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WELL-POSEDNESS OF DEGENERATE DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN HÖLDER CONTINUOUS FUNCTION SPACES*

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Abstract Using operator-valued $\dot{C}^\alpha$-Fourier multiplier results on vector- valued Hölder continuous function spaces, we give a characterization for the C^α-well-posedness of the first order degenerate differential equations with infinite delay $(Mu)'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t)$ $(t \in \mathbb{R})$, where A, M are closed operators on a Banach space X such that $D(A) \cap D(M) \neq \{0\}$, $a \in L^1_{\text{loc}}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+; t^\alpha dt)$.

Keywords C^α-well-posedness, degenerate differential equations, infinite delay, $\dot{C}^\alpha$-Fourier multiplier, Hölder continuous function spaces.

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1. Introduction

Recently, the problems of characterization of the well-posedness for degenerate differential equations with periodic boundary conditions have been studied extensively. See e.g. [7, 8, 13, 14, 16, 17] and the references therein. For example, Lizama and Ponce considered the first order degenerate equations:

$$(Mu)'(t) = Au(t) + f(t), \qquad (0 \le t \le 2\pi) \tag{1.1}$$

with periodic boundary condition $(Mu)(0) = (Mu)(2\pi)$, where A and M are closed linear operators on a complex Banach space X [13]. They obtained necessary and sufficient conditions to ensure the well-posedness of (1.1) in Lebesgue-Bochner spaces $L^p(\mathbb{T}; X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$ and periodic Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$ under suitable assumptions on the modified resolvent operator

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determined by (1.1), where $\mathbb{T} := [0, 2\pi]$. Moreover, Lizama and Ponce studied the following first order degenerate equations with infinite delay:

$$(Mu)'(t) = \alpha Au(t) + \int_{-\infty}^{\infty} a(t-s)Au(s)ds + f(t), \quad (0 \leq t \leq 2\pi) \tag{1.2}$$

with initial condition $(Mu)(0) = (Mu)(2\pi)$, where A and M are closed linear operators on a complex Banach space X, $\alpha \in \mathbb{R}\backslash\{0\}$ and $a \in L^1(\mathbb{R}_+)$ is an scalar kernel [14]. Using operator-valued Fourier multipliers techniques, they obtained necessary and sufficient conditions to guarantee the existence and uniqueness of periodic solutions for the equation (1.2) in $L^p(\mathbb{T}; X)$, $B^s_{p,q}(\mathbb{T}; X)$ and $F^s_{p,q}(\mathbb{T}; X)$.

Bu studied the second order degenerate equations:

$$(Mu')'(t) = Au(t) + f(t), \quad (0 \leq t \leq 2\pi) \tag{1.3}$$

with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$, where A and M are closed linear operators on a complex Banach space X satisfying $D(A) \subset D(M)$ [7]. Under suitable conditions on the modified resolvent operator determined by (1.3), he obtained necessary or sufficient conditions for the well-posedness of (1.3) in $L^p(\mathbb{T}; X)$, $B^s_{p,q}(\mathbb{T}; X)$ and $F^s_{p,q}(\mathbb{T}; X)$. See the monograph by Favini and Yagi [9] for detailed studies of abstract degenerate type differential equations.

On the other hand, existence and uniqueness of Hölder continuous solutions for differential equations have been extensively studied in the literature. See [1,8,10,16] and the references therein. The C^α-well-posedness of the first order degenerate differential equations on the line

$$(Mu)'(t) = Au(t) + f(t), \quad (t \in \mathbb{R}) \tag{1.4}$$

were recently studied independently in [16] by Ponce and in [8] by Bu. They have shown that (1.4) is C^α-well-posed if and only if $i\mathbb{R} \subset \rho_M(A)$ and $\sup_{s\in\mathbb{R}} \|sM(isM - A)^{-1}\| < \infty$ (Bu obtained this result under the assumption that $D(A) \subset D(M)$, while Ponce has independently shown that the result is true under the weaker assumption that $D(A) \cap D(M) \neq \{0\}$).

In this paper, we consider the well-posedness of the first order degenerate differential equations with infinite delay:

$$(Mu)'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad (t \in \mathbb{R}) \tag{P}$$

on Hölder continuous function spaces $C^\alpha(\mathbb{R}; X)$, where A and M are closed operators on a complex Banach space X such that $D(A) \cap D(M) \neq \{0\}$, $0 < \alpha < 1$, $a \in L^1_{\text{loc}}(\mathbb{R}_+) \cap L^1(\mathbb{R}; t^\alpha dt)$.

We say that (P) is C^α-well-posed, if for every $f \in C^\alpha(\mathbb{R}; X)$, there exists a unique $u \in C^\alpha(\mathbb{R}; D(A) \cap D(M))$, such that $Mu \in C^{1+\alpha}(\mathbb{R}; X)$, and (P) is satisfied for all $t \in \mathbb{R}$, here we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms, $D(A) \cap D(M)$ is equipped with the sum norm $\|x\| + \|Ax\| + \|Mx\|$, $C^{1+\alpha}(\mathbb{R}; X)$ is the space of all C^1-functions $u : \mathbb{R} \to X$ satisfying $u' \in C^\alpha(\mathbb{R}; X)$. Using known $\dot{C}^\alpha$-Fourier multiplier results by Arendt, Batty and Bu [1], we completely characterize the C^α-well-posedness of (P): when $0 < \alpha < 1$ and a is 2-regular, then (P) is C^α-well-posed if and only if $\mathbb{R} \subset \rho_{a,M}(A)$ and

$$\sup_{s\in\mathbb{R}} \left\| isM(isM - (1 + a_s)A)^{-1} \right\| < \infty,$$

where $\rho_{a,M}(A)$ is the (a, M)-modified resolvent set of A, a_s is the Fourier transform of a at s. Our result may be regarded as a generalization of the above mentioned results by in Ponce [16] and Bu [8] when $a = 0$. Thus our results also generalize the previous known results obtained in [1] when $M = I_X$ and $a = 0$.

We give concrete examples that our abstract may be applied. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and m be a non-negative bounded measurable function defined on Ω. Let X be the Hilbert space $H^{-1}(\Omega)$. Let M be the multiplication operator by m on $H^{-1}(\Omega)$ with domain of definition $D(M)$. We assume that $D(M) \cap D(\Delta) \neq \{0\}$, where Δ is the Laplacian operator on $H^{-1}(\Omega)$ with Dirichlet boundary condition. If the kernel $a \in L^1_{\text{loc}}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+; t^\alpha dt)$ is 1-regular and $\mathbb{R} \subset \rho_{a,M}(A)$. Then the corresponding problem (P) is C^α-well-posed.

Our abstract result may be also applied in the following situation: let H be a complex Hilbert space and let P be a densely defined positive selfadjoint operator on H with $P \geq \delta > 0$. Let $M = P - \epsilon$ with $\epsilon < \delta$, and let $A = \sum_{i=0}^k a_i P^i$ with $a_i \geq 0$, $a_k > 0$. If the kernel $a \in L^1_{\text{loc}}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+; t^\alpha dt)$ is 1-regular and $\mathbb{R} \subset \rho_{a,M}(A)$. Then the corresponding problem (P) is C^α-well-posed.

This paper is organized as follows: in the next section, we give some preliminaries; in section 3 we establish our main result; in the last section we give some concrete examples that our abstract results may be applied.

2. Preliminaries

Let X be a complex Banach space and let $0 < \alpha < 1$. We use $C^\alpha(\mathbb{R}; X)$ to denote the space of all X-valued functions u defined on $\mathbb{R}$ satisfying

$$\|u\|_\alpha := \sup_{s \neq t} \frac{\|u(s) - u(t)\|}{|s - t|^\alpha} < \infty.$$

Let

$$\|u\|_{C^\alpha(\mathbb{R};X)} := \|u(0)\| + \|u\|_\alpha.$$

It is easy to see that $C^\alpha(\mathbb{R}; X)$ equipped with norm $\|\cdot\|_{C^\alpha(\mathbb{R};X)}$ becomes a Banach space. The kernel of the seminorm $\|\cdot\|_\alpha$ on $C^\alpha(\mathbb{R}; X)$ is the space of all constant functions. The corresponding quotient space $\dot{C}^\alpha(\mathbb{R}; X)$ is also a Banach space under the quotient norm. We will identify a function $u \in C^\alpha(\mathbb{R}; X)$ with its equivalent class $\dot{u} := \{v \in C^\alpha(\mathbb{R}; X) : \ u - v \equiv \text{constant}\}$ in $\dot{C}^\alpha(\mathbb{R}; X)$.

Let X, Y be Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. When $X = Y$, we will simply denote it by $\mathcal{L}(X)$.

Now we introduce the notion of $\dot{C}^\alpha$-multiplier which has been studied in [1].

Definition 2.1. Let X, Y be complex Banach spaces, $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ be continuous. We say that m is a $\dot{C}^\alpha$-Fourier multiplier if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R}; X) \to \dot{C}^\alpha(\mathbb{R}; Y)$ such that

$$\int_{\mathbb{R}} (Lf)(s)\mathcal{F}\varphi(s)\, ds = \int_{\mathbb{R}} \mathcal{F}(\varphi m)(s) f(s)\, ds \tag{2.1}$$

for all $f \in C^\alpha(\mathbb{R}; X)$ and all $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$, where $C_c^\infty(\mathbb{R} \setminus \{0\})$ is the space of all C^∞-functions on $\mathbb{R} \setminus \{0\}$ with compact support in $\mathbb{R} \setminus \{0\}$, $\mathcal{F}$ is the Fourier transform given by

$$(\mathcal{F}h)(s) := \tilde{h}(s) := \int_{\mathbb{R}} h(t)e^{-ist} dt, \quad (s \in \mathbb{R})$$

when $h \in L^1(\mathbb{R}; X)$.

Remark 2.1. (1) By [1, Lemma 5.1], the right-hand side of (2.1) does not depend on the representative of $\dot{f}$ as

$$\int_{\mathbb{R}} \mathcal{F}(\varphi m)(s)ds = 2\pi(\varphi m)(0) = 0.$$

Moreover, the identity (2.1) defines $Lf \in C^\alpha(\mathbb{R}; X)$ uniquely up to an additive constant by [1, Lemma 5.1].

(2) The test function space $C_c^\infty(\mathbb{R} \setminus \{0\})$ in the definition 2.1 can be replaced by the lager space $C_c^2(\mathbb{R} \setminus \{0\})$ consisting of all C^2-functions on $\mathbb{R} \setminus \{0\}$ having compact support in $\mathbb{R} \setminus \{0\}$. This follows from the fact that if $\phi \in C_c^2(\mathbb{R} \setminus \{0\})$, then $\phi * \rho_n \in C_c^\infty(\mathbb{R} \setminus \{0\})$ when n is big enough, where ρ_n is a sequence of mollifying functions, and $f * \rho_n \to f$ in $L^1(\mathbb{R})$ for all $f \in L^1(\mathbb{R})$ [4, Théorème IV.22].

(3) Let X be a Banach space, let $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ be a $\dot{C}^\alpha$-Fourier multiplier, and let $h : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ be a C^2-function such that hI_Y is a $\dot{C}^\alpha$-Fourier multiplier. Then hm is a $\dot{C}^\alpha$-Fourier multiplier. This follows easily from the definition 2.1 and (2).

We need the following result which gives a sufficient condition for a C^2-function to be a $\dot{C}^\alpha$-Fourier multiplier [1].

Theorem 2.1. *Let X, Y be Banach spaces and $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ be a C^2-function satisfying:*

$$\sup_{s \neq 0} \Big(\|m(s)\| + \|sm'(s)\| + \|s^2 m''(s)\| \Big) < \infty, \tag{2.2}$$

then m is a $\dot{C}^\alpha$-Fourier multiplier whenever $0 < \alpha < 1$. If X has a non trivial Fourier type $r > 1$, $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ is a C^1-function satisfying the first order condition:

$$\sup_{s \neq 0} \Big(\|m(s)\| + \|sm'(s)\| \Big) < \infty, \tag{2.3}$$

then m is a $\dot{C}^\alpha$-Fourier multiplier whenever $0 < \alpha < 1$.

Recall that a Banach space X has Fourier type $r \in [1, 2]$ if there exists $C_r > 0$ such that

$$\|\mathcal{F}f\|_{r'} \leq C_r \|f\|_r$$

for all $f \in L^r(\mathbb{R}; X)$, where $\frac{1}{r} + \frac{1}{r'} = 1$ [15]. The trivial estimate $\|\mathcal{F}f\|_\infty \leq \|f\|_1$ shows that each Banach space has Fourier type 1. A Banach space has Fourier type 2 if and only if it is isomorphic to a Hilbert space [11] (see also [12, p 73 and p 74]). A space $L^q(\Omega, \Sigma, \mu)$ has Fourier type $r = min\{q, q'\}$ [15]. Each closed subspace and each quotient space of a Banach space X has the same Fourier type as X. Bourgain has shown that each B-convex Banach space (thus, in particular, each uniformly convex Banach space) has some non trivial Fourier type $r > 1$ [5, 6].

Let $0<\alpha<1$, we denote by $C^{1+\alpha}(\mathbb{R};X)$ the space of all X-valued functions u defined on $\mathbb{R}$, such that $u\in C^1(\mathbb{R};X)$ and $u'\in C^\alpha(\mathbb{R};X)$. The space $C^{1+\alpha}(\mathbb{R};X)$ is equipped with the following norm

$$\|u\|_{C^{1+\alpha}(\mathbb{R};X)} := \|u(0)\| + \|u'\|_{C^\alpha(\mathbb{R};X)},$$

and it is a Banach space. It follows from [1, Lemma 6.2] that if $u,v\in C^\alpha(\mathbb{R};X)$, then $u\in C^{1+\alpha}(\mathbb{R};X)$ and $u'=v+x$ for some $x\in X$ if and only if

$$\int_{\mathbb{R}} u(s)\mathcal{F}(\mathrm{id}\cdot\varphi)(s)ds = \int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds \tag{2.4}$$

whenever $\varphi\in C_c^\infty(\mathbb{R}\setminus\{0\})$, where $\mathrm{id}(s):=is$ when $s\in\mathbb{R}$.

Let $0<\alpha<1$, we denote by $L^1(\mathbb{R}_+,t^\alpha dt)$ the space of all scalar functions a defined on $\mathbb{R}_+$ such that $\int_0^\infty |a(t)|\,t^\alpha dt<\infty$. It is noted that if $a\in L^1(\mathbb{R}_+,t^\alpha dt)\cap L^1_{\mathrm{loc}}(\mathbb{R}_+)$, then $a\in L^1(\mathbb{R}_+)$.

Let $a\in L^1(\mathbb{R}_+,t^\alpha dt)\cap L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $u\in C^\alpha(\mathbb{R};X)$, we define

$$(a*u)(t) := \int_0^\infty a(s)u(t-s)ds = \int_{-\infty}^t a(t-s)u(s)ds,\ (t\in\mathbb{R}). \tag{2.5}$$

It is easy to verify that $a*u$ is well defined as

$$\|u(t-s)\| \le \|u(t)\| + \|u\|_\alpha |s|^\alpha$$

whenever $s,t\in\mathbb{R}$. $a*u\in C^\alpha(\mathbb{R};X)$ and

$$\|a*u\|_\alpha \le \|a\|_{L^1}\|u\|_\alpha . \tag{2.6}$$

Definition 2.2. We say that $a\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is 1-regular, if $\tilde a$ is differentiable on $\mathbb{R}$ and $\sup_{s\in\mathbb{R}}\left|s^n[\tilde a(s)]^{(n)}\right|<\infty$ when $n=0,1$. We say that $a\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is 2-regular, if $\tilde a$ is 2-times differentiable on $\mathbb{R}$ and $\sup_{s\in\mathbb{R}}\left|s^n[\tilde a(s)]^{(n)}\right|<\infty$ when $n=0,1,2$.

We notice that if a is 2-regular, then $\tilde a I_X$ is a $\dot C^\alpha$-Fourier multiplier whenever $0<\alpha<1$ by Theorem 2.1. When X has a non trivial Fourier type and a is 1-regular, then $\tilde a I_X$ is a $\dot C^\alpha$-Fourier multiplier whenever $0<\alpha<1$ by Theorem 2.1.

Let $u\in L^1_{\mathrm{loc}}(\mathbb{R};X)$. We say that u is of subexponential growth, if for all $\epsilon>0$

$$\int_{-\infty}^\infty e^{-\epsilon|t|}\|u(t)\|dt<\infty.$$

For such function u, we define its Carleman transform on $\mathbb{C}\setminus i\mathbb{R}$ by

$$\hat u(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t}u(t)dt, & Re\lambda>0,\\ -\int_0^\infty e^{\lambda t}u(-t)dt, & Re\lambda<0.\end{cases}$$

[2, page 292].

In what follows, we always assume that the scalar kernel a satisfies $\tilde a(s)\neq -1$ for all $s\in\mathbb{R}$, and we use the following notation:

$$a_s := \tilde a(s),\ (s\in\mathbb{R}). \tag{2.7}$$

3. The C^α-Well-Posednees of (P)

Let X be a complex Banach space, let $A : D(A) \to X$ and $M : D(M) \to X$ be closed linear operators on X satisfying $D(A) \cap D(M) \neq \{0\}$, $0 < \alpha < 1$ and let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+; t^\alpha dt)$. We consider the well-posedness of the first order degenerate differential equations with infinite delay

$$(Mu)'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad (t \in \mathbb{R}), \tag{P}$$

on Hölder continuous function spaces $C^\alpha(\mathbb{R}; X)$.

Definition 3.1. We say that (P) is C^α-well-posed, if for all $f \in C^\alpha(\mathbb{R}; X)$, there exists a unique $u \in C^\alpha(\mathbb{R}; D(A) \cap D(M))$, such that $Mu \in C^{1+\alpha}(\mathbb{R}; X)$, and (P) is satisfied for all $t \in \mathbb{R}$, here $D(A) \cap D(M)$ is equipped with the sum norm $\|x\| + \|Ax\| + \|Mx\|$ so that it becomes a Banach space.

We define the (a, M)-modified resolvent of A by

$$\begin{aligned}\rho_{a,M}(A) := \big\{s \in \mathbb{C} : isM - (1 + a_s)A : D(A) \cap D(M) \to X \\ \text{is a bijection and } (isM - (1 + a_s)A)^{-1} \in \mathcal{L}(X)\big\}.\end{aligned}$$

Let $s \in \rho_{a,M}(A)$. Then $(isM - (1 + a_s)A)^{-1} \in \mathcal{L}(X)$ is a bijection from X onto $D(A) \cap D(M)$ by definition. This implies that $M(isM - (1 + a_s)A)^{-1}$, $A(isM - (1 + a_s)A)^{-1} \in \mathcal{L}(X)$ by the closed graph theorem and the closedness of A and M. Consequently $(isM - (1 + a_s)A)^{-1} \in \mathcal{L}(X, D(A) \cap D(M))$.

The following results give a necessary and sufficient condition for (P) to be C^α-well-posed.

Theorem 3.1. *Let X be a complex Banach space, $0 < \alpha < 1$, $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, t^\alpha dt)$ and let A, M be closed linear operators on X satisfying $D(A) \cap D(M) \neq \{0\}$. Suppose that the kernel a is 2-regular. Then (P) is C^α-well-posed if and only if $\mathbb{R} \subset \rho_{a,M}(A)$ and*

$$\sup_{s \in \mathbb{R}} \big\|isM(isM - (1 + a_s)A)^{-1}\big\| < \infty.$$

Proof. Assume that $\mathbb{R} \subset \rho_{a,M}(A)$ and $\sup_{s\in\mathbb{R}} \big\|isM(isM - (1 + a_s)A)^{-1}\big\| < \infty$. Then $A : D(A) \cap D(M) \to X$ is invertible and its inverse $A^{-1} \in \mathcal{L}(X)$ as $0 \in \rho_{a,M}(A)$ by assumption and (2.7) (we should notice that the operator $A : D(A) \to X$ is necessarily surjective, it is injective if and only if $D(A) \subset D(M)$). Let

$$m(s) := (isM - (1 + a_s)A)^{-1},\ p(s) := isMm(s),\ q(s) := a_sAm(s),\ (s \in \mathbb{R}).$$

It is easy to verify that m, p and q are $\mathcal{L}(X)$-valued C^∞-functions on $\mathbb{R}$ [8]. We have

$$\sup_{s\in\mathbb{R}} \big\|p(s)\big\| < \infty, \quad \sup_{s\in\mathbb{R}} \big\|q(s)\big\| < \infty, \quad \sup_{s\in\mathbb{R}} \big\|Mm(s)\big\| < \infty \tag{3.1}$$

by assumption and the identity $isMm(s) - (1 + a_s)Am(s) = I_X$. Here we have used the fact that $\lim_{|s|\to\infty} a_s = 0$ by Riemann-Lebesgue Lemma as $a \in L^1(\mathbb{R}_+)$ by assumption, so that

$$\sup_{s\in\mathbb{R}} \big\|Am(s)\big\| < \infty, \quad \sup_{s\in\mathbb{R}} \big\|m(s)\big\| < \infty, \tag{3.2}$$

as $A^{-1} \in \mathcal{L}(X)$. We have

$$m'(s) = -m(s)(iM - a'_s A)m(s), \ (s \in \mathbb{R}) \tag{3.3}$$

and

$$m''(s) = 2m(s)(iM - a'_s A)m(s)(iM - a'_s A)m(s) + m(s)a''_s Am(s), \ (s \in \mathbb{R}). \tag{3.4}$$

It follows that

$$\begin{aligned} \sup_{s\in\mathbb{R}} \|sm'(s)\| < \infty, &\quad \sup_{s\in\mathbb{R}} \|s^2 m''(s)\| < \infty, \\ \sup_{s\in\mathbb{R}} \|sAm'(s)\| < \infty, &\quad \sup_{s\in\mathbb{R}} \|s^2 Am''(s)\| < \infty, \\ \sup_{s\in\mathbb{R}} \|sMm'(s)\| < \infty, &\quad \sup_{s\in\mathbb{R}} \|s^2 Mm''(s)\| < \infty \end{aligned}$$

by (3.1) and (3.2) and the assumption that a is 2-regular. Consequently, m, Am and Mm are $\dot{C}^\alpha$-Fourier multipliers by Theorem 2.1. This implies that, considering $m \in \mathcal{L}(X, D(A) \cap D(M))$, m is a $\dot{C}^\alpha$-Fourier multiplier. Since a is 2-regular, $a_s I_X$ is a $\dot{C}^\alpha$-Fourier multiplier by Theorem 2.1. So q is also a $\dot{C}^\alpha$-Fourier multiplier. Here we have used the second statement of Remark 2.1. For the function p, we have

$$p'(s) = iMm(s) - isMm(s)(iM - a'_s A)m(s), \tag{3.5}$$

and

$$\begin{aligned} p''(s) = &-2iMm(s)(iM - a'_s A)m(s) \\ &+ 2isMm(s)(iM - a'_s A)m(s)(iM - a'_s A)m(s) + isMm(s)a''_s Am(s). \end{aligned} \tag{3.6}$$

It follows that

$$\sup_{s\in\mathbb{R}} \|sp'(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2 p''(s)\| < \infty$$

by (3.1) and (3.2) and the assumption that a is 2-regular. Hence p is also a $\dot{C}^\alpha$-Fourier multiplier by Theorem 2.1.

Let $f \in C^\alpha(\mathbb{R}; X)$ be fixed. Then there exist $u \in C^\alpha(\mathbb{R}; D(A) \cap D(M))$ and $v, w \in C^\alpha(\mathbb{R}; X)$, such that

$$\int_{\mathbb{R}} u(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi m)(s)f(s)ds, \tag{3.7}$$

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\varphi p)(s)f(s)ds, \tag{3.8}$$

$$\int_{\mathbb{R}} w(s)(\mathcal{F}\psi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\psi q)(s)f(s)ds \tag{3.9}$$

for all $\phi, \varphi, \psi \in C_c^2(\mathbb{R} \setminus \{0\})$. We have Au, $Mu \in C^\alpha(\mathbb{R}; X)$ as $u \in C^\alpha(\mathbb{R}; D(A) \cap D(M))$. It follows from (3.7) and the closedness of A and M that

$$\int_{\mathbb{R}} Mu(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi Mm)(s)f(s)ds \tag{3.10}$$

and

$$\int_{\mathbb{R}} Au(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi Am)(s)f(s)ds \tag{3.11}$$

when $\phi \in C_c^2(\mathbb{R} \setminus \{0\})$. Choosing $\phi = \mathrm{id} \cdot \varphi$ in (3.10), where $\mathrm{id}(s) := is$ when $s \in \mathbb{R}$, we obtain from (3.8) that

$$\int_{\mathbb{R}} Mu(s)\mathcal{F}(\mathrm{id} \cdot \varphi)(s)ds = \int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds$$

whenever $\varphi \in C_c^2(\mathbb{R} \setminus \{0\})$. Thus $Mu \in C^{1+\alpha}(\mathbb{R}; X)$ and $(Mu)'(t) = v(t) + y_1$ for some $y_1 \in X$ by [1, Lemma 6.2].

Choosing $\phi(s) = a_s\psi(s)$ in (3.11), it follows from (3.9) that that

$$\int_{\mathbb{R}} Au(s)\mathcal{F}(a_s \cdot \psi)(s)ds = \int_{\mathbb{R}} w(s)(\mathcal{F}\psi)(s)ds$$

whenever $\psi \in C_c^2(\mathbb{R} \setminus \{0\})$. Thus $w(t) = (a * Au)(t) + y_2$ for some $y_2 \in X$ by [17, Lemma 3.3].

Now the identity $\mathrm{id} \cdot Mm(s) = (1 + a_s)Am(s) + I_X$ together with (3.8) and (3.9) implies that

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} Au(s)(\mathcal{F}\varphi)(s)ds + \int_{\mathbb{R}} w(s)(\mathcal{F}\varphi)(s)ds + \int_{\mathbb{R}} f(s)(\mathcal{F}\varphi)(s)ds$$

for all $\varphi \in C_c^2(\mathbb{R} \setminus \{0\})$. Therefore there exits $y_3 \in X$ such that $(Mu)'(t) = Au(t) + (a * Au)(t) + f(t) + y_1 + y_2 + y_3$ by [1, Lemma 6.2]. Let A^{-1} be the inverse of the bijection $A : D(A) \cap D(M) \to X$ and let $x = ((1 + a_0)A)^{-1}(y_1 + y_2 + y_3)$. Then $x \in D(A) \cap D(M)$ and $u_1 = u + x$ solves (P). We note that the vector x is well defined as we have assumed in (2.7) that $a_s \neq -1$ whenever $s \in \mathbb{R}$. This shows the existence.

To show the uniqueness, we let $u \in C^\alpha(\mathbb{R}; D(A) \cap D(M))$ be such that $Mu \in C^{1+\alpha}(\mathbb{R}; X)$ and

$$(Mu)'(t) = Au(t) + (a * Au)(t) \tag{3.12}$$

when $t \in \mathbb{R}$. We are going to show that $u = 0$ using a similar argument used in [10]. For $v \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$ of subexponential growth, we let

$$L_\sigma(v)(\rho) := \hat{v}(\sigma + i\rho) - \hat{v}(-\sigma + i\rho),$$

where $\sigma > 0$ and $\rho \in \mathbb{R}$. By [10, Proposition A.2], if $v \in C^{1+\alpha}(\mathbb{R}; X)$, then

$$L_\sigma(v')(\rho) = (\sigma + i\rho)L_\sigma(v)(\rho) + 2\sigma\hat{v}(-\sigma + i\rho).$$

Again by [10, Proposition A.2], if $v \in C^\alpha(\mathbb{R}; X)$ and $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, t^\alpha dt)$, then

$$L_\sigma(a * v)(\rho) = \hat{a}(\sigma + i\rho)L_\sigma(v)(\rho) + G_a^v(\sigma, \rho)$$

with

$$\lim_{\sigma \to 0^+} \int_{\mathbb{R}} G_a^v(\sigma, \rho)\phi(\rho)d\rho = 0$$

for all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$. It is clear from the closedness of M that $L_\sigma(u)(\rho) \in D(A) \cap D(M)$ and $L_\sigma(Mu)(\rho) = ML_\sigma(u)(\rho)$, $L_\sigma(Au)(\rho) = AL_\sigma(u)(\rho)$. It follows from (3.12) that

$$\begin{aligned}&(\sigma + i\rho)ML_\sigma(u)(\rho) + 2\sigma M\hat{u}(-\sigma + i\rho)\\ =&AL_\sigma(u)(\rho) + \hat{a}(\sigma + i\rho)AL_\sigma(v)(\rho) + G_a^{Av}(\sigma, \rho),\end{aligned}$$

which implies

$$\begin{aligned}&[(\sigma+i\rho)M-(1+\hat{a}(\sigma+i\rho)A]L_\sigma(u)(\rho)\\=&G_a^{Av}(\sigma,\rho)-2\sigma M\hat{u}(-\sigma+i\rho).\end{aligned}$$

Thus

$$\begin{aligned}L_\sigma(u)(\rho)=&R_\rho G_a^{Av}(\sigma,\rho)-2\sigma R_\rho M\hat{u}(-\sigma+i\rho)\\&+R_\rho[-\sigma M+(\hat{a}(\sigma+i\rho)-\hat{a}(i\rho))A]L_\sigma(u)(\rho),\end{aligned}$$

where $R_\rho := (i\rho M-(1+\hat{a}(i\rho))A)^{-1}$. Using Lebesgue dominated convergence Theorem, we obtain

$$\lim_{\sigma\to 0^+}\int_{-\infty}^{\infty}R_\rho G_a^{Av}(\sigma,\rho)\phi(\rho)d\rho=0$$

for all $\phi\in C_c^\infty(\mathbb{R}\setminus\{0\})$. A similar argument used in the proof of [10, Lemma A4] shows that

$$\lim_{\sigma\to 0^+}\int_{-\infty}^{\infty}R_\rho[-\sigma M+(\hat{a}(\sigma+i\rho)-\hat{a}(i\rho))A]L_\sigma(u)(\rho)\phi(\rho)d\rho=0$$

and

$$\lim_{\sigma\to 0^+}\int_{-\infty}^{\infty}\sigma R_\rho M\hat{u}(-\sigma+i\rho)\phi(\rho)d\rho=0$$

for all $\phi\in C_c^\infty(\mathbb{R}\setminus\{0\})$. We conclude that

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}L_\sigma(u)\phi(\rho)d\rho=\int_{\mathbb{R}}u(\rho)\mathcal{F}(\phi)(\rho)d\rho=0$$

for all $\phi\in C_c^\infty(\mathbb{R}\setminus\{0\})$. Consequently u is constant function by [2, Theorem 4.8.1, Theorem 4.8.2]. That is $u\equiv x$ for some $x\in D(A)\cap D(M)$. We have by (3.12)

$$0=(Mu)'(t)=Ax+(a*Ax)(t)=(1+a_0)x$$

which implies that $x=0$ as we have assumed that $a_s\neq 1$ for all $s\in\mathbb{R}$. We have shown the uniqueness. Hence (P) is C^α-well-posed.

Conversely, assume that (P) is C^α-well-posed. Let $L: C^\alpha(\mathbb{R};X)\to S(\mathbb{R};X)$ be the solution operator which associates to each $f\in C^\alpha(\mathbb{R};X)$ the unique solution of (P), where $S(\mathbb{R};X)$ is the solution space of (P) consisting of all $u\in C^\alpha(\mathbb{R};D(A)\cap D(M))$, such that $Mu\in C^{1+\alpha}(\mathbb{R};X)$. $S(\mathbb{R};X)$ equipped with the norm

$$\|u\|_{S(\mathbb{R};X)}:=\|u\|_{C^\alpha(\mathbb{R};D(A)\cap D(M))}+\|Mu\|_{C^{1+\alpha}(\mathbb{R};X)} \tag{3.13}$$

is a Banach space. L is linear and bounded by the closed graph theorem.

Let $s\in\mathbb{R}$, $s\neq 0$ be fixed , we are going to show that $s\in\rho_{a,M}(A)$. Let $x\in D(A)\cap D(M)$ be such that $isMx=(1+a_s)Ax$ and $u=e_s\otimes x$, where $(e_s\otimes x)(t):=e^{ist}x$ when $t\in\mathbb{R}$. Then $u\in C^\alpha(\mathbb{R};D(A)\cap D(M))$. It is clear that $Mu=e_s\otimes Mx\in C^{1+\alpha}(\mathbb{R};X)$ and $(Mu)'(t)=Au(t)+(a*Au)(t)$ for all $t\in\mathbb{R}$. This means that $u\in S(\mathbb{R};X)$ and u solves (P) when taking $f=0$. Hence $u=0$ by the uniqueness of the solution of (P). Consequently $x=0$. We have shown that $isM-(1+a_s)A: D(A)\cap D(M)\to X$ is injective.

To show that $isM-(1+a_s)A$ is also surjective, we let $y\in X$ be fixed and consider $f:=e_s\otimes y\in C^\alpha(\mathbb{R};X)$. Let $u\in S(\mathbb{R};X)$ be the unique solution of (P), that is

$$(Mu)'(t)=Au(t)+(a*Au)(t)+f(t)$$

for all $t\in\mathbb{R}$. For fixed $\xi\in\mathbb{R}$, we consider the function u_ξ given by $u_\xi(t)=u(t+\xi)$ when $t\in\mathbb{R}$. Then both functions u_ξ and $e^{i\xi s}u$ are in $S(\mathbb{R};X)$ and solve the problem

$$(Mv)'(t)=Av(t)+(a*Av)(t)+e^{i\xi s}f(t).$$

We deduce from the uniqueness that $u_\xi=e^{i\xi s}u$, that is $u(t+\xi)=e^{i\xi s}u(t)$ for $t,\xi\in\mathbb{R}$. Let $x=u(0)$. Then $x\in D(A)\cap D(M)$ and $u=e_s\otimes x$. Since u solves $(Mu)'(t)=Au(t)+(a*Au)(t)+f(t)$, we have $ise_s\otimes Mx=(1+a_s)Ae_s\otimes x+e_s\otimes y$. Letting $t=0$, we obtain $isMx-(1+a_s)Ax=y$. This shows that $isM-(1+a_s)A$ is surjective. Thus $isM-(1+a_s)A$ is a bijection from $D(A)\cap D(M)$ onto X and $x=(isM-(1+a_s)A)^{-1}y$. We have shown that $u=e_s\otimes(isM-(1+a_s)A)^{-1}y$. Therefore

$$\begin{aligned}\gamma_\alpha|s|^\alpha\|isM(isM-(1+a_s)A)^{-1}y\| &= \|ise_s\otimes M(isM-(1+a_s)A)^{-1}y\|_\alpha\\ &= \|(Mu)'\|_\alpha\le\|Mu\|_{C^{1+\alpha}}\\ &\le\|L\|\|f\|_{C^\alpha}=\|L\|(\|f\|_\alpha+\|f(0)\|)\\ &=\|L\|(\gamma_\alpha|s|^\alpha\|y\|+\|y\|),\end{aligned}\tag{3.14}$$

where $\gamma_\alpha:=\|e_1\|_\alpha=\sup_{t>0}t^{-\alpha}|\sin(t/2)|$ is a constant depending only on α [1, (3.1)]. It follows from (3.14) that when $s\neq0$

$$\|isM(isM-(1+a_s)A)^{-1}\|\le\|L\|(1+\gamma_\alpha^{-1}|s|^{-\alpha}).$$

It remains to show that $0\in\rho_{a,M}(A)$. For $s>0$

$$\begin{aligned}\|M(isM-(1+a_s)A)^{-1}y\| &= \|(Mu)(0)\|\le\|Mu\|_{C^{1+\alpha}}\\ &\le\|L\|\|f\|_{C^\alpha}=\|L\|(\gamma_\alpha|s|^\alpha\|y\|+\|y\|),\end{aligned}$$

which implies that

$$\sup_{0<s\le1}\|M(isM-(1+a_s)A)^{-1}y\|<\infty,\tag{3.15}$$

and

$$\sup_{0<s\le1}\|A(isM-(1+a_s)A)^{-1}y\|<\infty\tag{3.16}$$

as a_s is continuous on s and $a_s\neq-1$ for all $s\in\mathbb{R}$. For $s>0$, we have

$$(1+a_0)A=[1+((a_0-a_s)A+isM)((1+a_s)A-isM)^{-1}]((1+a_s)A-isM).\tag{3.17}$$

It follows from (3.15), (3.16) and the uniform continuity of a_s for $s\in[0,1]$ that, when $0<s\le1$ is small enough the operator

$$1+((a_0-a_s)A+isM)((1+a_s)A-isM)^{-1}$$

is invertible on X. Thus $(1+a_0)A$ is a bijection from $D(A)\cap D(M)$ onto X and $[(1+a_0)A]^{-1}\in\mathcal{L}(X)$. We have shown that $0\in\rho_{a,M}(A)$. The proof is complete. $\square$

When the underlying Banach spaces X and Y have non trivial Fourier types, the first order condition (2.3) is already sufficient for a C^1-function $m:\mathbb{R}\setminus\{0\}\to\mathcal{L}(X,Y)$ to be a $\dot{C}^\alpha$-Fourier multiplier. This observation together with the proof of Theorem 3.1 gives the following result.

Corollary 3.1. *Let X be a complex Banach space with a non trivial Fourier type, $0<\alpha<1$, $a\in L^1_{\text{loc}}(\mathbb{R}_+)\cap L^1(\mathbb{R}_+,t^\alpha dt)$ and let A,M be closed linear operators on X satisfying $D(A)\cap D(M)\neq\{0\}$. Suppose that the kernel a is 1-regular. Then (P) is C^α-well-posed if and only if $\mathbb{R}\subset\rho_{a,M}(A)$ and*

$$\sup_{s\in\mathbb{R}}\left\|isM(isM-(1+a_s)A)^{-1}\right\|<\infty.$$

4. Applications

In the last section, we give some examples that our abstract results (Theorem 3.1 and Corollary 3.1) may be applied.

Example 4.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and m be a non-negative bounded measurable function defined on Ω. Let X be the Hilbert space $H^{-1}(\Omega)$. We consider the following first order degenerate differential equations with infinite delay:

$$\begin{cases}\frac{\partial(m(x)u(t,x))}{\partial t}=\Delta u(t,x)+\int_{-\infty}^t a(t-s)(\Delta u)(s,x)ds+f(t,x),\ (t,x)\in\mathbb{R}\times\Omega,\\ u(t,x)=0,\qquad (t,x)\in\mathbb{R}\times\partial\Omega,\end{cases}$$

where $a\in L^1_{\text{loc}}(\mathbb{R}_+)\cap L^1(\mathbb{R}_+;t^\alpha dt)$ is a fixed 1-regular kernel and the Laplacian operator Δ acts on the second variable $x\in\Omega$.

Let M be the multiplication operator by m on $H^{-1}(\Omega)$ with domain of definition $D(M)$. We assume that $D(M)\cap D(\Delta)\neq\{0\}$, where Δ is the Laplacian operator on $H^{-1}(\Omega)$ with Dirichlet boundary condition. Then it follows from [9, Section 3.7] that there exists a constant $C\geq 0$ such that

$$\left\|M(zM-\Delta)^{-1}\right\|\leq\frac{C}{1+|z|},$$

when $\mathrm{Re(z)}\geq-\alpha(1+|\mathrm{Im(z)}|)$ for some positive constant α depending only on m, which implies that there exists $C_1\geq 0$, such that

$$\left\|M(isM-\Delta)^{-1}\right\|\leq\frac{C_1}{1+|s|},\quad \left\|\Delta(isM-\Delta)^{-1}\right\|\leq C_1 \tag{4.1}$$

when $s\in\mathbb{R}$. If we assume that m^{-1} is regular enough so that the multiplication operator by the function m^{-1} is bounded on $H^{-1}(\Omega)$, then there exists a constant $C_2\geq 0$ such that

$$\left\|(isM-\Delta)^{-1}\right\|\leq\frac{C_2}{1+|s|} \tag{4.2}$$

when $s \in \mathbb{R}$. Furthermore we assume that $\mathbb{R} \subset \rho_{a,M}(\Delta)$ so that for all $s \in \mathbb{R}$, the operator $isM - (1+a_s)A$ is a bijection from $D(M) \cap D(\Delta)$ onto X, and $(isM - (1+a_s)A)^{-1} \in \mathcal{L}(X)$. We observe that

$$sM(isM - (1+a_s)\Delta)^{-1} = sM(isM - \Delta)^{-1}[1 - a_s\Delta(isM - \Delta)^{-1}]^{-1},$$

which implies that

$$\sup_{s\in\mathbb{R}} \|sM(isM - (1+a_s)\Delta)^{-1}\| < \infty$$

by (4.1) and the fact that $\lim_{s\to\infty} a_s = 0$. We deduce from Corollary 3.1 that the above problem is C^α-well-posed. Here we have used the fact that the Hilbert space $H^{-1}(\Omega)$ has Fourier type 2.

Example 4.2. Let H be a complex Hilbert space and let P be a densely defined positive selfadjoint operator on H with $P \geq \delta > 0$. Let $M = P - \epsilon$ with $\epsilon < \delta$, and let $A = \sum_{i=0}^k a_i P^i$ with $a_i \geq 0,\ a_k > 0$. Then there exists a constant $C > 0$, such that

$$\left\|M(zM + A)^{-1}\right\| \leq \frac{C}{1+|z|}$$

whenever $Rez \geq -\alpha(1+|Imz|)$ for some positive constant α depending only on A and M by [9, page 73]. This implies in particular that

$$\sup_{s\in\mathbb{R}} \|sM(isM + A)^{-1}\| < \infty.$$

If we assume $0 \in \rho(M)$, then

$$\sup_{s\in\mathbb{R}} \|s(isM + A)^{-1}\| < \infty.$$

Furthermore we assume that $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+; t^\alpha dt)$ is 1-regular and $\mathbb{R} \subset \rho_{a,M}(A)$. Then the argument used in Example 4.1 shows that the degenerate differential equations with infinite delay

$$(Mu)'(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)ds + f(t),\ (t \in \mathbb{R})$$

is C^α-well-posed.

We can also give a concrete example that the above abstract result may be applied. We consider the following degenerate problem with infinite delay:

$$\frac{\partial}{\partial t}(1 - \frac{\partial^2}{\partial x^2})u(t,x) = \frac{\partial^4}{\partial x^4}u(t,x) + \int_{-\infty}^t a(t-s)\frac{\partial^4}{\partial x^4}u(s,x)ds + f(t,x),$$

where $t \in \mathbb{R}$, $x \in \Omega := (0,1)$. Let $X = L^2(\Omega)$ and let $P = -\frac{\partial^2}{\partial x^2}$ with domain $D(P) = H^2(\Omega) \cap H^1_0(\Omega)$, i.e., P is the Laplacian on $L^2(\Omega)$ with Dirichlet boundary conditions. Then P is positive selfadjoint on X. Let $M = P + I_X$ and $A = P^2$. It is clear that $-P$ generates an contraction semigroup on $L^2(\Omega)$ [2, Example 3.4.7], hence $1 \in \rho(-P)$, or equivalently $M = I_X + P$ has a bounded inverse, i.e. $0 \in \rho(M)$. Then the abstract results obtained above may be applied: if a is 1-regular and $\mathbb{R} \subset \rho_{a,M}(A)$, then the above degenerate differential equations is C^α-well-posed for all $0 < \alpha < 1$.

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