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FINITE-TIME SYNCHRONIZATION FOR COUPLED SYSTEMS WITH TIME DELAY AND STOCHASTIC DISTURBANCE UNDER FEEDBACK CONTROL

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Abstract This paper proposes a framework for finite-time synchronization of coupled systems with time delay and stochastic disturbance under feedback control. Combining Kirchhoff's Matrix Tree Theorem with Lyapunov method as well as stochastic analysis techniques, several sufficient conditions are derived. Differing from previous references, the finite time provided by us is related to topological structure of networks. In addition, two concrete applications about stochastic coupled oscillators with time delay and stochastic Lorenz chaotic coupled systems with time delay are presented, respectively. Besides, two synchronization criteria are provided. Ultimately, two numerical examples are given to illustrate the effectiveness and feasibility of the obtained results.

Keywords Finite-time synchronization, stochastic coupled systems, feedback control, time delay.

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1. Introduction

Since the pioneering work of Watts and Strogatz [31], interest on complex networks has grown rapidly for their applications are not only extended to physical, mechanical, ecological, and electronic engineering fields, but also existing in our daily life with examples including the Internet, the World Wide Web, the World Trade Web, disease transmission networks [5], etc. Besides, coupled systems, as a vital class of complex networks, have been investigated widely in recent years, and numerous valuable results have been obtained [27,43,46], which can be used in traffic systems, electric systems and so on. As a matter of fact, coupled systems are influenced inevitably by stochastic disturbance which exists extensively in external environment. Thus, taking stochastic disturbance into account in the process of researching coupled systems is crucial, and many scholars admitted it and published some useful results [18,28,30,40]. Furthermore, it is worthy of noting that time delay is unavoidable in real life. For example, in the telephone communication network, receiver hears at time t while the transmitter often makes a sound at time $t-\tau$. So time delay has been attached great importance, and lots of previous references took time

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delay into consideration [9,23]. And there are also some references about stochastic coupled systems with time delay (SCSTD) [6,34].

On the other hand, plenty of scholars have focused on a variety of dynamical properties of complex networks, such as stability [12,19,26], synchronization [38,45], stationary distribution [16], etc. Among them, synchronization, as a major dynamical property of SCSTD, is one of the key issues that has been extensively researched owing to its ubiquity in real life. For instance, in traffic system, synchronization can be used to reduce congestion by designing applicable regional traffic signal [14]. Besides, as is known, synchronization is the process between two or more dynamical systems to attain a common behavior, which is in connection with drive-response systems in different communities called external synchronization. Corresponding with external synchronization, synchronization of subsystems in one community is called internal synchronization, which is commonly used in electric system. Therefore, it is also of a significant meaning to study internal synchronization in practical applications. For example, in power system, when synchronization signals of the power grid hardware are lost, the system can run in accordance with the original grid by means of the designed synchronization system. Hence, many problems about internal synchronization have been investigated, see [3,4,42]. Moreover, synchronization problems can be classified into several types: exponential synchronization [10,36], asymptotical synchronization [33], finite-time synchronization [37], etc. However, from a practical point of view, finite-time synchronization, which means that synchronization can be achieved in a setting time, is optimal because machines and humans life spans are limited and one may expect to achieve synchronization as fast as possible. Therefore, it is meaningful to research finite-time synchronization, and many efficient methodologies to clarify the finite-time synchronization problem have been developed [8, 25, 37]. Compared with these references, the finite time provided by us is related to topological structure of the networks.

Moreover, influenced by stochastic disturbance and time delay, synchronization of complex networks is difficult to achieve, sometimes. In order to research this issue, some scholars determine to employ control technique to synchronize complex networks based on pinning control, quantized control, feedback control, etc. Among those control approaches, feedback control acts on the difference between the input value and output value of systems and it has been verified to be a serviceable tool to study synchronization problem of different kinds of systems [13, 21, 35, 41]. In addition, Lyapunov method, as a powerful tool to research dynamical properties of complex networks, has secured growing attention among the research community and plenty of scholars concentrate on utilizing Lyapunov method to solve problems. However, constructing a suitable Lyapunov function directly is an enormous challenge due to the intricate relatives between the topological structure of complex networks and diverse dynamical properties of the coupled nodes. Fortunately, in [15], Li et al. put forward a novel method, which combines Kirchhoff's Matrix Tree Theorem with Lyapunov method, to cope with this problem. By utilizing this method, each subsystem in coupled systems can be described as a vertex system of digraph while the coupling structures of two subsystems can be performed by directed arcs between two vertex systems of digraph. Inspired by this method, amounts of results were obtained in [7, 11, 16, 28, 44]. Though there are many references relating with finite-time internal synchronization of systems, none of them researched this problem by making use of the method mentioned above, which makes our work valuable.

Based on the discussion above, in this paper, we consider a general SCSTD under feedback control. And some sufficient criteria are derived to guarantee finite-time synchronization of SCSTD by combining Kirchhoff's Matrix Tree Theorem, Lyapunov method and stochastic analysis techniques. What is noteworthy is that finite time estimated by us is connected with topological structure of SCSTD. In addition, two concrete applications about stochastic coupled oscillators with time delay and stochastic Lorenz chaotic coupled systems with time delay are presented. Meanwhile, two numerical examples are given to demonstrate the effectiveness and practicability of our theoretical results.

The rest of this paper is arranged in the following. In Section 2, some preliminaries and our model description are introduced. Besides, our main results are discussed in Section 3. Besides, two applications and the corresponding numerical simulations are provided in Section 4.

2. Preliminaries and model formulation

2.1. Preliminaries

To begin with, for convenience, unless the special cases, some useful notations used throughout this paper are introduced in the following. Define notations $\mathbb{L} = \{1, 2, \ldots, N\}$, $\mathbb{Z}^+ = \{1, 2, \ldots\}$, $\mathbb{R}^+ = [0, +\infty)$. Besides, $\mathbb{R}$ and $\mathbb{R}^m$ stand for real numbers and m-dimensional Euclidean space, respectively. The superscript "T" denotes the transpose of a vector. We define $\|\cdot\|$ is the Euclidean norm. And the family of all nonnegative functions $V(x,t)$ on $\mathbb{R}^{(N-1)m} \times \mathbb{R}^+$, which are continuously twice differentiable in x and once in t, are denoted by $C^{2,1}\left(\mathbb{R}^{(N-1)m} \times \mathbb{R}^+; \mathbb{R}^+\right)$. Denote notations $S_\rho^{(N-1)m} = \left\{x \in \mathbb{R}^{(N-1)m} : ||x|| < \rho\right\}$ and $a \wedge b$ represents the minimum of a and b. Moreover, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). In addition, $B(t)$ is a one-dimensional Brownian motion defined on the probability space and the mathematical expectation with respect to the given probability measure $\mathbb{P}$ is denoted by $\mathbb{E}(\cdot)$. For $\tau > 0$, $C([-\tau, 0]; \mathbb{R}^{(N-1)m})$ is the family of continuous function x from $[-\tau, 0]$ to $\mathbb{R}^{(N-1)m}$. A digraph $\mathcal{G} = (\mathbb{V}, \mathbb{U})$, containing a set $\mathbb{V}$ of N vertices and a set $\mathbb{U}$ of arcs (i,j) leading from initial vertex i to terminal vertex j, is weighted if each arc (j,i) is assigned a positive weight a_{ij} and we call $A = (a_{ij})_{N\times N}$ as the weighted matrix. Besides, a weighted digraph $(\mathcal{G}, A)$ is strongly connected if and only if the weighted matrix A is irreducible. And $\mathcal{P}$ is the Laplacian matrix of $(\mathcal{G}, A)$, in which $\mathcal{P}$ is defined as $\mathcal{P} = (p_{kh})_{N\times N}$, where $p_{kh} = -a_{kh}$ for $k \neq h$, $p_{kh} = \sum_{j\neq k} a_{kj}$ for $k = h$. And some other basic concepts of graph theory are introduced in references [22,32].

Next, some vital lemmas are provided in the following.

Lemma 2.1 (Kirchhoff's Matrix Tree Theorem, [15])**.** *Assume that $N \geq 2$. Let c_i denote the cofactor of the ith diagonal element of $\mathcal{P}$. Then the following equlity holds:*

$$\sum_{i,j=1}^{N} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r)\in\mathbb{U}(\mathcal{C}_\mathcal{Q})} F_{rs}(x_r, x_s).$$

Here $F_{ij}(x_i, x_j) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $1 \leq i, j \leq N$ is an arbitrary function, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $\mathcal{C}_\mathcal{Q}$ denotes *the directed cycle of* $\mathcal{Q}$. *In particular, if* $(\mathcal{G}, A)$ *is strongly connected, then* $c_i > 0$ *for* $i \in \mathbb{L}$.

Lemma 2.2 (Lemma 2, [37]). *If* $c \in (0,1)$, *for positive constants* $a_1, a_2, \ldots, a_n$, *there is an inequality*

$$\sum_{k=1}^{n} a_k^c \geq \left(\sum_{k=1}^{n} a_k\right)^c.$$

2.2. Model formulation

To begin with, consider the following SCSTD established on a digraph $\mathcal{G}$ with N $(N \geq 2)$ vertices

$$\begin{aligned} \mathrm{d}x_i(t) = &\left(f(x_i(t), x_i(t-\tau), t) + \sum_{k=1}^{N} a_{ik} H_{ik}(x_i(t), x_k(t)) + I(t)\right)\mathrm{d}t \\ &+ g(x_i(t), t)\mathrm{d}B(t),\ \ i \in \mathbb{L},\ \ t \geq 0, \end{aligned} \tag{2.1}$$

where $x_i(t) = \left(x_i^{(1)}(t), x_i^{(2)}(t), \ldots, x_i^{(m)}(t)\right)^{\mathrm{T}} \in \mathbb{R}^m$ for $m \in \mathbb{Z}^+$ represents the state of the ith vertex system. Besides, $f : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^m$, $g : \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^m$ are continuous functions, in which g stands for the disturbance intensity. And $\tau \geq 0$ stands for time delay, and $I(t)$ is external input vector. In addition, $H_{ik} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is coupling term and $a_{ik} \geq 0$ is coupling coefficient.

For convenience, we denote

$$x_{ij}(t) = x_j(t) - x_i(t),$$

where $x_{ij}(t) = \left(x_{ij}^{(1)}(t), x_{ij}^{(2)}(t), \ldots, x_{ij}^{(m)}(t)\right)^{\mathrm{T}}$. Suppose i $(i \in \mathbb{L})$ is any fixed. For the sake of investigating finite-time synchronization of SCSTD, a feedback controller $u(x_{ij}(t), t)$ is designed in the jth $(j \in \mathbb{L}, j \neq i)$ vertex system. Then the jth vertex system with feedback control can be described as

$$\begin{aligned} \mathrm{d}x_j(t) = &\left(f(x_j(t), x_j(t-\tau), t) + \sum_{k=1}^{N} a_{jk} H_{jk}(x_j(t), x_k(t)) + I(t) + u(x_{ij}(t), t)\right)\mathrm{d}t \\ &+ g(x_j(t), t)\mathrm{d}B(t),\ \ t \geq 0, \end{aligned}$$

thereinto, the controller $u(x_{ij}(t), t)$ is designed as follows:

$$\begin{aligned} u(x_{ij}(t), t) = &-\alpha_{ij} x_{ij}(t) - \mathrm{sign}(x_{ij}(t))|x_{ij}(t)|^b \\ &- \left(\int_{t-\tau}^{t} \theta_{ij} \|x_{ij}(s)\|^2 \mathrm{d}s\right)^{\frac{1+b}{2}} \frac{x_{ij}(t)}{\|x_{ij}(t)\|^2}, \end{aligned} \tag{2.2}$$

where $\alpha_{ij},\ \theta_{ij} > 0$, $0 < b < 1$, and

$$\mathrm{sign}(x_{ij}(t)) = \mathrm{diag}\left\{\mathrm{sign}\left(x_{ij}^{(1)}(t)\right), \mathrm{sign}\left(x_{ij}^{(2)}(t)\right), \ldots, \mathrm{sign}\left(x_{ij}^{(m)}(t)\right)\right\},$$

in which $\mathrm{sign}(\cdot)$ is a sign function, and

$$|x_{ij}(t)|^b = \left(|x_{ij}^{(1)}(t)|^b, |x_{ij}^{(2)}(t)|^b, \ldots, |x_{ij}^{(m)}(t)|^b\right)^{\mathrm{T}}.$$

Hence finite-time synchronization problem of system (2.1) is substituted by finite-time stability problem of the following synchronization error system:

$$
\begin{aligned}
\mathrm{d}x_{ij}(t) = & \left(f(x_j(t), x_j(t-\tau), t) - f(x_i(t), x_i(t-\tau), t) + u(x_{ij}(t), t)\right) \mathrm{d}t \\
& + \left(\sum_{k=1}^{N} a_{jk} H_{jk}(x_j(t), x_k(t)) - \sum_{k=1}^{N} a_{ik} H_{ik}(x_i(t), x_k(t))\right) \mathrm{d}t \\
& + \left(g(x_j(t), t) - g(x_i(t), t)\right) \mathrm{d}B(t), \quad i, j \in \mathbb{L}, \quad j \neq i, \quad t \geq 0.
\end{aligned} \tag{2.3}
$$

Furthermore, define $x(t) = \left(x_{i1}^{\mathrm{T}}(t), \ldots, x_{i,i-1}^{\mathrm{T}}(t), x_{i,i+1}^{\mathrm{T}}(t), \ldots, x_{iN}^{\mathrm{T}}(t)\right)^{\mathrm{T}} \in \mathbb{R}^{(N-1)m}$ and the initial conditions for system (2.3) are given by

$$
x(t) = \psi(t), \quad t \in [-\tau, 0],
$$

where $\psi = \left(\psi_{i1}^{\mathrm{T}}, \ldots, \psi_{i,i-1}^{\mathrm{T}}, \psi_{i,i+1}^{\mathrm{T}}, \ldots, \psi_{iN}^{\mathrm{T}}\right)^{\mathrm{T}} \in C\left([-\tau, 0]; \mathbb{R}^{(N-1)m}\right)$.

We end this subsection by introducing two definitions.

Definition 2.1. System (2.1) is said to be finite-time synchronized in probability, if the trivial solution of system (2.3) is finite-time stable in probability, which means system (2.3) admits a unique solution for any initial data $\psi(t)$, denoted by $x(t, \psi(t))$, moreover, the following statements hold:

(i) Finite-time attractiveness in probability: for every initial value $\psi(t) \in \mathbb{R}^{(N-1)m} \setminus \{0\}$, the first hitting time $q_{\psi(t)} = \inf\{t : x(t, \psi(t)) = 0\}$, which is called the stochastic settling time, is finite almost surely, which is $\mathbb{P}(q_{\psi(t)} < \infty)$;

(ii) Stability in probability: for every pair of $\epsilon \in (0, 1)$ and $\kappa > 0$, there exists a constant $\delta = \delta(\epsilon, \kappa) > 0$ such that $\mathbb{P}\left(\|x(t, \psi(t))\| < 0, \ t > 0\right) \geq 1 - \epsilon$, whenever $\sup_{-\tau \leq t \leq 0} \|\psi(t)\| < \delta$.

Definition 2.2. For function $V^{(ij)}(x_{ij}, t) \in C^{2,1}(\mathbb{R}^m \times \mathbb{R}^+; \mathbb{R}^+)$, $i, j \in \mathbb{L}$, differential operator $\mathfrak{L}V^{(ij)}(x_{ij}, t)$ is defined by

$$
\begin{aligned}
\mathfrak{L}V^{(ij)}(x_{ij}, t) = & V_{x_{ij}}^{(ij)} \left(f(x_j, x_j(t-\tau), t) - f(x_i, x_i(t-\tau), t) + u(x_{ij}, t)\right) \\
& + V_{x_{ij}}^{(ij)} \left(\sum_{k=1}^{N} a_{jk} H_{jk}(x_j, x_k) - \sum_{k=1}^{N} a_{ik} H_{ik}(x_i, x_k)\right) \\
& + \frac{1}{2} \mathrm{trace}\left(\left(g(x_j, t) - g(x_i, t)\right)^{\mathrm{T}} V_{x_{ij} x_{ij}}^{(ij)} \left(g(x_j, t) - g(x_i, t)\right)\right) \\
& + \frac{\partial V^{(ij)}(x_{ij}, t)}{\partial t}
\end{aligned}
$$

where

$$
V_{x_{ij}}^{(ij)} = \left(\frac{\partial V^{(ij)}(x_{ij}, t)}{\partial x_{ij}^{(1)}}, \frac{\partial V^{(ij)}(x_{ij}, t)}{\partial x_{ij}^{(2)}}, \ldots, \frac{\partial V^{(ij)}(x_{ij}, t)}{\partial x_{ij}^{(m)}}\right)
$$

and

$$
V_{x_{ij} x_{ij}}^{(ij)} = \left(\frac{\partial^2 V^{(ij)}(x_{ij}, t)}{\partial x_{ij}^{(p)} \partial x_{ij}^{(q)}}\right)_{m \times m}.
$$

3. Finite-time synchronization of SCSTD

First of all, a theorem that combines Kirchhoff's Matrix Tree Theorem with Lyapunov method will be put forward as follows.

Theorem 3.1. *Assume that i ($i \in \mathbb{L}$) is any fixed, for $j \in \mathbb{L},\ j \neq i$, there exists a function $V^{(ij)}(x_{ij}, t) \in C^{2,1}(\mathbb{R}^m \times \mathbb{R}^+; \mathbb{R}^+)$. If the following conditions hold,*

A1 *There exist constants $\mu_{ij} > 0$, $0 < \eta < 1$, $\vartheta_{ijk} \geq 0$ and a function M_{ijk} such that*

$$\mathfrak{L}V^{(ij)}(x_{ij}, t) \leq -\mu_{ij}\left(V^{(ij)}(x_{ij}, t)\right)^{\eta} + \sum_{k=1, k\neq i}^{N} \vartheta_{ijk} M_{ijk}(x_{ik}, x_{ij}),$$

in which $V^{(ij)}(x_{ij}, t)$ is a positive-definite and radially unbounded function.

A2 *Digraph $(\mathcal{G}, C)$ is strongly connected in which $C = (\vartheta_{ijk})_{(N-1)\times(N-1)}$, and along each directed cycle $\mathcal{C}_{\mathcal{Q}}$ of weighted digraph $(\mathcal{G}, C)$, there is*

$$\sum_{(s,r)\in\mathbb{U}(\mathcal{C}_{\mathcal{Q}})} M_{irs}(x_{is}, x_{ir}) \leq 0.$$

then system (2.1) achieves synchronization in finite time t_1 and the finite time satisfies

$$\mathbb{E}(t_1) \leq \frac{V^{1-\eta}(x(0), 0)}{\mu(1-\eta)},$$

where $\mu = \min_{j\in\mathbb{L}, j\neq i}\left\{c_{ij}^{1-\eta}\mu_{ij}\right\}$ and $V(x(0), 0) = \sum_{j=1, j\neq i}^{N} c_{ij} V^{(ij)}(x_{ij}(0), 0)$, in which c_{ij} represents the cofactor of the jth diagonal element of Laplacian matrix of $(\mathcal{G}, C)$.

Proof. For ease of exposition, we split the proof into the following two steps.

Step 1: We will prove that the trivial solution of system (2.3) is globally stable in probability.

According to condition A1, we know that there are functions $d_{ij}(\cdot)$ $(j \in \mathbb{L},\ j \neq i)$ such that

$$V^{(ij)}(x_{ij}, t) \geq d_{ij}(||x_{ij}||).$$

Since digraph $(\mathcal{G}, C)$ is strongly connected, combined with Lemma 2.1, then we can obtain $c_{ij} > 0$. Define Lyapunov function in the following:

$$V(x, t) = \sum_{j=1, j\neq i}^{N} c_{ij} V^{(ij)}(x_{ij}, t). \tag{3.1}$$

Therefore, one can obtain

$$V(x, t) \geq \sum_{j=1, j\neq i}^{N} c_{ij} d_{ij}(||x_{ij}||) \geq cd\left(\frac{1}{N}||x||\right) \triangleq d^{\star}(||x||),$$

where $c = \min_{j\in\mathbb{L},j\neq i}\{c_{ij}\}$ and $d(\cdot) = \min_{j\in\mathbb{L},j\neq i}\{d_{ij}(\cdot)\}$. Let $\epsilon \in (0,1)$ and $\kappa \geq 0$ be arbitrary constants. Then, we choose a sufficiently small $\rho > 0$ such that

$$\frac{1}{\epsilon}\sup_{x\in S_\rho^{(N-1)m}} V(x,t) \leq d^\star(\kappa). \tag{3.2}$$

Define $q = \inf\left\{t \geq 0 : x(t) \notin S_\kappa^{(N-1)m}\right\}$. According to condition A1, we get

$$\begin{aligned}\mathfrak{L}V(x,t) &= \sum_{j=1,j\neq i}^{N} c_{ij}\mathfrak{L}V^{(ij)}(x_{ij},t)\\ &\leq \sum_{j=1,j\neq i}^{N} c_{ij}\left(-\mu_{ij}\left(V^{(ij)}(x_{ij},t)\right)^\eta + \sum_{k=1,k\neq i}^{N}\vartheta_{ijk}M_{ijk}(x_{ik},x_{ij})\right)\\ &\leq \sum_{j=1,j\neq i}^{N}\sum_{k=1,k\neq i}^{N} c_{ij}\vartheta_{ijk}M_{ijk}(x_{ik},x_{ij}). \end{aligned}\tag{3.3}$$

Owing to Lemma 2.1, it has

$$\sum_{j=1,j\neq i}^{N}\sum_{k=1,k\neq i}^{N} c_{ij}\vartheta_{ijk}M_{ijk}(x_{ik},x_{ij}) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q})\sum_{(s,r)\in\mathbb{U}(\mathcal{C}_\mathcal{Q})} M_{irs}(x_{ir},x_{is}). \tag{3.4}$$

Combining equality (3.4) with condition A2, we have

$$\sum_{j=1,j\neq i}^{N}\sum_{k=1,k\neq i}^{N} c_{ij}\vartheta_{ijk}M_{ijk}(x_{ik},x_{ij}) \leq 0. \tag{3.5}$$

Hence, $\mathfrak{L}V(x,t) \leq 0$. Besides, by the Itô's formula, we have

$$\mathbb{E}\left(V(x(q\wedge t),t)\right) \leq V(x(0),0). \tag{3.6}$$

Moreover, we attain

$$\mathbb{E}\left(V(x(q\wedge t),t)\right) \geq \mathbb{E}(\mathrm{I}_{q<t}V(x(q),q)) \geq d^\star(\kappa)\mathbb{E}(\mathrm{I}_{q<t}). \tag{3.7}$$

Deriving from inequalities (3.6) and (3.7), there is $\mathbb{P}\left(q<\infty\right) < \epsilon$, which means $\mathbb{P}\left(||x(t)|| < \kappa, t \geq 0\right) \geq 1-\epsilon$. Further, since function $V^{(ij)}(x_{ij},t)$ is radially unbounded, one has

$$\lim_{||x||\to\infty}\inf_{t\geq 0} V(x,t) = \lim_{||x||\to\infty}\inf_{t\geq 0}\left\{\sum_{j=1,j\neq i}^{N} c_{ij}V^{(ij)}(x_{ij},t)\right\} = \infty.$$

According to reference [20], we can get that the trivial solution of system (2.3) is globally stable in probability.

Step 2: We will prove that $\mathbb{E}(t_1) < +\infty$.
Considering the definition of $V(x,t)$ and condition A1, we obtain

$$\mathfrak{L}V(x,t)$$

$$
\begin{aligned}
&\leq \sum_{j=1,j\neq i}^{N} c_{ij}\left(-\mu_{ij}\left(V^{(ij)}(x_{ij},t)\right)^{\eta}+\sum_{k=1,k\neq i}^{N}\vartheta_{ijk}M_{ijk}(x_{ik},x_{ij})\right)\\
&\leq -\sum_{j=1,j\neq i}^{N} c_{ij}\mu_{ij}\left(V^{(ij)}(x_{ij},t)\right)^{\eta}+\sum_{j=1,j\neq i}^{N}\sum_{k=1,k\neq i}^{N} c_{ij}\vartheta_{ijk}M_{ijk}(x_{ik},x_{ij}). \quad (3.8)
\end{aligned}
$$

Combining inequality (3.5) with equality (3.1), inequality (3.8) can be substituted by

$$
\begin{aligned}
\mathfrak{L}V(x,t) &\leq -\sum_{j=1,j\neq i}^{N} c_{ij}\mu_{ij}\left(V^{(ij)}(x_{ij},t)\right)^{\eta}\\
&\leq -\min_{j\in\mathbb{L},j\neq i}\left\{c_{ij}^{1-\eta}\mu_{ij}\right\}V^{\eta}(x,t) \qquad (3.9)\\
&= -\mu V^{\eta}(x,t).
\end{aligned}
$$

Next, let us define a function

$$
Q(V(x,t)) = \int_0^{V(x,t)} \frac{1}{\mu s^{\eta}}\mathrm{d}s.
$$

Because of $x(0)\neq 0$, there exists $p\in\mathbb{Z}^{+}$ such that $\frac{1}{p} < \|x(0)\| < p$. Then, denote

$$
q_p = \inf\left\{t\geq 0 : \|x(t,x(0))\| \notin \left(\frac{1}{p},p\right)\right\}.
$$

It is clear that q_p is an increasing stopping time sequence. For $t\leq q_p$, according to Definition 2.2, one has

$$
\mathfrak{L}Q(V(x,t)) = \frac{\mathfrak{L}V(x,t)}{\mu V^{\eta}(x,t)} - \frac{\eta}{2\mu V^{\eta+1}(x,t)}\mathrm{trace}\left(\sum_{k=1}^{N}\left[\left(\frac{\partial V_k}{\partial x}g\right)^{\mathrm{T}}\cdot\left(\frac{\partial V_k}{\partial x}g\right)\right]\right).
$$

Combined with inequality (3.9), we derive that $\mathfrak{L}Q(V(x,t)) \leq -1$. Hence, under the theory of stochastic differential equations [20], we obtain that

$$
\mathbb{E}(Q(V(x(p\wedge q_p),t))) - \mathbb{E}(Q(V(x(0),0))) \leq -\mathbb{E}(p\wedge q_p),
$$

which means that

$$
\mathbb{E}(p\wedge q_p) \leq Q(V(x(0),0)).
$$

Let $p\to\infty$, then we have $p\wedge q_p \to t_1,\ a.s.$ Moreover,

$$
\mathbb{E}(t_1) \leq Q(V(x(0),0)) = \int_0^{V(x(0),0)} \frac{1}{\mu s^{\eta}}\mathrm{d}s \leq \frac{V^{1-\eta}(x(0),0)}{\mu(1-\eta)} < +\infty.
$$

Therefore, it is derived that system (2.3) is finite-time stable, which means synchronization of system (2.1) can be reached in finite time. The proof is complete. □

Remark 3.1. In [29], authors combined Lyapunov-Krasovskii functional method with matrix theory to research the finite-time global synchronization for a class of Markovian jump complex networks and obtain the estimation of finite-time t^{*}.

In this paper, by using Kirchhoff's Matrix Tree Theorem method and Lyapunov method, finite-time synchronization of SCSTD via feedback control is ensured by conditions A1 and A2 in Theorem 3.1, and the finite time t_1 is estimated by $\mathbb{E}(t_1) \leq \frac{V^{1-\eta}(x(0),0)}{\mu(1-\eta)}$. Compared with [29], the finite time derived by us is related to topological structure of SCSTD.

Next, for establishing the synchronization criteria employed easily, some general assumptions are provided as follows.

Assumption 1. For the vector-valued function f, there exist positive constants ν and λ, such that f satisfies the semi-Lipschitz condition:

$$(x_j - x_i)^{\mathrm{T}} \left(f(x_j, y_j, t) - f(x_i, y_i, t)\right) \leq \nu \|x_j - x_i\|^2 + \lambda \|y_j - y_i\|^2.$$

Assumption 2. For vector-valued function H_{ik} $(i, k \in \mathbb{L})$, there exists a positive constant δ_{ik} such that

$$\|H_{ik}(x_i, x_k)\| \leq \delta_{ik} \|x_{ik}\|.$$

Assumption 3. Suppose vector-valued function g is Lipschitz continuous with a Lipschitz constant $\varepsilon > 0$, such that

$$\|g(x_j, t) - g(x_i, t)\| \leq \varepsilon \|x_{ij}\|.$$

Then, a theorem based on Theorem 3.1 is provided to ensure the finite-time synchronization of system (2.1) in the following.

Theorem 3.2. *For any fixed i $(i \in \mathbb{L})$, if Assumptions 1, 2, 3 and the following conditions hold,*

B1 *For any $j \in \mathbb{L}$, $j \neq i$, the following inequalities hold,*

$$\theta_{ij} + 2\nu + 3 \sum_{k=1, k\neq i}^{N} \iota_{ijk} + 2a_{ji}\delta_{ji} - 2\alpha_{ij} + \varepsilon^2 < 0, \qquad 2\lambda - \theta_{ij} < 0,$$

where $\iota_{ijk} = 2(a_{jk}\delta_{jk} + a_{ik}\delta_{ik})$.

B2 *Assume that digraph $(\mathcal{G}, C)$ is strongly connected, where $C = (\vartheta_{ijk})_{(N-1)\times(N-1)}$, with* $\vartheta_{ijk} = \begin{cases} \iota_{ijk}, \ j \neq k, \\ 0, \quad j = k. \end{cases}$

then system (2.1) achieves finite-time synchronization. Denote $\eta = \frac{1+b}{2}$, $\mu = \min_{j\in\mathbb{L}, j\neq i} \left\{ c_{ij}^{1-\eta} \mu_{ij} \right\}$, and the finite time t_1 is estimated by

$$\mathbb{E}(t_1) \leq \frac{V^{1-\eta}(x(0), 0)}{\mu(1-\eta)},$$

where $V(x(0), 0) = \sum_{j=1, j\neq i}^{N} c_{ij} V^{(ij)}(x_{ij}(0), 0)$ in which $V^{(ij)}(x_{ij}(0), 0) = \|x_{ij}(0)\|^2 + \theta_{ij} \int_{-\tau}^{0} \|x_{ij}(s)\|^2 \mathrm{d}s$.

Proof. Define function

$$V^{(ij)}(x_{ij}(t),t)=\|x_{ij}(t)\|^2+\theta_{ij}\int_{t-\tau}^{t}\|x_{ij}(s)\|^2\mathrm{d}s.$$

It is obvious that $V^{(ij)}(x_{ij}(t),t)$ is a positive-definite and radially unbounded function. According to Definition 2.2, we derive

$$\begin{aligned}
&\mathfrak{L}V^{(ij)}(x_{ij}(t),t)\\
&=\theta_{ij}\|x_{ij}(t)\|^2-\theta_{ij}\|x_{ij}(t-\tau)\|^2+\|g(x_j(t),t)-g(x_i(t),t)\|^2+2x_{ij}^{\mathrm{T}}(t)u(x_{ij}(t),t)\\
&\quad+2x_{ij}^{\mathrm{T}}(t)\left(f(x_j(t),x_j(t-\tau),t)-f(x_i(t),x_i(t-\tau),t)+\sum_{k=1}^{N}a_{jk}H_{jk}(x_j(t),x_k(t))\right)\\
&\quad-2x_{ij}^{\mathrm{T}}(t)\sum_{k=1}^{N}a_{ik}H_{ik}(x_i(t),x_k(t)).
\end{aligned}$$

From Assumption 2, we have

$$\begin{aligned}
&2x_{ij}^{\mathrm{T}}(t)\left(\sum_{k=1}^{N}a_{jk}H_{jk}(x_j(t),x_k(t))-\sum_{k=1}^{N}a_{ik}H_{ik}(x_i(t),x_k(t))\right)\\
&\le 2\|x_{ij}(t)\|\sum_{k=1,k\neq i}^{N}\left(a_{jk}\|H_{jk}(x_j(t),x_k(t))\|+a_{ik}\|H_{ik}(x_i(t),x_k(t))\|\right)\\
&\quad+2\|x_{ij}(t)a_{ji}\|H_{ji}(x_j(t),x_i(t))\|\\
&\le 2\|x_{ij}(t)\|\left(\sum_{k=1,k\neq i}^{N}\left(a_{jk}\delta_{jk}\|x_{jk}(t)\|+a_{ik}\delta_{ik}\|x_{ik}(t)\|\right)+a_{ji}\delta_{ji}\|x_{ij}(t)\|\right)\\
&\le \sum_{k=1,k\neq i}^{N}\left(a_{jk}\delta_{jk}\left(\|x_{ij}(t)\|^2+\|x_{jk}(t)\|^2\right)+a_{ik}\delta_{ik}\left(\|x_{ij}(t)\|^2+\|x_{ik}(t)\|^2\right)\right)\\
&\quad+2a_{ji}\delta_{ji}\|x_{ij}(t)\|^2\\
&\le \sum_{k=1,k\neq i}^{N}\left(a_{jk}\delta_{jk}+a_{ik}\delta_{ik}\right)\|x_{ij}(t)\|^2+2a_{ji}\delta_{ji}\|x_{ij}(t)\|^2\\
&\quad+\sum_{k=1,k\neq i}^{N}\left(a_{jk}\delta_{jk}+a_{ik}\delta_{ik}\right)\left(\|x_{jk}(t)\|^2+\|x_{ik}(t)\|^2\right)\\
&\le \left(3\sum_{k=1,k\neq i}^{N}\iota_{ijk}+2a_{ji}\delta_{ji}\right)\|x_{ij}(t)\|^2+\sum_{k=1,k\neq i}^{N}\iota_{ijk}\left(\|x_{ik}(t)\|^2-|x_{ij}(t)\|^2\right).
\end{aligned}$$

Combine controller (2.2) with Lemma 2.2, then it has

$$\begin{aligned}
&x_{ij}^{\mathrm{T}}(t)u(x_{ij}(t),t)\\
&=-\alpha_{ij}\|x_{ij}(t)\|^2-x_{ij}^{\mathrm{T}}(t)\mathrm{sign}(x_{ij}(t))|x_{ij}(t)|^b-\left(\int_{t-\tau}^{t}\theta_{ij}\|x_{ij}(s)\|^2\mathrm{d}s\right)^{\frac{1+b}{2}}\\
&\le-\alpha_{ij}\|x_{ij}(t)\|^2-\left(\|x_{ij}(t)\|^2\right)^{\frac{1+b}{2}}-\left(\int_{t-\tau}^{t}\theta_{ij}\|x_{ij}(s)\|^2\mathrm{d}s\right)^{\frac{1+b}{2}}
\end{aligned}$$

$$\leq -\alpha_{ij}\|x_{ij}(t)\|^2 - \left(\|x_{ij}(t)\|^2 + \int_{t-\tau}^{t} \theta_{ij}\|x_{ij}(s)\|^2 \mathrm{d}s\right)^{\frac{1+b}{2}}.$$

Thus, by making use of Assumptions 1 and 3, we obtain

$$\begin{aligned}
&\mathfrak{L}V^{(ij)}(x_{ij}(t),t)\\
\leq\ & \theta_{ij}\|x_{ij}(t)\|^2 - \theta_{ij}\|x_{ij}(t-\tau)\|^2 + 2\nu\|x_{ij}(t)\|^2 + 2\lambda\|x_{ij}(t-\tau)\|^2\\
&+\left(3\sum_{k=1,k\neq i}^{N}\iota_{ijk} + 2a_{ji}\delta_{ji}\right)\|x_{ij}(t)\|^2 + \sum_{k=1,k\neq i}^{N}\iota_{ijk}\left(\|x_{ik}(t)\|^2 - \|x_{jk}(t)\|^2\right)\\
&-2\alpha_{ij}\|x_{ij}(t)\|^2 - 2\left(\|x_{ij}(t)\|^2 + \int_{t-\tau}^{t}\theta_{ij}\|x_{ij}(s)\|^2\mathrm{d}s\right)^{\frac{1+b}{2}} + \varepsilon^2\|x_{ij}(t)\|^2\\
=\ & \left(\theta_{ij} + 2\nu + 3\sum_{k=1,k\neq i}^{N}\iota_{ijk} + 2a_{ji}\delta_{ji} - 2\alpha_{ij} + \varepsilon^2\right)\|x_{ij}(t)\|^2\\
&+(2\lambda - \theta_{ij})\|x_{ij}(t-\tau)\|^2 + \sum_{k=1,k\neq i}^{N}\iota_{ijk}\left(\|x_{ik}(t)\|^2 - \|x_{ij}(t)\|^2\right)\\
&-2\left(\|x_{ij}(t)\|^2 + \int_{t-\tau}^{t}\theta_{ij}\|x_{ij}(s)\|^2\mathrm{d}s\right)^{\frac{1+b}{2}}.
\end{aligned}$$

Furthermore, using condition B1, it is easily obtained that

$$\begin{aligned}
&\mathfrak{L}V^{(ij)}(x_{ij}(t),t)\\
\leq\ & \sum_{k=1,k\neq i}^{N}\iota_{ijk}\left(\|x_{ik}(t)\|^2 - \|x_{ij}(t)\|^2\right) - 2\left(\|x_{ij}(t)\|^2 + \int_{t-\tau}^{t}\theta_{ij}\|x_{ij}(s)\|^2\mathrm{d}s\right)^{\frac{1+b}{2}}\\
\leq\ & -2\left(V^{(ij)}(x_{ij}(t),t)\right)^{\frac{1+b}{2}} + \sum_{k=1,k\neq i}^{N}\iota_{ijk}\left(\|x_{ik}(t)\|^2 - \|x_{ij}(t)\|^2\right).
\end{aligned}$$

Let $\mu_{ij} = 2$, $\eta = \frac{1+b}{2}$, and $M_{ijk}(x_{ik}(t), x_{ij}(t)) = \|x_{ik}(t)\|^2 - \|x_{ij}(t)\|^2$, then, it has

$$\mathfrak{L}V^{(ij)}(x_{ij}(t),t) \leq -\mu_{ij}\left(V^{(ij)}(x_{ij}(t),t)\right)^{\eta} + \sum_{k=1,k\neq i}^{N}\iota_{ijk}M_{ijk}(x_{ik}(t), x_{ij}(t)).$$

From Theorem 3.1, system (2.1) achieves synchronization in finite time t_1, here t_1 is estimated by

$$\mathbb{E}(t_1) \leq \frac{V^{1-\eta}(x(0),0)}{\mu(1-\eta)}.$$

This completes the proof. □

Remark 3.2. Considering the practical applications, it is difficult to construct a global Lyapunov function directly to study finite-time synchronization of stochastic coupled systems. In Theorem 3.1, by means of the Lyapunov function of each vertex system, a theoretical framework to construct a proper global Lyapunov function is obtained, which is proposed in [15]. Besides, we give Theorem 3.2 with novel sufficient conditions verified more easily to guarantee system (2.1) is synchronized in finite time.

4. Applications and numerical examples

4.1. An application to stochastic coupled oscillators with time delay

It is known that one type of stochastic oscillator equations [17] is

$$\ddot{x}(t) + \varphi\dot{x}(t) + \tilde{\rho}x(t) = g(x(t),t)\dot{B}(t),\ t \geq 0,$$

where $\varphi > 0$, and $\tilde{\rho} > 0$ is the damping constant, $g : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ represents the disturbance intensity. Taking time delay, external input and coupling factors into account, we arrive at stochastic coupled oscillators (SCO) with time delay of second-order differential equations:

$$\ddot{x}_i(t)+\varphi\dot{x}_i(t)+\tilde{\rho}x_i(t-\tau)+\sum_{k=1}^{N}a_{ik}H_{ik}(x_i(t),x_k(t))+I(t)=g(x_i(t),t)\dot{B}(t),\ i\in\mathbb{L},\ t\geq 0, \tag{4.1}$$

where $x_i(t) \in \mathbb{R}$ stands for the state of the ith vertex system, $\tau \geq 0$ is time delay, $I(t)$ is external input, $H_{ik} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the coupling term and $a_{ik} \geq 0$ is coupling coefficient.

By making a transform of $\tilde{x}_i(t) = \dot{x}_i(t) + \xi x_i(t)$, in which $\xi > 0$, system (4.1) is rewritten as

$$\begin{cases} \mathrm{d}x_i(t) = (\tilde{x}_i(t) - \xi x_i(t))\mathrm{d}t, \\ \mathrm{d}\tilde{x}_i(t) = \big((\xi-\varphi)\tilde{x}_i(t) + (\xi\varphi-\xi^2)x_i(t) - \tilde{\rho}x_i(t-\tau) - I(t)\big)\,\mathrm{d}t \\ \qquad\qquad - \displaystyle\sum_{k=1}^{N} a_{ik}H_{ik}(x_i(t),x_k(t))\mathrm{d}t + g(x_i(t),t)\mathrm{d}B(t), \end{cases} i\in\mathbb{L},\ t\geq 0. \tag{4.2}$$

Firstly, we make the following notations: $X_i(t) = (x_i(t),\ \tilde{x}_i(t))^{\mathrm{T}}$ and let

$$\tilde{f}(X_i(t),X_i(t-\tau),t) = \begin{pmatrix} \tilde{x}_i(t) - \xi x_i(t) \\ (\xi-\varphi)\tilde{x}_i(t) + (\xi\varphi-\xi^2)x_i(t) - \tilde{\rho}x_i(t-\tau) \end{pmatrix}, \tag{4.3}$$

$$\tilde{I}(t) = \begin{pmatrix} 0 \\ -I(t) \end{pmatrix}, \qquad \tilde{g}(X_i(t),t) = \begin{pmatrix} 0 \\ g(x_i(t),t) \end{pmatrix}$$

and

$$\tilde{H}_{ik}(X_i(t),X_k(t)) = \begin{pmatrix} 0 \\ -H_{ik}(x_i(t),x_k(t)) \end{pmatrix}.$$

Thus system (4.2) can be substituted by

$$\begin{aligned} \mathrm{d}X_i(t) = {}& \left(\tilde{f}(X_i(t),X_i(t-\tau),t) + \sum_{k=1}^{N} a_{ik}\tilde{H}_{ik}(X_i(t),X_k(t)) + \tilde{I}(t)\right)\mathrm{d}t \\ & + \tilde{g}(X_i(t),t)\mathrm{d}B(t),\ i\in\mathbb{L},\ t\geq 0. \end{aligned} \tag{4.4}$$

For convenience, set

$$Z_{ij}(t) \triangleq X_j(t) - X_i(t) = \begin{pmatrix} x_j(t) \\ \tilde{x}_j(t) \end{pmatrix} - \begin{pmatrix} x_i(t) \\ \tilde{x}_i(t) \end{pmatrix} = \begin{pmatrix} x_{ij}(t) \\ \tilde{x}_{ij}(t) \end{pmatrix}.$$

In order to make system (4.4) be synchronized in finite time, a controller $\tilde{u}(Z_{ij}(t), t)$ will be affiliated and the jth ($j \in \mathbb{L}$, $j \neq i$) vertex system is described as:

$$\begin{aligned} \mathrm{d}X_j(t) = &\left(\tilde{f}(X_j(t), X_j(t-\tau), t) + \sum_{k=1}^{N} a_{jk}\tilde{H}_{jk}(X_j(t), X_k(t)) + \tilde{I}(t) + \tilde{u}(Z_{ij}(t), t)\right) \mathrm{d}t \\ &+ \tilde{g}(X_j(t), t)\mathrm{d}B(t), \quad t \geq 0, \end{aligned}$$

where

$$\begin{aligned} &\tilde{u}(Z_{ij}(t), t) \\ &= -\alpha_{ij} Z_{ij}(t) - \mathrm{sign}(Z_{ij}(t))|Z_{ij}(t)|^b - \left(\int_{t-\tau}^{t} \theta_{ij}\|Z_{ij}(s)\|^2 \mathrm{d}s\right)^{\frac{1+b}{2}} \frac{Z_{ij}(t)}{\|Z_{ij}(t)\|^2}, \end{aligned}$$

where $\alpha_{ij} > 0$, $0 < b < 1$, $\theta_{ij} > 0$ and $\mathrm{sign}(Z_{ij}(t)) = \mathrm{diag}\,\{\mathrm{sign}(x_{ij}(t)), \mathrm{sign}(\tilde{x}_{ij}(t))\}$ in which $\mathrm{sign}(\cdot)$ is a sign function, and $|Z_{ij}(t)|^b = (|x_{ij}(t)|^b, |\tilde{x}_{ij}(t)|^b)^{\mathrm{T}}$. Then, we can get immediately the following synchronization error system:

$$\begin{aligned} \mathrm{d}Z_{ij}(t) = &\left(\tilde{f}(X_j(t), X_j(t-\tau), t) - \tilde{f}(X_i(t), X_i(t-\tau), t) + \tilde{u}(Z_{ij}(t), t)\right) \mathrm{d}t \\ &+ \left(\sum_{k=1}^{N} a_{jk}\tilde{H}_{jk}(X_j(t), X_k(t)) - \sum_{k=1}^{N} a_{ik}\tilde{H}_{ik}(X_i(t), X_k(t))\right) \mathrm{d}t \\ &+ \left(\tilde{g}(X_j(t), t) - \tilde{g}(X_i(t), t)\right) \mathrm{d}B(t), \quad i, j \in \mathbb{L}, \quad j \neq i, \quad t \geq 0. \end{aligned} \tag{4.5}$$

Furthermore, we will introduce a theorem about finite-time synchronization of system (4.4) in the following.

Theorem 4.1. *Assume that i ($i \in \mathbb{L}$) is any fixed and the following conditions hold,*

C1 *For any $j \in \mathbb{L}$ ($j \neq i$), there exists a positive constant β such that*

$$\|g(x_j, t) - g(x_i, t)\| \leq \beta\|x_{ij}\|.$$

C2 *For any $j,\ k \in \mathbb{L}$, there exists a positive constant γ_{jk} such that*

$$\|H_{jk}(x_j, x_k)\| \leq \gamma_{jk}\|x_{jk}\|.$$

C3 *Digraph $(\mathcal{G}, C)$ is strongly connected, where $C = (\varsigma_{ijk})_{(N-1)\times(N-1)}$ with $\varsigma_{ijk} = \begin{cases} 2(a_{jk}\gamma_{ik} + a_{ik}\gamma_{ik}), & j \neq k, \\ 0, & j = k, \end{cases}$ and the following inequalities hold.*

$$\theta_{ij} + 2\sigma + 3\sum_{k=1, k\neq i}^{N} \varsigma_{ijk} + 2a_{ji}\gamma_{ji} - 2\alpha_{ij} + \beta^2 < 0, \quad \textit{and} \quad \tilde{\rho} - \theta_{ij} < 0,$$

where

$$\sigma = \max\left\{\frac{|\xi\varphi-\xi^2+1|-2\xi}{2}, \frac{|\xi\varphi-\xi^2+1|+2(\xi-\varphi)+\tilde{\rho}}{2}\right\} > 0.$$

then system (4.4) *achieves synchronization in finite time.*

Proof. To begin with, owing to notation (4.3), we have

$$\begin{aligned}
&Z_{ij}^{\mathrm{T}}(t)\left(\tilde{f}(X_j(t),X_j(t-\tau),t)-\tilde{f}(X_i(t),X_i(t-\tau),t)\right)\\
&=(\xi\varphi-\xi^2+1)(x_j(t)-x_i(t))^{\mathrm{T}}(\tilde{x}_j(t)-\tilde{x}_i(t))-\xi\|x_j(t)-x_i(t)\|^2\\
&\quad+(\xi-\varphi)\|\tilde{x}_j(t)-\tilde{x}_i(t)\|^2-\tilde{\rho}(\tilde{x}_j(t)-\tilde{x}_i(t))^{\mathrm{T}}(x_j(t-\tau)-x_i(t-\tau))\\
&\le\frac{|\xi\varphi-\xi^2+1|}{2}\left(\|x_j(t)-x_i(t)\|^2+\|\tilde{x}_j(t)-\tilde{x}_i(t)\|^2\right)-\xi\|x_j(t)-x_i(t)\|^2\\
&\quad+(\xi-\varphi)\|\tilde{x}_j(t)-\tilde{x}_i(t)\|^2+\frac{\tilde{\rho}}{2}\left(\|\tilde{x}_j(t)-\tilde{x}_i(t)\|^2+\|x_j(t-\tau)-x_i(t-\tau)\|^2\right)\\
&\le\frac{|\xi\varphi-\xi^2+1|-2\xi}{2}\|x_j(t)-x_i(t)\|^2+\frac{|\xi\varphi-\xi^2+1|+2(\xi-\varphi)+\tilde{\rho}}{2}\|\tilde{x}_j(t)-\tilde{x}_i(t)\|^2\\
&\quad+\frac{\tilde{\rho}}{2}\|x_j(t-\tau)-x_i(t-\tau)\|^2\\
&\le\sigma\left(\|x_j(t)-x_i(t)\|^2+\|\tilde{x}_j(t)-\tilde{x}_i(t)\|^2\right)\\
&\quad+\frac{\tilde{\rho}}{2}\left(\|x_j(t-\tau)-x_i(t-\tau)\|^2+\|\tilde{x}_j(t-\tau)-\tilde{x}_i(t-\tau)\|^2\right).
\end{aligned}$$

That is to say

$$Z_{ij}^{\mathrm{T}}(t)\left(\tilde{f}(X_j(t),X_j(t-\tau),t)-\tilde{f}(X_i(t),X_i(t-\tau),t)\right)\le\sigma\|Z_{ij}(t)\|^2+\frac{\tilde{\rho}}{2}\|Z_{ij}(t-\tau)\|^2.$$

Besides, according to condition C1, we have

$$\|\tilde{g}(X_j(t),t)-\tilde{g}(X_i(t),t)\|^2=\|g(x_j(t),t)-g(x_i(t),t)\|^2\le\beta^2\|x_{ij}(t)\|^2\le\beta^2\|Z_{ij}(t)\|^2,$$

namely,

$$\|\tilde{g}(X_j(t),t)-\tilde{g}(X_i(t),t)\|\le\beta\|Z_{ij}(t)\|.$$

In addition, according to condition C2, it is obtained that

$$\|\tilde{H}_{ik}\left(X_i(t),X_k(t)\right)\|^2=\|H_{ik}(x_i(t),x_k(t))\|^2\le\gamma_{ik}^2\|x_i(t)-x_k(t)\|^2\le\gamma_{ik}^2\|Z_{ik}(t)\|^2.$$

Hence, it yields

$$\|\tilde{H}_{ik}\left(X_i(t),X_k(t)\right)\|\le\gamma_{ik}\|Z_{ik}(t)\|.$$

Based on the discussion above, we can easily obtain that Assumptions 1, 2 and 3 hold. In addition, from condition C3, we derive that conditions B1 and B2 in Theorem 3.2 are fulfilled. In other words, all conditions in Theorem 3.2 hold. Therefore, system (4.4) achieves synchronization in finite time. □

Remark 4.1. Stability of stochastic coupled oscillators has been widely studied recently [17, 19, 28, 39]. For example, authors in [17] have investigated global exponential stability for stochastic networks of coupled oscillators with variable delay. As is known, synchronization problem can be translated into stability problem of synchronization error system. However, synchronization, especially finite-time synchronization, plays a more important role in real life. Thus, it is more practical to consider finite-time synchronization of SCO. In this paper, it is worth noting that Theorem 4.1 provides us with an easily verifiable sufficient criterion for SCO. To the best of our knowledge, the approach combining Kirchhoff's Matrix Tree Theorem with Lyapunov method is first utilized to consider the finite-time synchronization problem of SCO.

Example 4.1. In succession, a simple example is provided to illustrate the effectiveness of our main results derived above.

For the sake of simplification, we consider aforementioned SCO with time delay (4.4). Let $N = 5$, $a_{11} = 0$, $a_{12} = 0.01$, $a_{13} = 0.008$, $a_{14} = 0.005$, $a_{15} = 0.003$, $a_{21} = 0.1$, $a_{22} = 0$, $a_{23} = 0.044$, $a_{24} = 0.023$, $a_{25} = 0.015$, $a_{31} = 0.2$, $a_{32} = 0.03$, $a_{33} = 0$, $a_{34} = 0.009$, $a_{35} = 0.019$, $a_{41} = 0.3$, $a_{42} = 0.15$, $a_{43} = 0.006$, $a_{44} = 0$, $a_{45} = 0.025$, $a_{51} = 0.4$, $a_{52} = 0.02$, $a_{53} = 0.006$, $a_{54} = 0.019$, $a_{55} = 0$. Then, we choose time delay $\tau = 1$ and let $H_{ik}(x_i, x_k) = x_i - x_k$, $g(x_i, t) = \sin(\pi x_i) + x_i + t$ and $I(t) = 20\cos(\pi t)$. We can derive that $\beta = 2$, $\gamma_{ik} = 1$. Besides, for fixed $i = 1$, and $j, k = 2, 3, 4, 5$, digraph $(\mathcal{G}, C)$ is strongly connected since matrix

$$C = (\varsigma_{1jk})_{4\times 4} = \begin{pmatrix} 0 & 0.104 & 0.056 & 0.036 \\ 0.08 & 0 & 0.028 & 0.044 \\ 0.32 & 0.028 & 0 & 0.056 \\ 0.06 & 0.028 & 0.048 & 0 \end{pmatrix}$$

is irreducible. In addition, we make $\xi = 0.1$, $\varphi = 0.05$ and $\tilde{\rho} = 2.4$. Thus, $\sigma = 1.7475$.

The synchronization error system without control is shown by Figure 1, from which, we can get that system (4.4) does not achieve finite-time synchronization.

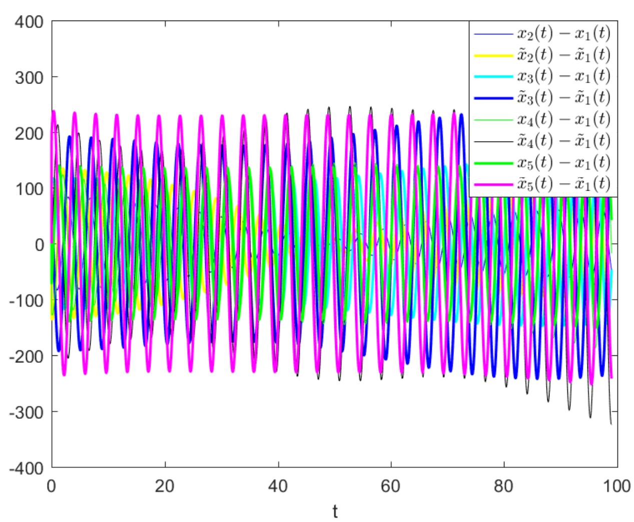

Figure 1. The trajectory of synchronization error system without control.

In order to make system (4.4) achieve synchronization within finite time, the controller is designed as follows ($j = 2, 3, 4, 5$)

$$\begin{aligned}&\tilde{u}(Z_{1j}(t), t)\\&= -\alpha_{1j} Z_{1j}(t) - \text{sign}(Z_{1j}(t))|Z_{1j}(t)|^{0.4} - \left(\int_{t-\tau}^{t} \theta_{ij} \|Z_{1j}(s)\|^2 \text{d}s\right)^{0.7} \frac{Z_{1j}(t)}{\|Z_{1j}(t)\|^2},\end{aligned}$$

where $\alpha_{12} = 5.5$, $\alpha_{13} = 5.5$, $\alpha_{14} = 5.7$, $\alpha_{15} = 5.4$, $\theta_{12} = 2.8$, $\theta_{13} = 3$, $\theta_{14} = 2.5$ and $\theta_{15} = 2.7$. Then

$$\begin{aligned}&\max_{j=2,3,4,5} \left\{\theta_{1j} + 2\sigma + 3\sum_{k=2}^{5} \varsigma_{1jk} + 2a_{j1}\gamma_{j1} - 2\alpha_{1j} + \beta^2\right\}\\&= \max\{-0.097, -0.033, -0.183, -0.194\} \le 0.\end{aligned}$$

It is easy to check that all conditions from Theorem 4.1 are satisfied.

Figure 2 shows the trajectories of synchronization error system (4.5). It is obvious that the synchronization errors tend to zero. Therefore, system (4.4) achieves finite-time synchronization. The corresponding numerical results illustrate the effectiveness and feasibility of our theoretical results.

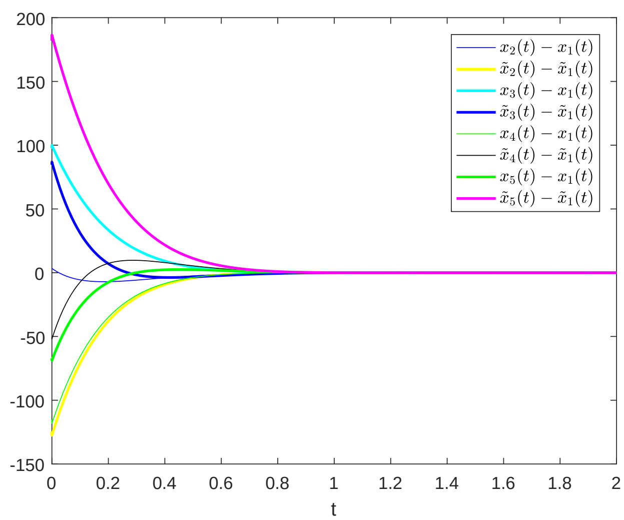

Figure 2. The trajectories of synchronization error system (4.5).

4.2. An application to stochastic Lorenz chaotic coupled systems with time delay

Consider the following time-delay Lorenz chaotic systems [24] as below

$$\begin{cases}\dot{x}_{i1}(t) = 10x_{i2}(t-\tau) - 10x_{i1}(t), \\ \dot{x}_{i2}(t) = 28x_{i1}(t) - x_{i2}(t) - x_{i1}(t)x_{i3}(t), & i \in \mathbb{L}, \\ \dot{x}_{i3}(t) = x_{i1}(t)x_{i2}(t) - \frac{8}{3}x_{i3}(t-\tau),\end{cases}$$

where $x_{i1}(t)$, $x_{i2}(t)$, $x_{i3}(t) \in \mathbb{R}$ are state variables, $\tau \geq 0$ stands for time delay. In the following, for convenience, set $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^{\mathrm{T}}$ and $\hat{f}(x_i(t), x_i(t-\tau), t) = (10x_{i2}(t-\tau) - 10x_{i1}(t), 28x_{i1}(t) - x_{i2}(t) - x_{i1}(t)x_{i3}(t), x_{i1}(t)x_{i2}(t) - \frac{8}{3}x_{i3}(t-\tau))^{\mathrm{T}}$.

Taking coupling factors and stochastic disturbance into consideration, the following stochastic Lorenz chaotic coupled systems with time delay can be expressed as

$$\begin{aligned} \mathrm{d}x_i(t) = &\left(\hat{f}(x_i(t), x_i(t-\tau), t) + \sum_{k=1}^{N} a_{ik}\hat{H}_{ik}(x_i(t), x_k(t)) + I(t)\right)\mathrm{d}t \\ &+ \hat{g}(x_i(t), t)\mathrm{d}B(t), \ \ i \in \mathbb{L}, \ \ t \geq 0. \end{aligned} \tag{4.6}$$

where $x_i(t)$ represents the state of the ith vertex system, $\hat{g} : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^3$ is a continuous function, which stands for the disturbance intensity. Function $\hat{H}_{ik} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is coupling term and $a_{ik} \geq 0$ is coupling strength. And $I(t)$ is external input vector. As we all know, the attractor of stochastic Lorenz chaotic coupled systems with time delay is bounded, that is to say, there are positive constants M_{is}, such that $|x_{is}(t)| \leq M_{is}, s = 1, 2, 3, i \in \mathbb{L}$.

Define error vector as

$$y_{ij}(t) = x_j(t) - x_i(t) = \begin{pmatrix} x_{j1}(t) - x_{i1}(t) \\ x_{j2}(t) - x_{i2}(t) \\ x_{j3}(t) - x_{i3}(t) \end{pmatrix},$$

where $y_{ij}(t) = (y_{ij1}(t), y_{ij2}(t), y_{ij3}(t))^{\mathrm{T}}$, $y_{ijs}(t) = x_{js}(t) - x_{is}(t)$, $s = 1, 2, 3$.

For the sake of achieving the finite-time synchronization of system (4.6), a suitable feedback control $\hat{u}(y_{ij}(t), t)$ is utilized, and the jth vertex system can be described as

$$\begin{aligned} \mathrm{d}x_j(t) = &\left(\hat{f}(x_j(t), x_j(t-\tau), t) + \sum_{k=1}^{N} a_{jk}\hat{H}_{jk}(x_j(t), x_k(t)) + I(t)\right)\mathrm{d}t \\ &+ \hat{g}(x_j(t), t)\mathrm{d}B(t), \ \ t \geq 0, \end{aligned}$$

in which

$$\begin{aligned} &\hat{u}(y_{ij}(t), t) \\ &= -\alpha_{ij}y_{ij}(t) - \mathrm{sign}(y_{ij}(t))|y_{ij}(t)|^b - \left(\int_{t-\tau}^{t} \theta_{ij}\|y_{ij}(s)\|^2\mathrm{d}s\right)^{\frac{1+b}{2}} \frac{y_{ij}(t)}{\|y_{ij}(t)\|^2}, \end{aligned}$$

where $\alpha_{ij} > 0$, $0 < b < 1$, $\theta_{ij} > 0$. Then, the following synchronization error system can be derived:

$$\begin{aligned} \mathrm{d}y_{ij}(t) = &\left(\hat{f}(x_j(t), x_j(t-\tau), t) - \hat{f}(x_i(t), x_i(t-\tau), t) + \hat{u}(y_{ij}(t), t)\right)\mathrm{d}t \\ &+ \left(\sum_{k=1}^{N} a_{jk}\hat{H}_{jk}(x_j(t), x_k(t)) - \sum_{k=1}^{N} a_{ik}\hat{H}_{ik}(x_i(t), x_k(t))\right)\mathrm{d}t \\ &+ \left(\hat{g}(x_j(t), t) - \hat{g}(x_i(t), t)\right)\mathrm{d}B(t), \ \ \ i, j \in \mathbb{L}, \ \ \ j \neq i, \ \ \ t \geq 0. \end{aligned} \tag{4.7}$$

Subsequently, a theorem which ensures system (4.6) to realize finite-time synchronization is presented.

Theorem 4.2. *Assume that i ($i \in \mathbb{L}$) is any fixed and the following conditions hold,*

D1 *For any $j \in \mathbb{L}$ ($j \neq i$), there is a positive constant $\hat{\beta}$ such that*

$$\|\hat{g}(x_j, t) - \hat{g}(x_i, t)\| \leq \hat{\beta}\|y_{ij}\|.$$

D2 *For any $j,\ k \in \mathbb{L}$, there is a positive constant $\hat{\gamma}_{jk}$ such that*

$$\|\hat{H}_{jk}(x_j, x_k)\| \leq \hat{\gamma}_{jk}\|y_{jk}\|.$$

D3 *Digraph $(\mathcal{G}, D)$ is strongly connected, where $D = (\epsilon_{ijk})_{(N-1)\times(N-1)}$ with $\epsilon_{ijk} = \begin{cases} 2(a_{jk}\hat{\gamma}_{ik} + a_{ik}\hat{\gamma}_{ik}), & j \neq k, \\ 0, & j = k, \end{cases}$ and the following inequalities are satisfied.*

$$\theta_{ij} + 2\hat{\sigma} + 3\sum_{k=1, k\neq i}^{N} \epsilon_{ijk} + 2a_{ji}\hat{\gamma}_{ji} - 2\alpha_{ij} + \hat{\beta}^2 < 0, \quad \textit{and} \quad 10 - \theta_{ij} < 0,$$

where

$$\hat{\sigma} = \max\left\{9 + \frac{M_{i2} + M_{i3}}{2}, 14 + \frac{M_{i3}}{2}, \frac{8 + 3M_{i3}}{6}\right\}.$$

then system (4.6) achieves finite-time synchronization.

Proof. According to condition D1 and D2, it is easy to observe that Assumptions 2 and 3 are fulfilled. In the next, we will prove that Assumption 1 can be satisfied.

In fact, notice that

$$\begin{aligned}
& (x_j - x_i)^{\mathrm{T}}(\hat{f}(t, x_j) - \hat{f}(t, x_i)) \\
&= (y_{ij1}, y_{ij2}, y_{ij3}) \begin{pmatrix} 10(y_{ij2}(t-\tau) - y_{ij1}) \\ 28y_{ij1} - y_{ij2} - x_{j1}x_{j3} + x_{i1}x_{i3} \\ x_{j1}x_{j2} - x_{i1}x_{i2} - \frac{8}{3}y_{ij3}(t-\tau) \end{pmatrix} \\
&\leq \left(9 + \frac{M_{i2} + M_{i3}}{2}\right) y_{ij1}^2 + \left(14 + \frac{M_{i3}}{2}\right) y_{ij2}^2 + \frac{8 + 3M_{i3}}{6} y_{ij3}^2 \\
&\quad + 5y_{ij2}^2(t-\tau) + \frac{4}{3}y_{ij3}^2(t-\tau) \\
&\leq \hat{\sigma}\|y_{ij}(t)\|^2 + 5\|y_{ij}(t-\tau)\|^2.
\end{aligned} \tag{4.8}$$

Hence, from above (4.8), Assumption 1 holds. Combined with condition D3, all conditions in Theorem 3.2 are satisfied. Therefore, system (4.6) achieves finite-time synchronization. This ends the proof. □

Remark 4.2. Currently, with the development of chaotic synchronization in secure communication, synchronization of chaotic systems has attracted the increasing attention of researchers. For example, in [1], finite-time synchronization of two different chaotic systems with unknown parameters was studied by sliding mode technique. Robust finite-time anti-synchronization of chaotic systems with different dimensions was investigated in [2]. In this paper, we consider not only time delay, but also stochastic disturbance into Lorenz systems, which is more practical. It is worth pointing out that finite-time synchronization of stochastic Lorenz chaotic coupled systems with time delay is studied for the first time.

Example 4.2. Next, a numerical example is also given to illustrate the effectiveness of theoretic results. Consider stochastic Lorenz chaotic coupled systems with time delay (4.6). Let $N = 6$, $a_{11} = 0$, $a_{12} = 0.1$, $a_{13} = 0.8$, $a_{14} = 0.5$, $a_{15} = 0.3$, $a_{16} = 0.2$, $a_{21} = 0.1$, $a_{22} = 0$, $a_{23} = 0.044$, $a_{24} = 0.52$, $a_{25}0.23$, $a_{26} = 0.15$, $a_{31} = 0.2$, $a_{32} = 0.03$, $a_{33} = 0$, $a_{34} = 0.9$, $a_{35} = 0.2$, $a_{36} = 0.13$, $a_{41} = 0.3$, $a_{42} = 0.15$, $a_{43} = 0.34$, $a_{44} = 0$, $a_{45} = 0.17$, $a_{46} = 0.29$, $a_{51} = 0.4$, $a_{52} = 0.3$, $a_{53} = 0.37$, $a_{54} = 0.9$, $a_{55} = 0$, $a_{56} = 0.02$, $a_{61} = 0.4$, $a_{62} = 0.02$, $a_{63} = 0.06$, $a_{64} = 0.01$, $a_{65} = 0.02$, $a_{66} = 0$. We set time delay as $\tau = 0.01$, and choose $H_{ik}(x_i, x_k) = \sin(x_i - x_k)$, $g(x_i, t) = \sin(\pi x_i)$. Then we can obtain that $\hat{\beta} = 1$, $\hat{\gamma}_{ik} = 1$. Besides, for convenience, we fixed $i = 1$. It is obvious that digraph $(\mathcal{G}, D)$ is strongly connected because

$$D = (\epsilon_{1jk})_{5\times 5} = \begin{pmatrix} 0 & 1.688 & 2.04 & 1.06 & 0.7 \\ 0.26 & 0 & 2.8 & 1 & 0.66 \\ 0.5 & 2.28 & 0 & 0.94 & 0.98 \\ 0.8 & 2.34 & 2.8 & 0 & 0.44 \\ 0.24 & 1.72 & 1.02 & 0.64 & 0 \end{pmatrix}$$

is irreducible.

When $i = 1$, the following vertex system can be derived as

$$\begin{aligned} \mathrm{d}x_1(t) = &\left(\hat{f}(x_1(t), x_1(t-\tau), t) + \sum_{k=1}^{N} a_{1k} \hat{H}_{1k}(x_1(t), x_k(t)) + I(t) \right) \mathrm{d}t \\ &+ \hat{g}(x_1(t), t)\mathrm{d}B(t), \ t \geq 0, \end{aligned} \tag{4.9}$$

whose attractor is shown in Figure 3. Hence, we can get $M_{11} = 21.36$, $M_{12} = 28.7443$, $M_{13} = 54.024$ and $\hat{\sigma} = \max\{50.3841, 41.012, 28.3453\} = 50.3841$.

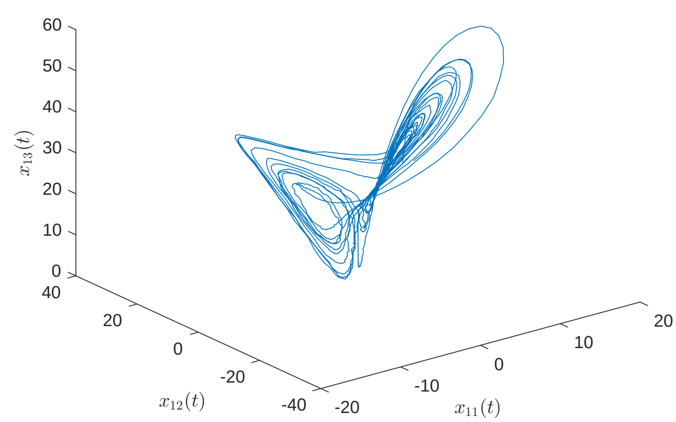

Figure 3. The attractor of vertex system (4.9).

Besides, the trajectories of synchronization error system (4.7) without control is shown by Figure 4. It is evident that system (4.6) is not synchronized.

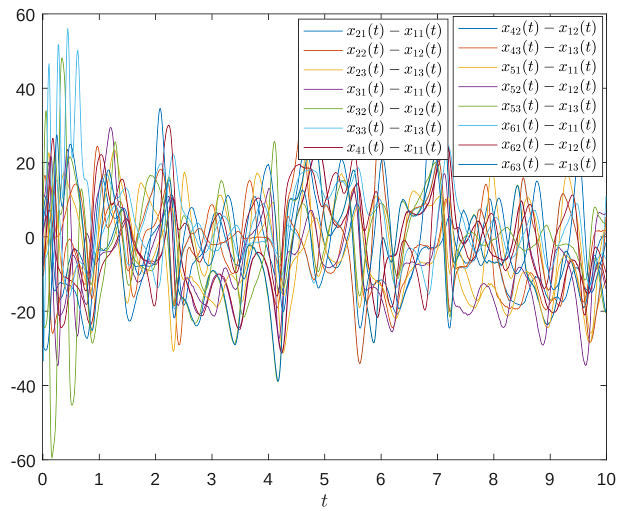

Figure 4. The trajectories of synchronization error system (4.7) without control.

For the sake of achieving the finite-time synchronization of system (4.6), feedback control $\hat{u}(y_{ij}(t),t)$ is utilized as

$$\begin{aligned}&\hat{u}(y_{1j}(t),t)\\&=-\alpha_{1j}y_{1j}(t)-\mathrm{sign}(y_{1j}(t))|y_{1j}(t)|^{0.4}-\left(\int_{t-\tau}^{t}\theta_{1j}\|y_{1j}(s)\|^2\mathrm{d}s\right)^{0.7}\frac{y_{1j}(t)}{\|y_{1j}(t)\|^2},\end{aligned}$$

where $\alpha_{1j}=66$, $\theta_{1j}=10.02$.

By simple calculation, we have

$$\begin{aligned}&\max_{j=2,3,4,5,6}\left\{\theta_{1j}+2\hat{\sigma}_1+3\sum_{k=2}^{N}\epsilon_{1jk}+2a_{j1}\hat{\gamma}_{j1}-2\alpha_{1j}+\hat{\beta}^2\right\}\\&=\max\{-3.5478,-5.8518,-5.9118,-0.8718,-9.1518\}<0,\end{aligned}$$

and

$$10-\theta_{1j}=-0.02<0.$$

It is easy to know that all conditions of Theorem 4.2 are satisfied.

Figure 5 shows the trajectories of synchronization error system (4.7), which implies that system (4.6) achieves finite-time synchronization. The simulation results demonstrate the effectiveness and feasibility of our theoretical results.

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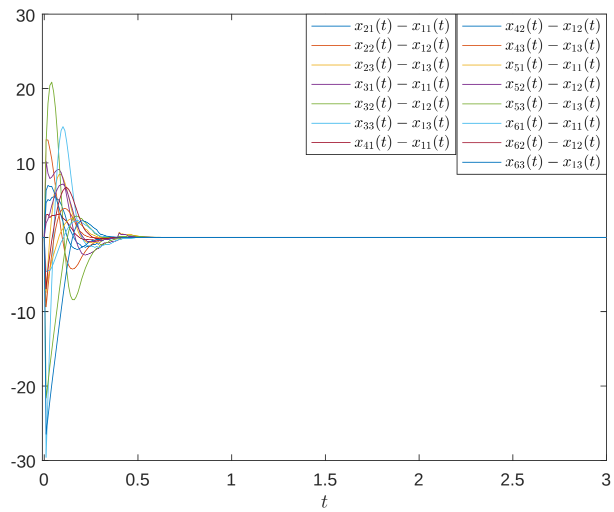

Figure 5. The trajectories of synchronization error system (4.7).

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