

# HIGH-ORDER COPOSITIVE TENSORS AND ITS APPLICATIONS

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**Abstract** With the coming of the big data era, high-order high-dimensional structured tensors received much attentions of researchers' in recent years, and now they are developed into a new research branch in mathematics named multilinear algebra. As a special kind of structured tensor, the copositive tensor receives a special concern due to its wide applications in vacuum stability of a general scalar potential, polynomial optimization, tensor complementarity problem and tensor eigenvalue complementarity problem. In this review, we will give a simple survey on recent advances of high-order copositive tensors and its applications. Some potential research directions in the future are also listed in the paper.

**Keywords** Copositive tensor, tensor complementarity problem, homogeneous polynomial, tensor eigenvalue, hypergraphs.

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## 1. Introduction

A tensor is a multidimensional array and a physical quantity which is independent from co-ordinate system changes. A zero order tensor is a scalar, a first order tensor is a vector and a second-order tensor is a matrix, and tensors of order three or higher are called higher-order tensors. Normally, an  $m$ -order  $n$ -dimensional tensor is an element of the tensor product of  $m$   $n$ -dimensional vector spaces, each of which has its own coordinate system. This notion of tensors is not confused with tensors in physics and engineering (such as stress tensors) [57], which are generally referred to as tensor fields in mathematics [73]. It should be noted that, in the very beginning of the 20th century, Ricci, Levi-Civita, etc., developed tensor analysis as a mathematical discipline. And then, it was Einstein who applied tensor analysis in his study of general relativity in 1916, which made tensor analysis an important tool in theoretical physics, continuum mechanics and many other areas of science and engineering [11, 14, 21, 23–25, 52, 59, 60, 69, 70, 81–87, 90, 118]. Furthermore, tensor theory has a close connection with matrix equation [53, 54], nonlinear analysis [80, 102–117], and partial differential equation theory [29–34, 56, 91–96]. More details about tensors and its applications can be found in books [27, 65].

Recently, high-order high-dimension tensors have attracted much attention of

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researchers', and this makes it to be a useful tool in data analysis [22, 35, 89, 97, 98, 119, 120], Tensor analysis and its computing find applications in such as approximation algorithms, computational biology, computer graphics, computer vision, data analysis, graph theory, pattern recognition, phylogenetics, quantum computing, scientific computing, signal processing, spectroscopy, and wireless communication, among other areas. Unfortunately, Hillar and Lim [36] proved that most tensor problems are NP-hard. For example, determining the feasibility of a system of bilinear equations, deciding whether a third order tensor possesses a given eigenvalue, singular value, or spectral norm; approximating an eigenvalue, eigenvector, singular vector, and determining the rank or best rank-1 approximation of a third order tensor are all NP-hard problems. However, most tensors from practical problems have some special structures, and we call it structured tensors. For this special kind tensors, these problems may be not NP-hard. Hence, many structured tensors such as nonnegative tensors, M-tensors, Hankel tensors, Hilbert tensors, Cauchy tensors, B-tensors, diagonal dominant tensors, copositive tensors, completely positive tensors and so on, are concerned in the literature [15–18, 42, 55, 63, 67, 118].

In the last five years, high-order copositive tensors have received a growing amount of interest in polynomial optimization problems [61, 77], vacuum stability of a general scalar potential [39], tensor complementarity problem (TCP) [1, 12, 75, 76, 88], hypergraph theory [17] and tensor eigenvalue complementarity problems (TECP) [28, 51]. The notion of copositive tensor is a natural extension of copositive matrix. A symmetric tensor is called copositive if it generates a multivariate form taking nonnegative values over the nonnegative orthant [63]. Copositive tensors include nonnegative tensors  $M$ -tensors in the even order symmetric case, diagonally dominant tensors as special cases [15, 19, 38, 42–44, 64, 67, 100]. In this paper, we will give a simple survey on recent advances of high-order copositive tensors and its applications. Furthermore, some potential research directions in the future will be raised.

To end this section, we briefly mention the notations to be used in the paper. Let  $\mathbb{R}^n$  denote the  $n$  dimensional real Euclidean space and the set of all nonnegative (positive) vectors be denoted by  $\mathbb{R}_+^n$  ( $\mathbb{R}_{++}^n$ ). The set of all positive integers is denoted by  $\mathbb{N}$ . For positive integers  $m, n$ , we use  $[n]$  to denote set  $\{1, 2, \dots, n\}$ . Vectors are denoted by bold lowercase letters such as  $\mathbf{x}, \mathbf{y}, \dots$ , matrices are denoted by capital letters such as  $A, B, \dots$ , and tensors are written as calligraphic capitals such as  $\mathcal{A}, \mathcal{T}, \dots$ . The  $i$ -th unit coordinate vector in  $\mathbb{R}^n$  is denoted by  $\mathbf{e}_i$ . All one tensor and all one vector are denoted by  $\mathcal{E}$  and  $\mathbf{e}$  respectively. If the symbol  $|\cdot|$  is used on a tensor  $\mathcal{A} = (a_{i_1 \dots i_m})_{1 \leq i_j \leq n, j = 1, \dots, m}$ , it denotes another tensor  $|\mathcal{A}| = (|a_{i_1 \dots i_m}|)_{1 \leq i_j \leq n, j \in [m]}$ . If  $\mathcal{B} = (b_{i_1 \dots i_m})_{1 \leq i_j \leq n, j \in [m]}$  is another tensor, then  $\mathcal{A} \leq \mathcal{B}$  means  $a_{i_1 \dots i_m} \leq b_{i_1 \dots i_m}$  for all  $i_1, \dots, i_m \in [n]$ .

## 2. Preliminaries

In this section, we recall some symbols and basic facts about tensors and the corresponding homogeneous polynomials. Here, we use the notations given in [62] and [66].

### 2.1. Tensors and basic multiplications

A real  $m$ -th order  $n$ -dimensional tensor  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  is a multi-array of real entries  $a_{i_1 i_2 \dots i_m}$ , where  $i_j \in [n]$  for  $j \in [m]$ . Let  $\mathbb{T}_{m,n}$  denote the set of all  $m$ -order  $n$ -dimensional real tensors. A tensor is said to be nonnegative if its all entries are nonnegative. If the entries  $a_{i_1 i_2 \dots i_m}$  are invariant under any permutation of their indices, then tensor  $\mathcal{A}$  is called symmetric. We use  $\mathbb{S}_{m,n}$  to denote the set of all  $m$ -order  $n$ -dimensional symmetric tensors. Clearly,  $\mathbb{S}_{m,n} \subseteq \mathbb{T}_{m,n}$  is a vector space under the addition and multiplication defined as below: for any  $t \in \mathbb{R}$ ,  $\mathcal{A} = (a_{i_1 \dots i_m})_{1 \leq i_1, \dots, i_m \leq n}$  and  $\mathcal{B} = (b_{i_1 \dots i_m})_{1 \leq i_1, \dots, i_m \leq n}$ ,

$$\mathcal{A} + \mathcal{B} = (a_{i_1 \dots i_m} + b_{i_1 \dots i_m})_{1 \leq i_1, \dots, i_m \leq n} \quad \text{and} \quad t\mathcal{A} = (ta_{i_1 \dots i_m})_{1 \leq i_1, \dots, i_m \leq n}.$$

In this review, we always consider real symmetric tensors [58]. All one tensor  $\mathcal{E}$  (all one vector  $\mathbf{e}$ ) is a tensor (vector) with all entries equal one, and the identity tensor  $\mathcal{I} = (I_{i_1 \dots i_m}) \in \mathbb{S}_{m,n}$  is given by

$$I_{i_1 \dots i_m} = \begin{cases} 1 & i_1 = \dots = i_m, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose  $\mathcal{A}, \mathcal{B} \in \mathbb{S}_{m,n}$ , the inner product of  $\mathcal{A}$  and  $\mathcal{B}$  is defined as

$$\langle \mathcal{A}, \mathcal{B} \rangle := \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} b_{i_1 \dots i_m},$$

and the norm of tensor  $\mathcal{A}$  is given by

$$\|\mathcal{A}\| = \langle \mathcal{A}, \mathcal{A} \rangle^{1/2} = \left( \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m}^2 \right)^{1/2}.$$

Let  $x_i$  denote the  $i$ th component of a given vector  $\mathbf{x} \in \mathbb{R}^n$  and use  $\|\mathbf{x}\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$  to denote the  $p$ -norm of  $\mathbf{x}$ . For  $m$  vectors  $\mathbf{x}, \mathbf{y}, \dots, \mathbf{z} \in \mathbb{R}^n$ , we use  $\mathbf{x} \circ \mathbf{y} \circ \dots \circ \mathbf{z}$  to denote the  $m$ -th order  $n$ -dimensional rank one tensor with

$$(\mathbf{x} \circ \mathbf{y} \circ \dots \circ \mathbf{z})_{i_1 i_2 \dots i_m} = x_{i_1} y_{i_2} \dots z_{i_m}, \quad \forall i_1, \dots, i_m \in [n].$$

Then the inner product of a symmetric tensor and the rank one tensor is defined as

$$\langle \mathcal{A}, \mathbf{x} \circ \mathbf{y} \circ \dots \circ \mathbf{z} \rangle := \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1} y_{i_2} \dots z_{i_m}.$$

Particularly, if  $\mathbf{x} = \mathbf{y} = \dots = \mathbf{z}$ , then  $\mathbf{x}^m = \mathbf{x} \circ \mathbf{x} \circ \dots \circ \mathbf{x}$  is a symmetric rank one tensor.

Based on this, we may denote

$$\mathcal{A}\mathbf{x}^k \mathbf{y}^{m-k} = \langle \mathcal{A}, \underbrace{\mathbf{x} \circ \dots \circ \mathbf{x}}_k \circ \underbrace{\mathbf{y} \circ \dots \circ \mathbf{y}}_{m-k} \rangle \quad \text{and} \quad \mathcal{A}\mathbf{x}^m = \langle \mathcal{A}, \underbrace{\mathbf{x} \circ \dots \circ \mathbf{x}}_m \rangle,$$

for  $m \in \mathbb{N}$  and  $k \in [m]$ . For any  $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{S}_{m,n}$  and  $\mathbf{x} \in \mathbb{R}^n$ , we define  $\mathcal{A}\mathbf{x}^{m-1}$  as a vector in  $\mathbb{R}^n$  with

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2, i_3, \dots, i_m \in [n]} a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_m}, \quad \forall i \in [n].$$

Another useful multiplication between tensors is given by Shao *et al.* [72].

**Definition 2.1** ([72]). Let  $\mathcal{A}$  ( $\mathcal{B}$ ) be an order  $m \geq 2$  (an order  $k \geq 1$ ) dimension  $n$  tensor. The product  $\mathcal{A}\mathcal{B}$  is the following tensor  $\mathcal{C}$  of order  $(m-1)(k-1)+1$  with entries:

$$c_{i\alpha_1\alpha_2\cdots\alpha_{m-1}} = \sum_{i_2, \dots, i_m \in [n_2]} a_{ii_2\cdots i_m} b_{i_2\alpha_1} \cdots b_{i_m\alpha_{m-1}},$$

where  $i \in [n], \alpha_1, \alpha_2, \dots, \alpha_{m-1} \in [n]^{k-1}$ .

When tensor  $\mathcal{B}$  reduces to a 1-order tensor, i.e., vector of  $\mathbb{R}^n$ , then

$$\mathcal{A}\mathbf{x} = \sum_{i_1, i_2, \dots, i_m \in [n]} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

which coincides with the usual notation  $\mathcal{A}\mathbf{x}^m$  discussed above. Moreover, when  $k=2$ , tensor  $\mathcal{B}$  reduces to a matrix  $B = (b_{ij})$ . By Definition 2.1, one has

$$(B^\top \mathcal{A}B)_{i_1 i_2 \dots i_m} = \sum_{j_1, \dots, j_m \in [n]} b_{j_1 i_1} a_{j_1 j_2 \dots j_m} b_{j_2 i_2} \cdots b_{j_m i_m}.$$

## 2.2. Tensor eigenvalues and eigenvectors

With different ways of extension from the matrix case, several types of tensor eigenvalues were defined and their properties and applications were discussed [66]. The definition of tensor eigenvalue was first introduced for higher order symmetric tensors by Qi [62].

Let  $\mathbb{C}$  and  $\mathbb{C}^n$  denote the set of all complex numbers and  $n$  dimensional complex vectors respectively. Suppose  $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{T}_{m,n}$  is a given tensor. Then  $\lambda \in \mathbb{C}$  is called an eigenvalue of  $\mathcal{A}$  if there is a nonzero vector  $\mathbf{x} \in \mathbb{C}^n$  satisfying the following system

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \lambda x_i^{m-1}, \quad \forall i = 1, 2, \dots, n, \quad (2.1)$$

and the vector  $\mathbf{x}$  is called an eigenvector of  $\mathcal{A}$  associated with the eigenvalue  $\lambda$ . For the sake of simplicity, denote  $\mathbf{x}^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1})$ , then (2.1) can be simply expressed as

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}.$$

In [62], Qi gave the definition of H-eigenvalue and H-eigenvector. Eigenvalue  $\lambda$  of  $\mathcal{A}$  is called an H-eigenvalue of  $\mathcal{A}$  if it has real eigenvector  $\mathbf{x}$ . In this case,  $\mathbf{x}$  is called an H-eigenvector associated with  $\lambda$ . Similarly, Qi [62] also gave the definitions of E-eigenvalue and Z-eigenvalue for tensors. If  $\lambda \in \mathbb{C}$  ( $\lambda \in \mathbb{R}$ ) is called an E-eigenvalue (Z-eigenvalue) of  $\mathcal{A}$  if there is a vector  $\mathbf{x} \in \mathbb{C}^n$  ( $\mathbf{x} \in \mathbb{R}^n$ ) such that

$$\begin{cases} \mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}, \\ \mathbf{x}^\top \mathbf{x} = 1. \end{cases}$$

In this case,  $\mathbf{x}$  is called an E-eigenvector (Z-eigenvector) of  $\mathcal{A}$  accordingly.

Furthermore, based on the notions of H-eigenvalue and Z-eigenvalues for tensors, Qi [64] and Song *et al.* [74] gave the definitions of H<sup>+</sup>-eigenvalue, H<sup>++</sup>-eigenvalue, Z<sup>+</sup>-eigenvalue and Z<sup>++</sup>-eigenvalue. An H-eigenvalue  $\lambda$  of  $\mathcal{A}$  is called an H<sup>+</sup>-eigenvalue (H<sup>++</sup>-eigenvalue) of  $\mathcal{A}$ , if the associated H-eigenvector  $\mathbf{x} \in \mathbb{R}_+^n$  ( $\mathbf{x} \in \mathbb{R}_{++}^n$ ); A Z-eigenvalue  $\mu$  of  $\mathcal{A}$  is called a Z<sup>+</sup>-eigenvalue (Z<sup>++</sup>-eigenvalue) of  $\mathcal{A}$ , if the associated Z-eigenvector  $\mathbf{x} \in \mathbb{R}_+^n$  ( $\mathbf{x} \in \mathbb{R}_{++}^n$ ).

### 2.3. Corresponding polynomials and copositive tensors

It is known that an  $m$ -th order  $n$ -dimensional symmetric tensor defines uniquely an  $m$ -th degree homogeneous polynomial  $f_{\mathcal{A}}(\mathbf{x})$  on  $\mathbb{R}^n$ : for all  $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ ,

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \sum_{i_1, i_2, \dots, i_m \in [n]} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}; \tag{2.2}$$

and conversely, any  $m$ -th degree homogeneous polynomial function  $f(\mathbf{x})$  on  $\mathbb{R}^n$  also corresponds uniquely a symmetric tensor. Based on this, we can define the positive semi-definite (positive definite) tensor. Tensor  $\mathcal{A} \in \mathbb{S}_{m,n}$  is called positive semi-definite (positive definite) if and only if

$$f_{\mathcal{A}}(\mathbf{x}) \geq 0 \text{ (} f_{\mathcal{A}}(\mathbf{x}) > 0 \text{)}, \quad \forall \mathbf{x} \in \mathbb{R}^n \text{ (} \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \text{)}.$$

However, except for the trivial case, positive semi-definite tensors always have an even order (details see references [62]). From this point of view, copositive tensors may be seen as generalization of positive semi-definite tensors.

To end this section, we list the definition of copositive tensor defined by Qi in [63].

**Definition 2.2** ([63]). Let  $\mathcal{A} \in \mathbb{S}_{m,n}$  be given. If  $\mathcal{A}\mathbf{x}^m \geq 0$  ( $\mathcal{A}\mathbf{x}^m > 0$ ) for any  $\mathbf{x} \in \mathbb{R}_+^n$  ( $\mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ ), then  $\mathcal{A}$  is called a copositive (strictly copositive) tensor. All copositive tensors in  $\mathbb{S}_{m,n}$  constitute the copositive tensor cone  $\text{COP}_{m,n}$ .

## 3. Basic properties of copositive tensors

By the definitions of copositive tensors, it is easy to know that nonnegative tensors and positive semi-definite tensors are all copositive tensors. Thus, all properties for those two kinds of tensors are valid for copositive tensors. Here, we mainly describe some intrinsic basic properties of copositive tensors such as sufficient or necessary conditions for copositive tensors, spectral properties and so on.

### 3.1. Necessary or sufficient conditions for copositive tensors

Although copositive tensors have many practical applications, it is generally difficult to know whether the given tensor is strictly copositive or not. So, it is meaningful if one can find some equivalent conditions or some checkable numerical methods for copositivity detection of a given symmetric tensor. For this, we have the following conclusions [63, 74].

**Theorem 3.1** ([63]). Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{S}_{m,n}$  be a given symmetric tensor. Then

(1)  $\mathcal{A}$  is copositive if and only if

$$\min \{ \mathcal{A}\mathbf{x}^m \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i \in [n] \} \geq 0.$$

(2)  $\mathcal{A}$  is strictly copositive if and only if

$$\min \{ \mathcal{A}\mathbf{x}^m \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i \in [n] \} > 0.$$

Based on this conclusion, Song and Qi [74] established several sufficient and necessary conditions for copositive tensors or strictly copositive tensors by two new nonnegative vectors  $\mathbf{x}^+$  and  $\mathbf{x}^-$ , where

$$\mathbf{x}^+ = (x_1^+, x_2^+, \dots, x_n^+), \mathbf{x}^- = (x_1^-, x_2^-, \dots, x_n^-), x_i^+ = \max\{x_i, 0\}, x_i^- = \max\{-x_i, 0\}$$

for all  $i \in [n]$  and for any  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ .

**Theorem 3.2** ([74]). *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{S}_{m,n}$  be a given symmetric tensor. Then, the followings hold.*

- (1)  $\mathcal{A}$  is copositive if and only if  $\mathcal{A}\mathbf{x}^m \geq 0$  for all  $\mathbf{x} \in \mathbb{R}_+^n$  with  $\|\mathbf{x}\| = 1$ ;
- (2)  $\mathcal{A}$  is strictly copositive if and only if  $\mathcal{A}\mathbf{x}^m > 0$  for all  $\mathbf{x} \in \mathbb{R}_+^n$  with  $\|\mathbf{x}\| = 1$ ;
- (3)  $\mathcal{A}$  is strictly copositive if and only if  $\mathcal{A}$  is copositive and

$$\mathcal{A}\mathbf{x}^m = 0 \text{ for } \mathbf{x} \in \mathbb{R}_+^n \Rightarrow \mathbf{x} = \mathbf{0};$$

- (4)  $\mathcal{A}$  is strictly copositive if and only if there is a real number  $\gamma \geq 0$  such that

$$\mathcal{A}\mathbf{x}^m + \gamma \|\mathbf{x}^-\|^m > 0, \text{ for all } \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\};$$

- (5) If  $m$  is even, then  $\mathcal{A}$  is strictly copositive if and only if there is a real number  $\gamma \geq 0$  such that

$$\mathcal{A}\mathbf{x}^m + \gamma \|\mathbf{x}^+\|^m > 0, \text{ for all } \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

Next, we recall some necessary or sufficient conditions with the help of principal sub-tensor of tensors introduced by Qi [62].

For  $\mathcal{A} \in \mathbb{S}_{m,n}$ , by the homogeneous polynomial (2.2), if we let some  $x_i$  be zero, then we have a less variable homogeneous polynomial, which defines a lower dimensional tensor. Such a lower dimensional tensor is called a principal sub-tensor of  $\mathcal{A}$  i.e. an  $m$ th-order  $r$ -dimensional principal sub-tensor  $\mathcal{B}$  of an  $m$ th-order  $n$ -dimensional tensor  $\mathcal{A}$  consists of  $r^m$  elements in  $\mathcal{A} = (a_{i_1 \dots i_m})$  such that, for any set  $N$  that composed of  $r$  elements in  $[n]$ ,

$$\mathcal{B} = (a_{i_1 \dots i_m}), \quad i_1, i_2, \dots, i_m \in N.$$

For copositive tensors, Song and Qi [74] presented the following conclusions.

**Theorem 3.3.** *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{S}_{m,n}$  be a given symmetric tensor. Then the following conditions are equivalent each other.*

- (1)  $\mathcal{A}$  is a copositive tensor;
- (2) Every principal sub-tensor of  $\mathcal{A}$  has no negative  $H^{++}$ -eigenvalue;
- (3) Every principal sub-tensor of  $\mathcal{A}$  has no eigenvector  $\mathbf{v} > \mathbf{0}$  with associated  $H$ -eigenvalue  $\lambda < 0$ ;
- (4) Every principal sub-tensor of  $\mathcal{A}$  has no negative  $Z^{++}$ -eigenvalue;
- (5) Every principal sub-tensor of  $\mathcal{A}$  has no eigenvector  $\mathbf{v} > \mathbf{0}$  with associated  $Z$ -eigenvalue  $\lambda < 0$ ;
- (6) For every principal sub-tensor  $\mathcal{B}$  of  $\mathcal{A}$ , the fact that  $\lambda$  is  $H^{++}$  (or  $Z^{++}$ )-eigenvalue of  $\mathcal{B}$  implies  $\lambda \geq 0$ .

For strictly copositive tensors, Song and Qi [74] gave the following results.

**Theorem 3.4** ([74]). Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{S}_{m,n}$  be a given symmetric tensor. Then the following conditions are equivalent each other.

- (1)  $\mathcal{A}$  is a strictly copositive tensor;
- (2) Every principal sub-tensor of  $\mathcal{A}$  has no non-positive  $H^{++}$ -eigenvalue;
- (3) Every principal sub-tensor of  $\mathcal{A}$  has no eigenvector  $\mathbf{v} > \mathbf{0}$  with associated  $H$ -eigenvalue  $\lambda \leq 0$ ;
- (4) Every principal sub-tensor of  $\mathcal{A}$  has no non-positive  $Z^{++}$ -eigenvalue;
- (5) Every principal sub-tensor of  $\mathcal{A}$  has no eigenvector  $\mathbf{v} > \mathbf{0}$  with associated  $Z$ -eigenvalue  $\lambda \leq 0$ ;
- (6) For every principal sub-tensor  $\mathcal{B}$  of  $\mathcal{A}$ , the fact that  $\lambda$  is  $H^{++}$  (or  $Z^{++}$ )-eigenvalue of  $\mathcal{B}$  implies  $\lambda > 0$ .

Recently, Chen, Huang and Qi [16, 17] studied the copositivity detection for symmetric tensors, and several necessary conditions are established given below.

**Theorem 3.5** ([16, 17]). Let  $\mathcal{A} \in \mathbb{S}_{m,n}$  be a copositive tensor. Then the followings hold.

- (1) If there is  $\mathbf{x} \in \mathbb{R}_+^n$  such that  $\mathcal{A}\mathbf{x}^m = 0$ , then  $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$ .
- (2) For any principal sub-tensor  $\mathcal{B}$  of  $\mathcal{A}$  with dimension  $r$ ,  $\mathcal{B}\mathbf{x}^{m-1} \geq \mathbf{0}$  admits a nonzero solution  $\mathbf{x} \in \mathbb{R}_+^n$ .
- (3) For any symmetric tensor  $\mathcal{D}$ , if there exists  $t \in [0, 1]$  such that  $(1-t)\mathcal{A} + t\mathcal{D}$  is copositive, then  $\max\{\mathcal{A}\mathbf{u}^m + \mathcal{A}\mathbf{v}^m, \mathcal{D}\mathbf{u}^m + \mathcal{D}\mathbf{v}^m\} \geq 0$  for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^n$ .

### 3.2. Pareto H(Z)-eigenvalues of copositive tensors

Following the definitions of H-eigenvalue, Z-eigenvalue in [62] and the Pareto eigenvalue for matrix [71], Song and Qi [77] gave the concepts of Pareto H-eigenvalue (Pareto Z-eigenvalue) for symmetric tensors and proved that the minimum Pareto H-eigenvalue (Pareto Z-eigenvalue) is equivalent to the optimal value of a polynomial optimization problem. It is proved that symmetric tensor  $\mathcal{A}$  is strictly copositive if and only if every Pareto H-eigenvalue (Z-eigenvalue) of  $\mathcal{A}$  is positive, and  $\mathcal{A}$  is copositive if and only if every Pareto H-eigenvalue (Z-eigenvalue) of  $\mathcal{A}$  is nonnegative [77]. Note that, it is NP-hard to compute the minimum Pareto H-eigenvalue or Pareto Z-eigenvalue of a general symmetric tensor.

Suppose  $\mathcal{A} \in \mathbb{T}_{m,n}$ , a real number  $\lambda$  is called Pareto H-eigenvalue of  $\mathcal{A}$  if there exists a non-zero vector  $\mathbf{x} \in \mathbb{R}^n$  satisfying the system

$$\begin{cases} \mathcal{A}\mathbf{x}^m = \lambda \mathbf{x}^\top \mathbf{x}^{[m-1]} \\ \mathcal{A}\mathbf{x}^{m-1} - \lambda \mathbf{x}^{[m-1]} \geq \mathbf{0} \\ \mathbf{x} \geq \mathbf{0}, \end{cases}$$

and the non-zero vector  $\mathbf{x}$  is called a Pareto H-eigenvector of  $\mathcal{A}$  associated to  $\lambda$ . Similarly, a real number  $\mu$  is said to be Pareto Z-eigenvalue of the tensor  $\mathcal{A}$  if there is a non-zero vector  $\mathbf{x} \in \mathbb{R}^n$  such that

$$\begin{cases} \mathcal{A}\mathbf{x}^m = \mu (\mathbf{x}^\top \mathbf{x})^{\frac{m}{2}} \\ \mathcal{A}\mathbf{x}^{m-1} - \mu (\mathbf{x}^\top \mathbf{x})^{\frac{m}{2}-1} \mathbf{x} \geq \mathbf{0} \\ \mathbf{x} \geq \mathbf{0}, \end{cases}$$

and the non-zero vector  $\mathbf{x}$  is called a Pareto Z-eigenvector of  $\mathcal{A}$  associated to  $\mu$ . By the notions above, necessary and sufficient condition for copositive tensors and strictly copositive tensors are listed below.

**Theorem 3.6** ([77]). *Let  $\mathcal{A} \in \mathbb{S}_{m,n}$  be a given symmetric tensor. Then the following conclusions hold.*

- (1)  $\mathcal{A}$  always have Pareto H-eigenvalue;
- (2)  $\mathcal{A}$  always have Pareto Z-eigenvalue;
- (3)  $\mathcal{A}$  is copositive if and only if all of its Pareto H-eigenvalues are nonnegative.
- (4)  $\mathcal{A}$  is strictly copositive if and only if all of its Pareto H-eigenvalues are positive.
- (5)  $\mathcal{A}$  is copositive if and only if all of its Pareto Z-eigenvalues are nonnegative.
- (6)  $\mathcal{A}$  is strictly copositive if and only if all of its Pareto Z-eigenvalues are positive.

## 4. Copositive tensors in TCP and TEiCP

The tensor complementarity problem [12, 20, 75, 76], denoted by  $\text{TCP}(\mathbf{q}, \mathcal{A})$  such that

$$\mathbf{x} \geq \mathbf{0}, \quad \mathbf{A}\mathbf{x}^{m-1} + \mathbf{q} \geq \mathbf{0}, \quad \mathbf{x}^T(\mathbf{A}\mathbf{x}^{m-1} + \mathbf{q}) = 0, \quad (4.1)$$

is a special case of nonlinear complementarity problem [13, 101], which is also a generalization of linear complementarity problem.

For this problem, Che, Qi and Wei [12] showed that the tensor complementarity problem with a strictly copositive tensor has a nonempty and compact solution set. Song and Qi [76] proved that a real symmetric tensor is a (strictly) semi-positive if and only if it is (strictly) copositive. Song and Qi [75, 76] obtained several results for the tensor complementarity problem with a (strictly) semi-positive tensor. Huang and Qi [37] formulated an  $n$ -person noncooperative game as a tensor complementarity problem with the involved tensor being nonnegative. Thus, copositive tensors play an important role in the tensor complementarity problem. The existence of the solution to TCP is addressed in the following theorem.

**Theorem 4.1** ([12, 75, 76]). *Let  $\mathcal{A} \in \mathbb{S}_{m,n}$  be a given symmetric tensor. Then the following results hold.*

- (1) If  $\mathcal{A}$  is strictly copositive, then the  $\text{TCP}(\mathbf{q}, \mathcal{A})$  has a nonempty, compact solution set;
- (2)  $\mathcal{A}$  is copositive if and only if the  $\text{TCP}(\mathbf{q}, \mathcal{A})$  has a unique solution for every  $\mathbf{q} > \mathbf{0}$ ;
- (3)  $\mathcal{A}$  is strictly copositive if and only if the  $\text{TCP}(\mathbf{q}, \mathcal{A})$  has a unique solution for every  $\mathbf{q} \geq \mathbf{0}$ ;
- (4)  $\mathcal{A}$  is (strictly) copositive if and only if it is (strictly) semi-positive.

Besides, Ling and Fan *et al.* [28, 51] discussed the solution existence of the tensor generalized eigenvalue complementarity problem, especially in strictly copositive case.

Mathematically, the TEiCP is to find scalar  $\lambda \in \mathbb{R}$  and vector  $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  such that

$$\mathbf{x} \geq \mathbf{0}, \quad \lambda \mathcal{B}\mathbf{x}^{m-1} - \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \quad \mathbf{x}^T(\lambda \mathcal{B}\mathbf{x}^{m-1} - \mathcal{A}\mathbf{x}^{m-1}) = 0, \quad (4.2)$$

where  $\mathcal{A}, \mathcal{B} \in \mathbb{T}_{m,n}$ . To present the related result in [51], we denote

$$\lambda_{\mathcal{A}, \mathcal{B}}^{\max} = \max\{\lambda \mid \exists \mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\} \text{ such that } (\lambda, \mathbf{x}) \text{ is a solution of (1.4)}\} \quad (4.3)$$



and

$$\rho(\mathcal{A}, \mathcal{B}) = \max_{\mathbf{x}} \{ \lambda(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}_+^n, \sum_{i=1}^n x_i = 1 \}, \tag{4.4}$$

where  $\lambda(\mathbf{x}) = \frac{\mathcal{A}\mathbf{x}^m}{\mathcal{B}\mathbf{x}^m}$ ,  $\mathcal{B}\mathbf{x}^m \neq 0$ .

**Theorem 4.2** ([51]). *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m}), \mathcal{B} = (b_{i_1 i_2 \dots i_m}) \in \mathbb{T}_{m,n}$ .*

- (1) *If  $\mathcal{B}$  is strictly copositive, then (4.2) has at least one solution;*
- (2) *If  $\mathcal{A}$  and  $\mathcal{B}$  are symmetric and  $\mathcal{B}$  is strictly copositive. Let  $\bar{\mathbf{x}}$  be a stationary point of (4.4). Then  $(\lambda(\bar{\mathbf{x}}), \bar{\mathbf{x}})$  is a solution of TEiCP;*
- (3) *If  $\mathcal{A}$  and  $\mathcal{B}$  are symmetric tensors and  $\mathcal{B}$  is strictly copositive, then*

$$\lambda_{\mathcal{A}, \mathcal{B}}^{\max} = \rho(\mathcal{A}, \mathcal{B}).$$

More recently, Ling *et al.* [51] established the bounds of the number of eigenvalues of tensor generalized eigenvalue complementarity problem, which further enrich the theory of TEiCP. They also developed an implementable projection algorithm for TEiCP and some preliminary computational results were reported.

In [28], Fan *et al.* gave a solution method for computing all Pareto-eigenvalues in which they formulated TEiCPs (4.2) equivalently as polynomial optimization problems. Then the related polynomial problem can be solved by Lasserre type semi-definite relaxations. It should be noted that one of a algorithm is proposed under assumption that  $\mathcal{B}$  is strictly copositive.

## 5. Copositive tensors in polynomial optimization

A polynomial optimization problem (POP) is an optimization problem that has both polynomial objective and constraints. It can be viewed as a generalization of a quadratically constrained quadratic program to higher order polynomials. There is a well established body of research on polynomial optimization problems based on reformulations of the original problem as a conic program over the cone of completely positive tensors, or the cone of copositive tensors [41, 61, 99]. As a result, novel solution schemes for polynomial optimization problems have been designed by drawing on conic programming tools. To show the copositive tensors in polynomial optimization problems, we first define the completely positive tensor cone, which is the dual cone of copositive tensor cone [55, 68].

A tensor  $\mathcal{A} \in \mathbb{S}_{m,n}$  is called a completely positive tensor if there are finite vectors  $\mathbf{u}_1, \dots, \mathbf{u}_r \in \mathbb{R}_+^n$  such that

$$\mathcal{A} = \mathbf{u}_1^m + \mathbf{u}_2^m + \dots + \mathbf{u}_r^m.$$

Let  $\mathbb{CP}_{m,n}$  denote the cone of all complete positive tensors with order  $m$  dimension  $n$ .

In [61], Peña *et al.* provided a general characterization of polynomial optimization problems that can be formulated as a conic program over the cone of completely positive tensors. By the dual relationship between  $\mathbb{CP}_{m,n}$  and  $\mathbb{COP}_{m,n}$ , we know that any completely positive program stated in [61] has a natural dual conic program over the cone of copositive tensors. Furthermore, as a consequence of this characterization, it follows that recent related results for quadratic problems can

be further strengthened and generalized to higher order polynomial optimization problems.

Let  $\mathbb{R}_d[\mathbf{x}]$  be the set of all polynomials with degree less than  $d$ . Consider the following polynomial optimization problem:

$$\begin{aligned} \inf \quad & q(\mathbf{x}) \\ \text{s.t.} \quad & h_i(\mathbf{x}) = 0, \quad i = 1, 2, \dots, m, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{5.1}$$

where  $q(\mathbf{x}), h_i(\mathbf{x}) \in \mathbb{R}_d[\mathbf{x}]$  are given polynomials. For this problem, Peña *et al.* [61] presented another optimization problem with conic constraints in the completely positive tensor cone:

$$\begin{aligned} \inf \quad & \langle C_d(q(\mathbf{x})), \mathcal{A} \rangle \\ \text{s.t.} \quad & \langle C_d(h_i(\mathbf{x})), \mathcal{A} \rangle = 0, \quad i = 1, 2, \dots, m, \\ & \langle C_d(1), \mathcal{A} \rangle = 1, \\ & \mathcal{A} \in \mathbb{CP}_{d,n+1}, \end{aligned} \tag{5.2}$$

where  $C_d : \mathbb{R}_d[\mathbf{x}] \rightarrow \mathbb{S}_{d,n+1}$  is a map defined as

$$C_d \left( \sum_{|\beta| \leq d} p_\beta \mathbf{x}^\beta \right)_{i_1 i_2 \dots i_d} = \frac{\alpha_1! \cdots \alpha_n!}{|\alpha|!} p_\alpha,$$

and  $\alpha$  is the (unique) exponent such that  $x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_{i_1} \cdots x_{i_d}$  (i.e.,  $\alpha_k$  is the number of times  $k$  appears in the multi-set  $\{i_1, \dots, i_d\}$ ).

Actually, problem (5.2) is a relaxation problem of (5.1).

**Theorem 5.1** ([61]). *Let  $q(\mathbf{x}), h_1(\mathbf{x}), \dots, h_m(\mathbf{x}) \in \mathbb{R}_d[\mathbf{x}]$  in (5.1) be given polynomials. Then the optimal value of (5.2) is a lower bound for the optimal value of (5.1).*

Based on Theorem 5.1, we want to know how to characterize conditions under which the relaxation (5.2) is tight, which means that one of the problems (5.1) and (5.2) attains its optimal value if and only the other one does. To answer this question, the following definition is needed.

Given a nonempty set  $E \subseteq \mathbb{R}^n$ , the horizon cone  $E^\infty$  is defined as

$$E^\infty := \{\mathbf{y} \in \mathbb{R}^n \mid \exists \mathbf{x}^{(k)} \in E, \lambda_k \in \mathbb{R}_+, k \in \mathbb{N} \text{ such that } \lambda_k \rightarrow 0 \text{ and } \lambda_k \mathbf{x}^{(k)} \rightarrow \mathbf{y}\}.$$

More properties about the cone  $E^\infty$  can be found in Proposition 3 of [61]. Moreover, by the notion of the horizon cone, Peña *et al.* [61] introduced several equivalent conditions for problems (5.1) and (5.2).

For function  $h(\mathbf{x}) \in \mathbb{R}_d[\mathbf{x}]$ , let  $\tilde{h}(\mathbf{x})$  denote the homogeneous part of  $h(\mathbf{x})$  of highest total degree, which means that  $\tilde{h}(\mathbf{x})$  is obtained by dropping from  $h$  the terms whose total degree is less than  $\deg(h)$ . Then we have the following.

**Theorem 5.2** ([61]). *Suppose  $q(\mathbf{x}), h_1(\mathbf{x}), \dots, h_m(\mathbf{x}) \in \mathbb{R}_d[x]$  are given polynomials in (5.1). Then problems (5.1) and (5.2) are equivalent if it satisfies that*

- (1)  $\deg(h_i) = d, h_i(\mathbf{x}) \geq 0$  for all  $\mathbf{x} \in E_{i-1}$ ,
  - (2)  $\{\mathbf{x} \in E_{i-1}^\infty \mid \tilde{h}_i(\mathbf{x}) = 0\} \subseteq E_i^\infty$ ,
- where  $E_0 = \mathbb{R}_+^n$  and  $E_i = \{\mathbf{x} \in E_{i-1} \mid h_i(\mathbf{x}) = 0\}$ .

Similarly, we can give another equivalent conditions.

**Theorem 5.3** ([61]). *Suppose  $q(\mathbf{x}), h_1(\mathbf{x}), \dots, h_m(\mathbf{x}) \in \mathbb{R}_d[x]$  are given polynomials in (5.1). Then problems (5.1) and (5.2) are equivalent if it satisfies that*

- (1)  $\deg(h_i) = d, h_i(\mathbf{x}) \geq 0$  for all  $\mathbf{x} \in \mathbb{R}_+^n$ , and
- (2)  $\tilde{q}(\mathbf{x}) \geq 0$  for all  $\mathbf{x} \in \{\mathbf{x} \in \mathbb{R}_+^n \mid \tilde{h}_i(\mathbf{x}) = 0, i = 1, 2, \dots, m\}$ .

It should be noted that the reformulation procedures presented in (5.2) for the equality constrained polynomial optimization problem (5.1) can be applied to inequality constrained polynomial optimization problems by adding slack variables [61]. More details for completely positive tensors, copositive tensors and their applications can be found in related references [40, 41, 55, 68, 99].

## 6. Copositive tensor detection

From previous sections, we have know that copositive tensors play an important role in some problems, and many interesting theoretical equivalent conditions for copositive tensors have been established. Then, a challenging question is posed naturally: how to check the copositivity of a given symmetric tensor efficiently? In other words, can we propose some numerical methods to check the copositivity of a given symmetric tensor? In this section, we will review some existing solution methods for identifying the copositivity of a symmetric tensor.

Very recently, by Theorem 3.2, Chen *et al.* [16, 17] give several new sufficient conditions or necessary conditions with the help of simplicial partition for a standard simplex. Then an efficient numerical methods is proposed to check the copositivity of tensors. As applications of the proposed method, it is proved that an upper bound of the coclique number of a uniform hypergraph can be computed through an equivalent optimization problem. The proposed algorithm is also applied in testing copositivity of some potential fields.

To move on, we first present some notions about simplex and its simplicial partitions. The standard simplex with vertices  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$  is denoted by  $S_0 = \{\mathbf{x} \in \mathbb{R}_+^n \mid \|\mathbf{x}\|_1 = 1\}$ .

Let  $S, S_1, S_2, \dots, S_r$  be finite simplices in  $\mathbb{R}^n$ . The set  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  is called a simplicial partition of  $S$  if it satisfies that

$$S = \bigcup_{i=1}^r S_i \quad \text{and} \quad \text{int}S_i \cap \text{int}S_j = \emptyset \text{ for any } i, j \in [r] \text{ with } i \neq j,$$

where  $\text{int}S_i$  denotes the interior of  $S_i$  for any  $i \in [r]$ . Let  $d(\tilde{S})$  denote the maximum diameter of a simplex in  $\tilde{S}$ , which is given by

$$d(\tilde{S}) = \max_{k \in [r]} \max_{i, j \in [n]} \|\mathbf{u}_i^k - \mathbf{u}_j^k\|_2.$$

Based on the notions above, Chen *et al.* [16] gave several conclusions, which is useful for proposing numerical algorithms to check the copositivity of tensors.

**Theorem 6.1** ([16]). Let  $\mathcal{A} \in \mathbb{S}_{m,n}$  be given. Suppose  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  is a simplicial partition of simplex  $S_0$ ; and the vertices of simplex  $S_k$  are denoted by  $\mathbf{u}_1^k, \mathbf{u}_2^k, \dots, \mathbf{u}_n^k$  for any  $k \in [r]$ . Let  $V_{S_k} = (\mathbf{u}_1^k \ \mathbf{u}_2^k \ \dots \ \mathbf{u}_n^k)$  be the matrix corresponding to simplex  $S_k$  for any  $k \in [r]$ . Then, the followings hold.

- (1) if  $\langle \mathcal{A}, \mathbf{u}_{i_1}^k \circ \mathbf{u}_{i_2}^k \circ \dots \circ \mathbf{u}_{i_m}^k \rangle \geq 0$  for all  $k \in [r], i_j \in [n], j \in [m]$ , then  $\mathcal{A}$  is copositive;
- (2) if  $\langle \mathcal{A}, \mathbf{u}_{i_1}^k \circ \mathbf{u}_{i_2}^k \circ \dots \circ \mathbf{u}_{i_m}^k \rangle > 0$  for all  $k \in [r], i_j \in [n], j \in [m]$ , then  $\mathcal{A}$  is strictly copositive;
- (3) if  $V_{S_k}^T \mathcal{A} V_{S_k}$  is copositive for all  $k \in [r]$ , then  $\mathcal{A}$  is copositive;
- (4) if  $V_{S_k}^T \mathcal{A} V_{S_k}$  is strictly copositive for all  $k \in [r]$ , then  $\mathcal{A}$  is strictly copositive.

On the other hand, to show the simplicial partition is fine enough, Chen *et al.* [16] give a necessary condition for strictly copositive tensor, and an equivalent condition for a symmetric tensor which is not copositive.

**Theorem 6.2** ([16]). Let  $\mathcal{A} \in \mathbb{S}_{m,n}$  be given. Suppose  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  is a simplicial partition of simplex  $S_0$ ; and the vertices of simplex  $S_k$  are denoted by  $\mathbf{u}_1^k, \mathbf{u}_2^k, \dots, \mathbf{u}_n^k$  for any  $k \in [r]$ . Then the following two assertions hold.

- (1) If  $\mathcal{A}$  is strictly copositive, then there exists  $\varepsilon > 0$  such that  $d(\tilde{S}) < \varepsilon$ , it follows that

$$\langle \mathcal{A}, \mathbf{u}_{i_1}^k \circ \mathbf{u}_{i_2}^k \circ \dots \circ \mathbf{u}_{i_m}^k \rangle > 0$$

for all  $k \in [r], i_j \in [n], j \in [m]$ .

- (2)  $\mathcal{A}$  is not copositive if and only if there exists  $\varepsilon > 0$  such that  $d(\tilde{S}) < \varepsilon$ , there are at least one  $k \in [r]$  and one  $i \in [n]$  satisfying  $\mathcal{A}(\mathbf{u}_i^k)^m < 0$ .

Based on Theorems 6.1-6.2, Chen *et al.* [16] develop an algorithm to verify whether a tensor is copositive or not, as stated below.

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**Algorithm 6.1: Test whether a given symmetric tensor is copositive or not**

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**Input:**  $\mathcal{A} \in \mathbb{S}_{m,n}$   
 Set  $\tilde{S} := \{S_0\}$ , where  $S_0 = \text{conv}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  is the standard simplex  
 while  $\tilde{S} \neq \emptyset$  do  
     choose  $S = \text{conv}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} \in \tilde{S}$   
     if there exists  $i \in [n]$  such that  $\mathcal{A}\mathbf{u}_i^m < 0$ , then return “ $\mathcal{A}$  is not copositive”  
     else if  $\langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \dots \circ \mathbf{u}_{i_m} \rangle \geq 0$  for all  $i_1, i_2, \dots, i_m \in [n]$ , then  $\tilde{S} = \tilde{S} \setminus \{S\}$   
     else simplicial partition  $S = S_1 \cup S_2$ ; and set  $\tilde{S} := \tilde{S} \setminus \{S\} \cup \{S_1, S_2\}$   
     end if  
 end while  
 return “ $\mathcal{A}$  is copositive.”

---

**Output:** “ $\mathcal{A}$  is copositive” or “ $\mathcal{A}$  is not copositive”.

---

By this algorithm, the numerical performance given in [16] show that Algorithm 6.1 can capture all strictly copositive tensors and non-copositive tensors. Unfortunately, when the input symmetric tensor is copositive but not strictly copositive, it is possible that the partition procedure of the algorithm leads to  $d(\tilde{S}) \rightarrow 0$ ; and in this case, the algorithm dose not terminate in general. The reason for this is listed below.

**Theorem 6.3** ([16]). Suppose  $\mathcal{A} \in \mathbb{S}_{m,n}$  is copositive. Let  $S = \text{conv}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  be a simplex with  $\mathcal{A}\mathbf{u}_i^m > 0$  for all  $i \in [n]$ . If there exists  $\mathbf{x} \in S \setminus \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$

such that  $\mathcal{A}\mathbf{x}^m = 0$ , then there are  $i_1, i_2, \dots, i_m \in [n]$  such that  $\langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \dots \circ \mathbf{u}_{i_m} \rangle < 0$ .

Similar to Algorithm 6.1, Chen, Huang and Qi [17] make a modification for the method in [16], where the updated algorithm is proposed based on a convex cone  $\mathbb{M}$ . Here we do not say more about the updated algorithm, we pay attention to several practical applications in the following analysis.

### 6.1. An upper bound for the coclique number of an uniform hypergraph

Very recently, Chen *et al.* [17] showed that computing the coclique number of a uniform hypergraph can be reformulated as a linear program over the cone of completely positive tensors. By the fact that copositive tensor cone is the dual cone of completely positive tensor cone, it is presented that an upper bound for the coclique number can be computed by Algorithm 6.1.

We first recall some notions of hypergraph [26, 64], which are generalized from the graph theory [2–10, 48–50, 78, 79]. A hypergraph means an undirected simple  $m$ -uniform hypergraph  $G = (V, E)$  with vertex set  $V = \{1, 2, \dots, n\}$ , and edge set  $E = \{e_1, e_2, \dots, e_k\}$  with  $e_p \subseteq V$  for  $p \in [k]$ . By  $m$ -uniformity, for every edge  $e \in E$ , the cardinality  $|e|$  of  $e$  is equal to  $m$ . A 2-uniform hypergraph is typically called graph. Throughout this review, we mainly focus on  $m \geq 3$  and  $n \geq m$ . Furthermore, since the trivial hypergraph (i.e.,  $E = \emptyset$ ) is of less interest, the hypergraphs considered here has at least one edge (i.e., nontrivial).

**Definition 6.1** (Coclique number of a hypergraph). The coclique of an  $m$ -uniform hypergraph  $G$  is a set of vertices such that any of its  $m$  vertex subset is not an edge of  $G$ , and the largest cardinality of a coclique of  $G$  is called the coclique number of  $G$ , denoted by  $\omega(G)$ .

The following definition for the adjacency tensor is first introduced by Cooper and Dutle [26]

**Definition 6.2** (Adjacency tensor of a hypergraph). Let  $G = (V, E)$  be an  $m$ -uniform hypergraph where  $V = \{1, 2, \dots, n\}$ . The adjacency tensor of  $G$  is defined as the  $m$ -th order  $n$ -dimensional tensor  $\mathcal{A}$  with

$$a_{i_1 i_2 \dots i_m} = \begin{cases} \frac{1}{(m-1)!} & \{i_1, i_2, \dots, i_m\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\mathcal{E}$  be a all one tensor with order  $m$  and dimension  $n$ , and  $\mathcal{X}$  is a completely positive tensor such as

$$\mathcal{X} = \sum_{i=1}^s \mathbf{v}_i^m, \text{ for some } \mathbf{v}_i \in \mathbb{R}^n, \mathbf{v}_i \geq \mathbf{0}, i \in [s] \text{ and } s \in \mathbb{N}.$$

Based on Definitions 6.1-6.2, we have the following conclusions.

**Theorem 6.4** ([17]). *Let  $G = (V, E)$  be an  $m$ -uniform hypergraph. Suppose  $|V| = n$  and  $G$  is nontrivial. Let  $\omega(G)$  denote the coclique number of  $G$ . Then we have the following conclusions.*

(1) The value  $\omega(G)^{m-1}$  is equal to the optimal value of the following problem:

$$\begin{aligned}
 \text{(P)} \quad & \max \langle \mathcal{X}, \mathcal{E} \rangle \\
 \text{s.t.} \quad & \mathcal{X}_{i_1 i_2 \dots i_m} = 0, \quad \{i_1, i_2, \dots, i_m\} \in E, \\
 & \langle \mathcal{X}, \mathcal{I} \rangle = 1, \\
 & \mathcal{X} \in \mathbb{CP}_{m,n}.
 \end{aligned}$$

(2) It holds that

$$\omega(G)^{m-1} \leq \min_{\lambda \in \mathbb{N}} \{ \lambda \mid \lambda(\mathcal{A} + \mathcal{I}) - \mathcal{E} \in \mathbb{COP}_{m,n} \}.$$

By Theorem 6.4 and Definition 6.1, one can try finitely many iterations to get an upper bound for the coclique number of a given uniform hypergraph by Algorithms 6.1. For example, for an  $m$ -uniform hypergraph  $G = (V, E)$  with  $V = [n]$ , if there is  $k \in [n]$  such that

$$k^{m-1}(\mathcal{A} + \mathcal{I}) - \mathcal{E} \in \mathbb{COP}_{m,n}, \quad (k-1)^{m-1}(\mathcal{A} + \mathcal{I}) - \mathcal{E} \notin \mathbb{COP}_{m,n},$$

then we know that the coclique number of  $G$  satisfies  $\omega(G) \leq k$ .

### 6.2. Checking vacuum stability for $\mathbb{Z}_3$ scalar dark matter

Recently, Kannike [39] studied the vacuum stability of a general scalar potential of a few fields. With the help of copositive tensors and its relationship to orbit space variables, Kannike showed that how to find positivity conditions for more complicated potentials. Then, he discussed the vacuum stability conditions of the general potential of two real scalars, without and with the Higgs boson included in the potential [39]. Furthermore, explicit vacuum stability conditions for the two Higgs doublet model were given, and a short overview of positivity conditions for tensors of quadratic couplings were established via tensor eigenvalues.

In [39], one important physical example is given by scalar dark matter stable under  $\mathbb{Z}_3$  discrete group. The most general scalar quartic potential of the standard model(SM) Higgs  $H_1$ , an inert doublet  $H_2$  and a complex singlet  $S$  which is symmetric under a  $\mathbb{Z}_3$  group is

$$\begin{aligned}
 & V(h_1, h_2, S) \\
 & = \lambda_1 |H_1|^4 + \lambda_2 |H_2|^4 + \lambda_3 |H_1|^2 |H_2|^2 + \lambda_4 (H_1^\dagger H_2)(H_2^\dagger H_1) + \lambda_S |S|^4 \\
 & \quad + \lambda_{S1} |S|^2 |H_1|^2 + \lambda_{S2} |S|^2 |H_2|^2 + \frac{1}{2} (\lambda_{S12} S^2 H_1^\dagger H_2 + \lambda_{S12}^* S^{\dagger 2} H_2^\dagger H_1) \\
 & = \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_3 h_1^2 h_2^2 + \lambda_4 \rho^2 h_1^2 h_2^2 + \lambda_S s^4 + \lambda_{S1} s^2 h_1^2 + \lambda_{S2} s^2 h_2^2 \\
 & \quad - |\lambda_{S12}| \rho s^2 h_1 h_2 \\
 & \equiv \lambda_S s^4 + M^2(h_1, h_2) s^2 + V(h_1, h_2),
 \end{aligned} \tag{6.1}$$

where  $M^2(h_1, h_2) := \lambda_{S1} s^2 h_1^2 + \lambda_{S2} s^2 h_2^2 - |\lambda_{S12}| \rho s^2 h_1 h_2$  and  $V(h_1, h_2) := V(h_1, h_2, 0)$ . In physical sense, the variables  $h_1, h_2$  and  $s$  should be nonnegative since they are magnitudes of scalar fields, the coupling tensor  $\mathcal{V}$  of coefficients of (6.1) has to be copositive, and this has to hold for all values of the extra parameter  $\rho \in [0, 1]$ . Hence, the potential has to be minimized or scanned over it.

By the analysis above, in [17], an explicit form for the coupling tensor of (6.1) is given such as  $\mathcal{V} = (V_{i_1 i_2 i_3 i_4})$ , which is a 4-order 3-dimensional real symmetric tensor:

$$V_{1111} = \lambda_1, \quad V_{2222} = \lambda_2, \quad V_{3333} = \lambda_S,$$

$$V_{1122} = \frac{1}{6}(\lambda_3 + \lambda_4 \rho^2), V_{1133} = \frac{1}{6}\lambda_{S1}, V_{2233} = \frac{1}{6}\lambda_{S2}, V_{1233} = -\frac{1}{12}|\lambda_{S12}|$$

and  $V_{i_1 i_2 i_3 i_4} = 0$  for the others.

As to  $\lambda$ 's in the entries of  $\mathcal{V}$ , in particle physics all calculated quantities are expanded in series of  $\lambda_i/(4\pi)$ . Due to the perturbativity requirement of these series, the absolute values of the  $\lambda$  coefficients must be no larger than  $4\pi$ . On the other hand, for the coupling tensor to be copositive, the diagonal entries are nonnegative. Hence, we can take from the beginning that  $0 \leq V_{1111}, V_{2222}, V_{3333} \leq 4\pi$ .

Then, by the fact that the rest of the entries of  $\mathcal{V}$  are a  $\lambda$  parameter times some coefficients, their lower and upper bounds should be accordingly changed. So

$$-2 \times 4\pi/6 \leq V_{1122} \leq 2 \times 4\pi/6$$

with an extra factor 2 because it is the sum of two  $\lambda$ 's, and

$$-4\pi/6 \leq V_{1133} \leq 4\pi/6, \quad -4\pi/6 \leq V_{1133} \leq 4\pi/6,$$

$$-4\pi/6 \leq V_{2233} \leq 4\pi/6, \quad -4\pi/12 \leq V_{1233} \leq 0.$$

When  $\rho \neq 0$ , Kannike [39] obtained that the conditions for the potential (6.1) symmetric under a  $\mathbb{Z}_3$  to be bounded from below are

$$\begin{cases} \lambda_S > 0, \\ V(h_1, h_2) > 0, \\ 0 < h_1^2 < 1, 0 < h_2^2 < 1, 0 < s^2 < 1, \text{ and } 0 < \rho^2 < 1 \implies V_{\min} > 0, \end{cases} \quad (6.2)$$

where

$$\rho = (|\lambda_{S12}|s^2) / (2\lambda_4 h_1 h_2),$$

$$h_1^2 = \frac{1}{2} \{ (2\lambda_2 - \lambda_3)(4\lambda_S \lambda_4 - |\lambda_{S12}|^2) + 2\lambda_4 [(\lambda_3 + \lambda_{S1})\lambda_{S2} - 2\lambda_2 \lambda_{S1} - \lambda_{S2}^2] \} / t,$$

$$h_2^2 = \frac{1}{2} \{ (2\lambda_1 - \lambda_3)(4\lambda_S \lambda_4 - |\lambda_{S12}|^2) + 2\lambda_4 [(\lambda_3 + \lambda_{S2})\lambda_{S1} - 2\lambda_1 \lambda_{S2} - \lambda_{S1}^2] \} / t, \quad (6.3)$$

$$s^2 = \lambda_4 (4\lambda_1 \lambda_2 - \lambda_3^2 - 2\lambda_1 \lambda_{S2} - 2\lambda_2 \lambda_{S1} + \lambda_3(\lambda_{S1} + \lambda_{S2})) / t,$$

$$V_{\min} = \frac{1}{4} [(4\lambda_1 \lambda_2 - \lambda_3^2)(4\lambda_S \lambda_4 - |\lambda_{S12}|^2) - 4\lambda_4 (\lambda_1 \lambda_{S2}^2 + \lambda_2 \lambda_{S1}^2 - \lambda_3 \lambda_{S1} \lambda_{S2})] / t$$

with

$$t := (\lambda_1 + \lambda_2 - \lambda_3) \times (4\lambda_S \lambda_4 - |\lambda_{S12}|^2)$$

$$+ \lambda_4 [4\lambda_1 \lambda_2 - \lambda_3^2 - 4\lambda_1 \lambda_{S2} - 4\lambda_2 \lambda_{S1} + 2\lambda_3(\lambda_{S1} + \lambda_{S2}) - (\lambda_{S1} - \lambda_{S2})^2],$$

where the third formula in (6.2) is replaced by  $V_{\rho=0} > 0$  when  $\rho = 0$ ; and by  $V_{\rho=1} > 0$  when  $\rho = 1$ .

From the analysis above and the algorithms raised in [16, 17], one can easily check the copositivity of the tensor defined by the potential (6.1), and the numerical results of [17] verify that the algorithm is efficient and applicable to such physical problems.

## 7. Conclusions

In this survey, we have provided an overview of high order copositive tensors theory and its applications. We mainly focus on sufficient or necessary conditions for symmetric copositive tensors and their applications in tensor complementarity problem, tensor eigenvalue complementarity problem, hypergraphs and so on.

Although we have mentioned applications ranging from optimization problems to hypergraphs and particle physics, the study about high order copositive tensors is still at the starting stage. There are some more interesting problems need to study in the future. Here, we list some potential problems below:

1. Are there any better and efficient numerical methods to test the copositivity of symmetric tensors? Can we do the problem by some traditional optimization method such as ADMM, penalty function method [45–47] and so on?
2. How to update the current methods to make it available for copositive tensors but not strictly copositive?
3. It would be interesting to derive a copositive formulation for the coclique number of a uniform hypergraph to be able to derive stronger bounds.
4. Another interesting direction of future work is to find other completely positive reformulation results for quadratic constraints quadratic problems that could be potentially generalized to apply for polynomial optimization problems.

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