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A TIME FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION DRIVEN BY THE FRACTIONAL BROWNIAN MOTION*

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Abstract Let B^H be a fractional Brownian motion with Hurst index $H > \frac{1}{2}$. In this paper, we prove the global existence and uniqueness of the equation

$$\begin{cases} {}^C D_t^{\gamma} x(t) = f(x_t) + G(x_t)\frac{d}{dt}B^H(t), & t \in (0,T], \\ x(t) = \eta(t), & t \in [-r,0], \end{cases}$$

where $\max\{H, 2-2H\} < \gamma < 1$, ${}^C D_t^{\gamma}$ is the Caputo derivative, and $x_t \in \mathcal{C}_r = \mathcal{C}([-r,0],\mathbb{R})$ with $x_t(u) = x(t+u), u \in [-r,0]$. We also study the dependence of the solution on the initial condition.

Keywords Fractional Brownian motion, the caputo derivative, stochastic functional differential equation, time delay.

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1. Introduction

Recall that a mean zero Gaussian process $B^H = \{B_t^H, t \geq 0\}$ is called a fractional Brownian motion(fBm) with Hurst index $H \in (0,1)$ if $B_0^H = 0$ and

$$E\left[B_t^H B_s^H\right] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

for all $t, s \geq 0$. When $H = 1/2$, B^H coincides with the standard Brownian motion B. B^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with B^H. As a Gaussian process, one can construct the stochastic calculus of variations with respect to B^H. Some surveys and complete literatures for fractional Brownian motion could be found in Biagini *et al* [3], Hu [9], Mandelbrot and Ness [11], Mishura [12], Nourdin [13], Nualart [15] and the references therein.

On the other hand, differential equation with fractional derivative can be used to describe the hereditary character of various kinds of materials and processes. Compare to the classical differential equation, the fractional order models are better to fit the models of the real life. The advantages of fractional derivatives become apparent in various fields of science and engineering such as control theory, porous

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media, viscoelasticity, image and signal processing etc. As a result, the study of time fractional differential equations attracts a lot of interest by many researchers (see, for examples, [1,2,10,16]). For the stochastic functional equation, in Boufoussi and S Hajji [4], the authors prove a global existence and uniqueness for the solution of a stochastic functional differential equation driven by a fBm with Hurst parameter $H > \frac{1}{2}$. Thereafter, Boufoussi *et al* [5] extend this to functional differential equation in Hilbert space by the properties of semigroup. Although there are a few references (see [7,8]) studying stochastic functional differential equation driven by fBm, such researches are not amply and it is worthwhile to study more. In this paper, we consider the time fractional functional differential equation of the form

$$\begin{cases} {}^C D_t^{\gamma} x(t) = f(x_t) + G(x_t)\frac{d}{dt}B^H(t), & t \in (0,T], \\ x(t) = \eta(t), & t \in [-r,0], \end{cases} \tag{1.1}$$

with $\frac{1}{2} < H < 1$ and $\max\{H, 2-2H\} < \gamma < 1$ where

- ${}^C D_t^{\gamma}$ denotes the Caputo derivative;
- $x_t \in \mathcal{C}_r$ with $x_t(u) = x(t+u)$ for $u \in [-r,0]$, $\mathcal{C}_r = C([-r,0])$ is the space of continuous functions f from $[-r,0]$ to $\mathbb{R}$ endowed by the uniform norm $\|\cdot\|_{\mathcal{C}_r}$;
- $f, G : \mathcal{C}_r \to \mathbb{R}$ are proper functions;
- $\eta : [-r,0] \to \mathbb{R}$ is a smooth function.

It is important to note that (1.1) can be written as the integral form

$$\begin{cases} x(t) = \eta(0) + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(x_s)ds \\ \qquad\qquad + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} G(x_s)dB^H(s), \quad t \in (0,T], \\ x(t) = \eta(t), \quad t \in [-r,0]. \end{cases} \tag{1.2}$$

In this paper, we assume that the integral with respect to B^H is a pathwise Riemann-Stieltjes (R-S) integral in the sense of Zähle [18,19]. Similar to the result given by Nualart and Rascanu [14], we extend it to the time fractional cases. In fact, by Young [17] if we have a stochastic processes $\{u(t), t \geq 0\}$ whose trajectories are λ-Hölder continuous with $\lambda > 1-H$, then the R-S integral $\int_0^T u(s)dB^H(s)$ exists for each trajectory. Then, by fractional calculus, Zähle [18] introduced a generalized Stieltjes integral, which is expressed in terms of fractional derivative operators. This integral coincides with R-S integral $\int_0^T f dg$ where f, g are both Hölder continuous of order λ and ω, respectively, with $\lambda + \omega > 1$.

This paper is organized as follows. In Section 2, we present assumptions on the coefficients and some necessary preliminaries on the fractional integral and derivative are given. The basic knowledge of extended Stieltjes integrals is also introduced. In Section 3, we derive some useful estimates for these indefinite integrals. In Section 4, we obtain the existence, uniqueness and dependence on the initial data for the solution of the deterministic equations as some preliminaries studying the solution of (1.1). In the last section, we apply the results of Section 4 to our time fractional stochastic equations driven by a fBm and give the proofs of our main results.

2. Preliminaries

Throughout this paper we fix a time interval $[0,T]$ and a complete probability space $(\Omega, \mathscr{F}, P)$. We assume that $\frac{1}{2} \leq H < 1$ is arbitrary but fixed and we let $B^H = \{B_t^H, t \geq 0\}$ be a one-dimensional fBm with Hurst index H defined on $(\Omega, \mathscr{F}, P)$. Consider the equivalent equation (1.2) in the later sections and let us consider the following assumptions on the coefficients.

(**H.f**) The function f is Lipschitz continuous and has linear growth; that is, there exist constants C_1 and C_2 such that for all $\xi, \eta \in \mathcal{C}_r$

$$|f(\xi) - f(\eta)| \leq C_1 \|\xi - \eta\|_{\mathcal{C}_r} \text{ and } |f(\xi)| \leq C_2(1 + \|\xi\|_{\mathcal{C}_r}).$$

(**H.G**) The function G is Fréchet differentiable. Moreover, there exist constants C_3 and C_4 such that for all $\xi, \eta \in \mathcal{C}_r$

$$|DG(\xi)|_{\mathfrak{L}(\mathcal{C}_r, \mathbb{R})} \leq C_3 \text{ and } |DG(\xi) - DG(\eta)|_{\mathfrak{L}(\mathcal{C}_r, \mathbb{R})} \leq C_4 \|\xi - \eta\|_{\mathcal{C}_r}.$$

It is important to note that assumption (H.G) implies that the linear growth property, i.e., there exists a constant $C_5 > 0$ such that

$$|G(\xi)| \leq C_5(1 + \|\xi\|_{\mathcal{C}_r})$$

for all $\xi, \eta \in \mathcal{C}_r$. Given real numbers $a < b$ and $\mu \in (0,1]$, we will denote by $\mathcal{C}^\mu([a,b])$ the space of μ-Hölder continuous functions $f : [a,b] \to \mathbb{R}$, endowed with the norm

$$\|f\|_\mu := \|f\| + \sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{(t-s)^\mu},$$

where $\|f\| = \sup_{t \in [a,b]} |f(t)|$.

Now, we recall some definitions and notions of fractional calculus.

Definition 2.1. Let $\alpha > 0$. The fractional integral of order α for a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \quad t > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the classical Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

Definition 2.2. Let $n > 0$ be an integer number and let $n - 1 < \alpha < n$. The Caputo derivative of order α for a function $f \in C^n([0, \infty))$ is defined as

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds = I^{n-\alpha} f^{(n)}(t) \quad t > 0.$$

Based on fractional integrals and derivatives, Zähle [18] has introduced the Riemann-Stieltjes integral. We refer the reader to the papers of Zähle [18, 19] and Nualart and Rascanu [14] for the general theory of this integral. Fix a parameter $0 < \alpha < \frac{1}{2}$, and denote by $W^{\alpha,1}(0, T; \mathbb{R})$ the space of measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\|f\|_{\alpha,1} := \int_0^T \left(\frac{|f(s)|}{s^\alpha} + \int_0^s \frac{|f(s) - f(t)|}{(s-t)^{\alpha+1}} dt \right) ds < \infty.$$

We also denote by $W^{1-\alpha,\infty}(0,T;\mathbb{R})$ the space of measurable functions $g:[0,T]\to\mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty} := \sup_{0<s<t<T} \frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(u)-g(s)|}{(u-s)^{2-\alpha}} du < \infty.$$

Clearly,

$$\mathcal{C}^{1-\alpha+\varepsilon}(0,T;\mathbb{R}) \subset W^{1-\alpha,\infty}(0,T;\mathbb{R}) \subset \mathcal{C}^{1-\alpha}(0,T;\mathbb{R}), \forall \varepsilon > 0.$$

Given two functions $f \in W^{\alpha,1}(0,T;\mathbb{R})$ and $g \in W^{1-\alpha,\infty}(0,T;\mathbb{R})$, the generalized Stieltjes integral $\int_0^T f(s)dg(s)$ is defined by

$$\int_0^T f(s)dg(s) = (-1)^\alpha \int_0^T D_{0+}^\alpha f(t) D_{T-}^{1-\alpha} g_{T-}(t)dt,$$

where

$$D_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{t^\alpha} + \alpha \int_0^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}} ds \right)$$

and

$$D_{T-}^{1-\alpha} g_{T-}(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{g(t)-g(T)}{(T-t)^\alpha} + \alpha \int_t^T \frac{g(t)-g(s)}{(s-t)^{\alpha+1}} ds \right).$$

Furthermore, we have the estimate

$$\left\| \int_0^t f dg \right\| \le \Lambda_\alpha(g) \|f\|_{\alpha,1}, \tag{2.1}$$

where

$$\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |D_{t-}^{1-\alpha} g_{t-}(s)| \le \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty}.$$

3. Some priori estimates

Throughout this paper we assume that $\frac{1}{2} < \nu < 1$ is arbitrary but fixed and we let $\max\{\nu, 2-2\nu\} < \gamma < 1$, $\alpha \in (2-\nu-\gamma, \nu)$. For $g \in \mathcal{C}^\nu([0,T])$ and $x \in \mathcal{C}^{1-\alpha}([-r,T])$, we denote

$$I(x)(t) = \int_0^t (t-s)^{\gamma-1} f(x_s) ds$$

and

$$J(x)(t) = \int_0^t (t-s)^{\gamma-1} G(x_s) dg(s).$$

The following proposition provides the basic estimate for iterative calculus in Banach fixed point theorem applied to the time fractional differential equation considered in this paper. We will use M to denote a generic constant which may change from line to line.

Proposition 3.1. *Let $g \in \mathcal{C}^\nu([0,T])$, $x \in \mathcal{C}^{1-\alpha}([-r,T])$. Under conditions (H.f) and (H.G) we have*

$$I(x),\ J(x) \in \mathcal{C}^{1-\alpha}([0,T]).$$

Proof. Let $\alpha_0=\min\{1-\alpha,\alpha+\gamma-1\}$ and fix $\beta\in(1-\nu,\alpha_0)$. It follows that

$$\begin{aligned}
&|I(x)(t)-I(x)(s)| = \left|\int_0^t (t-u)^{\gamma-1}f(x_u)du-\int_0^s (s-u)^{\gamma-1}f(x_u)du\right| \\
&\le \int_0^s \left|(t-u)^{\gamma-1}f(x_u)-(s-u)^{\gamma-1}f(x_u)\right|du+\int_s^t \left|(t-u)^{\gamma-1}f(x_u)\right|du \\
&\le M\int_0^s (1+\|x_u\|_{\mathcal{C}_r})\left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]du \\
&\quad + M\int_s^t (1+\|x_u\|_{\mathcal{C}_r})(t-u)^{\gamma-1}du \\
&\le M(t-s)^{\gamma}(1+\|x\|_{1-\alpha}) \le M(t-s)^{1-\alpha}(1+\|x\|_{1-\alpha}),
\end{aligned}$$

for $s,t\in[0,T]$ with $s<t$, which implies that $I(x)\in\mathcal{C}^{1-\alpha}([0,T])$. For the second point, we have

$$\begin{aligned}
&|J(x)(t)-J(x)(s)| \\
&\le \left|\int_s^t (t-u)^{\gamma-1}G(x_u)dg(u)\right|+\left|\int_0^s \left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]G(x_u)dg(u)\right| \\
&\equiv J_{11}+J_{12}
\end{aligned}$$

for $s,t\in[0,T]$ with $s<t$. From (2.1) we can write

$$\begin{aligned}
J_{11} \le &\Lambda_\beta(g)\left\{\int_s^t \frac{\left|(t-u)^{\gamma-1}G(x_u)\right|}{(u-s)^\beta}du\right. \\
&\left.+\int_s^t\int_s^u \frac{\left|(t-u)^{\gamma-1}G(x_u)-(t-v)^{\gamma-1}G(x_v)\right|}{(u-v)^{\beta+1}}dvdu\right\}.
\end{aligned} \tag{3.1}$$

By the condition (H.G) and $\gamma-\beta>1-\alpha$ we get

$$\begin{aligned}
&\int_s^t \frac{\left|(t-u)^{\gamma-1}G(x_u)\right|}{(u-s)^\beta}du \le \int_s^t \frac{M(t-u)^{\gamma-1}(1+\|x_u\|_{\mathcal{C}_r})}{(u-s)^\beta}du \\
&= M(t-s)^{\gamma-\beta}(1+\|x\|_{1-\alpha}) \le M(t-s)^{1-\alpha}(1+\|x\|_{1-\alpha}).
\end{aligned} \tag{3.2}$$

for $s,t\in[0,T]$ with $s<t$. Using again the condition (H.G) we have

$$\begin{aligned}
&\int_s^t\int_s^u \frac{\left|(t-u)^{\gamma-1}G(x_u)-(t-v)^{\gamma-1}G(x_v)\right|}{(u-v)^{\beta+1}}dvdu \\
&\le \int_s^t\int_s^u \frac{\left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]G(x_v)\right|+\left|(t-u)^{\gamma-1}[G(x_u)-G(x_v)]\right|}{(u-v)^{\beta+1}}dvdu \\
&\le \int_s^t\int_s^u \frac{M\left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]\right|(1+\|x\|_{1-\alpha})}{(u-v)^{\beta+1}}dvdu \\
&\quad +\int_s^t\int_s^u \frac{M(t-u)^{\gamma-1}(u-v)^{1-\alpha}\|x\|_{1-\alpha}}{(u-v)^{\beta+1}}dvdu \\
&\le M(t-s)^{\gamma-\beta}(1+\|x\|_{1-\alpha})+M(t-s)^{\gamma+1-\alpha-\beta}\|x\|_{1-\alpha} \\
&\le M(t-s)^{1-\alpha}(1+\|x\|_{1-\alpha})
\end{aligned} \tag{3.3}$$

by the next fact

$$\begin{aligned}
&\int_s^t\int_s^u \frac{\left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]\right|}{(u-v)^{\beta+1}}dvdu\\
&=\int_s^t\int_0^{u-s} \frac{(t-u)^{\gamma-1}-(t-u+w)^{\gamma-1}}{w^{\beta+1}}dwdu\\
&=\int_0^{t-s}\int_0^{t-v-s} \frac{v^{\gamma-1}-(v+w)^{\gamma-1}}{w^{\beta+1}}dwdu\\
&=\int_0^{t-s}\int_0^{t-s-w} \frac{v^{\gamma-1}-(v+w)^{\gamma-1}}{w^{\beta+1}}dvdw\\
&=\int_0^{t-s}\frac{1}{w^{\beta+1}}\left(\frac{(t-s-w)^{\gamma}}{\gamma}-\frac{(t-s)^{\gamma}}{\gamma}+\frac{w^{\gamma}}{\gamma}\right)dw\\
&\le\int_0^{t-s}\frac{1}{w^{\beta+1}}\frac{w^{\gamma}}{\gamma}dw=\frac{1}{\gamma}(t-s)^{\gamma-\beta}
\end{aligned}\tag{3.4}$$

for $s,t\in[0,T]$ with $s<t$. Combining this with (3.2) and (3.3), we see that

$$J_{11}\le M(t-s)^{1-\alpha}\left(1+\|x\|_{1-\alpha}\right)$$

for $s,t\in[0,T]$ with $s<t$. Now, we estimate J_{12}. Form (2.1), we have

$$\begin{aligned}
&J_{12}\le\Lambda_\beta(g)\left\{\int_0^s\frac{\left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right]G(x_u)\right|}{u^{\beta}}du\right.\\
&+\int_0^s\int_0^u\frac{\left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right][G(x_u)-G(x_v)]\right|}{(u-v)^{\beta+1}}dvdu\\
&\left.+\int_0^s\int_0^u\frac{\left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}-(s-u)^{\gamma-1}+(s-v)^{\gamma-1}\right]G(x_v)\right|}{(u-v)^{\beta+1}}dvdu\right\}.
\end{aligned}\tag{3.5}$$

For the first term, by the condition (H.G)

$$\begin{aligned}
&\int_0^s\frac{\left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right]G(x_u)\right|}{u^{\beta}}du\\
&\le M\left(1+\|x\|_{1-\alpha}\right)\int_0^s\left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]u^{-\beta}du\\
&=M\left(1+\|x\|_{1-\alpha}\right)\left\{\int_0^s(s-u)^{\gamma-1}u^{-\beta}du-\int_0^t(t-u)^{\gamma-1}u^{-\beta}du+\int_s^t(t-u)^{\gamma-1}u^{-\beta}du\right\}\\
&=M\left(1+\|x\|_{1-\alpha}\right)\left\{\left(s^{\gamma-\beta}-t^{\gamma-\beta}\right)B(1-\beta,\gamma)+\int_s^t(t-u)^{\gamma-1}u^{-\beta}du\right\}\\
&\le M\left(1+\|x\|_{1-\alpha}\right)\int_s^t(t-u)^{\gamma-1}u^{-\beta}du\le M\left(1+\|x\|_{1-\alpha}\right)(t-s)^{\gamma-\beta}.
\end{aligned}\tag{3.6}$$

For the second term,

$$\begin{aligned}
&\int_0^s\int_0^u\frac{\left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right][G(x_u)-G(x_v)]\right|}{(u-v)^{\beta+1}}dvdu\\
&\le\int_0^s\int_0^u\frac{\left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]\|x\|_{1-\alpha}}{(u-v)^{\beta+\alpha}}dvdu\\
&\le M(t-s)^{\gamma}\|x\|_{1-\alpha}.
\end{aligned}\tag{3.7}$$

For the last term, some elementary calculations may show that

$$
\begin{aligned}
&\int_0^s\int_0^u \frac{\left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}-(s-u)^{\gamma-1}+(s-v)^{\gamma-1}\right]\right|}{(u-v)^{\beta+1}}dvdu\\
=&\int_0^s\int_0^u \frac{(s-u)^{\gamma-1}-(s-v)^{\gamma-1}}{(u-v)^{\beta+1}}dvdu-\int_0^t\int_0^u \frac{(t-u)^{\gamma-1}-(t-v)^{\gamma-1}}{(u-v)^{\beta+1}}dvdu\\
&+\int_s^t\int_0^u \frac{(t-u)^{\gamma-1}-(t-v)^{\gamma-1}}{(u-v)^{\beta+1}}dvdu\\
\le&\int_s^t\int_0^u \frac{(t-u)^{\gamma-1}-(t-v)^{\gamma-1}}{(u-v)^{\beta+1}}dvdu \qquad (3.8)\\
=&\int_s^t\int_0^s \frac{(t-u)^{\gamma-1}-(t-v)^{\gamma-1}}{(u-v)^{\beta+1}}dvdu+\int_s^t\int_s^u \frac{(t-u)^{\gamma-1}-(t-v)^{\gamma-1}}{(u-v)^{\beta+1}}dvdu\\
\le&\int_s^t\int_0^s \frac{(t-u)^{\gamma-1}}{(u-v)^{\beta+1}}dvdu+M(t-s)^{\gamma-\beta}\\
=&\int_s^t (t-u)^{\gamma-1}\left[(u-s)^{-\beta}-u^{-\beta}\right]du+M(t-s)^{\gamma-\beta}\\
\le& M(t-s)^{\gamma-\beta}
\end{aligned}
$$

for $s,t\in[0,T]$ with $s<t$. It follows from the condition (H.G) that

$$
\begin{aligned}
&\int_0^s\int_0^u \frac{\left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}-(s-u)^{\gamma-1}+(s-v)^{\gamma-1}\right]G(x_v)\right|}{(u-v)^{\beta+1}}dvdu\\
\le& M\left(1+\|x\|_{1-\alpha}\right)(t-s)^{\gamma-\beta}
\end{aligned} \tag{3.9}
$$

for $s,t\in[0,T]$ with $s<t$. Combining this with (3.6) and (3.7), we get that

$$J_{12}\le M(t-s)^{1-\alpha}\left(1+\|x\|_{1-\alpha}\right)$$

for $s,t\in[0,T]$ with $s<t$. Thus, we have showed that

$$|J(x)(t)-J(x)(s)|=J_{11}+J_{12}\le M(t-s)^{1-\alpha}\left(1+\|x\|_{1-\alpha}\right)$$

for $s,t\in[0,T]$ with $s<t$, and the proposition follows. □

For any $\lambda\ge 0$, we introduce the following equivalent norm in the space $\mathcal{C}^{1-\alpha}([a,b])$ defined by

$$\|x\|_{1-\alpha,\lambda}=\sup_{t\in[a,b]}e^{-\lambda t}|x(t)|+\sup_{a\le s<t\le b}e^{-\lambda t}\frac{|x(t)-x(s)|}{(t-s)^{1-\alpha}}.$$

Proposition 3.2. *Let $g\in\mathcal{C}^{\nu}([0,T])$ and $x\in\mathcal{C}^{1-\alpha}([-r,T])$. Under the conditions (H.f) and (H.G), there exist some constants $M_1,M_2,c_1(\lambda),c_2(\lambda)>0$ such that*

$$\|I(x)\|_{1-\alpha,\lambda}\le M_1+c_1(\lambda)\|x\|_{1-\alpha,\lambda}$$

and

$$\|J(x)\|_{1-\alpha,\lambda}\le M_2+c_2(\lambda)\|x\|_{1-\alpha,\lambda},$$

where constants M_1 and M_2 are independent of λ, and $c_1(\lambda),c_2(\lambda)\to 0$ as $\lambda\to\infty$.

Proof. We first prove the first assertion. For $t, s \in [0, T]$ with $s < t$, we have

$$e^{-\lambda t}\frac{|I(x)(t)-I(x)(s)|}{(t-s)^{1-\alpha}} \le e^{-\lambda t}\frac{\left|\int_0^s \left[(t-u)^{\gamma-1}f(x_u)-(s-u)^{\gamma-1}f(x_u)\right]du\right|}{(t-s)^{1-\alpha}} + e^{-\lambda t}\frac{\left|\int_s^t (t-u)^{\gamma-1}f(x_u)du\right|}{(t-s)^{1-\alpha}}. \tag{3.10}$$

By the condition (H.f), we get

$$\begin{aligned} e^{-\lambda t}\frac{\left|\int_s^t (t-u)^{\gamma-1}f(x_u)du\right|}{(t-s)^{1-\alpha}} &\le e^{-\lambda t}\frac{\left|\int_s^t (t-u)^{\gamma-1}\left(1+\|x_u\|_{\mathcal{C}_r}\right)du\right|}{(t-s)^{1-\alpha}} \\ &\le MT^{\gamma+\alpha-1}+\frac{M\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}}\int_0^\infty \frac{e^{-z}}{z^{2-\alpha-\gamma}}dz \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & e^{-\lambda t}\frac{\left|\int_0^s \left[(t-u)^{\gamma-1}f(x_u)-(s-u)^{\gamma-1}f(x_u)\right]du\right|}{(t-s)^{1-\alpha}} \\ &\le \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\int_0^s \left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]\left(1+\|x_u\|_{\mathcal{C}_r}\right)du \\ &\le MT^{\alpha+\gamma-1}+\frac{\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}}\int_0^s \left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]e^{-\lambda(t-u)}du \\ &\le MT^{\alpha+\gamma-1}+\frac{\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}}\left\{\int_0^s (s-u)^{\gamma-1}e^{-\lambda(s-u)}du\right. \\ &\qquad\qquad \left.-\left(\int_0^t-\int_s^t\right)(t-u)^{\gamma-1}e^{-\lambda(t-u)}du\right\} \\ &\le MT^{\alpha+\gamma-1}+\frac{\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}}\int_s^t (t-u)^{\gamma-1}e^{-\lambda(t-u)}du \\ &\le MT^{\alpha+\gamma-1}+\frac{M\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}}\int_0^\infty \frac{e^{-z}}{z^{2-\alpha-\gamma}}dz. \end{aligned} \tag{3.12}$$

Substituting (3.11) and (3.12) in (3.10) to lead the first result since $\alpha+\gamma>1$.

For the second assertion, let $\beta \in (1-\nu, \alpha_0)$ as in the proof of Proposition 3.1. Then, by (2.1) we have that

$$\begin{aligned} & e^{-\lambda t}\frac{|J(x)(t)-J(x)(s)|}{(t-s)^{1-\alpha}} \\ &\le e^{-\lambda t}\frac{\left|\int_0^s \left[(t-u)^{\gamma-1}G(x_u)-(s-u)^{\gamma-1}G(x_u)\right]dg\right|}{(t-s)^{1-\alpha}} + e^{-\lambda t}\frac{\left|\int_s^t (t-u)^{\gamma-1}G(x_u)dg\right|}{(t-s)^{1-\alpha}} \\ &\le \frac{\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\int_0^s \frac{\left|(t-u)^{\gamma-1}G(x_u)-(s-u)^{\gamma-1}G(x_u)\right|}{u^\beta}du\right. \\ &\quad + \int_0^s\int_0^u \frac{\left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}-(t-v)^{\gamma-1}+(s-v)^{\gamma-1}\right]G(x_v)\right|}{(u-v)^{\beta+1}}dvdu \\ &\quad + \int_0^s\int_0^u \frac{\left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right]\left[G(x_u)-G(x_v)\right]\right|}{(u-v)^{\beta+1}} + \int_s^t \frac{\left|(t-u)^{\gamma-1}G(x_u)\right|}{(u-s)^\beta}du \end{aligned} \tag{3.13}$$

$$+\int_s^t\int_s^u \frac{\left|(t-u)^{\gamma-1}G(x_u)-(t-v)^{\gamma-1}G(x_v)\right|}{(u-v)^{\beta+1}}dvdu\Bigg\}$$

$$:=\sum_{i=1}^{5}J_{2i}.$$

We need to estimate $J_{2i}, i=1,2,\dots,5$. From the condition (H.G), we get

$$\begin{aligned}
J_{24}\leq&\frac{M\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}}\int_s^t\frac{\left|(t-u)^{\gamma-1}\left(1+\|x_u\|_{\mathcal{C}_r}\right)\right|}{(u-s)^\beta}du\\
\leq& M\Lambda_\beta(g)T^{\gamma-\beta+\alpha-1}+\frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}}\int_s^t\frac{(t-u)^{\gamma-1}e^{-\lambda(t-u)}}{(u-s)^\beta}du\\
\leq& M\Lambda_\beta(g)T^{\gamma-\beta+\alpha-1}+M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}\int_s^t\frac{e^{-\lambda(t-u)}}{(t-u)^{2-\alpha-\gamma}(u-s)^\beta}du\\
\leq& M\Lambda_\beta(g)T^{\gamma-\beta+\alpha-1}+M\frac{\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}}{\lambda^{\gamma-1+\alpha-\beta}}\sup_{k>0}\int_0^k\frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta}dz
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
J_{25}\leq&\frac{M\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\int_s^t\int_s^u\frac{\left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]G(x_v)\right|}{(u-v)^{\beta+1}}dvdu\right.\\
&\left.+\int_s^t\int_s^u\frac{\left|(t-u)^{\gamma-1}\left[G(x_u)-G(x_v)\right]\right|}{(u-v)^{\beta+1}}dvdu\right\}\\
&\leq\frac{M\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\int_s^t\int_s^u\frac{\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]\left(1+e^{\lambda v}\|x\|_{1-\alpha,\lambda}\right)}{(u-v)^{\beta+1}}dvdu\right.\\
&\left.+\int_s^t\int_s^u\frac{(t-u)^{\gamma-1}e^{\lambda u}\|x\|_{1-\alpha,\lambda}}{(u-v)^{\beta+\alpha}}dvdu\right\}\\
\leq& M\Lambda_\beta(g)T^{\gamma-\beta+\alpha-1}+\frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}}\\
&\times\left\{\int_s^t\int_s^u\frac{\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]e^{-\lambda(t-v)}}{(u-v)^{\beta+1}}dvdu+\int_s^t\int_s^u\frac{(t-u)^{\gamma-1}e^{-\lambda(t-u)}}{(u-v)^{\beta+\alpha}}dvdu\right\}\\
\leq& M\Lambda_\beta(g)T^{\gamma-\beta+\alpha-1}+\frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}}\\
&\times\left\{\int_0^{t-s}\int_0^{t-s-w}\frac{\left[v^{\gamma-1}-(v+w)^{\gamma-1}\right]e^{-\lambda(v+w)}}{w^{\beta+1}}dvdw\right.\\
&\left.+\int_s^t(t-u)^{\gamma-1}e^{-\lambda(t-u)}(u-s)^{1-\beta-\alpha}du\right\}\\
\leq& M\Lambda_\beta(g)\left\{1+\frac{\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-\beta-1}}\int_0^\infty e^{-z}z^{\alpha+\gamma-\beta-2}dz+\frac{\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}}\int_0^\infty z^{\alpha+\gamma-2}e^{-z}dz\right\}.
\end{aligned}\tag{3.15}$$

For the term J_{21}, we have

$$\frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\int_0^s\frac{\left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]e^{\lambda u}}{u^\beta}du$$

$$= \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \left\{ \int_0^s \frac{(s-u)^{\gamma-1}e^{\lambda u}}{u^\beta} du - \left(\int_0^t - \int_s^t \right) \frac{(t-u)^{\gamma-1}e^{\lambda u}}{u^\beta} du \right\}$$
$$\le \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_s^t \frac{(t-u)^{\gamma-1}e^{\lambda u}}{u^\beta} du$$

for all $s,t \in [0,T]$ with $s<t$. It follows that

$$J_{21} \le \frac{M\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \frac{\left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right]\left(1+e^{\lambda u}\|x\|_{1-\alpha,\lambda}\right)}{u^\beta} du$$
$$\le M\Lambda_\beta(g)T^{\gamma-1+\alpha-\beta} + \frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \frac{\left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right]e^{\lambda u}}{u^\beta} du$$
$$\le M\Lambda_\beta(g)T^{\gamma-1+\alpha-\beta} + \frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}}{\lambda^{\gamma-1+\alpha-\beta}} \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz \tag{3.16}$$

for $t,s \in [0,T]$ with $s<t$. For the term J_{22}, we have

$$\frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \int_0^u \frac{\left\{\left[(s-u)^{\gamma-1}-(s-v)^{\gamma-1}\right]-\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]\right\}e^{\lambda u}}{(u-v)^{\beta+1}} dvdu$$
$$= \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \left\{ \int_0^s \int_0^u \frac{\left[(s-u)^{\gamma-1} - (s-v)^{\gamma-1}\right]e^{\lambda v}}{(u-v)^{\beta+1}} dvdu \right.$$
$$\left. - \left(\int_0^t \int_0^u - \int_s^t \int_0^u \right) \frac{\left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right]e^{\lambda u}}{(u-v)^{\beta+1}} dvdu \right\}$$
$$\le \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \left(\int_s^t \int_s^u + \int_s^t \int_0^s \right) \frac{\left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right]e^{\lambda u}}{(u-v)^{\beta+1}} dvdu$$

for $t,s \in [0,T]$ with $s<t$, which implies that

$$J_{22} \le \frac{M\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \int_0^u \frac{\left[(s-u)^{\gamma-1} - (s-v)^{\gamma-1}\right] - \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right]}{(u-v)^{\beta+1}}$$
$$\times (1+\|x_v\|_{\mathcal{C}_r})\, dvdu$$
$$\le M\Lambda_\beta(g)T^{\gamma-\beta+\alpha-1} + \frac{M\Lambda_\beta(g)e^{-\lambda t}\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}} \tag{3.17}$$
$$\times \int_0^s \int_0^u \frac{\left\{\left[(s-u)^{\gamma-1} - (s-v)^{\gamma-1}\right] - \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right]\right\}e^{\lambda \nu}}{(u-v)^{\beta+1}} dvdu$$
$$\le M\Lambda_\beta(g)\left\{ 1 + \frac{\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-\beta-1}} \int_0^\infty e^{-z} z^{\alpha+\gamma-\beta-2} dz \right.$$
$$\left. + \frac{\|x\|_{1-\alpha,\lambda}}{\lambda^{\gamma-\beta+\alpha-1}} \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz \right\}$$

for $t,s \in [0,T]$ with $s<t$. For the last term J_{23}, we have

$$J_{23} \le \frac{M\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \int_0^u \frac{\left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right]e^{\lambda u}\|x\|_{1-\alpha,\lambda}}{(u-v)^{\beta+\alpha}}$$
$$\le \frac{M\Lambda_\beta(g)e^{-\lambda t}\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}} \int_0^s \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right] e^{\lambda u} u^{1-\alpha-\beta} du \tag{3.18}$$
$$\le \frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}} \int_0^\infty e^{-z} z^{\alpha+\gamma-2} dz$$

for $t, s \in [0, T]$ with $s < t$. Thus, we have showed that there exist some constants $M_2, c_2(\lambda) > 0$ such that

$$\|J(x)\|_{1-\alpha,\lambda} \le M_2 + c_2(\lambda)\|x\|_{1-\alpha,\lambda},$$

and M_2 is independent of λ, and $c_2(\lambda) \to 0$ as $\lambda \to \infty$. This completes the proof. □

In order to prove main results by using the Banach fixed point theorem we need an additional estimate.

Proposition 3.3. *Let $g \in \mathcal{C}^{\nu}([0,T]), x, y \in \mathcal{C}^{1-\alpha}([-r,T])$. Under the conditions (H.f) and (H.G), there exist some constants $c_1(\lambda)$ and $c_2(\lambda) > 0$ such that $c_1(\lambda), c_2(\lambda) \to 0$ as $\lambda \to \infty$, and*

$$\|I(x) - I(y)\|_{1-\alpha,\lambda} \le c_1(\lambda)\|x - y\|_{1-\alpha,\lambda}$$

and

$$\|J(x) - J(y)\|_{1-\alpha,\lambda} \le c_2(\lambda)\left(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha}\right)\|x - y\|_{1-\alpha,\lambda}.$$

Proof. By the Lipschitz property of f, we have

$$\begin{aligned}
&e^{-\lambda t}\frac{|I(x)(t) - I(y)(t) - I(x)(s) + I(y)(s)|}{(t-s)^{1-\alpha}}\\
&\le \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\int_0^s \left|(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right| |f(x_u) - f(y_u)|\, du\right.\\
&\qquad\left. + \int_s^t \left|(t-u)^{\gamma-1}\left[f(x_u) - f(y_u)\right]\right| du\right\}\\
&\le \frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}}\int_0^{\infty}\frac{e^{-z}}{z^{2-\alpha-\gamma}}dz
\end{aligned}$$

for any $t, s \in [0,T]$ with $s < t$, which implies that the first assertion holds.

To prove the second assertion, let $\beta \in (1-\nu, \alpha_0)$ as in the proof of Proposition 3.1. We have

$$\begin{aligned}
&\frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\left|J(x)(t) - J(y)(s) - J(x)(s) + J(y)(s)\right|\\
&\le \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\left|\int_0^s \left[(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right]\left[G(x_u) - G(y_u)\right] dg\right|\right.\\
&\qquad\left. + \left|\int_s^t (t-u)^{\gamma-1}\left[G(x_u) - G(y_u)\right] dg\right|\right\}\\
&\equiv J_{31} + J_{32}
\end{aligned}$$

for $t, s \in [0,T]$ with $s < t$. From (2.1), we have

$$\begin{aligned}
J_{31} \le &\frac{\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\int_0^s \frac{\left|\left[(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right]\left[G(x_u) - G(y_u)\right]\right|}{u^{\beta}}du\right.\\
&+\int_0^s\int_0^u \frac{\left|\left[(t-u)^{\gamma-1} - (s-u)^{\gamma-1} - (t-v)^{\gamma-1} + (s-v)^{\gamma-1}\right]\left[G(x_v) - G(y_v)\right]\right|}{(u-v)^{\beta+1}}dvdu\\
&\left.+\int_0^s\int_0^u \frac{\left|\left[(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right]\left[G(x_u) - G(x_v) - G(y_u) + G(y_v)\right]\right|}{(u-v)^{\beta+1}}dvdu\right\}
\end{aligned}$$

$$\equiv \Lambda_\beta(g)\sum_{i=1}^{3} K_i. \tag{3.19}$$

Similar to the proof of Proposition 3.2, we have

$$K_1 \le \frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\gamma-\beta+\alpha-1}} \sup_{k>0}\int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta}dz, \tag{3.20}$$

and

$$\begin{aligned} K_2 \le M\|x-y\|_{1-\alpha,\lambda}\Bigg\{&\frac{1}{\lambda^{\gamma-\beta+\alpha-1}}\int_0^\infty e^{-z}z^{\alpha+\gamma-\beta-2}dz \\ &+\frac{1}{\lambda^{\gamma-1+\alpha-\beta}}\sup_{k>0}\int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta}dz\Bigg\}. \end{aligned} \tag{3.21}$$

By the mean value theorem and the condition (H.G), we have for all $u, v \in [0,T]$ (see Boufoussi *et al* [5] for detail)

$$\begin{aligned} &\|G(x_u)-G(y_u)-G(x_v)+G(y_v)\| \\ \le& C_3\|x_u-y_u-x_v+y_v\|_{\mathcal{C}_r} + \left[C_4\|x_u-x_v\|_{\mathcal{C}_r} + C_4\|y_u-y_v\|_{\mathcal{C}_r}\right]\|x_u-y_u\|_{\mathcal{C}_r}. \end{aligned}$$

It follows that

$$\begin{aligned} K_3 \le& \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\Bigg\{\int_0^s\int_0^u \frac{\left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right]\right| C_3\|x_u-y_u-x_v+y_v\|_{\mathcal{C}_r}}{(u-v)^{\beta+1}}dvdu \\ &+\int_0^s\int_0^u \frac{\left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right]\right|\left[C_4\|x_u-x_v\|_{\mathcal{C}_r}+C_4\|y_u-y_v\|_{\mathcal{C}_r}\right]\|x_u-y_u\|_{\mathcal{C}_r}}{(u-v)^{\beta+1}}dvdu\Bigg\} \\ \equiv& L_1+L_2. \end{aligned} \tag{3.22}$$

Some elementary calculations may show that

$$\begin{aligned} L_1 \le& \frac{Me^{-\lambda t}}{(t-s)^{1-\alpha}}\int_0^s\int_0^u \frac{\left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]e^{\lambda u}\|x-y\|_{1-\alpha,\lambda}}{(u-v)^{\beta+\alpha}}dvdu \\ \le& M\frac{\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}}, \end{aligned}$$

and

$$\begin{aligned} L_2 \le& \frac{Me^{-\lambda t}}{(t-s)^{1-\alpha}}\int_0^s\int_0^u \frac{\left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]\|x_u-y_u\|_{\mathcal{C}_r}\left(\|x\|_{1-\alpha}+\|y\|_{1-\alpha}\right)e^{\lambda u}}{(u-v)^{\beta+\alpha}}dvdu \\ \le& \frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}}\left(\|x\|_{1-\alpha}+\|y\|_{1-\alpha}\right), \end{aligned}$$

for all $s, t \in [0,T]$ with $s < t$. According to inequalities (3.19)–(3.22), we get

$$J_{31} \le c(\lambda)(1+\|x\|_{1-\alpha}+\|y\|_{1-\alpha})\|x-y\|_{1-\alpha,\lambda} \tag{3.23}$$

and $\lim_{\lambda\to\infty} c(\lambda)=0$. Now, let us estimate the term J_{32}. From (2.1), we have that

$$
\begin{aligned}
J_{32} \le & \frac{\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\int_s^t \frac{\left|(t-u)^{\gamma-1}\left[G(x_u)-G(y_u)\right]\right|}{(u-s)^\beta}du\right.\\
&+\int_s^t\int_s^u \frac{\left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]\left[G(x_v)-G(y_v)\right]\right|}{(u-v)^{\beta+1}}dvdu\\
&\left.+\int_s^t\int_s^u \frac{\left|(t-u)^{\gamma-1}\left[G(x_u)-G(x_v)-G(y_u)+G(y_v)\right]\right|}{(u-v)^{\beta+1}}dvdu\right\}\\
\equiv & \Lambda_\beta(g)\sum_{i=1}^3 K_i'.
\end{aligned}
\tag{3.24}
$$

Obviously, we have

$$
K_1' \le \frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\gamma-\beta+\alpha-1}}\sup_{k>0}\int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta}dz, \tag{3.25}
$$

and

$$
K_2' \le \frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\gamma-\beta+\alpha-1}}\int_0^\infty e^{-z}z^{\alpha+\gamma-\beta-2}dz. \tag{3.26}
$$

In the same arguments as in the term K_3, we get

$$
\begin{aligned}
K_3' \le & \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\int_s^t\int_s^u \frac{(t-u)^{\gamma-1}C_3\|x_u-y_u-x_v+y_v\|_{\mathcal{C}_r}}{(u-v)^{\beta+1}}dvdu\right.\\
&\left.+\int_s^t\int_s^u \frac{(t-u)^{\gamma-1}\left[C_4\|x_u-x_v\|_{\mathcal{C}_r}+C_4\|y_u-y_v\|_{\mathcal{C}_r}\right]\|x_u-y_u\|_{\mathcal{C}_r}}{(u-v)^{\beta+1}}dvdu\right\}\\
\le & \frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}}\left(1+\|x\|_{1-\alpha}+\|y\|_{1-\alpha}\right),
\end{aligned}
\tag{3.27}
$$

where we have used the next estimates:

$$
\begin{aligned}
&\frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\int_s^t\int_s^u \frac{(t-u)^{\gamma-1}\|x_u-y_u\|_{\mathcal{C}_r}}{(u-v)^{\beta+\alpha}}dvdu\\
&\le \frac{M\|x-y\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}}\int_s^t\int_s^u \frac{(t-u)^{\gamma-1}e^{-\lambda(t-u)}}{(u-v)^{\beta+\alpha}}dvdu\\
&\le M\|x-y\|_{1-\alpha,\lambda}\int_s^t \frac{e^{-\lambda(t-u)}(u-s)^{1-\alpha-\beta}}{(t-u)^{2-\alpha-\gamma}}du\\
&\le \frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}}\int_0^\infty z^{\alpha+\gamma-2}e^{-z}dz.
\end{aligned}
$$

It follows from (3.24)-(3.27) that

$$
J_{32} \le c(\lambda)(1+\|x\|_{1-\alpha}+\|y\|_{1-\alpha})\,\|x-y\|_{1-\alpha,\lambda} \tag{3.28}
$$

and $\lim_{\lambda\to\infty} c(\lambda)=0$. Thus, we have showed that the estimate

$$
\|J(x)-J(y)\| \le c(\lambda)\left(1+\|x\|_{1-\alpha}+\|y\|_{1-\alpha}\right)\|x-y\|_{1-\alpha,\lambda}
$$

holds and the constant $c(\lambda)$ satisfied $\lim_{\lambda\to\infty} c(\lambda)=0$, and the second assertion follows. □

Define the functions

$$\Psi : \mathcal{C}^{1-\alpha}([-r,T]) \to \mathcal{C}^{1-\alpha}([-r,T])$$

by $\Psi(x)(t) = x(t)$ for $t \in [-r,0]$ and

$$\Psi(x)(t) = x(0) + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(x_s)ds + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} G(x_s)dg(s)$$

for $t \geq 0$. According to Proposition 3.2 and Proposition 3.3, we get the following result.

Corollary 3.1. *Let $g \in \mathcal{C}^{\nu}([0,T])$ and $x, y \in \mathcal{C}^{1-\alpha}([-r,T])$. Under the conditions (H.f) and (H.G), there exist some constants $M, c_1(\lambda), c_2(\lambda) > 0$ such that $c_1(\lambda), c_2(\lambda) \to 0$ as $\lambda \to \infty$, and*

$$\|\Psi(x)\|_{1-\alpha,\lambda} \leq M + \|x_0\|_{1-\alpha,\lambda} + c_1(\lambda)\|x\|_{1-\alpha,\lambda}$$

and

$$\|\Psi(x) - \Psi(y)\|_{1-\alpha,\lambda} \leq \|x_0 - y_0\|_{1-\alpha,\lambda} + c_2(\lambda)\left(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha}\right)\|x-y\|_{1-\alpha,\lambda}.$$

4. Deterministic time fractional functional differential equation

Keep the notations in Section 3. For $g \in \mathcal{C}^{\nu}([0,T])$, we consider the deterministic time functional differential equation of the form

$$\begin{cases} x(t) = \eta(0) + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(x_s)ds + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} G(x_s)dg(s), \ t \in (0,T], \\ x(t) = \eta(t), \qquad t \in [-r,0]. \end{cases} \tag{4.1}$$

Theorem 4.1. *Let f and G satisfy the conditions (H.f) and (H.G). If $\eta \in \mathcal{C}^{1-\alpha}([-r,0])$, then the equation (4.1) admits a unique solution in $\mathcal{C}^{1-\alpha}([-r,T])$.*

Proof. We first prove existence of the solution. Let

$$\mathcal{H}^{1-\alpha}([-r,T],\eta) = \left\{x \in \mathcal{C}^{1-\alpha}([-r,T]) \middle| x = \eta \text{ on } [-r,0]\right\}$$

and define an operator

$$\Phi : \mathcal{H}^{1-\alpha}([-r,T],\eta) \to \mathcal{H}^{1-\alpha}([-r,T],\eta)$$

by $\Phi(x)(t) = \eta(t)$ for $t \in [-r,0]$ and

$$\Phi(x)(t) = \eta(0) + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(x_s)ds + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} G(x_s)dg(s)$$

for $t \geq 0$. By Corollary 3.1, we have

$$\|\Phi(x)\|_{1-\alpha,\lambda} \leq M + \|\eta\|_{1-\alpha,\lambda} + c(\lambda)\|x\|_{1-\alpha,\lambda},$$

where M is a constant and $\lim_{\lambda\to\infty} c(\lambda) = 0$. Let λ_0 be sufficiently large such that $c(\lambda_0) \le \frac{1}{2}$ and let $M_0 = 2(\|\eta\|_{1-\alpha,\lambda_0} + M)$ and

$$B_{\lambda_0} = \left\{x \in \mathcal{H}^{1-\alpha}([-r,T],\eta) \middle| \|x\|_{1-\alpha,\lambda_0} \le M_0\right\}.$$

Then, we have that Φ maps B_{λ_0} into itself. We now need to show there exists $\lambda > \lambda_0$ s.t. Φ is a contraction on B_{λ_0} under the norm $\|\cdot\|_{1-\alpha,\lambda}$. By Corollary 3.1, we have

$$\|\Phi(x) - \Phi(y)\|_{1-\alpha,\lambda} \le c(\lambda)(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha})\|x - y\|_{1-\alpha,\lambda}$$

for all $x, y \in \mathcal{H}^{1-\alpha}([-r,T],\eta)$. It follows from proper choice of λ (see Boufoussi and Hajji [4]) such that Φ is a contraction on the B_λ of the complete metric space $\mathcal{C}^{1-\alpha}([-r,T])$, which implies that Φ has a fixed point x in B_{λ_0}. From the definition of Φ, the fixed point x is a solution of (4.1) in $\mathcal{C}^{1-\alpha}([-r,T])$.

We now prove uniqueness of the solution. Let x an y be two solutions of (4.1) in $\mathcal{C}^{1-\alpha}([-r,T])$. By using Corollary 3.1, we obtain that

$$\|x - y\|_{1-\alpha,\lambda} \le \frac{1}{2}\|x - y\|_{1-\alpha,\lambda}$$

for λ large enough, which shows that $x = y$. This completes the proof. □

Lemma 4.1 (Nualart and Rascanu [14])**.** *Given* $0 \le \alpha < 1, a, b \ge 0$. *Let* $x : [0,+\infty) \to [0,+\infty)$ *be a continuous function such that*

$$x(t) \le a + bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} x(s)ds$$

for each t. *Then*

$$x(t) \le a d_\alpha e^{c_\alpha b^{\frac{1}{1-\alpha}} t},$$

where c_α *and* d_α *are two positive constants depending only on* α.

Proposition 4.1. *Under the same condition of Theorem 4.1, if* $\gamma + \nu > \frac{3}{2}$, *the solution* x *of* (4.1) *satisfies*

$$\|x\|_{1-\alpha} \le c_1(1 + \|\eta\|_{1-\alpha}) exp\left(c_2 \Lambda_\beta(g)^{\frac{1}{2\alpha+2\gamma-3}}\right)$$

for any $\alpha \in (\frac{3}{2} - \gamma, \nu)$, *where* $c_1, c_2 > 0$ *are two constants depending only on* α, β, γ, T.

Proof. For $t \ge 0$ we assume that

$$h(t) = \sup_{s\in[-r,t]} |x(s)| + \sup_{-r\le s<u\le t} \frac{|x(u) - x(s)|}{(u-s)^{1-\alpha}}.$$

We have

$$\begin{aligned}\frac{|x(u) - x(s)|}{(u-s)^{1-\alpha}} \le \frac{1}{(u-s)^{1-\alpha}} &\left\{\int_0^s \left|\left[(u-v)^{\gamma-1} - (s-v)^{\gamma-1}\right] f(x_v)\right| dv\right. \\ &+ \int_s^u \left|(u-v)^{\gamma-1} f(x_v)\right| dv + \left|\int_0^s \left[(u-v)^{\gamma-1} - (s-v)^{\gamma-1}\right] G(x_v)dg\right| \\ &\left. + \left|\int_s^u (u-v)^{\gamma-1} G(x_v)dg\right|\right\} \\ \equiv \sum_{i=1}^4 K_{2i}\end{aligned} \tag{4.2}$$

for $0 \le s < u \le t$. By the condition (H.f), we have

$$\begin{aligned} K_{21} \le & \frac{M}{(u-s)^{1-\alpha}} \int_0^s \left[(s-v)^{\gamma-1} - (u-v)^{\gamma-1}\right] (1 + \|x_v\|_{\mathcal{C}_r})\, dv \\ \le & \frac{M}{(u-s)^{1-\alpha}} \int_s^u (u-v)^{\gamma-1} (1 + h(v))\, dv \le M \left(1 + \int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma}} dv\right) \end{aligned} \tag{4.3}$$

and

$$K_{22} \le M \left(1 + \int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma}} dv\right) \tag{4.4}$$

for $0 \le s < u \le t$. For $\beta \in (1-\nu, \alpha_0)$, from the inequality (2.1) we get

$$\begin{aligned} K_{24} \le & \frac{\Lambda_\beta(g)}{(u-s)^{1-\alpha}} \left\{ \int_s^u \frac{\left|(u-v)^{\gamma-1} G(x_v)\right|}{(v-s)^\beta} dv \right. \\ & \left. + \int_s^u \int_s^v \frac{\left|(u-v)^{\gamma-1} G(x_v) - (u-\theta)^{\gamma-1} G(x_\theta)\right|}{(v-\theta)^{\beta+1}} d\theta dv \right\} \\ \le & \frac{\Lambda_\beta(g)}{(u-s)^{1-\alpha}} \int_s^u \frac{(u-v)^{\gamma-1} \left[|G(x_v) - G(x_s)| + |G(x_s)|\right]}{(v-s)^\beta} dv + \frac{\Lambda_\beta(g)}{(u-s)^{1-\alpha}} \\ & \times \int_s^u \int_s^v \frac{\left|\left[(u-v)^{\gamma-1} - (u-\theta)^{\gamma-1}\right] G(x_\theta) + (u-v)^{\gamma-1} \left[G(x_v) - G(x_\theta)\right]\right|}{(v-\theta)^{\beta+1}} d\theta dv \end{aligned}$$

for $0 \le s < u \le t$. By the condition (H.G), we have that

$$\begin{aligned} & \frac{1}{(u-s)^{1-\alpha}} \int_s^u \frac{(u-v)^{\gamma-1} |G(x_v) - G(x_s)|}{(v-s)^\beta} dv \\ \le & \frac{M}{(u-s)^{1-\alpha}} \int_s^u \frac{h(v)(v-s)^{1-\alpha-\beta}}{(u-v)^{1-\gamma}} dv \le M T^{1-\alpha-\beta} \int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma}} dv \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{(u-s)^{1-\alpha}} \int_s^u \frac{(u-v)^{\gamma-1} |G(x_s)|}{(v-s)^\beta} dv \\ \le & \frac{M}{(u-s)^{1-\alpha}} \int_s^u \frac{(u-v)^{\gamma-1} (1 + \|x_s\|_{\mathcal{C}_r})}{(v-s)^\beta} dv \\ \le & M(T^{\gamma-\beta+\alpha-1} + h(s)(u-s)^{\gamma-\beta+\alpha-1}) B(1-\beta, \gamma) \\ \le & M \left\{ T^{\gamma-\beta+\alpha-1} + (\gamma-\beta+\alpha-1) \int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma+\beta}} dv \right\} \end{aligned}$$

for $0 \le s < u \le t$. Similarly, we also have that

$$\begin{aligned} & \frac{1}{(u-s)^{1-\alpha}} \int_s^u \int_s^v \frac{\left|\left[(u-v)^{\gamma-1} - (u-\theta)^{\gamma-1}\right] G(x_\theta)\right|}{(v-\theta)^{\beta+1}} d\theta dv \\ \le & M + \frac{1}{(u-s)^{1-\alpha}} \int_s^u \int_s^v \frac{\left[(u-v)^{\gamma-1} - (u-\theta)^{\gamma-1}\right] h(\theta)}{(v-\theta)^{\beta+1}} d\theta dv \\ = & M + \frac{1}{(u-s)^{1-\alpha}} \int_0^{u-s} \int_0^{u-y-s} \frac{\left[y^{\gamma-1} - (y+x)^{\gamma-1}\right] h(u-y-x)}{x^{\beta+1}} dx dy \end{aligned}$$

$$
\begin{aligned}
&= M + \frac{1}{(u-s)^{1-\alpha}} \int_0^{u-s} \int_0^{u-x-s} \frac{\left[y^{\gamma-1} - (y+x)^{\gamma-1}\right] h(u-y-x)}{x^{\beta+1}} dy dx \\
&\le M + \frac{1}{(u-s)^{1-\alpha}} \int_0^{u-s} \int_0^{u-x-s} \frac{\left[y^{\gamma-1} - (y+x)^{\gamma-1}\right] h(u-x)}{x^{\beta+1}} dy dx \\
&\le M + \frac{1}{(u-s)^{1-\alpha}} \int_0^{u-s} \frac{h(u-x)}{x^{\beta+1-\gamma}} dx \le M + \int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma+\beta}} dv
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{(u-s)^{1-\alpha}} \int_s^u \int_s^v \frac{\left|(u-v)^{\gamma-1}\left[G(x_v) - G(x_\theta)\right]\right|}{(v-\theta)^{\beta+1}} d\theta dv \\
&\le \frac{M}{(u-s)^{1-\alpha}} \int_s^u \int_s^v \frac{(u-v)^{\gamma-1}(v-\theta)^{1-\alpha} h(v)}{(v-\theta)^{\beta+1}} d\theta dv \\
&\le M \int_0^t \frac{h(v)}{(t-v)^{1-\gamma+\beta}} dv
\end{aligned}
$$

for $0 \le s < u \le t$. Consequently

$$
K_{24} \le M \Lambda_\beta(G) \left\{ 1 + \int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma+\beta}} dv \right\} \tag{4.5}
$$

for $0 \le s < u \le t$.

For K_{23}, by (2.1), we have

$$
\begin{aligned}
K_{23} \le & \frac{\Lambda_\beta(g)}{(u-s)^{1-\alpha}} \left\{ \int_0^s \frac{\left|\left[(u-v)^{\gamma-1} - (s-v)^{\gamma-1}\right] G(x_v)\right|}{v^\beta} dv \right. \\
& \left. + \int_0^s \int_0^v \frac{\left|\left[(u-v)^{\gamma-1} - (s-v)^{\gamma-1}\right] G(x_v) - \left[(u-\theta)^{\gamma-1} - (s-\theta)^{\gamma-1}\right] G(x_\theta)\right|}{(v-\theta)^{\beta+1}} d\theta dv \right\} \\
\le & \frac{\Lambda_\beta(g)}{(u-s)^{1-\alpha}} \int_0^s \frac{\left|\left[(u-v)^{\gamma-1} - (s-v)^{\gamma-1}\right] G(x_v)\right|}{v^\beta} dv \\
& + \frac{\Lambda_\beta(g)}{(u-s)^{1-\alpha}} \left\{ \int_0^s \int_0^v \frac{\left|\left[(u-v)^{\gamma-1} - (s-v)^{\gamma-1} - (u-\theta)^{\gamma-1} + (s-\theta)^{\gamma-1}\right] G(x_\theta)\right|}{(v-\theta)^{\beta+1}} d\theta dv \right. \\
& \left. + \int_0^s \int_0^v \frac{\left|\left[(u-v)^{\gamma-1} - (s-v)^{\gamma-1}\right] \left[G(x_v) - G(x_\theta)\right]\right|}{(v-\theta)^{\beta+1}} d\theta dv \right\}
\end{aligned}
\tag{4.6}
$$

for $0 \le s < u \le t$. By the condition (H.G), we have

$$
\begin{aligned}
&\frac{\Lambda_\beta(g)}{(u-s)^{1-\alpha}} \int_0^s \frac{\left|\left[(u-v)^{\gamma-1} - (s-v)^{\gamma-1}\right] G(x_v)\right|}{v^\beta} dv \\
&\le \frac{M \Lambda_\beta(g)}{(u-s)^{1-\alpha}} \int_0^s \frac{\left[(s-v)^{\gamma-1} - (u-v)^{\gamma-1}\right] (1 + h(v))}{v^\beta} dv \\
&\le M \Lambda_\beta(g) \left\{ 1 + \frac{1}{(u-s)^{1-\alpha}} \int_s^u \frac{(u-v)^{\gamma-1} h(v)}{v^\beta} dv \right\} \\
&\le M \Lambda_\beta(g) \left\{ 1 + \int_0^t \frac{h(v)}{v^{2\beta}} dv + \int_0^t \frac{h(v)}{(t-v)^{2(2-\alpha-\gamma)}} dv \right\}
\end{aligned}
\tag{4.7}
$$

for $0 \le s < u \le t$. Similarly, by the fact

$$
\frac{1}{(u-s)^{1-\alpha}} \int_s^u \int_0^v \frac{\left[(u-v)^{\gamma-1} - (u-\theta)^{\gamma-1}\right] h(\theta)}{(v-\theta)^{\beta+1}} d\theta dv
$$

$$\le \int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma+\beta}}dv + \frac{1}{(u-s)^{1-\alpha}}\int_s^u \frac{h(v)}{(u-v)^{1-\gamma}}dv\int_0^s (v-\theta)^{-(\beta+1)}d\theta$$
$$\le \int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma+\beta}}dv$$

for $0 \le s < u \le t$, we have

$$\frac{1}{(u-s)^{1-\alpha}}\int_0^s\int_0^v \frac{\left|\left[(u-v)^{\gamma-1}-(s-v)^{\gamma-1}-(u-\theta)^{\gamma-1}+(s-\theta)^{\gamma-1}\right]G(x_\theta)\right|}{(v-\theta)^{\beta+1}}d\theta dv$$
$$\le \frac{M}{(u-s)^{1-\alpha}}\int_0^s\int_0^v \frac{\left\{\left[(s-v)^{\gamma-1}-(s-\theta)^{\gamma-1}\right]-\left[(u-v)^{\gamma-1}-(u-\theta)^{\gamma-1}\right]\right\}(1+h(\theta))}{(v-\theta)^{\beta+1}}d\theta dv$$
$$\le M\left\{1+\frac{1}{(u-s)^{1-\alpha}}\int_s^u\int_0^v \frac{\left[(u-v)^{\gamma-1}-(u-\theta)^{\gamma-1}\right]h(\theta)}{(v-\theta)^{\beta+1}}d\theta dv\right\}$$
$$\le M\left\{1+\int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma+\beta}}dv\right\}$$

and

$$\frac{1}{(u-s)^{1-\alpha}}\int_0^s\int_0^v \frac{\left|\left[(u-v)^{\gamma-1}-(s-v)^{\gamma-1}\right]\left[G(x_v)-G(x_\theta)\right]\right|}{(v-\theta)^{\beta+1}}d\theta dv$$
$$\le \frac{M}{(u-s)^{1-\alpha}}\int_0^s\int_0^v \frac{\left|\left[(u-v)^{\gamma-1}-(s-v)^{\gamma-1}\right]h(v)\right|}{(v-\theta)^{\beta+\alpha}}d\theta dv$$
$$\le \frac{M}{(u-s)^{1-\alpha}}\int_0^s \left[(s-v)^{\gamma-1}-(u-v)^{\gamma-1}\right]h(v)v^{1-\alpha-\beta}dv$$
$$\le MT^{1-\alpha-\beta}\int_0^t \frac{h(v)}{(t-v)^{2-\alpha-\gamma}}dv$$

for $0 \le s < u \le t$. It follows that

$$K_{23} \le M\Lambda_\beta(g)\left\{1+\int_0^t \frac{h(v)}{v^{2\beta}}dv + \int_0^t \frac{h(v)}{(t-v)^{2(2-\alpha-\gamma)}}dv\right\} \tag{4.8}$$

for $0 \le s < u \le t$. Thus, we have gotten the desired estimate

$$\sup_{0\le s<u\le t}\frac{|x(u)-x(s)|}{(u-s)^{1-\alpha}} \le M(1+\Lambda_\beta(g))\left\{1+\int_0^t h(v)\left[(t-v)^{-2(2-\alpha-\gamma)}+v^{-2\beta}\right]dv\right\}$$

for all $t \in [0,T]$, which implies immediately that

$$h(t) \le \|\eta\|_{1-\alpha} + M\left(1+\Lambda_\beta(g)\right)\left\{1+\int_0^t h(v)[(t-v)^{-2(2-\alpha-\gamma)}+v^{-2\beta}]dv\right\}$$
$$\le \|\eta\|_{1-\alpha} + M\left(1+\Lambda_\beta(g)\right)$$
$$\times\left\{1+\left(1+T^{2(2-\alpha-\gamma-\beta)}\right)\int_0^t h(v)(t-v)^{-2(2-\alpha-\gamma)}v^{-2(2-\alpha-\gamma)}t^{2(2-\alpha-\gamma)}dv\right\}$$

for all $t \in [0,T]$. Combining this with Lemma 4.1, we obtain

$$\|x\|_{1-\alpha} \le c_1(1+\|\eta\|_{1-\alpha})\exp(c_2\Lambda_\beta(g)^{\frac{1}{2\alpha+2\gamma-3}}),$$

and the proposition follows. □

Now, we study the dependance of the solution of (4.1) on the initial condition.

Proposition 4.2. *Let $\nu+\gamma>\frac{3}{2}, \alpha\in(\frac{3}{2}-\nu,\nu)$ and let the conditions (H.f) and (H.G) be satisfied. Suppose that $\eta,\eta^n\in\mathcal{C}^{1-\alpha}([-r,0])$. If x is the solution of* (4.1) *and x^n is the solution of the same equation with η^n in place of η, then we have*

$$\begin{aligned}\|x-x^n\|_{1-\alpha}\leq & c_1\|\eta-\eta^n\|_{1-\alpha}\exp\left\{c_2\left(\|x\|_{1-\alpha}^{\frac{1}{2\alpha+2\gamma-3}}+\|x^n\|_{1-\alpha}^{\frac{1}{2\alpha+2\gamma-3}}\right)\right\}\\ &\times\exp\left(c_3\Lambda_\beta(g)^{\frac{1}{2\alpha+2\gamma-3}}\right),\end{aligned}$$

where $c_1,c_2,c_3>0$ are constants depending only on α,β,γ,T.

Proof. For $t\geq 0$, we set

$$h^n(t)=\sup_{s\in[-r,t]}\|x(s)-x^n(s)\|+\sup_{-r\leq s<u\leq t}\frac{|x(u)-x^n(u)-x(s)+x^n(s)|}{(u-s)^{1-\alpha}}.$$

By the similar calculus as in the proof of Proposition 4.1, we get

$$\begin{aligned}h^n(t)\leq & \|\eta-\eta^n\|_{1-\alpha}+M\left\{1+\Lambda_\beta(g)\left(1+\|x\|_{1-\alpha}^{\frac{1}{2\alpha+2\gamma-3}}+\|x^n\|_{1-\alpha}^{\frac{1}{2\alpha+2\gamma-3}}\right)\right\}\\ &\cdot\left\{1+\left(1+T^{2(2-\alpha-\gamma-\beta)}\right)\int_0^t h^n(v)t^{2(2-\alpha-\gamma)}(t-v)^{-2(2-\alpha-\gamma)}v^{-2(2-\alpha-\gamma)}dv\right\}\end{aligned}$$

for $t\geq 0$. It follows from Lemma 4.1 that

$$\begin{aligned}\|x-x^n\|_{1-\alpha}\leq & c_1\|\eta-\eta^n\|_{1-\alpha}\exp\left\{c_2\left(\|x\|_{1-\alpha}^{\frac{1}{2\alpha+2\gamma-3}}+\|x^n\|_{1-\alpha}^{\frac{1}{2\alpha+2\gamma-3}}\right)\right\}\\ &\times\exp\left(c_3\Lambda_\beta(g)^{\frac{1}{2\alpha+2\gamma-3}}\right),\end{aligned}$$

where $c_1,c_2,c_3>0$ are constants depending only on α,β,γ,T, which completes the proof of the proposition. □

5. Time fractional stochastic functional differential equation driven by fBm

In this section we will apply the results of the previous sections in order to prove the main theorems of this paper. It is well known that the fractional Brownian motion B^H has ν-Hölder continuous trajectories of order $\nu<H$. Then for all $\beta\in\left(1-H,\frac{1}{2}\right)$, the trajectories of B^H belong to the space $W^{1-\beta,\infty}(0,T;\mathbb{R})$. As a consequence, if $u=\{u(t),t\in[0,T]\}$ is a stochastic process whose trajectories belong a.s. to the space $W^{\beta,1}(0,T;\mathbb{R})$ with $\beta\in\left(1-H,\frac{1}{2}\right)$, the Riemann-Stieltjes integral $\int_0^T u(s)dB^H(s)$ exists and

$$\left\|\int_0^T u(s)dB^H(s)\right\|\leq\Lambda_\beta(B^H)\|u\|_{\beta,1}.$$

As a simple consequence of these facts, we expound and prove the following theorems which introduce the uniqueness, existence and dependence of the solution of (1.1) on the initial condition.

Theorem 5.1. *Let $\frac{1}{2} < H < 1$ and $\max\{H, 2-2H\} < \gamma < 1$. Assume that the coefficients f, G satisfy the assumptions $(H.f)$ and $(H.G)$, respectively. If $\alpha \in (2-H-\gamma, H)$ and $\eta \in \mathcal{C}^{1-\alpha}([-r,0])$, almost surely, then there exists a unique solution x of (1.1) with paths in $\mathcal{C}^{1-\alpha}([-r,T])$, $P-a.s.$ If in addition $\alpha+\gamma > \frac{7}{4}$, then the solution x satisfies*

$$E\left(\|x\|_{1-\alpha}^p\right) < \infty$$

for all $p \geq 1$.

Theorem 5.2. *Assume that $\alpha \in \left(\frac{3}{2}-H, v\right), \gamma + H > \frac{3}{2}$, and that the coefficients f, G satisfy the assumptions $(H.f)$ and $(H.G)$, respectively. Let $\eta, \eta^n \in \mathcal{C}^{1-\alpha}([-r,0])$ for $n \geq 1$. Suppose that x is the solution of (1.1) and that x^n is the solution of the same equation with η^n in place of η for every $n \geq 1$. If*

$$\lim_n \|\eta^n - \eta\|_{1-\alpha} = 0$$

almost surely, then for P-almost all $\omega \in \Omega$, we have

$$\lim_n \|x^n(\omega,\cdot) - x(\omega,\cdot)\|_{1-\alpha} = 0.$$

If in addition $\alpha + \gamma > \frac{7}{4}$, then

$$\lim_n \mathrm{E}\|x^n - x\|_{1-\alpha}^p = 0$$

for all $p \geq 1$.

Proof of Theorem 5.1. The existence and uniqueness of the solution follows directly from the deterministic Theorem 4.1. Then, by Proposition 4.1, we obtain

$$\|x\|_{1-\alpha} \leq c_1(1 + \|\eta\|_{1-\alpha}) \exp\left(c_2 \Lambda_\beta(B)^{\frac{1}{2\alpha+2\gamma-3}}\right)$$

for any $\alpha \in \left(\frac{3}{2}-\gamma, H\right)$, where $c_1, c_2 > 0$ are constants depending only on α, β, γ, T. Hence, for all $p \geq 1$

$$\mathrm{E}\|x\|_{1-\alpha}^p \leq \frac{1}{2} c_1^{2p} E(1+\|\eta\|_{1-\alpha})^{2p} + \frac{1}{2} E \exp\left(2pc_2 \Lambda_\beta(B)^{\frac{1}{2\alpha+2\gamma-3}}\right).$$

By the classical Fernique's theorem (see Fernique [6]), we have

$$E\exp\left(\Lambda_\beta(B)^\delta\right) < \infty$$

for any $0 < \delta < 2$. Consequently, $\mathrm{E}\|x\|_{1-\alpha}^p < \infty$ for all $p \geq 1$, as $\frac{1}{2\alpha+2\gamma-3} < 2$. □

Proof of Theorem 5.2. It suffices to apply Proposition 4.2 to obtain the almost sure convergence. The convergence in L^p can be obtained by dominated convergence argument since by Proposition 4.1. We have that

$$\|x^n - x\|_{1-\alpha} \leq \|x^n\|_{1-\alpha} + \|x\|_{1-\alpha} \leq c_1\left(2 + \|\eta\|_{1-\alpha} + \|\eta^n\|_{1-\alpha}\right) \exp\left(c_2 \Lambda_\beta(B)^{\frac{1}{2\alpha+2\gamma-3}}\right)$$

for any $n \in \mathbb{N}$. Note that $\|\eta^n\|_{1-\alpha}$ is bounded, we can write

$$\|x^n - x\|_{1-\alpha} \leq Y \equiv M \exp\left(c_2 \Lambda_\beta(B)^{\frac{1}{2\alpha+2\gamma-3}}\right)$$

and $EY^p < \infty$ for all $p \geq 1$. □

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