω and by parts. This gives

$$\begin{aligned} &\frac{d}{dt}(\|A^{\frac{1}{2}}u\|^2 + 2\int_\Omega F(u)\mathrm{d}x + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\|^2) \\ &+ 2\|\frac{\partial u}{\partial t}\|^2 + 2\|\nabla\frac{\partial\alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial\alpha}{\partial t}\|^2 = 0 \end{aligned} \tag{3.2}$$

(note that $\|\frac{\partial\alpha}{\partial t}\|^2 + 2\|\nabla\frac{\partial\alpha}{\partial t}\|^2 + \|\Delta\frac{\partial\alpha}{\partial t}\|^2 = \|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\|^2$).

We then multiply (2.7) by u and have, owing to (2.11),

$$\frac{d}{dt}\|u\|^2 + c(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)\mathrm{d}x) \leqslant c'(\|\frac{\partial\alpha}{\partial t}\|^2 + \|\Delta\frac{\partial\alpha}{\partial t}\|^2) + c'', \quad c > 0. \tag{3.3}$$

Multipling (2.2) by $-\Delta\alpha$, we then obtain

$$\begin{aligned} &\frac{d}{dt}(\|\Delta\alpha\|^2 - 2((\frac{\partial\alpha}{\partial t}, \Delta\alpha)) + 2((\Delta\frac{\partial\alpha}{\partial t}, \Delta\alpha))) + \|\Delta\alpha\|^2 \\ \leqslant &\|\frac{\partial u}{\partial t}\|^2 + 2\|\nabla\frac{\partial\alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial\alpha}{\partial t}\|^2. \end{aligned} \tag{3.4}$$

Summing finaly (3.2), δ_1 times (3.3) and δ_2 times (3.4), where $\delta_1, \delta_2 > 0$ are chosen small enough, we have a differential inequality of the form

$$\frac{dE_1}{dt} + c(E_1 + \|\frac{\partial u}{\partial t}\|^2) \leqslant c', \quad c > 0, \tag{3.5}$$

where

$$\begin{aligned} E_1 = &\|A^{\frac{1}{2}}u\|^2 + 2\int_\Omega F(u)\mathrm{d}x + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\|^2 \\ &+ \delta_1\|u\|^2 + \delta_2(\|\Delta\alpha\|^2 - 2((\frac{\partial\alpha}{\partial t}, \Delta\alpha)) + 2((\Delta\frac{\partial\alpha}{\partial t}, \Delta\alpha))), \end{aligned}$$

satisfies

$$E_1 \geqslant c(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)\mathrm{d}x + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial\alpha}{\partial t}\|^2_{H^2(\Omega)}) - c', \quad c > 0. \tag{3.6}$$

Next, we multiply (2.7) by Au and find, employing (2.10),

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2 + c\|u\|^2_{H^2(\Omega)} \leqslant c'(\|u\|^2_{H^1(\Omega)} + \|\frac{\partial\alpha}{\partial t}\|^2 + \|\Delta\frac{\partial\alpha}{\partial t}\|^2), \quad c > 0. \tag{3.7}$$

Summing (3.5) and δ_3 times (3.7), where $\delta_3 > 0$ is chosen small enough, we have a differential inequality of the form

$$\frac{dE_2}{dt} + c(E_2 + \|u\|^2_{H^2(\Omega)} + \|\frac{\partial u}{\partial t}\|^2) \leqslant c', \quad c > 0, \tag{3.8}$$

where

$$E_2 = E_1 + \delta_3 \|A^{\frac{1}{2}} u\|^2$$

satisfies

$$E_2 \geqslant c(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)\mathrm{d}x + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}) - c', \quad c > 0. \tag{3.9}$$

We now differentiate (2.7) with respect to time to have, owing to (2.2),

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = \Delta\frac{\partial \alpha}{\partial t} + \Delta\alpha - \frac{\partial u}{\partial t}, \tag{3.10}$$

$$\frac{\partial u}{\partial t} = 0 \quad \text{on} \quad \Gamma, \tag{3.11}$$

$$\frac{\partial u}{\partial t}|_{t=0} = -Au_0 - f(u_0) + \alpha_1 - \Delta\alpha_1. \tag{3.12}$$

In particular, if $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ (=D(A)) and $\alpha_1 \in H^2(\Omega) \cap H^1_0(\Omega)$, then $\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$ and

$$\|\frac{\partial u}{\partial t}(0)\| \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}). \tag{3.13}$$

Indeed, if follows from the continuity of f and the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$ that

$$\|f(u_0)\| \leqslant Q(\|u_0\|_{H^2(\Omega)}). \tag{3.14}$$

Multiplying (3.10) by $\frac{\partial u}{\partial t}$, we obtain, owing to (2.10),

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 + c\|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} \leqslant c'(\|\frac{\partial u}{\partial t}\|^2 + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}), \quad c > 0. \tag{3.15}$$

Summing finally (3.8) and δ_4 times (3.15), where $\delta_4 > 0$ is chosen small enough, we obtain an inequality of the form

$$\frac{dE_3}{dt} + c(E_3 + \|u\|^2_{H^2(\Omega)} + \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)}) \leqslant c', \quad c > 0, \tag{3.16}$$

where

$$E_3 = E_2 + \delta_4\|\frac{\partial u}{\partial t}\|^2$$

satisfies

$$E_3 \geqslant c(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)\mathrm{d}x + \|\frac{\partial u}{\partial t}\|^2 + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}) - c', \quad c > 0, \tag{3.17}$$

hence

$$u \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

and

$$\alpha, \frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega)).$$

Rewriting then (2.7) as an elliptic equation, for $t > 0$ fixed,

$$Au + f(u) = -\frac{\partial u}{\partial t} + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \tag{3.18}$$

we find, multiplying (3.18) by Au and employing (2.10),

$$\|u\|^2_{H^2(\Omega)} \leqslant c(\|u\|^2_{H^1(\Omega)} + \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}),$$

hence

$$u \in L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega)).$$

We rewrite (2.7) as

$$\frac{\partial u}{\partial t} + Au + f(u) = g, \tag{3.19}$$

where

$$g = g(t,x) = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \in L^2((0,T) \times \Omega).$$

We multiply (3.19) by $f(u)$. Integrating over Ω, we have

$$\frac{d}{dt}\int_\Omega F(u)\mathrm{d}x + \|f(u)\|^2 \leqslant c\|A^{\frac{1}{2}}u\|^2 + ((g, f(u))).$$

As a consequence, we find

$$\frac{d}{dt}\int_\Omega F(u)\mathrm{d}x + \frac{1}{2}\|f(u)\|^2 \leqslant c(\|A^{\frac{1}{2}}u\|^2 + \|g\|^2), \tag{3.20}$$

hence

$$f(u) \in L^2((0,T) \times \Omega).$$

We finally multiply (2.2) by $\Delta^2 \frac{\partial \alpha}{\partial t}$ and easily find

$$\frac{d}{dt}(\|\nabla\Delta\alpha\|^2 + \|\nabla\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2) + \|\nabla\Delta\frac{\partial \alpha}{\partial t}\|^2 \leqslant \|\nabla\frac{\partial u}{\partial t}\|^2, \tag{3.21}$$

hence

$$\alpha \in L^\infty(0,T;H_0^1(\Omega) \cap H^3(\Omega)),$$
$$\frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H_0^1(\Omega) \cap H^3(\Omega)) \cap L^2(0,T;H_0^1(\Omega) \cap H^3(\Omega)),$$

and

$$g = \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H^1(\Omega)).$$

4. Separation property

Our aim now is to prove that u a priori satisfies

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant 1-\delta, \quad t \in [0,T]\,, \tag{4.1}$$

where $\delta \in (0,1)$ depends only on the initial data and the final time T.

In one space dimension, we have the

Proposition 4.1. *Let* $\dim \Omega = 1$ *and let the nonlinearity* f *satisfy assumtions* (2.9). *Then any solution* $u(t)$ *of equation* (2.7) *satisfies* (4.1).

Proof. In one space dimension, we have, owing to the embedding $H^1(\Omega) \subset L^\infty(\Omega)$,

$$\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \in L^\infty((0,T)\times\Omega).$$

It is then not difficult to prove the separation property (4.1) for solutions to the parabolic equation

$$\frac{\partial u}{\partial t} + Au + f(u) = g$$

with right-hand side $g \in L^\infty((0,T)\times\Omega)$.

Indeed, let $\delta \in (0,1)$ be such that

$$\|u_0\|_{L^\infty(\Omega)} \leqslant \delta, \quad \|g\|_{L^\infty((0,T)\times(\Omega))} - f(\delta) \leqslant 0 \tag{4.2}$$

(note that $\lim\limits_{s\to 1^-} f(s) = \infty$).

We set $v = u - \delta$ and have

$$\frac{\partial v}{\partial t} + Av + f(u) - f(\delta) = g - f(\delta). \tag{4.3}$$

We multiply (4.3) by $v^+ = \max(v,0)$ and obtain, owing to (2.10) and (4.2),

$$\frac{d}{dt}\|v^+\|^2 \leqslant c\|v^+\|^2, \tag{4.4}$$

which yields, owing to Growall's lemma and noting that $v^+(0) = 0$, that

$$\|v^+(t)\|^2 \leqslant 0, \quad \forall t \in [0,T]\,,$$

hence

$$u(t,x) \leqslant \delta, \quad \forall t \in [0,T]\,, \quad \text{a.e.} \quad x \in \Omega.$$

Proceeding as above and noting that f is odd, we also prove that

$$u(t,x) \geqslant -\delta, \quad \forall t \in [0,T]\,, \quad \text{a.e.} \quad x \in \Omega,$$

hence, finally,

$$\|u\|_{L^\infty((0,T)\times\Omega)} \leqslant \delta(<1). \tag{4.5}$$

$\square$

We now consider the case $\dim \Omega = 2$. In that case, we do not have the embedding $H^1(\Omega) \subset L^\infty(\Omega)$ and, consequently, we are not able to obtain estimate (4.1) for all the nonlinearities satisfying (2.9). Nevertheless, using the embedding of $H^1(\Omega)$ into an appropriate Orlicz space, we obtain this result for a wide class of nonlinearities, which includes the thermodynamically relevant logarithmic nonlinearities.

Theorem 4.1. *We assume that* $\dim \Omega = 2$ *and that the nonlinearity* f *satisfies assumptions* (2.9) *and the following additional condition:*

$$|f'(u)| \leqslant e^{c|f(u)|+c'}, \tag{4.6}$$

with some positive constants c *and* c'. *Then every solution* $u(t)$ *of equation* (2.7) *satisfies estimate* (4.1).

The proof of this result is based on the the following lemma.

Lemma 4.1. *We assume that* $\dim \Omega = 2$ *and that the nonlinearity* f *satisfies assumptions* (2.9). *Then, for every* $L > 0$, *the following estimate holds:*

$$\int_{(0,T)\times\Omega} e^{L|f(u(x,t))|} \mathrm{d}x\mathrm{d}t \leqslant c, \tag{4.7}$$

where $c = c(L)$ *depends only on the initial data and the final time* T.

Proof. We proceed as in [17] (see also [24]).

We rewrite (2.7) in the form

$$\frac{\partial u}{\partial t} + Au + f(u) = g, \tag{4.8}$$

where

$$\|g(t)\|_{H_0^1(\Omega)} \leqslant c, \quad t \in [0,T], \tag{4.9}$$

where c depends only on the initial data and T. We can also assume, without loss of generality, that

$$f'(s) \geqslant 0, \quad s \in (-1,1) \tag{4.10}$$

(i.e., $\lambda_1 = 0$ in f; indeed, $f + 2\lambda_1 I$ satisfes (4.10) and $u \in L^\infty(0,T;H_0^1(\Omega))$).

We fix $L > 0$ and multiply (4.8) by $f(u)e^{L|f(u)|}$ to have

$$\frac{d}{dt}\int_\Omega F_L(u)\mathrm{d}x + \int_\Omega |A^{\frac{1}{2}}u|^2 f'(u)(1+L|f(u)|)e^{L|f(u)|}\mathrm{d}x + \int_\Omega |f(u)|^2 e^{L|f(u)|}\mathrm{d}x$$
$$= \int_\Omega g f(u) e^{L|f(u)|}\mathrm{d}x, \tag{4.11}$$

where

$$F_L(s) = \int_0^s \tau e^{L|\tau|}\mathrm{d}\tau,$$

which yields, by integrating (4.11) between 0 and T,

$$\int_\Omega F_L(u(T))\mathrm{d}x + \int_{(0,T)\times\Omega} |A^{\frac{1}{2}}u|^2 f'(u)(1+L|f(u)|)e^{L|f(u)|}\mathrm{d}x\mathrm{d}t$$
$$+ \int_{(0,T)\times\Omega} |f(u)|^2 e^{L|f(u)|}\mathrm{d}x\mathrm{d}t$$
$$= \int_\Omega F_L(u_0)\mathrm{d}x + \int_{(0,T)\times\Omega} g f(u) e^{L|f(u)|}\mathrm{d}x\mathrm{d}t. \tag{4.12}$$

We thus deduce from (2.13), (4.10) and (4.12) that

$$\int_{(0,T)\times\Omega} |f(u)|^2 e^{L|f(u)|}\mathrm{d}x\mathrm{d}t \leqslant c + \int_{(0,T)\times\Omega} |g||f(u)|e^{L|f(u)|}\mathrm{d}x\mathrm{d}t, \tag{4.13}$$

where c depends on the initial data.

In order to estimate the second term on the right-hand side of (4.13), we use the following Young's inequality (see [32]):

$$ab \leqslant \varphi(a) + \psi(b), \quad a, b \geqslant 0, \tag{4.14}$$

where

$$\varphi(s) = e^s - s - 1, \quad \psi(s) = (1+s)\ln(1+s) - s, \quad s \geqslant 0. \tag{4.15}$$

Taking $a = N|g|$ and $b = N^{-1}|f(u)|e^{L|f(u)|}$, where $N > 0$ is to be fixed later, in (4.14), we obtain

$$|g||f(u)|e^{L|f(u)|} \leqslant e^{N|g|} + (1 + N^{-1}|f(u)|e^{L|f(u)|})\ln(1 + N^{-1}|f(u)|e^{L|f(u)|}).$$

Now, if $|f(u)| \leqslant 1$, then

$$|g||f(u)|e^{L|f(u)|} \leqslant e^{N|g|} + (1 + N^{-1}e^L)\ln(1 + N^{-1}e^L).$$

Furthermore, if $|f(u)| \geqslant 1$, then $|f(u)|e^{L|f(u)|} \geqslant 1$ and

$$\begin{aligned}|g||f(u)|e^{L|f(u)|} &\leqslant e^{N|g|} + (1 + N^{-1}|f(u)|e^{L|f(u)|})\ln((1 + N^{-1})|f(u)|e^{L|f(u)|})\\ &= e^{N|g|} + LN^{-1}|f(u)|^2e^{L|f(u)|} + N^{-1}(1+N^{-1})|f(u)|e^{L|f(u)|}\\ &\quad + N^{-1}|f(u)|\ln(|f(u)|)e^{L|f(u)|} + L|f(u)| + \ln(|f(u)|) + \ln(1+N^{-1})\\ &\leqslant e^{N|g|} + N^{-1}(L + 1 + \ln(1+N^{-1}))|f(u)|^2e^{L|f(u)|}\\ &\quad + (1+L)|f(u)| + \ln(1+N^{-1})\\ &\leqslant e^{N|g|} + N^{-1}(L+1+\ln(1+N^{-1}))|f(u)|^2e^{L|f(u)|} + \frac{1}{4}|f(u)|^2e^{L|f(u)|} + c,\end{aligned}$$

because $(1+L)|f(u)| \leqslant \frac{1}{4}|f(u)|^2 + (1+L)^2 \leqslant \frac{1}{4}|f(u)|^2e^{L|f(u)|} + (1+L)^2$, where c depends on N and L. Choosing finally $N = N(L)$ large enough, we find, in both cases,

$$|g||f(u)|e^{L|f(u)|} \leqslant e^{N|g|} + \frac{1}{2}|f(u)|^2e^{L|f(u)|} + c, \tag{4.16}$$

where c depens only on L. We thus deduce from (4.13) and (4.16) the following inequality:

$$\int_{(0,T)\times\Omega} |f(u)|^2e^{L|f(u)|}\mathrm{d}x\mathrm{d}t \leqslant c + 2\int_{(0,T)\times\Omega} e^{N|g|}\mathrm{d}x\mathrm{d}t, \tag{4.17}$$

where c depends only on the initial data, T and L.

To conclude, we use the following Orlicz's embedding inequality ([32]):

$$\int_\Omega e^{N|v|}\mathrm{d}x \leqslant e^{c(\|v\|^2_{H^1(\Omega)}+1)}, \quad \forall v \in H^1(\Omega), \tag{4.18}$$

where c depends only on Ω and N. It then follows from (4.9), (4.17) and (4.18) that

$$\int_{(0,T)\times\Omega} |f(u)|^2e^{L|f(u)|}\mathrm{d}x\mathrm{d}t \leqslant c, \tag{4.19}$$

where c depends only on the initial data, T and L. Noting finally that

$$\int_{(0,T)\times\Omega} e^{L|f(u)|}\mathrm{d}x \leqslant \int_{|f(u)|\leqslant 1} e^{L|f(u)|}\mathrm{d}x + \int_{|f(u)|\geqslant 1} e^{L|f(u)|}\mathrm{d}x$$

$$\leqslant c + \int_{|f(u)|\geqslant 1} |f(u)|^2 e^{L|f(u)|} \mathrm{d}x$$
$$\leqslant c + \int_{(0,T)\times\Omega} |f(u)|^2 e^{L|f(u)|} \mathrm{d}x,$$

where c depends on T and L, (4.19) yields the desired inequality (4.7).

It is not difficult to show, by comparing growths, that the logarithmic function f satisfies:

$$|f'(s)| \leqslant e^{c|f(s)|+c'}, \quad s \in (-1,1), \quad c, c' \geqslant 0. \tag{4.20}$$

Therefore,

$$\int_{(0,T)\times\Omega} |f'(u)|^p \mathrm{d}x\mathrm{d}t \leqslant \int_{(0,T)\times\Omega} e^{cp|f(u)|+c'p} \mathrm{d}x\mathrm{d}t,$$

whence, owing to (4.7),

$$\|f'(u)\|_{L^p((0,T)\times\Omega)} \leqslant c, \quad \forall p \geqslant 1, \tag{4.21}$$

where c depends only on the initial data and T (and p).

We then rewrite (2.7) in the form

$$\frac{\partial u}{\partial t} + Au = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} - f(u) \tag{4.22}$$

and have, differentiating with respect to time,

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) + A\frac{\partial u}{\partial t} = h, \tag{4.23}$$

where

$$h = \Delta \frac{\partial \alpha}{\partial t} + \Delta\alpha - \frac{\partial u}{\partial t} - f'(u)\frac{\partial u}{\partial t} \tag{4.24}$$

satisfies, owing to (4.21) (for $p = 4$) and the above a priori estimates (which imply that

$$\begin{aligned} &\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \subset L^4(0,T;H^{\frac{1}{2}}(\Omega)) \subset L^4((0,T)\times\Omega))), \\ &\|h\|_{L^2((0,T)\times\Omega)} \leqslant c, \end{aligned} \tag{4.25}$$

where c depends only on the initial data and T.

Multipling (4.23) by $A\frac{\partial u}{\partial t}$, we find, owing to (4.25),

$$\|A^{\frac{1}{2}}\frac{\partial u}{\partial t}(t)\|^2 + \int_{(0,T)\times\Omega} \|A\frac{\partial u}{\partial t}\|^2 \mathrm{d}x\mathrm{d}t \leqslant c, \quad t \in [0,T], \tag{4.26}$$

where c depends only on the initial data and T (recall that $u_0 \in H^3(\Omega)$), hence

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)).$$

We finally rewrite (2.2) in the (functional) form

$$\frac{d^2\alpha}{\mathrm{d}t^2} + B\frac{d^2\alpha}{\mathrm{d}t^2} + B\frac{d\alpha}{\mathrm{d}t} + B\alpha = -\frac{du}{\mathrm{d}t} \quad \text{in} \quad L^2(\Omega), \tag{4.27}$$

where B denotes the minus Laplace operator with Dirichlet boundary conditions.

Taking the scalar product (in $L^2(\Omega)$) of (4.27) by $B^3\frac{d\alpha}{\mathrm{d}t}$, we have

$$\frac{d}{dt}(\|B^2\alpha\|^2+\|B^{\frac{3}{2}}\frac{d\alpha}{\mathrm{d}t}\|^2+\|B^2\frac{d\alpha}{\mathrm{d}t}\|^2)+\|B^2\frac{d\alpha}{\mathrm{d}t}\|^2\leqslant\|B\frac{du}{\mathrm{d}t}\|^2, \tag{4.28}$$

and we deduce from (4.26) and (4.28) that

$$\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\in L^\infty(0,T;H^2(\Omega)).$$

Rewriting again (2.7) in the form

$$\frac{\partial u}{\partial t}+Au+f(u)=g, \tag{4.29}$$

we have, owing to the above estimates,

$$g\in L^\infty((0,T)\times\Omega) \tag{4.30}$$

and the separation property follows as in the one-dimensional case. $\square$

Remark 4.1. In three space dimensions, one can also prove the strict separation property, but under growth assumptions on the singular nonlinear term which are not satisfied by the thermodynamically relevant ones, see, e.g., [17, 22].

5. Existence and uniqueness of solutions

We restrict ourselves to the one- and two-dimensional cases (note however that one can prove an existence result also in three space dimensions, but without the strict separation property, see, e.g., [14]). We have the following result.

Theorem 5.1. *We assume that* (2.12)–(2.14) *hold. Then,* (2.1)–(2.4) *possesses a unique solution* $(u,\alpha,\frac{\partial\alpha}{\partial t})$ *with the above regularity, such that,* $\forall\ T>0$,

$$u,\alpha,\frac{\partial\alpha}{\partial t}\in L^\infty(0,T;H^3(\Omega)\cap H^1_0(\Omega)),$$
$$\frac{\partial u}{\partial t}\in L^\infty(0,T;H^1_0(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))$$

and

$$\|u(t)\|_{L^\infty(\Omega)}\leqslant\delta,\quad t\in[0,T],$$

where

$$\delta=\delta(T,u_0)\in(0,1),\quad T>0.$$

Proof. a)Existence:

The proof of existence is standard, once we have the separation property (4.1), since the problem then reduces to one with a regular nonlinearity.

Indeed, we define the approximated function f_δ by

$$f_\delta(s)=\begin{cases}s+\delta+f(-\delta), & s<-\delta,\\ f(s), & |s|\leqslant\delta,\\ s-\delta+f(\delta), & s>\delta,\end{cases}$$

where δ is the same constant as in (4.1) (note that f_δ is not C^1, but we can make a further regularization, see [10]). We choose δ such that

$$f(\delta) \geqslant \delta \quad (\text{resp.,} \quad f(-\delta) \leqslant -\delta).$$

We then consider the following approximated problem (P_δ) defined by

$$(P_\delta) \begin{cases} \dfrac{\partial u^\delta}{\partial t} + Au^\delta + f(u^\delta) = \dfrac{\partial \alpha^\delta}{\partial t} - \Delta \dfrac{\partial \alpha^\delta}{\partial t}, \\ \dfrac{\partial^2 \alpha^\delta}{\partial t^2} - \Delta \dfrac{\partial^2 \alpha^\delta}{\partial t^2} - \Delta \dfrac{\partial \alpha^\delta}{\partial t} - \Delta \alpha^\delta = -\dfrac{\partial u^\delta}{\partial t}, \\ u^\delta = \alpha^\delta = 0 \quad \text{on} \quad \partial\Omega, \\ u^\delta|_{t=0} = u_0, \quad \alpha^\delta|_{t=0} = \alpha_0, \quad \dfrac{\partial \alpha^\delta}{\partial t}|_{t=0} = \alpha_1. \end{cases}$$

We know that (P_δ) possesses one solution $(u^\delta, \alpha^\delta, \dfrac{\partial \alpha^\delta}{\partial t})$ if the functions f_δ and F_δ satisfie the same properties as those satisfied by f and F (see [17]). We have the

Lemma 5.1. *We set*

$$F_\delta(s) = \int_0^s f_\delta(\tau) \mathrm{d}\tau.$$

The functions f_δ and F_δ possess the following properties:

$$f'_\delta(s) \geqslant -c_0, \quad s \in \mathbb{R} \tag{5.1}$$

and

$$-c_1 \leqslant F_\delta(s) \leqslant f_\delta(s)s + c_1, \quad s \in \mathbb{R}, \tag{5.2}$$

where c_0 and c_1 are the same constants as those in (2.10) *and* (2.11).

Proof. We only detail the case:

$$f_\delta(s) = s - \delta + f(\delta), \quad s > \delta.$$

The other are very similar and are omitted.

It follows from the definition of f_δ that

$$f'_\delta(s) = 1 \geqslant -c_0, \quad s > \delta. \tag{5.3}$$

Moreover, since f satisfie (2.11) and $f(\delta) > \delta$, we obtain

$$\begin{aligned} F_\delta(s) &= \int_0^s f_\delta(\tau) \mathrm{d}\tau \\ &= \int_0^\delta f_\delta(\tau) \mathrm{d}\tau + \int_\delta^s f_\delta(\tau) \mathrm{d}\tau \\ &= \int_0^\delta f(\tau) \mathrm{d}\tau + \int_\delta^s (\tau - \delta + f(\delta)) \mathrm{d}\tau \\ &= F(\delta) + \frac{s^2 - \delta^2}{2} + (f(\delta) - \delta)(s - \delta) \geqslant -c_1. \end{aligned}$$

Then, since $f(\delta) = f_\delta(s) - s + \delta$ for $s > \delta$, (2.11) also leads to

$$\begin{aligned} F_\delta(s) &= F(\delta) + (s-\delta)(\frac{s-\delta}{2} + f(\delta)) \\ &\leqslant f(\delta)\delta + c_1 + (s-\delta)(\frac{s-\delta}{2} + f(\delta)) \\ &\leqslant f(\delta)s + c_1 + \frac{(s-\delta)^2}{2} \\ &\leqslant f_\delta(s)s + c_1 + (\delta - s)(\frac{s-\delta}{2}) \leqslant f_\delta(s)s + c_1. \end{aligned}$$

Hence,

$$-c_1 \leqslant F_\delta(s) \leqslant f_\delta(s)s + c_1. \tag{5.4}$$

Consequently, we know that problem P_δ possesses at least one solution $(u^\delta, \alpha^\delta, \frac{\partial \alpha^\delta}{\partial t})$ such that the estimates derived above are valid here. In particular,

$$\|u^\delta(t)\|_{L^\infty(\Omega)} \leqslant 1 - \delta, \quad \forall t \in [0, T],$$

as a result,

$$f_\delta(u^\delta) = f(u^\delta).$$

Hence, $(u^\delta, \alpha^\delta, \frac{\partial \alpha^\delta}{\partial t})$ is also a solution to the original problem. □

b)Uniqueness:

We actually prove a more general result, namely, the uniqueness of solutions such that $|u(t,x)| < 1$ almost everywhere in $(0,T) \times \Omega$ and which do not necessarily satisfy the separation property (4.5) (when this property is satisfied, the proof of uniqueness is straightforward).

Let $(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t})$ and $(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$ be two solutions to (2.1)–(2.3) with initial data $(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$, respectively. We set

$$(u, \alpha, \frac{\partial \alpha}{\partial t}) = (u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}) - (u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}) - (u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}).$$

Then, (u, α) satisfies

$$\frac{\partial u}{\partial t} + Au + (f(u^{(1)}) - f(u^{(2)})) = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \tag{5.5}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{5.6}$$

$$u = \alpha = 0 \quad \text{on} \quad \partial\Omega, \tag{5.7}$$

$$u|_{t=0} = u_0, \alpha|_{t=0} = \alpha_0, \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{5.8}$$

We multiply (5.5) by u and have, owing to (2.10),

$$\frac{d}{dt}\|u\|^2 + c\|u\|^2_{H^1(\Omega)} \leqslant c'(\|u\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}), \quad c > 0. \tag{5.9}$$

Next, we integrate (5.6) between 0 and t to obtain, taking, for simplicity, $(u_0,\alpha_0,\alpha_1)=(0,0,0)$,

$$\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha-\Delta\int_0^t\alpha(\tau)\mathrm{d}\tau=-u. \tag{5.10}$$

Multiplying (5.10) by $-\Delta\alpha$, we find

$$\frac{d}{dt}(\|\nabla\alpha\|^2+\|\Delta\alpha\|^2+\|\Delta\int_0^t\alpha(\tau)\mathrm{d}\tau\|^2)+c\|\alpha\|_{H^2(\Omega)}\leqslant\|u\|^2,\quad c>0. \tag{5.11}$$

Multiplying then (5.10) by $-\Delta\frac{\partial\alpha}{\partial t}$, we have

$$\frac{d}{dt}(\|\Delta\alpha\|^2+2((\Delta\int_0^t\alpha(\tau)\mathrm{d}\tau,\Delta\alpha)))+c\|\frac{\partial\alpha}{\partial t}\|_{H^2(\Omega)}\leqslant c'(\|u\|^2+\|\alpha\|^2_{H^2(\Omega)}),\quad c>0. \tag{5.12}$$

Summing finally δ_5 times (5.9), (5.11) and δ_6 times (5.12), we obtain a differential inequality of the form, taking $\delta_5,\delta_6>0$ small enough,

$$\frac{dE_4}{dt}\leqslant cE_4, \tag{5.13}$$

where

$$\begin{aligned}E_4=&\delta_6\|u\|^2+\|\nabla\alpha\|^2+\|\Delta\alpha\|^2+\|\Delta\int_0^t\alpha(\tau)\mathrm{d}\tau\|^2\\&+\delta_7(\|\Delta\alpha\|^2+2((\Delta\int_0^t\alpha(\tau)\mathrm{d}\tau,\Delta\alpha)))\end{aligned}$$

satisfies

$$E_4\geqslant c(\|u\|^2+\|\alpha\|^2_{H^2(\Omega)}),\quad c>0. \tag{5.14}$$

Gronwall's lemma, together with (5.14), then yields the uniqueness. We can note that this would not give a continuity result (with respect to the initial data) for $\frac{\partial\alpha}{\partial t}$, but such a continuity would then follow from (5.10). □

Owing to these results, we can define the semigroup

$$S(t):\Phi\to\Phi,\quad S(t)(u_0,\alpha_0,\alpha_1)=(u(t),\alpha(t),\frac{\partial\alpha}{\partial t}(t)),$$

where $(u,\alpha,\frac{\partial\alpha}{\partial t})$ is the unique solution to the problem (2.1) – (2.4) with initial data (u_0,α_0,α_1) and

$$\Phi=\{(u,\alpha,\frac{\partial\alpha}{\partial t})\in(H^3(\Omega)\cap H^1_0(\Omega))^3,\quad\|u\|_{L^\infty}<1\}.$$

6. Existence of global attractor

We saw in Section 4 that the key estimate, in view of the well-posedness, is a separation property of the form

$$\|u\|_{L^\infty((0,T)\times\Omega)}\leqslant\delta(<1). \tag{6.1}$$

However, here, δ depends on u_0; more precisely, one has

$$\delta \geqslant \|u_0\|_{L^\infty},$$

meaning, in particular, that (6.1) cannot be a dissipative estimate, i.e., independent of the initial data for large times and for initial data belonging to some bounded set (thus, (6.1) cannot be used to prove the existence of a bounded absorbing set). Actually, in order to have a dissipative estimate, one has to be more accurate. To do so, we fix $R > 0$ and assume that the initial data satisty

$$\frac{1}{1-\|u_0\|_{L^\infty}} + \|u_0\|_{H^3} + \|\alpha_0\|_{H^3} + \|\alpha_1 - \Delta\alpha_1\|_{H^1} \leqslant R.$$

Then, one can prove that there exists $t_0 = t_0(R)$ such that

$$\|(\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t})(t)\|_{H^1} \leqslant c, \quad \forall t \geqslant t_0,$$

where c is independent of R, hence, for $\dim\Omega = 1$, owing to the continuous embedding $H^1(\Omega) \subset L^\infty(\Omega)$,

$$\|(\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t})(t)\|_{L^\infty} \leqslant c_0, \quad \forall t \geqslant t_0, \tag{6.2}$$

where c_0 is independent of R. Furthemore, one has

$$\|(\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t})(t)\|_{L^\infty} \leqslant c_1, \quad \forall t \geqslant 0, \tag{6.3}$$

where $c_1 = c_1(R)$ (we can assume, without loss of generality, that $c_0 \leqslant c_1$). We now choose δ_0 (independent of R) and $t_1 \geqslant t_0$ such that
⋆ $f(\delta_0) \geqslant c_0 + 1$,
⋆ $\delta_0 \in [\gamma, 1)$, where $\gamma > 0$ is such that $f' \geqslant 0$ on $[\gamma, 1)$,
⋆ $\lambda(= \lambda(R)) = \frac{1-\delta_0}{t_1}$ is such that

$$0 < \lambda \leqslant 1, \quad f(1-\lambda t_0) \geqslant c_1 + 1.$$

We set

$$y_+(t) = \max(\delta_0, 1-\lambda t) = \begin{cases} 1-\lambda t \text{ if } & 0 \leqslant t \leqslant t_1, \\ \delta_0 \qquad \text{if } & t \geqslant t_1. \end{cases}$$

Note in particular that

$$\delta_0 \leqslant y_+(t) < 1, \quad \forall t > 0, \quad y_+(0) = 1.$$

Consider now the equation

$$\frac{\partial u}{\partial t} + Au + f(u) = \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t},$$

$$u = 0, \quad \text{on} \quad \Gamma.$$

We set $\theta = u - y_+$. We then have

$$\frac{\partial\theta}{\partial t} + A\theta + f(u) - f(y_+) = G, \tag{6.4}$$

where
$$G = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} - f(y_+) - \frac{\partial y_+}{\partial t}, \quad t > 0, \quad t \neq t_1.$$
It follows from the above assumptions that
$$G(t) \leqslant \begin{cases} c_1 + 1 - f(1 - \lambda t_0) & \text{if} \quad 0 < t \leqslant t_0, \\ c_0 + 1 - f(\delta_0) & \text{if} \quad t \geqslant t_0, \quad t \neq t_1, \end{cases}$$
hence
$$G(t) \leqslant 0, \quad \forall t > 0, \quad t \neq t_1.$$
Then, proceeding as in Section 4, i.e., multiplying (6.4) by $\theta^+ = \max(\theta, 0)$, we have, noting that $y_+(0) = 1$, so that
$$\theta(0) \leqslant 0$$
and
$$\theta \leqslant 0 \quad \text{on} \quad \Gamma,$$
$$u(t) \leqslant y_+(t), \quad \forall t \geqslant 0.$$
Noting that $y_+(t) = \delta_0$ for $t \geqslant t_1$, where δ_0 is independent of R, and proceeding similarly for a lower bound (note that f is odd), we obtain
$$\|u(t)\|_{L^\infty} \leqslant \delta_0, \quad \forall t \geqslant t_1,$$
hence a dissipative estimate.

Corollary 6.1. *The semigroup $S(t)$ possesses the compact global attractor $\mathcal{A}$ on Φ.*

Remark 6.1. It is now not difficult to prove, in view of the strict separation property of u, that $\mathcal{A}$ has finite dimension (in the sens of the Hausdorff or the fractal dimension, see, e.g., [23, 28]); to do so, we essentially proceed as in the case of regular potentials (see, e.g., [25]).

Remark 6.2. When $\dim \Omega = 2$, the situation is more involved and will be studied elsewhere.

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