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GLOBAL HIGHER INTEGRABILITY OF SOLUTIONS TO SUBELLIPTIC DOUBLE OBSTACLE PROBLEMS*

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Abstract In this paper we consider the double obstacle problems associated with nonlinear subelliptic equation

$$X^*A(x,u,Xu)+B(x,u,Xu)=0, \quad x\in\Omega,$$

where $X=(X_1,\dots,X_m)$ is a system of smooth vector fields defined in $\mathbb{R}^n$ satisfying Hörmander's condition. The global higher integrability for the gradients of the solutions is obtained under a capacitary assumption on the complement of the domain Ω.

Keywords Global higher integrability, subelliptic equation, double obstacle problems.

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1. Introduction

Suppose that Ω is a bounded domain of $\mathbb{R}^n$. For two given obstacle functions ψ_1,ψ_2 and a boundary value function u_0 with $\psi_1(x)\le u_0(x)\le\psi_2(x)$, we define

$$\mathcal{K}^{u_0,p}_{\psi_1,\psi_2}(\Omega)=\left\{u\in W^{1,p}_X(\Omega): u-u_0\in W^{1,p}_{X,0}(\Omega), \psi_1\le u\le\psi_2 \text{ a.e. in } \Omega\right\}.$$

Here $W^{1,p}_X(\Omega)$ and $W^{1,p}_{X,0}(\Omega)$ are Sobolev type spaces defined in next section.

This paper is concerned with the double obstacle problems for nonlinear subelliptic equations of the form:

$$X^*A(x,u,Xu)+B(x,u,Xu)=0, \tag{1.1}$$

where $X=(X_1,\dots,X_m)$ $(m\le n)$ is a system of smooth vector fields in $\mathbb{R}^n$ satisfying Hörmander's condition and $X^*=(X_1^*,\dots,X_m^*)$ is a family of operators formal adjoint to X_j in L^2. We assume that the functions $A=(A_1,\dots,A_m):\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^m$ and $B:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$ are both Carathéodory functions

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satisfying

$$A(x,u,\xi)\cdot\xi \geq \alpha|\xi|^p, \tag{1.2}$$
$$|A(x,u,\xi)| \leq \beta(|u|^{p-1}+|\xi|^{p-1}), \tag{1.3}$$
$$|B(x,u,\xi)| \leq \beta(|\xi|^{p(1-\frac{1}{\gamma})}+g(x)), \tag{1.4}$$

where α,β are positive constants; $1<p<\infty$;

$$\gamma = \begin{cases} \frac{pQ}{Q-p}, & \text{if } 1<p<Q, \\ \text{any } \gamma \geq p, & \text{if } p \geq Q; \end{cases}$$

$g(x)\in L^\nu$ with $\nu > \frac{\gamma}{\gamma-1}$. Here Q is the homogenous dimension of Ω, see the next section.

Definition 1.1. We say that u is a solution to the double obstacle problem for (1.1), provided $u \in \mathcal{K}^{u_0,p}_{\psi_1,\psi_2}(\Omega)$ and

$$\int_\Omega A(x,u,Xu)\cdot X\varphi dx + \int_\Omega B(x,u,Xu)\varphi dx \geq 0 \tag{1.5}$$

whenever $\varphi \in C_0^\infty(\Omega)$ with $\psi_1 \leq \varphi + u \leq \psi_2$ a.e. in Ω.

In the Euclidean setting, the global higher integrability of solutions to obstacle problems was first considered by Li and Martio in [21]. Under the assumption that the boundary of the domain is p-Poincaré thick, they proved the global higher integrability of solutions to single obstacle problems associated with elliptic equation

$$\text{div}A(x,\nabla u)=0,\ x\in\Omega. \tag{1.6}$$

They also derived the same result for solutions of double obstacle problems to (1.6) in [22]. In [16], Kilpeläinen and Koskela established the global integrability of the weak solutions to (1.6) with a uniform fatness condition on $\partial\Omega$ and pointed out that this condition is sharper than the p-Poincaré thickness condition in [21]. For more related papers, we also refer the readers to see [11–13, 17–20, 23–25, 31].

Since Hörmander's celebrated paper [15], there has been a tremendous amount of work on the study of regularity for solutions to the subelliptic obstacle problems (see for instance [1, 5, 9, 10, 27]) due to their important applications in mechanical engineering, mathematical finance, image reconstruction and neurophysiology. In [8, 10], Du et al. investigated the global higher integrability for very weak solutions to (1.1) and the corresponding single obstacle problems by proving a refined Sobolev type inequality with capacity term. Based on these results, we can obtain better properties of solutions, e.g. Hölder continuity (see [7, 29]).

Motivated by the above work, the aim of this paper is to study the global higher integrability of solutions to double obstacle problems related to (1.1). Before stating our main result, we first recall the definition of uniform fatness, a condition that has to be imposed on the complement of Ω. A set $E\subset\mathbb{R}^n$ is called uniformly (X,p)-fat if there exist constants $C_0, R_0>0$ such that

$$\text{cap}_p(E\cap\bar{B}(x,R),B(x,2R)) \geq C_0\ \text{cap}_p(\bar{B}(x,R),B(x,2R))$$

for all $x\in\partial E$ and $0<R<R_0$, where cap_p is the variational p-capacity defined in Section 2.

Theorem 1.1. *Suppose that the complement $\mathbb{R}^n \setminus \Omega$ of Ω is uniformly (X,p)-fat. Assume that* (1.2)-(1.4) *hold and $\psi_1, \psi_2, u_0 \in W_X^{1,r}(\Omega)$, $r > p$, u is a solution to the double obstacle problem for (1.1). Then there exists $\delta > 0$ such that $u \in W_X^{1,p+\delta}(\Omega)$.*

The remainder of the paper is organized as follows. In Section 2, we recall some basic facts of Hörmander vector fields and some preliminary results concerning the Carnot-Carathéodory metric. In Section 3, Theorem 1.1 is proved by using the refined Sobolev inequality (Lemma 2.3) and the Gehring lemma on the metric measure space (Lemma 2.4).

2. Preliminaries

Give in $\mathbb{R}^n (n \geq 3)$ a system $X = (X_1, \ldots, X_m)$ of vector fields

$$X_j = \sum_{k=1}^{n} b_{jk} \frac{\partial}{\partial x_k}, \qquad j = 1, 2, \ldots, m$$

with real-valued, smooth coefficients b_{jk}. For a multi-index $\alpha = (i_1, \ldots, i_k)$, denote by $X_\alpha = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]] \ldots]$ the commutator of $\{X_1, ..., X_m\}$ with length $k = |\alpha|$. We say that the vector fields $X_1, \ldots, X_m$ satisfy Hörmander's condition at step s (see [15]) provided there exists a positive integer s such that $\{X_\alpha\}_{|\alpha| \leq s}$ span the tangent space at each point in $\mathbb{R}^n$. We consider X_j as a first order differential operator acting on $u \in \mathrm{Lip}(\mathbb{R}^n)$ of the form

$$X_j u(x) = \langle X_j(x), \nabla u(x) \rangle, \quad j = 1, 2, \ldots, m.$$

The generalized gradient is denoted by $Xu = (X_1 u, ..., X_m u)$ and its length is given by

$$|Xu(x)| = \left(\sum_{j=1}^{m} |X_j u(x)|^2 \right)^{\frac{1}{2}}.$$

The vector fields $X_1, \ldots, X_m$ are said to be free up to step s (see [2]) if the X_j's and their commutators up to step s do not satisfy any linear relation other than those which hold automatically as a consequence of antisymmetry of the Lie bracket and Jacobi identity.

An absolutely continuous curve $\gamma : [a,b] \to \mathbb{R}^n$ is said to be admissible with respect to the system of vector fields X, if there exist functions $c_i(t)$, $a \leq t \leq b$, satisfying

$$\sum_{i=1}^{m} c_i(t)^2 \leq 1 \text{ and } \gamma'(t) = \sum_{i=1}^{m} c_i(t) X_i(\gamma(t))$$

for almost every $t \in [a,b]$. The Carnot-Carathéodory distance $d(x,y)$ generated by X is defined by

$$d(x,y) = \inf \{ T > 0 : \text{there is an admissible curve } \gamma, \text{ with } \gamma(0) = x, \ \gamma(T) = y \}.$$

In the remaining part of the paper we always assume that the vector fields $X_1, \ldots, X_m$ are free up to step s and satisfy Hörmander's condition at step s. Then by the accessibility theorem of Chow [3], the distance d is a metric and therefore

$(\mathbb{R}^n, d)$ is a metric space which is called the Carnot-Carathéodory space. The ball is denoted by

$$B(x,R) = \{y \in \mathbb{R}^n : d(y,x) < R\}.$$

If $\sigma > 0$ and $B = B(x,R)$ we write σB to indicate $B(x, \sigma R)$. Furthermore, if $E \subset \mathbb{R}^n$ is a Lebesgue measurable set with Lebesgue measure $|E|$, we set $u_E = \fint_E u dx$ the integral average of u on E. We also define $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$.

It was proved (see [2,28]) that for every connected $K \subset \Omega$ there exist constants $C_1, C_2 > 0$ such that

$$C_1|x-y| \le d(x,y) \le C_2|x-y|^{\frac{1}{s}}, \ x, y \in K.$$

Moreover there are $R_d > 0$ and $C_d \ge 1$ such that for any $x \in K$ and $R \le R_d$,

$$|B(x, 2R)| \le C_d |B(x,R)|. \tag{2.1}$$

In particular, we have

$$C_1 R^Q \le |B(x,R)| \le C_2 R^Q.$$

Property (2.1) is the so-called "doubling condition" which is assumed to hold on the spaces of homogeneous type. The best constant C_d in (2.1) is called the doubling constant and $Q = \log_2 C_d$ is the homogeneous dimension relative to Ω.

Now, we introduce the Sobolev spaces associated with $X = (X_1, ..., X_m)$. Given $1 \le p < \infty$, the Sobolev space $W_X^{1,p}(\Omega)$ is the Banach space

$$W_X^{1,p}(\Omega) = \{u \in L^p(\Omega) : X_j u \in L^p(\Omega), \ j = 1, 2, \dots, m\},$$

endowed with the norm

$$\|u\|_{W_X^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)}.$$

Here, $X_j u$ $(j = 1, 2, \dots, m)$ is the distributional derivative of $u \in L^1_{\text{loc}}(\Omega)$ defined by the identity

$$\langle X_j u, \varphi \rangle = \int_\Omega u X_j^* \varphi dx, \quad \varphi \in C_0^\infty(\Omega),$$

where $X_j^* = -\sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{jk} \cdot)$ is the formal adjoint of X_j, not necessarily a vector field in general. The local version of $W_X^{1,p}(\Omega)$ is denoted by $W_{X,\text{loc}}^{1,p}(\Omega)$. We also define $W_{X,0}^{1,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ under the above norm $\|\cdot\|_{W_X^{1,p}(\Omega)}$. The following Sobolev-Poincaré inequalities on vector fields can be found in [7,14,26].

Lemma 2.1. *For every compact set $K \subset \Omega$, there exist constants $C > 0$ and $\bar{R} > 0$ such that for any metric ball $B = B(x_0, R)$ with $x_0 \in K$ and $0 < R \le \bar{R}$, it holds that for any $f \in W_X^{1,p}(B)$,*

$$\left(\fint_B |f - f_B|^{\kappa p} dx\right)^{\frac{1}{\kappa p}} \le CR \left(\fint_B |Xf|^p dx\right)^{\frac{1}{p}},$$

where $1 \le \kappa \le Q/(Q-p)$, if $1 \le p < Q$; $1 \le \kappa < \infty$, if $p \ge Q$. Moreover,

$$\left(\fint_B |f|^{\kappa p} dx\right)^{\frac{1}{\kappa p}} \le CR \left(\fint_B |Xf|^p dx\right)^{\frac{1}{p}},$$

whenever $f \in W_{X,0}^{1,p}(B)$.

The (X,p)-capacity of a compact set $K \subset \Omega$ in Ω is defined by

$$\mathrm{cap}_p(K,\Omega) = \inf\left\{ \int_\Omega |Xu|^p dx : u \in C_0^\infty(\Omega),\ u = 1 \text{ on } K \right\}$$

and for an arbitrary set $E \subset \Omega$, the (X,p)-capacity of E is

$$\mathrm{cap}_p(E,\Omega) = \inf_{\substack{G\subset\Omega \text{ open} \\ E\subset G}} \ \sup_{\substack{K\subset G \\ K \text{ compact}}} \mathrm{cap_p}(K,\Omega).$$

The following two-sided estimate of (X,p)-capacity can be found in [4]: For $x \in \Omega$ and $0 < R < \mathrm{diam}\Omega$, there exist $C_1, C_2 > 0$ such that

$$C_1 \frac{|B(x,R)|}{R^p} \le \mathrm{cap}_p(\bar{B}(x,R), B(x,2R)) \le C_2 \frac{|B(x,R)|}{R^p}.$$

Lemma 2.2 (see [6], Theorem 3.3)**.** *If $\mathbb{R}^n \backslash \Omega$ is uniformly (X,p)-fat, then there exists $1 < q < p$ such that $\mathbb{R}^n \backslash \Omega$ is also uniformly (X,q)-fat.*

The uniform (X,q)-fatness also implies uniform (X,p)-fatness for all $p \ge q$, which is a simple consequence of Hölder's and Young's inequality.

Lemma 2.3 (see [8], Lemma 2.8)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the homogeneous dimension Q, $1 < q < \infty$ and $0 < R < \mathrm{diam}\Omega$. For any $x \in \Omega$, denote $B = B(x,R)$ and $N(\varphi) = \{y \in \bar{B} : \varphi(y) = 0\}$. Then there exists a constant $C = C(Q,q) > 0$ such that for all $\varphi \in C^\infty(2B) \cap W_X^{1,q}(2B)$,*

$$\left(\fint_{2B} |\varphi|^{\kappa q} dx \right)^{\frac{1}{\kappa q}} \le C \left(\frac{1}{\mathrm{cap}_q(N(\varphi), 2B)} \int_{2B} |X\varphi|^q dx \right)^{\frac{1}{q}}, \tag{2.2}$$

where $1 \le \kappa \le Q/(Q-q)$ if $1 \le q < Q$ and $1 \le \kappa < \infty$ if $q \ge Q$.

At the end of this section we recall a Gehring lemma on the metric measure space (Y,d,μ), where d is a metric and μ is a doubling measure.

Lemma 2.4 (see [30], Theorem 3.3)**.** *Let $q \in [q_0, 2Q]$, where $q_0 > 1$ is fixed. Assume that functions f, g are nonnegative and $g \in L^q_{\mathrm{loc}}(Y,\mu)$, $f \in L^{r_0}_{\mathrm{loc}}(Y,\mu)$, for some $r_0 > q$. If there exist constants $b > 1$ and θ such that for every ball $B \subset \sigma B \subset Y$ the following inequality holds*

$$\fint_B g^q d\mu \le b\left[\left(\fint_{\sigma B} g d\mu \right)^q + \fint_{\sigma B} f^q d\mu \right] + \theta \fint_{\sigma B} g^q d\mu,$$

then there exist nonnegative constants $\theta_0 = \theta_0(q_0, Q, C_d, \sigma)$ and $\varepsilon_0 = \varepsilon_0(b, q_0, Q, C_d, \sigma)$ such that if $0 < \theta < \theta_0$ then $g \in L^p_{\mathrm{loc}}(Y,\mu)$ for $p \in [q, q+\varepsilon_0)$ and moreover

$$\left(\fint_B g^p d\mu \right)^{\frac{1}{p}} \le C\left[\left(\fint_{\sigma B} g^q d\mu \right)^{\frac{1}{q}} + \left(\fint_{\sigma B} f^p d\mu \right)^{\frac{1}{p}} \right]$$

for some positive constant $C = C(q_0, Q, C_d, \sigma)$.

3. Global higher integrability

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. Since Ω is bounded, we can choose a ball B_0 such that $\overline{\Omega} \subset \frac{1}{2}B_0$. Fix $0 < R < 1$ and let B be a ball of radius R with $2B \subset B_0$. It leads to two cases: (i) $2B \subset \Omega$ or (ii) $2B \backslash \Omega \neq \emptyset$.

In the case (i) we let η be a smooth cut-off function on $2B$, i.e. $\eta \in C_0^\infty(2B)$ such that

$$0 \leq \eta \leq 1,\ \eta = 1 \text{ on } B \text{ and } |X\eta| \leq \frac{c}{R}.$$

For a solution u to the double obstacle problem for (1.1), consider the function

$$w = u - u_{2B} - \eta^p \left(u - u_{2B} - m\right),$$

where $m = (\psi_2 - u_{2B})^- + \min\{(\psi_1 - u_{2B})^+, (\psi_2 - u_{2B})^+\}$. Since

$$m = \begin{cases} (\psi_1 - u_{2B})^+, & \psi_2 \geq u_{2B}, \\ \psi_2 - u_{2B}, & \psi_2 < u_{2B}, \end{cases}$$

we have that $\psi_1 - u_{2B} \leq w \leq \psi_2 - u_{2B}$ and

$$|m| \leq \begin{cases} |\psi_1 - (\psi_1)_{2B}|, & \psi_2 \geq u_{2B}, \\ |\psi_2 - (\psi_2)_{2B}|, & \psi_2 < u_{2B}. \end{cases}$$

Let $v = w - (u - u_{2B}) = -\eta^p(u - u_{2B} - m)$ then $v \in W^{1,p}_{X,0}(2B)$ and $\psi_1 \leq v + u \leq \psi_2$ a.e. in $2B$. Taking $\varphi = v$ in (1.1) yields

$$\int_{2B} A(x,u,Xu) \cdot X(\eta^p(u - u_{2B} - m))dx + \int_{2B} B(x,u,Xu)\eta^p(u - u_{2B} - m)dx \leq 0.$$

then

$$\begin{aligned} \int_{2B} \eta^p A(x,u,Xu) \cdot Xudx \leq & \int_{2B} \eta^p A(x,u,Xu) \cdot Xmdx \\ & - p\int_{2B} \eta^{p-1} A(x,u,Xu) \cdot X\eta(u - u_{2B} - m)dx \\ & - \int_{2B} \eta^p B(x,u,Xu)(u - u_{2B} - m)dx. \end{aligned} \tag{3.1}$$

Next we estimate the integrals on the right-hand side of (3.1) one by one. First observe that $|Xm| \leq |X\psi_1| + |X\psi_2|$. Then the first integral on the right-hand side can be estimated by using (1.3) and Hölder's inequality that

$$\begin{aligned} & \int_{2B} \eta^p A(x,u,Xu) \cdot Xmdx \\ \leq & \beta \int_{2B} \eta^p (|u|^{p-1} + |Xu|^{p-1})|Xm|dx \\ \leq & \frac{\varepsilon}{3} \int_{2B} \eta^p |Xu|^p dx + c\int_{2B} \eta^p |u|^p dx + c_\varepsilon \int_{2B} (|X\psi_1|^p + |X\psi_2|^p)dx. \end{aligned} \tag{3.2}$$

Assumption (1.3) and Poincaré's inequality imply

$$\begin{aligned}
& p\int_{2B}\eta^{p-1}A(x,u,Xu)\cdot X\eta(u-u_{2B}-m)dx \\
\le\ & p\beta\int_{2B}\eta^{p-1}(|u|^{p-1}+|Xu|^{p-1})|X\eta|(|u-u_{2B}|+|m|)dx \\
\le\ & \frac{\varepsilon}{3}\int_{2B}\eta^p|Xu|^pdx+c\int_{2B}\eta^p|u|^pdx+c_\varepsilon R^{-p}\int_{2B}(|u-u_{2B}|^p+|m|^p)dx \\
\le\ & \frac{\varepsilon}{3}\int_{2B}\eta^p|Xu|^pdx+c\int_{2B}\eta^p|u|^pdx \\
& +c_\varepsilon|2B|\left(\fint_{2B}|Xu|^{\frac{pQ}{p+Q}}dx\right)^{\frac{p+Q}{Q}}+c_\varepsilon\int_{2B}(|X\psi_1|^p+|X\psi_2|^p)dx. \qquad (3.3)
\end{aligned}$$

On the other hand, by (1.4) and Lemma 2.1, we infer

$$\begin{aligned}
& \int_{2B}\eta^pB(x,u,Xu)(u-u_{2B}-m)dx \\
\le\ & \beta\int_{2B}\eta^p(|Xu|^{p(1-\frac{1}{\gamma})}+|g|)(|u-u_{2B}|+|m|)dx \\
\le\ & \frac{\varepsilon}{3}\int_{2B}\eta^p|Xu|^pdx+c_\varepsilon\int_{2B}(|u-u_{2B}|^\gamma+|m|^\gamma)dx+c\int_{2B}|g|^{\frac{\gamma}{\gamma-1}}dx \\
\le\ & \frac{\varepsilon}{3}\int_{2B}\eta^p|Xu|^pdx+c_\varepsilon R^{\gamma+Q(1-\frac{\gamma}{p})}\left(\int_{2B}|Xu|^pdx\right)^{\frac{\gamma}{p}} \\
& +c(\varepsilon,\|X\psi_1\|_{L^p}^{\gamma-p},\|X\psi_1\|_{L^p}^{\gamma-p})\int_{2B}(|X\psi_1|^p+|X\psi_2|^p)dx+c\int_{2B}|g|^{\frac{\gamma}{\gamma-1}}dx, \quad (3.4)
\end{aligned}$$

where $\gamma\ge p$ is the constant defined in Section 1 satisfying $\gamma+Q(1-\frac{\gamma}{p})\ge 0$.

Applying (1.2), Combining (3.2)-(3.4) with (3.1) and taking $\varepsilon=\frac{\alpha}{2}$, we find

$$\begin{aligned}
\int_{2B}\eta^p|Xu|^pdx\le\ & c\int_{2B}(|u|^p+|g|^{\frac{\gamma}{\gamma-1}}+|X\psi_1|^p+|X\psi_2|^p)dx \\
& +cR^{\gamma+Q(1-\frac{\gamma}{p})}\left(\int_{2B}|Xu|^pdx\right)^{\frac{\gamma}{p}}+c|2B|\left(\fint_{2B}|Xu|^{\frac{pQ}{p+Q}}dx\right)^{\frac{p+Q}{Q}}.
\end{aligned}$$

Dividing by $|2B|$ on both sides and noting that $\eta=1$ on B, we obtain

$$\begin{aligned}
\fint_B|Xu|^pdx\le\ & c\fint_{2B}(|u|^p+|g|^{\frac{\gamma}{\gamma-1}}+|X\psi_1|^p+|X\psi_2|^p)dx+c\left(\fint_{2B}|Xu|^{\frac{pQ}{p+Q}}dx\right)^{\frac{p+Q}{Q}} \\
& +\frac{c}{|2B|}R^{\gamma+Q(1-\frac{\gamma}{p})}\left(\int_{2B}|Xu|^pdx\right)^{\frac{\gamma}{p}}. \qquad (3.5)
\end{aligned}$$

In the case (ii), we let $w=u-\eta^p(u-u_0)$, where η is the cut-off function on $2B$. Since $\psi_1\le u,u_0\le\psi_2$, we have

$$\psi_1=(1-\eta^p)\psi_1+\eta^p\psi_1\le w=(1-\eta^p)u+\eta^pu_0\le(1-\eta^p)\psi_2+\eta^p\psi_2=\psi_2,$$

which implies $w \in \mathcal{K}^{u_0,p}_{\psi_1,\psi_2}$. If we set $v = w - u = -\eta^p(u - u_0)$, then v is admissible as a test function in the definition of solutions. Taking $\varphi = v$ in (1.5), it follows

$$\int_\Omega A(x,u,Xu)\cdot \eta^p X(u-u_0)dx + p\int_\Omega A(x,u,Xu)\cdot \eta^{p-1}(u-u_0)X\eta dx + \int_\Omega B(x,u,Xu)\eta^p(u-u_0)dx \le 0.$$

If $D = 2B\cap\Omega$, it follows from the structure conditions (1.2)-(1.4) that

$$\begin{aligned}\alpha\int_D \eta^p|Xu|^p dx &\le \beta\int_D \eta^p(|u|^{p-1}+|Xu|^{p-1})|Xu_0|dx \\ &\quad + \beta\int_D \eta^p(|Xu|^{p(1-\frac{1}{\gamma})}+|g|)|u-u_0|dx \\ &\quad + p\int_D \eta^{p-1}(|u|^{p-1}+|Xu|^{p-1})|u-u_0||X\eta|dx \\ &=: I_1+I_2+I_3. \end{aligned} \tag{3.6}$$

Using Young's inequality with ε, we derive

$$I_1 \le \frac{\varepsilon}{3}\int_D \eta^p|Xu|^p dx + c\int_D \eta^p|u|^p dx + c_\varepsilon\int_D \eta^p|Xu_0|^p dx;$$

$$I_2 \le \frac{\varepsilon}{3}\int_D \eta^p|Xu|^p dx + c\int_D |g|^{\frac{\gamma}{\gamma-1}}dx + c_\varepsilon\int_{2B}|u-u_0|^\gamma dx;$$

and

$$I_3 \le \frac{\varepsilon}{3}\int_D \eta^p|Xu|^p dx + c\int_D \eta^p|u|^p dx + c_\varepsilon\int_{2B}|u-u_0|^p|X\eta|^p dx.$$

Substituting the above estimates into (3.6) and letting $\varepsilon = \frac{\alpha}{2}$ give

$$\begin{aligned}\int_D \eta^p|Xu|^p dx &\le c\int_D (|u|^p+|Xu_0|^p dx+|g|^{\frac{\gamma}{\gamma-1}})dx \\ &\quad + c\int_{2B}\left|\frac{u-u_0}{R}\right|^p dx + c\int_{2B}|u-u_0|^\gamma dx. \end{aligned}\tag{3.7}$$

To estimate the second integral on the right-hand side of (3.7), let $s = p(1-\epsilon)$, where $0<\epsilon<\frac{p}{p+Q}$ if $p\le Q$ and $0<\epsilon<\min\left\{\frac{p-Q}{p},\frac{1}{2}\right\}$ if $p>Q$. If

$$\kappa = \begin{cases}\frac{Q}{Q-s}, & s<Q,\\ 2, & s>Q,\end{cases}$$

then $\kappa s \ge p$, and it follows from Lemma 2.3 that

$$\begin{aligned}\left(\fint_{2B}\left|\frac{u-u_0}{R}\right|^p dx\right)^{\frac{1}{p}} &\le cR^{-1}\left(\fint_{2B}|u-u_0|^{\kappa s}dx\right)^{\frac{1}{\kappa s}} \\ &\le cR^{-1}\left(\frac{1}{\mathrm{cap}_s(N(u-u_0),2B)}\int_{2B}|X(u-u_0)|^s dx\right)^{\frac{1}{s}}, \end{aligned}$$

where $N(u-u_0)=\{x\in \bar{B}: u(x)=u_0(x)\}$. Since $u-u_0$ vanishes outside Ω, we know that $\mathbb{R}^n\setminus\Omega\subset\{u-u_0=0\}$. On the other hand, by Lemma 2.2, there exists ϵ_0 such that if $0<\epsilon<\epsilon_0$, $\mathbb{R}^n\setminus\Omega$ is uniformly (X,s)-fat, and hence, for ϵ small enough

$$\operatorname{cap}_s(N(u-u_0),2B)\ge \operatorname{cap}_s(\bar{B}\cap(\mathbb{R}^n\setminus\Omega),2B)\ge c\operatorname{cap}_s(\bar{B},2B)\ge c|B|R^{-s}. \tag{3.8}$$

From (3.8) and the doubling condition (2.1) we obtain

$$\begin{aligned}\int_{2B}\left|\frac{u-u_0}{R}\right|^p dx &\le c|2B|\left(\frac{1}{|2B|}\int_D |Xu-Xu_0|^s\right)^{\frac{p}{s}}\\ &\le c|2B|\left(\frac{1}{|2B|}\int_D |Xu|^s\right)^{\frac{p}{s}}+c\int_D |Xu_0|^p dx. \end{aligned}\tag{3.9}$$

For the last term on the right-hand side of (3.7), we have by Lemma 2.3 that

$$\begin{aligned}\left(\fint_{2B}|u-u_0|^\gamma dx\right)^{\frac{1}{\gamma}} &\le c\left(\frac{1}{\operatorname{cap}_p(N(u-u_0),2B)}\int_{2B}|X(u-u_0)|^p dx\right)^{\frac{1}{p}}\\ &\le cR\left(\frac{1}{|2B|}\int_D |X(u-u_0)|^p dx\right)^{\frac{1}{p}},\end{aligned}$$

where the last inequality comes from an argument similar to (3.8). Then it follows

$$\begin{aligned}\int_{2B}|u-u_0|^\gamma dx &\le cR^{\gamma+Q(1-\frac{\gamma}{p})}\left(\int_D |X(u-u_0)|^p dx\right)^{\frac{\gamma}{p}}\\ &\le cR^{\gamma+Q(1-\frac{\gamma}{p})}\left(\int_D |Xu|^p dx\right)^{\frac{\gamma}{p}}+c(\|Xu_0\|^{\gamma-p}_{L^p(\Omega)})\int_D |Xu_0|^p dx.\end{aligned}\tag{3.10}$$

Inserting (3.9) and (3.10) into (3.7), we find

$$\begin{aligned}\frac{1}{|B|}\int_{B\cap\Omega}|Xu|^p dx &\le \frac{c}{|2B|}\int_D(|u|^p+|Xu_0|^p+|g|^{\frac{\gamma}{\gamma-1}})dx+c\left(\frac{1}{|2B|}\int_D |Xu|^s\right)^{\frac{p}{s}}\\ &\quad+\frac{c}{|2B|}R^{\gamma+Q(1-\frac{\gamma}{p})}\left(\int_D |Xu|^p dx\right)^{\frac{\gamma}{p}},\end{aligned}\tag{3.11}$$

where $s=p(1-\epsilon)<p$.

To combine the above two cases, we let

$$g(x)=\begin{cases}|Xu|^t, & x\in\Omega,\\ 0, & x\in\mathbb{R}^n\backslash\Omega,\end{cases}$$

and

$$f(x)=\begin{cases}\left(|u-u_0|+|u_0|+|Xu_0|+|X\psi_1|+|X\psi_2|+|g|^{\frac{\gamma}{p(\gamma-1)}}\right)^t, & x\in\Omega,\\ 0, & x\in\mathbb{R}^n\backslash\Omega,\end{cases}$$

where $t=\max\{s,\frac{pQ}{p+Q}\}<p$. From (3.5) and (3.11) we obtain

$$\fint_B g^q dx \le b\left[\left(\fint_{2B} g dx\right)^q+\fint_{2B} f^q dx\right]+\theta(R)\fint_{2B} g^q dx,$$

where $q=\frac{p}{t}$, $\theta(R)=cR^{\gamma+Q(1-\frac{\gamma}{p})}\left(\int_{2B}|Xu|^p dx\right)^{\frac{\gamma}{p}-1}$ and $b>1$. By the absolute continuity of the Lebesgue integral, we know that $\theta(R)\to 0$ as $R\to 0$. Choosing $R>0$ small enough, we have by Lemma 2.4 that there exists $t_1>p$, such that $|Xu|\in L^{t_1}(\Omega)$.

Furthermore, we show that there exists $t_2>p$ such that $u\in L^{t_2}(\Omega)$. Since $u-u_0\in W^{1,p}_{X,0}(\Omega)$, we obtain from Sobolev's inequality that for $p<Q$, $p^*=Qp/(Q-p)$,

$$\left(\int_\Omega |u-u_0|^{p^*}dx\right)^{\frac{1}{p^*}}\le C(\Omega)\left(\int_\Omega |Xu-Xu_0|^p dx\right)^{\frac{1}{p}}<\infty.$$

Noting that $u_0\in L^r(\Omega)$ $(r>p)$, we can take $t_2=\min\{r,p^*\}>p$ to get

$$\begin{aligned}\left(\int_\Omega |u|^{t_2}dx\right)^{\frac{1}{t_2}}&\le C\left(\int_\Omega |u-u_0|^{t_2}dx\right)^{\frac{1}{t_2}}+C\left(\int_\Omega |u_0|^{t_2}dx\right)^{\frac{1}{t_2}}\\&\le C\left(\int_\Omega |u-u_0|^{p^*}dx\right)^{\frac{1}{p^*}}+C\left(\int_\Omega |u_0|^{t_2}dx\right)^{\frac{1}{t_2}}\end{aligned}$$

and then $u\in L^{t_2}(\Omega)$. If $p\ge Q$ we can apply the above reasoning for any $p^*<\infty$ to obtain $u\in L^{t_2}(\Omega)$.

Setting $\tilde p=\min\{t_1,t_2\}>p$, it follows that $u\in W^{1,\tilde p}_X(\Omega)$ and the proof is thereby complete.

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