Journal of Applied Analysis and Computation
Volume 8, Number 4, August 2018, 1260–1281
Website:http://jaac-online.com/
DOI:10.11948/2018.1260

OSCILLATORY AND ASYMPTOTIC CRITERIA OF THIRD ORDER NONLINEAR DELAY DYNAMIC EQUATIONS WITH DAMPING TERM ON TIME SCALES*

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Abstract In this paper, we are concerned with oscillatory and asymptotic behavior of third order nonlinear delay dynamic equations with damping term on time scales. By using a generalized Riccati function and inequality technique, we establish some new oscillatory and asymptotic criteria. The established results on one hand extend some known results in the literature, on the other hand unify continuous and discrete analysis as two special cases of an arbitrary time scale. We also present some applications for the established results.

Keywords Oscillation, asymptotic behavior, third order nonlinear delay dynamic equations, time scales.

MSC(2010) 34N05, 26E70, 34C10.

1. Introduction

In the research of the theory of differential equations, if their exact solutions can not be expressed in usual form, then it is necessary to research the qualitative properties of the solutions. In the investigations of qualitative properties of solutions of differential equations, research for existence, stability and oscillation of solutions has been a hot topic, which has been paid much attention by many authors so far. For example, in [11, 12, 25, 27, 33, 35, 36, 38], existence of solutions of various differential equations were researched, while in [40, 41], orbital stability of solitary wave solutions and periodic traveling wave solutions of two nonlinear evolution equations were investigated. In [8, 19–24, 26, 34, 37, 39, 42], oscillation of solutions of various differential equations and systems were researched, and a lot of new oscillation criteria for these equations have been established therein.

On the other hand, the theory of time scale, which was initiated by Hilger [15], trying to treat continuous and discrete analysis in a consistent way, have received a lot of attention in recent years. Various investigations have been done by many authors. Among these investigations, some authors have taken research in oscillation of dynamic equations on time scales, and there has been increasing interest in obtaining sufficient conditions for oscillatory and asymptotic behavior of solutions

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*The author was partially supported by Natural Science Foundation of Shandong Province (China) (ZR2017BA003) and the development supporting plan for young teachers in Shandong University of Technology.

of first order or second order dynamic equations on time scales (for example, we refer the reader to [2, 5, 9, 28, 30]). Recently, there have also been much attention paid to the research of oscillation of third order or higher order dynamic equations on time scales. For such results, we refer the reader to [6, 7, 10, 14, 16, 18, 29, 31, 32].

We notice that in the research mentioned above, relatively less attention has been paid to oscillatory and asymptotic behavior of third order nonlinear delay dynamic equations with damping term on time scales, in which the damping term brings new difficulty in establishing oscillatory and asymptotic criteria for them.

Motivated by the analysis above, in this paper, we are concerned with oscillatory and asymptotic behavior of third order nonlinear delay dynamic equation with damping term on time scales of the following form:

$$(a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma})^{\Delta}+p(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}+q(t)f(x(\theta(t)))=0,\ t\in\mathbb{T}_0, \quad (1.1)$$

where $\mathbb{T}$ is an arbitrary time scale, $\mathbb{T}_0=[t_0,\infty)\bigcap\mathbb{T}$, $a,\ r,\ p,\ q\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$, $f\in C(\mathbb{R},\mathbb{R})$ satisfying $xf(x)>0$, $\frac{f(x)}{x^{\gamma}}\geq L>0$ for $x\neq 0$, $\theta\in C_{rd}(\mathbb{R},\mathbb{R})$ satisfying $\theta(t)\leq t$, $\theta^{\Delta}(t)\geq 0$ and $\lim\limits_{t\to\infty}\theta(t)=\infty$, $\gamma\geq 1$ is a quotient of two odd positive integers.

A solution of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1.1) is said to be oscillatory in case all its solutions are oscillatory.

We will establish some new oscillatory and asymptotic criteria for Eq. (1.1) by a generalized Riccati function and inequality technique in Section 2, and present some applications for our results in Section 3. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+=(0,\infty)$, while $\mathbb{Z}$ denotes the set of integers. $\mathbb{T}$ denotes an arbitrary time scale, and we always assume $\sup\mathbb{T}=\infty$. For an interval $[a,b]$, $[a,b]_{\mathbb{T}}:=[a,b]\bigcap\mathbb{T}$. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma\in(\mathbb{T},\mathbb{T})$ and $\rho\in(\mathbb{T},\mathbb{T})$ such that $\sigma(t)=\inf\{s\in\mathbb{T},s>t\}$, $\rho(t)=\sup\{s\in\mathbb{T},s<t\}$. A point $t\in\mathbb{T}$ with $t>\inf\mathbb{T}$ is said to be left-dense if $\rho(t)=t$ and right-dense if $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$. A function $f\in(\mathbb{T},\mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1+\mu(t)f(t)\neq 0$, where $\mu(t)=\sigma(t)-t$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+=\{f|f\in\mathfrak{R},\ 1+\mu(t)f(t)>0,\ \forall t\in\mathbb{T}\}$.

Definition 1.1. For $p\in\mathfrak{R}$, the *exponential function* is defined by

$$e_p(t,s)=\exp(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau)$$

for $s,t\in\mathbb{T}$.

Remark 1.1. If $\mathbb{T}=\mathbb{R}$, then $e_p(t,s)=\exp(\int_s^t p(\tau)d\tau)$ for $s,\ t\in\mathbb{R}$. If $\mathbb{T}=\mathbb{Z}$, then $e_p(t,s)=\prod\limits_{\tau=s}^{t-1}[1+p(\tau)]$ for $s,\ t\in\mathbb{Z}$ and $s<t$.

The following two theorems include some known properties on the *exponential function.*

Theorem 1.1 (Agarwal etc [1, Theorem 5.1])**.** *If $p\in\mathfrak{R}$, and fix $t_0\in\mathbb{T}$, then the exponential function $e_p(t,t_0)$ is the unique solution of the following initial value* *problem*

$$\begin{cases} y^{\Delta}(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

Theorem 1.2 (Agarwal etc [1, Theorem 5.2])**.** *If $p \in \Re^+$, then $e_p(t,s) > 0$ for $\forall s,\ t \in \mathbb{T}$.*

For more details about the calculus of time scales, we refer to Bohner and Peterson [4].

2. Main Results

For the sake of convenience, in the rest of the paper, set $\delta_1(t,a) = \int_a^t \frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta s$, $\delta_2(t,a) = \int_a^t \frac{\delta_1(s,a)}{r(s)}\Delta s$, and we always assume $t_i \in \mathbb{T},\ i = 1,2,...,6$, and $\theta\circ\sigma = \sigma\circ\theta$.

Lemma 2.1. *Suppose $-\frac{p}{a} \in \Re_+$, and assume that*

$$\int_{t_0}^{\infty} \frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta s = \infty, , \tag{2.1}$$

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta s = \infty, \tag{2.2}$$

and x is eventually a positive solution of Eq. (1.1)*. Then there exists a sufficiently large T_1^* such that*

$$\Big(\frac{a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)}\Big)^{\Delta} < 0,\ \ [r(t)x^{\Delta}(t)]^{\Delta} > 0$$

on $[T_1^*,\infty)_{\mathbb{T}}$.

Proof. By $-\frac{p}{a} \in \Re_+$, from Theorem 1.2 we have $e_{-\frac{p}{a}}(t,t_0) > 0$. Since x is eventually a positive solution of (1.1), there exists a sufficiently large t_1 such that $x(t) > 0,\ x(\theta(t)) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, and for $t \in [t_1,\infty)_{\mathbb{T}}$, by Theorem 1.1 we obtain that

$$\begin{aligned}
&\Big(\frac{a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)}\Big)^{\Delta} \\
=&\frac{e_{-\frac{p}{a}}(t,t_0)(a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma})^{\Delta} - (e_{-\frac{p}{a}}(t,t_0))^{\Delta}a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)e_{-\frac{p}{a}}(\sigma(t),t_0)} \\
=&\frac{(a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma})^{\Delta} + p(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(\sigma(t),t_0)} \\
=&\frac{-q(t)f(x(\theta(t)))}{e_{-\frac{p}{a}}(\sigma(t),t_0)} < 0.
\end{aligned} \tag{2.3}$$

Then $\dfrac{a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$, and together with $a(t)>0$, $e_{-\frac{p}{a}}(t,t_0)>0$ we deduce that $[r(t)x^{\Delta}(t)]^{\Delta}$ is eventually of one sign. We claim $[r(t)x^{\Delta}(t)]^{\Delta}>0$ on $[t_2,\infty)_{\mathbb{T}}$, where $t_2>t_1$ is sufficiently large. Otherwise, assume there exists a sufficiently large $t_3>t_2$ such that $[r(t)x^{\Delta}(t)]^{\Delta}<0$ on $[t_3,\infty)_{\mathbb{T}}$. Then

$$\begin{aligned}
&r(t)x^{\Delta}(t)-r(t_3)x^{\Delta}(t_3)\\
=&\int_{t_3}^{t}\frac{[e_{-\frac{p}{a}}(s,t_0)a(s)]^{\frac{1}{\gamma}}[r(s)x^{\Delta}(s)]^{\Delta}}{[e_{-\frac{p}{a}}(s,t_0)a(s)]^{\frac{1}{\gamma}}}\Delta s\\
\le&\frac{a^{\frac{1}{\gamma}}(t_3)[r(t_3)x^{\Delta}(t_3)]^{\Delta}}{[e_{-\frac{p}{a}}(t_3,t_0)]^{\frac{1}{\gamma}}}\int_{t_3}^{t}\frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta s. \qquad (2.4)
\end{aligned}$$

By (2.1), we have $\lim_{t\to\infty} r(t)x^{\Delta}(t)=-\infty$, and thus there exists a sufficiently large $t_4\in[t_3,\infty)_{\mathbb{T}}$ such that $r(t)x^{\Delta}(t)<0$ on $[t_4,\infty)_{\mathbb{T}}$. By the assumption $[r(t)x^{\Delta}(t)]^{\Delta}<0$ one can see $r(t)x^{\Delta}(t)$ is strictly decreasing on $[t_4,\infty)_{\mathbb{T}}$, and then

$$x(t)-x(t_4)=\int_{t_4}^{t}\frac{r(s)x^{\Delta}(s)}{r(s)}\Delta s\le r(t_4)x^{\Delta}(t_4)\int_{t_4}^{t}\frac{1}{r(s)}\Delta s.$$

Using (2.2), we have $\lim_{t\to\infty} x(t)=-\infty$, which leads to a contradiction. So $[r(t)x^{\Delta}(t)]^{\Delta}>0$ on $[t_2,\infty)_{\mathbb{T}}$. Setting $T_1^*=t_2$, we complete the proof.

□

Lemma 2.2. *Suppose* $-\frac{p}{a}\in\mathfrak{R}_+$, *and assume that*

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)^{\frac{1}{\gamma}}\Delta\tau]\Delta\xi=\infty. \qquad (2.5)$$

Then either there exists a sufficiently large T_2^* *such that* $x^{\Delta}(t)>0$ *on* $[T_2^*,\infty)_{\mathbb{T}}$ *or* $\lim_{t\to\infty} x(t)=0$.

Proof. By Lemma 2.1, we deduce that $x^{\Delta}(t)$ is eventually of one sign. So there exists a sufficiently large $t_5>t_4$ such that either $x^{\Delta}(t)>0$ or $x^{\Delta}(t)<0$ on $[t_5,\infty)_{\mathbb{T}}$, where t_4 is defined as in Lemma 2.1. If $x^{\Delta}(t)<0$, together with $x(t)$ is eventually a positive solution of Eq. (1.1), we obtain $\lim_{t\to\infty} x(t)=\alpha\ge 0$ and $\lim_{t\to\infty} r(t)x^{\Delta}(t)=\beta\le 0$. We claim $\alpha=0$. Otherwise, assume $\alpha>0$. Then $x(t)\ge\alpha$ on $[t_5,\infty)_{\mathbb{T}}$. Since $\lim_{t\to\infty}\theta(t)=\infty$, there exists $t_6>t_5$ such that $\theta(t)>t_5$ on $[t_6,\infty)_{\mathbb{T}}$, and then $x(\theta(t))\ge\alpha$ on $[t_6,\infty)_{\mathbb{T}}$. On the other hand, for $t\in[t_6,\infty)\bigcap\mathbb{T}$, an integration for (2.3) from t to ∞ yields

$$\begin{aligned}
-\frac{a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)}=&-\lim_{t\to\infty}\frac{a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)}+\int_{t}^{\infty}\frac{-q(s)f(x(\theta(s)))}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s\\
\le&-\lim_{t\to\infty}\frac{a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)}+\int_{t}^{\infty}\frac{-Lq(s)x^{\gamma}(\theta(s))}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s\\
\le&-L\int_{t}^{\infty}\frac{q(s)x^{\gamma}(\theta(s))}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s
\end{aligned}$$

$$\leq -L\alpha^{\gamma}\int_t^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s,$$

which is followed by

$$-[r(t)x^{\Delta}(t)]^{\Delta} \leq -[L\alpha^{\gamma}(\frac{e_{-\frac{p}{a}}(t,t_0)}{a(t)}\int_t^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)]^{\frac{1}{\gamma}}. \quad (2.6)$$

Substituting t with τ in (2.6), an integration for (2.6) with respect to τ from t to ∞ yields

$$\begin{aligned} r(t)x^{\Delta}(t) =& \lim_{t\to\infty} r(t)x^{\Delta}(t) - \alpha L^{\frac{1}{\gamma}}\int_t^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)^{\frac{1}{\gamma}}\Delta\tau \\ =&\beta - \alpha L^{\frac{1}{\gamma}}\int_t^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)^{\frac{1}{\gamma}}\Delta\tau \\ \leq& -\alpha L^{\frac{1}{\gamma}}\int_t^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)^{\frac{1}{\gamma}}\Delta\tau, \end{aligned}$$

which implies

$$x^{\Delta}(t) \leq -\alpha L^{\frac{1}{\gamma}}\frac{1}{r(t)}\int_t^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)^{\frac{1}{\gamma}}\Delta\tau. \quad (2.7)$$

Substituting t with ξ in (2.7), an integration for (2.7) with respect to ξ from t_6 to t yields

$$x(t) - x(t_6) \leq -\alpha L^{\frac{1}{\gamma}}\int_{t_6}^{t}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}\Delta s)^{\frac{1}{\gamma}}\Delta\tau]\Delta\xi. \quad (2.8)$$

By (2.8) and (2.5) we have $\lim_{t\to\infty} x(t) = -\infty$, which leads to a contradiction. So we have $\alpha = 0$, and the proof is complete by setting $T_2^* = t_5$. □

Lemma 2.3. *Suppose* $-\frac{p}{a} \in \mathfrak{R}_+$, *and assume that* x *is eventually a positive solution of Eq.* (1.1) *such that*

$$[r(t)x^{\Delta}(t)]^{\Delta} > 0, \ x^{\Delta}(t) > 0$$

on $[T_3^*,\infty)_{\mathbb{T}}$, *where* $T_3^* \geq t_0$ *is sufficiently large. Then we have*

$$x^{\Delta}(t) \geq \frac{\delta_1(t,T_3^*)}{r(t)}\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{\Delta}(t)]^{\Delta}}{[e_{-\frac{p}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\}$$

and

$$x(t) \geq \delta_2(t,T_3^*)\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{\Delta}(t)]^{\Delta}}{[e_{-\frac{p}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\}$$

on $[T_3^*,\infty)_{\mathbb{T}}$.

Proof. Take $T_3^* > \max(T_1^*, T_2^*)$, where T_1^*, T_2^* are defined as in Lemmas 2.1 and 2.2 respectively. By Lemma 2.1 we have $\frac{a(t)([r(t)x^\Delta(t)]^\Delta)^\gamma}{e_{-\frac{p}{a}}(t,t_0)}$ is strictly decreasing on $[T_3^*, \infty)_{\mathbb{T}}$. So

$$\begin{aligned} r(t)x^\Delta(t) \geq & r(t)x^\Delta(t) - r(T_3^*)x^\Delta(T_3^*) \\ = & \int_{T_3^*}^t \frac{[e_{-\frac{p}{a}}(s,t_0)a(s)]^{\frac{1}{\gamma}}[r(s)x^\Delta(s)]^\Delta}{[e_{-\frac{p}{a}}(s,t_0)a(s)]^{\frac{1}{\gamma}}}\Delta s \\ \geq & \frac{a^{\frac{1}{\gamma}}(t)[r(t)x^\Delta(t)]^\Delta}{[e_{-\frac{p}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\int_{T_3^*}^t \frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta s \\ = & \delta_1(t,T_3^*)\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^\Delta(t)]^\Delta}{[e_{-\frac{p}{a}}(t,t_0)]^{\frac{1}{\gamma}}}, \end{aligned}$$

and then

$$x^\Delta(t) \geq \frac{\delta_1(t,T_3^*)}{r(t)}\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^\Delta(t)]^\Delta}{[e_{-\frac{p}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\}.$$

Furthermore,

$$\begin{aligned} x(t) \geq & x(t) - x(T_3^*) = \int_{T_3^*}^t x^\Delta(s)\Delta s \\ \geq & \int_{T_3^*}^t \frac{\delta_1(s,T_3^*)}{r(s)}\{\frac{a^{\frac{1}{\gamma}}(s)[r(s)x^\Delta(s)]^\Delta}{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}\}\Delta s \\ \geq & \{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^\Delta(t)]^\Delta}{[e_{-\frac{p}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\}\int_{T_3^*}^t \frac{\delta_1(s,T_3^*)}{r(s)}\Delta s \\ = & \delta_2(t,T_3^*)\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^\Delta(t)]^\Delta}{[e_{-\frac{p}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\}, \end{aligned}$$

which is the desired result. □

Lemma 2.4 (Hardy etc [17, Theorem 41])**.** *Assume that X and Y are nonnegative real numbers. Then*

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda-1)Y^\lambda$$

for all $\lambda > 1$.

Theorem 2.1. *Suppose $-\frac{p}{a} \in \mathfrak{R}_+$, and assume that* (2.1), (2.2), (2.5) *hold, and for all sufficiently large $T \in \mathbb{T}$,*

$$\limsup_{t\to\infty}\int_T^t \{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^\Delta$$
$$+ \frac{\zeta_1(s)\theta^\Delta(s)\delta_1(\theta(s),T)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))}$$

$$-[\frac{r(\theta(s))\zeta_1^{\Delta}(s)+(\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}\Delta s\}$$
$$=\infty, \tag{2.9}$$

where $\zeta_1,\ \zeta_2$ *are two given nonnegative functions on* $\mathbb{T}$ *with* $\zeta_1(t)>0$. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero.*

Proof. Assume (1.1) has a nonoscillatory solution x on $\mathbb{T}_0$. Without loss of generality, we may assume $x(t)>0,\ x(\theta(t))>0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemmas 2.1 and 2.2, there exists sufficiently large t_2 such that $[r(t)x^{\Delta}(t)]^{\Delta}>0$ on $[t_2,\infty)_{\mathbb{T}}$, and either $x^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty}x(t)=0$. Now we assume $x^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$. Since $\lim_{t\to\infty}\theta(t)=\infty$, there exists $t_3>t_2$ such that $\theta(t)>t_2$ on $[t_3,\infty)_{\mathbb{T}}$. So $x^{\Delta}(\theta(t))>0$ on $[t_3,\infty)_{\mathbb{T}}$. Define a generalized Riccati function:

$$\omega(t)=\zeta_1(t)a(t)[\frac{([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{x^{\gamma}(\theta(t))e_{-\frac{p}{a}}(t,t_0)}+\zeta_2(t)].$$

Then for $t\in[t_3,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
&\omega^{\Delta}(t)\\
=&\frac{\zeta_1(t)}{x^{\gamma}(\theta(t))}\{\frac{a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)}\}^{\Delta}\\
&+[\frac{\zeta_1(t)}{x^{\gamma}(\theta(t))}]^{\Delta}\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}\\
&+\zeta_1(t)[a(t)\zeta_2(t)]^{\Delta}+\zeta_1^{\Delta}(t)a(\sigma(t))\zeta_2(\sigma(t))\\
=&\frac{\zeta_1(t)}{x^{\gamma}(\theta(t))}\{\frac{e_{-\frac{p}{a}}(t,t_0)(a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma})^{\Delta}-(e_{-\frac{p}{a}}(t,t_0))^{\Delta}a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)e_{-\frac{p}{a}}(\sigma(t),t_0)}\}\\
&+[\frac{x^{\gamma}(\theta(t))\zeta_1^{\Delta}(t)-(x^{\gamma}(\theta(t)))^{\Delta}\zeta_1(t)}{x^{\gamma}(\theta(t))x^{\gamma}(\theta(\sigma(t)))}]\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}\\
&+\zeta_1(t)[a(t)\zeta_2(t)]^{\Delta}+\zeta_1^{\Delta}(t)a(\sigma(t))\zeta_2(\sigma(t))\\
=&\frac{\zeta_1(t)}{x^{\gamma}(\theta(t))}[\frac{(a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma})^{\Delta}+p(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}]+\frac{\zeta_1^{\Delta}(t)}{\zeta_1(\sigma(t))}\omega(\sigma(t))\\
&-[\frac{\zeta_1(t)(x^{\gamma}(\theta(t)))^{\Delta}}{x^{\gamma}(\theta(t))}]\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{x^{\gamma}(\theta(\sigma(t)))e_{-\frac{p}{a}}(\sigma(t),t_0)}+\zeta_1(t)[a(t)\zeta_2(t)]^{\Delta}\\
=&-\frac{\zeta_1(t)}{x^{\gamma}(\theta(t))}[\frac{q(t)f(x(\theta(t)))}{e_{-\frac{p}{a}}(\sigma(t),t_0)}]+\frac{\zeta_1^{\Delta}(t)}{\zeta_1(\sigma(t))}\omega(\sigma(t))\\
&-[\frac{\zeta_1(t)(x^{\gamma}(\theta(t)))^{\Delta}}{x^{\gamma}(\theta(t))}]\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{x^{\gamma}(\theta(\sigma(t)))e_{-\frac{p}{a}}(\sigma(t),t_0)}+\zeta_1(t)[a(t)\zeta_2(t)]^{\Delta}\\
\le&-L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\zeta_1^{\Delta}(t)}{\zeta_1(\sigma(t))}\omega(\sigma(t))\\
&-[\frac{\zeta_1(t)(x^{\gamma}(\theta(t)))^{\Delta}}{x^{\gamma}(\theta(t))}]\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{x^{\gamma}(\theta(\sigma(t)))e_{-\frac{p}{a}}(\sigma(t),t_0)}+\zeta_1(t)[a(t)\zeta_2(t)]^{\Delta}.
\end{aligned}$$

By [4, Theorem 1.93], and $\theta\circ\sigma=\sigma\circ\theta$, we have

$$(x^{\gamma}(\theta(t)))^{\Delta}\geq\gamma x^{\gamma-1}(\theta(t))(x(\theta(t)))^{\Delta}=\gamma x^{\gamma-1}(\theta(t))x^{\Delta}(\theta(t))\theta^{\Delta}(t).$$

Then

$$\begin{aligned}\omega^{\Delta}(t)\leq&-L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\zeta_1^{\Delta}(t)}{\zeta_1(\sigma(t))}\omega(\sigma(t))+\zeta_1(t)[a(t)\zeta_2(t)]^{\Delta}\\&-\zeta_1(t)[\frac{\gamma x^{\gamma-1}(\theta(t))x^{\Delta}(\theta(t))\theta^{\Delta}(t)}{x^{\gamma}(\theta(t))}]\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{x^{\gamma}(\theta(\sigma(t)))e_{-\frac{p}{a}}(\sigma(t),t_0)}.\end{aligned}\tag{2.10}$$

By Lemma 2.3 and $x^{\Delta}(t)>0$, we have

$$\begin{aligned}\omega^{\Delta}(t)\leq&-L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\zeta_1^{\Delta}(t)}{\zeta_1(\sigma(t))}\omega(\sigma(t))+\zeta_1(t)[a(t)\zeta_2(t)]^{\Delta}\\&-[\frac{\gamma\zeta_1(t)\theta^{\Delta}(t)}{x(\theta(\sigma(t)))}]\{\frac{\delta_1(\theta(t),t_3)}{r(\theta(t))}[\frac{a^{\frac{1}{\gamma}}(\theta(t))[r(\theta(t))x^{\Delta}(\theta(t))]^{\Delta}}{[e_{-\frac{p}{a}}(\theta(t),t_0)]^{\frac{1}{\gamma}}}]\}\\&\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{x^{\gamma}(\theta(\sigma(t)))e_{-\frac{p}{a}}(\sigma(t),t_0)}.\end{aligned}$$

By Lemma 2.1, $\dfrac{a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(t,t_0)}$ is strictly decreasing on $[t_2,\infty)_{\mathbb{T}}$. So

$$\frac{(a(\theta(t)))^{\frac{1}{\gamma}}[r(\theta(t))x^{\Delta}(\theta(t))]^{\Delta}}{[e_{-\frac{p}{a}}(\theta(t),t_0)]^{\frac{1}{\gamma}}}>\frac{(a(\sigma(t)))^{\frac{1}{\gamma}}[r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta}}{[e_{-\frac{p}{a}}(\sigma(t),t_0)]^{\frac{1}{\gamma}}}$$

for $t\in[t_3,\infty)_{\mathbb{T}}$, and we obtain that

$$\begin{aligned}\omega^{\Delta}(t)\leq&-L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\zeta_1^{\Delta}(t)}{\zeta_1(\sigma(t))}\omega(\sigma(t))+\zeta_1(t)[a(t)\zeta_2(t)]^{\Delta}\\&-\gamma\frac{\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)}{r(\theta(t))}[\frac{\omega(\sigma(t))}{\zeta_1(\sigma(t))}-a(\sigma(t))\zeta_2(\sigma(t))]^{1+\frac{1}{\gamma}}.\end{aligned}\tag{2.11}$$

Using the following inequality (see Hassan [13, Eq. (2.17)]):

$$(u-v)^{1+\frac{1}{\gamma}}\geq u^{1+\frac{1}{\gamma}}+\frac{1}{\gamma}v^{1+\frac{1}{\gamma}}-(1+\frac{1}{\gamma})v^{\frac{1}{\gamma}}u,$$

we obtain

$$\begin{aligned}&[\frac{\omega(\sigma(t))}{\zeta_1(\sigma(t))}-a(\sigma(t))\zeta_2(\sigma(t))]^{1+\frac{1}{\gamma}}\\\geq&\frac{\omega^{1+\frac{1}{\gamma}}(\sigma(t))}{\zeta_1^{1+\frac{1}{\gamma}}(\sigma(t))}+\frac{1}{\gamma}[a(\sigma(t))\zeta_2(\sigma(t))]^{1+\frac{1}{\gamma}}-(1+\frac{1}{\gamma})\frac{[a(\sigma(t))\zeta_2(\sigma(t))]^{\frac{1}{\gamma}}\omega(\sigma(t))}{\zeta_1(\sigma(t))}.\end{aligned}\tag{2.12}$$

A combination of (2.11) and (2.12) yields:

$$\begin{aligned}\omega^{\Delta}(t) \leq &- L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)} + \zeta_1(t)[a(t)\zeta_2(t)]^{\Delta} \\ &- \frac{\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)[a(\sigma(t))\zeta_2(\sigma(t))]^{1+\frac{1}{\gamma}}}{r(\theta(t))} \\ &+ \frac{r(\theta(t))\zeta_1^{\Delta}(t) + (\gamma+1)\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)[a(\sigma(t))\zeta_2(\sigma(t))]^{\frac{1}{\gamma}}}{r(\theta(t))\zeta_1(\sigma(t))}\omega(\sigma(t)) \\ &- \gamma\frac{\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)}{r(\theta(t))}\frac{\omega^{1+\frac{1}{\gamma}}(\sigma(t))}{\zeta_1^{1+\frac{1}{\gamma}}(\sigma(t))}. \end{aligned} \tag{2.13}$$

Setting

$$\begin{aligned}\lambda =& 1+\frac{1}{\gamma}, X^{\lambda} = \gamma\frac{\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)}{r(\theta(t))}\frac{\omega^{1+\frac{1}{\gamma}}(\sigma(t))}{\zeta_1^{1+\frac{1}{\gamma}}(\sigma(t))}, \\ Y^{\lambda-1} =& \gamma^{\frac{1}{\gamma+1}}[\frac{r(\theta(t))\zeta_1^{\Delta}(t) + (\gamma+1)\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)[a(\sigma(t))\zeta_2(\sigma(t))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(t))\zeta_1^{\frac{\gamma}{\gamma+1}}(t)(\theta^{\Delta}(t))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(t),t_2)}].\end{aligned}$$

Using Lemma 2.4 in (2.13) we get that

$$\begin{aligned}\omega^{\Delta}(t) \leq &- L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)} + \zeta_1(t)[a(t)\zeta_2(t)]^{\Delta} \\ &- \frac{\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)[a(\sigma(t))\zeta_2(\sigma(t))]^{1+\frac{1}{\gamma}}}{r(\theta(t))} \\ &+ [\frac{r(\theta(t))\zeta_1^{\Delta}(t) + (\gamma+1)\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)[a(\sigma(t))\zeta_2(\sigma(t))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(t))\zeta_1^{\frac{\gamma}{\gamma+1}}(t)(\theta^{\Delta}(t))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(t),t_2)}]^{\gamma+1}.\end{aligned} \tag{2.14}$$

Substituting t with s in (2.14), an integration for (2.14) with respect to s from t_3 to t yields

$$\begin{aligned}&\int_{t_3}^{t}\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta} \\ &+ \frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))} \\ &- [\frac{r(\theta(s))\zeta_1^{\Delta}(s) + (\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),t_2)}]^{\gamma+1}\}\Delta s \\ \leq&\omega(t_3) - \omega(t) \leq \omega(t_3) < \infty,\end{aligned}$$

which contradicts (2.9), and the proof is complete. □

In Theorem 2.1, if we take $\mathbb{T}$ for some special cases, then we can obtain the following corollaries:

Corollary 2.1. *Let $\mathbb{T}=\mathbb{R}$. Assume that*

$$\int_{t_0}^{\infty}\frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}ds=\infty, \tag{2.15}$$

$$\int_{t_0}^{\infty}\frac{1}{r(s)}ds=\infty, \tag{2.16}$$

$$\int_{t_0}^{\infty}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(s,t_0)}ds)^{\frac{1}{\gamma}}d\tau]d\xi=\infty, \tag{2.17}$$

and for all sufficiently large $T\in\mathbb{R}$,

$$\lim_{t\to\infty}\sup\{\int_T^t\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s,t_0)}-\zeta_1(s)[a(s)\zeta_2(s)]'+\frac{\zeta_1(s)\theta'(s)\delta_1(\theta(s),T)[a(s)\zeta_2(s)]^{1+\frac{1}{\gamma}}}{r(\theta(s))}$$
$$-[\frac{r(\theta(s))\zeta_1'(s)+(\gamma+1)\zeta_1(s)\theta'(s)\delta_1(\theta(s),T)[a(s)\zeta_2(s)]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta'(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}ds\}=\infty. \tag{2.18}$$

Then every solution of Eq. (1.1) *is oscillatory or tends to zero.*

Corollary 2.2. *Let $\mathbb{T}=\mathbb{Z}$ and $-\frac{p}{a}\in\mathfrak{R}_+$. Assume that*

$$\sum_{s=t_0}^{\infty}\frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}=\infty, \tag{2.19}$$

$$\sum_{s=t_0}^{\infty}\frac{1}{r(s)}=\infty, \tag{2.20}$$

$$\sum_{\xi=t_0}^{\infty}[\frac{1}{r(\xi)}\sum_{\tau=\xi}^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(s+1,t_0)})^{\frac{1}{\gamma}}]=\infty, \tag{2.21}$$

and for all sufficiently large $T\in\mathbb{Z}$,

$$\lim_{t\to\infty}\sup\{\sum_{s=T}^{t-1}\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s+1,t_0)}-\zeta_1(s)[a(s+1)\zeta_2(s+1)-a(s)\zeta_2(s)]$$
$$+\frac{\zeta_1(s)(\theta(s+1)-\theta(s))\delta_1(\theta(s),T)[a(s+1)\zeta_2(s+1)]^{1+\frac{1}{\gamma}}}{r(\theta(s))}$$
$$-\frac{1}{[(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta(s+1)-\theta(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)]^{\gamma+1}}$$
$$\times[r(\theta(s))(\zeta_1(s+1)-\zeta_1(s))+(\gamma+1)\zeta_1(s)(\theta(s+1)-\theta(s))\delta_1(\theta(s),T)$$
$$[a(s+1)\zeta_2(s+1)]^{\frac{1}{\gamma}}]^{\gamma+1}\}\}=\infty. \tag{2.22}$$

Then every solution of Eq. (1.1) *is oscillatory or tends to zero.*

Theorem 2.2. *Suppose $-\frac{p}{a}\in\mathfrak{R}_+$, and* (2.1), (2.2), (2.5) *hold. If for all sufficiently* *large $T \in \mathbb{T}$,*

$$\lim_{t\to\infty}\sup\{\int_T^t\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}$$
$$+\frac{\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)\delta_2^{\gamma-1}(\theta(\sigma(s),T))a^2(\sigma(s))\zeta_2^2(\sigma(s))}{r(\theta(s))}$$
$$-\frac{[r(\theta(s))\zeta_1^{\Delta}(s)+2\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)\delta_2^{\gamma-1}(\theta(\sigma(s),T))a(\sigma(s))\zeta_2(\sigma(s))]^2}{4\gamma r(\theta(s))\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)\delta_2^{\gamma-1}(\theta(\sigma(s),T))}\}\Delta s\}$$
$$=\infty, \tag{2.23}$$

where ζ_1, ζ_2 are defined as in Theorem 2.1. Then every solution of Eq. (1.1) *is oscillatory or tends to zero.*

Proof. Assume (1.1) has a nonoscillatory solution x on $\mathbb{T}_0$. Similar to Theorem 2.1, we may assume $x(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemmas 2.1 and 2.2, there exists sufficiently large t_2 such that $[r(t)x^{\Delta}(t)]^{\Delta}>0$ on $[t_2,\infty)_{\mathbb{T}}$, and either $x^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty}x(t)=0$. Now we assume $x^{\Delta}(t)>0$, $x^{\Delta}(\theta(t))>0$ on $[t_3,\infty)_{\mathbb{T}}$, where $t_3>t_2$ is sufficiently large. Let $\omega(t)$ be defined as in Theorem 2.1. By Lemma 2.3, for $t\in[t_3,\infty)_{\mathbb{T}}$, we have the following observation:

$$\begin{aligned}\frac{x^{\Delta}(\theta(t))}{x(\theta(t))}\geq&\frac{x^{\Delta}(\theta(t))}{x(\theta(\sigma(t)))}=\frac{x^{\Delta}(\theta(t))}{x^{\gamma}(\theta(\sigma(t)))}x^{\gamma-1}(\theta(\sigma(t)))\\
\geq&\frac{\delta_1(\theta(t),t_3)}{r(\theta(t))x^{\gamma}(\theta(\sigma(t)))}\{\frac{a^{\frac{1}{\gamma}}(\theta(t))[r(\theta(t))x^{\Delta}(\theta(t))]^{\Delta}}{[e_{-\frac{p}{a}}(\theta(t),t_0)]^{\frac{1}{\gamma}}}\}x^{\gamma-1}(\theta(\sigma(t)))\\
\geq&\frac{\delta_1(\theta(t),t_3)}{r(\theta(t))x^{\gamma}(\theta(\sigma(t)))}\{\frac{a^{\frac{1}{\gamma}}(\theta(t))[r(\theta(t))x^{\Delta}(\theta(t))]^{\Delta}}{[e_{-\frac{p}{a}}(\theta(t),t_0)]^{\frac{1}{\gamma}}}\}\\
&\times\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)\{\frac{a^{\frac{1}{\gamma}}(\theta(\sigma(t)))[r(\theta(\sigma(t)))x^{\Delta}(\theta(\sigma(t)))]^{\Delta}}{[e_{-\frac{p}{a}}(\theta(\sigma(t)),t_0)]^{\frac{1}{\gamma}}}\}^{\gamma-1}\\
\geq&\frac{\delta_1(\theta(t),t_3)}{r(\theta(t))x^{\gamma}(\theta(\sigma(t)))}\{\frac{a^{\frac{1}{\gamma}}(\theta(t))[r(\theta(t))x^{\Delta}(\theta(t))]^{\Delta}}{[e_{-\frac{p}{a}}(\theta(t),t_0)]^{\frac{1}{\gamma}}}\}\\
&\times\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)\{\frac{a^{\frac{1}{\gamma}}(\sigma(t))[r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta}}{[e_{-\frac{p}{a}}(\sigma(t),t_0)]^{\frac{1}{\gamma}}}\}^{\gamma-1}\\
\geq&\frac{\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)}{r(\theta(t))}\{\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(\sigma(t),t_0)x^{\gamma}(\theta(\sigma(t)))}\}.\end{aligned} \tag{2.24}$$

Using (2.24) in (2.10) we get that

$$\begin{aligned}&\omega^{\Delta}(t)\\
\leq&-L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\zeta_1^{\Delta}(t)}{\zeta_1(\sigma(t))}\omega(\sigma(t))+\zeta_1(t)[a(t)\zeta_2(t)]^{\Delta}\\
&-\zeta_1(t)[\frac{\gamma x^{\gamma-1}(\theta(t))x^{\Delta}(\theta(t))\theta^{\Delta}(t)}{x^{\gamma}(\theta(t))}]\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{x^{\gamma}(\theta(\sigma(t)))e_{-\frac{p}{a}}(\sigma(t),t_0)}\end{aligned}$$

$$
\begin{aligned}
\leq & - L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)} + \frac{\zeta_1^{\Delta}(t)}{\zeta_1(\sigma(t))}\omega(\sigma(t)) + \zeta_1(t)[a(t)\zeta_2(t)]^{\Delta} \\
& - \gamma\zeta_1(t)\theta^{\Delta}(t)\frac{\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)}{r(\theta(t))}\{\frac{a(\sigma(t))([r(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta})^{\gamma}}{e_{-\frac{p}{a}}(\sigma(t),t_0)x^{\gamma}(\theta(\sigma(t)))}\}^2 \\
= & - L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)} + \frac{\zeta_1^{\Delta}(t)}{\zeta_1(\sigma(t))}\omega(\sigma(t)) + \zeta_1(t)[a(t)\zeta_2(t)]^{\Delta} \\
& - \gamma\zeta_1(t)\theta^{\Delta}(t)\frac{\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)}{r(\theta(t))}[\frac{\omega(\sigma(t))}{\zeta_1(\sigma(t))} - a(\sigma(t))\zeta_2(\sigma(t))]^2 \\
= & - L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)} + \zeta_1(t)[a(t)\zeta_2(t)]^{\Delta} \\
& - \frac{\gamma\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)a^2(\sigma(t))\zeta_2^2(\sigma(t))}{r(\theta(t))} \\
& + [\frac{r(\theta(t))\zeta_1^{\Delta}(t) + 2\gamma\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)a(\sigma(t))\zeta_2(\sigma(t))}{r(\theta(t))\zeta_1(\sigma(t))}]\omega(\sigma(t)) \\
& - \frac{\gamma\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)}{r(\theta(t))\zeta_1^2(\sigma(t))}\omega^2(\sigma(t)) \\
\leq & - L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)} + \zeta_1(t)[a(t)\zeta_2(t)]^{\Delta} \\
& - \frac{\gamma\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)a^2(\sigma(t))\zeta_2^2(\sigma(t))}{r(\theta(t))} \\
& + \frac{[r(\theta(t))\zeta_1^{\Delta}(t) + 2\gamma\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)a(\sigma(t))\zeta_2(\sigma(t))]^2}{4\gamma r(\theta(t))\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)}.
\end{aligned} \tag{2.25}
$$

Substituting t with s in (2.25), an integration for (2.25) with respect to s from t_3 to t yields

$$
\begin{aligned}
& \int_{t_3}^{t}\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta} \\
& + \frac{\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)a^2(\sigma(s))\zeta_2^2(\sigma(s))}{r(\theta(s))} \\
& - \frac{[r(\theta(s))\zeta_1^{\Delta}(s) + 2\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)a(\sigma(s))\zeta_2(\sigma(s))]^2}{4\gamma r(\theta(s))\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)}\}\Delta s \\
\leq & \omega(t_3) - \omega(t) \leq \omega(t_3) < \infty,
\end{aligned}
$$

which contradicts (2.23). So the proof is complete. □

Based on Theorems 2.1 and 2.2, we will establish some Philos-type oscillation criteria for Eq. (1.1).

Theorem 2.3. *Suppose* $-\frac{p}{a} \in \mathfrak{R}_+$, *and assume that* (2.1), (2.2), (2.5) *hold, and define* $\mathbb{D} = \{(t,s)|t \geq s \geq t_0,\ t,s \in \mathbb{T}\}$. *If there exists a function* $H \in C_{rd}(\mathbb{D},\mathbb{R})$ *such that*

$$
H(t,t) = 0,\ \textit{for}\ t \geq t_0,\quad H(t,s) > 0, \tag{2.26}
$$

for $t > s \geq t_0$, and H has a nonpositive $\Delta-$ partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable, and for all sufficiently large $T \in \mathbb{T}$,

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)}\{\int_{t_0}^{t} H(t,s)\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta} + \frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))} - [\frac{r(\theta(s))\zeta_1^{\Delta}(s) + (\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}\Delta s\} = \infty. \tag{2.27}$$

Then every solution of Eq. (1.1) *is oscillatory or tends to zero.*

Proof. Assume (1.1) has a nonoscillatory solution x on $\mathbb{T}_0$. Similar to Theorem 2.1, we may assume $x(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemmas 2.1 and 2.2, there exists sufficiently large t_2 such that $[r(t)x^{\Delta}(t)]^{\Delta} > 0$ on $[t_2,\infty)_{\mathbb{T}}$, and either $x^{\Delta}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$. Now we assume $x^{\Delta}(t) > 0$, $x^{\Delta}(\theta(t)) > 0$ on $[t_3,\infty)_{\mathbb{T}}$, where $t_3 > t_2$ is sufficiently large. Let $\omega(t)$ be defined as in Theorem 2.1. Then by (2.14), for $t \in [t_3,\infty)_{\mathbb{T}}$, we have

$$L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)} - \zeta_1(t)[a(t)\zeta_2(t)]^{\Delta} + \frac{\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)[a(\sigma(t))\zeta_2(\sigma(t))]^{1+\frac{1}{\gamma}}}{r(\theta(t))} - [\frac{r(\theta(t))\zeta_1^{\Delta}(t) + (\gamma+1)\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)[a(\sigma(t))\zeta_2(\sigma(t))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(t))\zeta_1^{\frac{\gamma}{\gamma+1}}(t)(\theta^{\Delta}(t))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(t),t_2)}]^{\gamma+1} \leq -\omega^{\Delta}(t). \tag{2.28}$$

Substituting t with s in (2.28), multiplying both sides by $H(t,s)$ and then integrating with respect to s from t_3 to t yields

$$\begin{aligned}&\int_{t_3}^{t} H(t,s)\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta} + \frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))} \\ &- [\frac{r(\theta(s))\zeta_1^{\Delta}(s) + (\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),t_2)}]^{\gamma+1}\}\Delta s \\ &\leq -\int_{t_3}^{t} H(t,s)\omega^{\Delta}(s)\Delta s \\ &= H(t,t_3)\omega(t_3) + \int_{t_3}^{t} H^{\Delta_s}(t,s)\omega(\sigma(s))\Delta s \\ &\leq H(t,t_3)\omega(t_3) \leq H(t,t_0)\omega(t_3).\end{aligned}$$

Then

$$\int_{t_0}^{t} H(t,s)\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}$$

$$
\begin{aligned}
&+\frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))}\\
&-[\frac{r(\theta(s))\zeta_1^{\Delta}(s)+(\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),t_2)}]^{\gamma+1}\}\Delta s\\
=&\int_{t_0}^{t_3}H(t,s)\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}\\
&+\frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))}\\
&-[\frac{r(\theta(s))\zeta_1^{\Delta}(s)+(\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),t_2)}]^{\gamma+1}\}\Delta s\\
&+\int_{t_3}^{t}H(t,s)\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}\\
&+\frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))}\\
&-[\frac{r(\theta(s))\zeta_1^{\Delta}(s)+(\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),t_2)}]^{\gamma+1}\}\Delta s\\
\leq& H(t,t_0)\omega(t_3)+H(t,t_0)\int_{t_0}^{t_3}|L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}\\
&+\frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))}\\
&-[\frac{r(\theta(s))\zeta_1^{\Delta}(s)+(\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),t_2)}]^{\gamma+1}|\Delta s.
\end{aligned}
$$

So

$$
\begin{aligned}
&\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\{\int_{t_0}^{t}H(t,s)\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}\\
&+\frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))}\\
&-[\frac{r(\theta(s))\zeta_1^{\Delta}(s)+(\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),t_2)}]^{\gamma+1}\}\Delta s\}\\
\leq&\omega(t_3)+\int_{t_0}^{t_3}|L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}\\
&+\frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))}\\
&-[\frac{r(\theta(s))\zeta_1^{\Delta}(s)+(\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),t_2)}]^{\gamma+1}|\Delta s\\
<&\infty,
\end{aligned}
$$

which contradicts (2.27). So the proof is complete. □

Theorem 2.4. *Suppose* $-\frac{p}{a} \in \Re_+$, *and assume that* (2.1), (2.2), (2.5) *hold. Let* H *be defined as in Theorem 2.3, and for all sufficiently large* $T \in \mathbb{T}$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\{\int_{t_0}^{t} H(t,s)\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta} \\
&+ \frac{\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)\delta_2^{\gamma-1}(\theta(\sigma(s),T))a^2(\sigma(s))\zeta_2^2(\sigma(s))}{r(\theta(s))} \\
&- \frac{[r(\theta(s))\zeta_1^{\Delta}(s) + 2\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)\delta_2^{\gamma-1}(\theta(\sigma(s),T))a(\sigma(s))\zeta_2(\sigma(s))]^2}{4\gamma r(\theta(s))\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)\delta_2^{\gamma-1}(\theta(\sigma(s),T))}\}\Delta s\} \\
=&\infty. \qquad (2.29)
\end{aligned}$$

Then every solution of Eq. (1.1) *is oscillatory or tends to zero.*

Proof. Assume (1.1) has a nonoscillatory solution x on $\mathbb{T}_0$. Similar to Theorem 2.1, we may assume $x(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemmas 2.1 and 2.2, there exists sufficiently large t_2 such that $[r(t)x^{\Delta}(t)]^{\Delta} > 0$ on $[t_2,\infty)_{\mathbb{T}}$, and either $x^{\Delta}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$. Now we assume $x^{\Delta}(t) > 0$, $x^{\Delta}(\theta(t)) > 0$ on $[t_3,\infty)_{\mathbb{T}}$, where $t_3 > t_2$ is sufficiently large. Let $\omega(t)$ be defined as in Theorem 2.1. Then by (2.25), for $t \in [t_3,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
&L\frac{q(t)\zeta_1(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)} - \zeta_1(t)[a(t)\zeta_2(t)]^{\Delta} \\
&+ \frac{\gamma\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)a^2(\sigma(t))\zeta_2^2(\sigma(t))}{r(\theta(t))} \\
&- \frac{[r(\theta(t))\zeta_1^{\Delta}(t) + 2\gamma\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)a(\sigma(t))\zeta_2(\sigma(t))]^2}{4\gamma r(\theta(t))\zeta_1(t)\theta^{\Delta}(t)\delta_1(\theta(t),t_3)\delta_2^{\gamma-1}(\theta(\sigma(t)),t_3)} \\
\leq& -\omega^{\Delta}(t). \qquad (2.30)
\end{aligned}$$

Substituting t with s in (2.30), multiplying both sides by $H(t,s)$ and then integrating with respect to s from t_3 to t yields

$$\begin{aligned}
&\int_{t_3}^{t} H(t,s)\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta} \\
&+ \frac{\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)a^2(\sigma(s))\zeta_2^2(\sigma(s))}{r(\theta(s))} \\
&- \frac{[r(\theta(s))\zeta_1^{\Delta}(s) + 2\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)a(\sigma(s))\zeta_2(\sigma(s))]^2}{4\gamma r(\theta(s))\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)}\}\Delta s \\
\leq& -\int_{t_3}^{t} H(t,s)\omega^{\Delta}(s)\Delta s \\
=&H(t,t_3)\omega(t_3) + \int_{t_3}^{t} H^{\Delta_s}(t,s)\omega(\sigma(s))\Delta s \\
\leq&H(t,t_3)\omega(t_3) \leq H(t,t_0)\omega(t_3).
\end{aligned}$$

Then similar to Theorem 2.3, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\{\int_{t_0}^{t} H(t,s)\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}\\
&+\frac{\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)a^2(\sigma(s))\zeta_2^2(\sigma(s))}{r(\theta(s))}\\
&-\frac{[r(\theta(s))\zeta_1^{\Delta}(s)+2\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)a(\sigma(s))\zeta_2(\sigma(s))]^2}{4\gamma r(\theta(s))\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)}\}\Delta s\}\\
\le&\omega(t_3)+\int_{t_0}^{t_3}|L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}\\
&+\frac{\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)a^2(\sigma(s))\zeta_2^2(\sigma(s))}{r(\theta(s))}\\
&-\frac{[r(\theta(s))\zeta_1^{\Delta}(s)+2\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)a(\sigma(s))\zeta_2(\sigma(s))]^2}{4\gamma r(\theta(s))\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),t_3)\delta_2^{\gamma-1}(\theta(\sigma(s)),t_3)}|\Delta s\\
<&\infty,
\end{aligned}$$

which contradicts (2.29). So the proof is complete. □

In Theorems 2.3 and 2.4, if we take $H(t,s)$ for some special functions such as $(t-s)^m$ or $\ln\frac{t}{s}$, then we can obtain some corollaries. For example, if we take $H(t,s)=(t-s)^m,\ m\ge 1$, then we have the following corollaries.

Corollary 2.3. *Suppose* $-\frac{p}{a}\in\Re_+$, *and assume that* (2.1), (2.2), (2.5) *hold, and for all sufficiently large* $T\in\mathbb{T}$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{(t-t_0)^m}\{\int_{t_0}^{t}(t-s)^m\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}\\
&+\frac{\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)[a(\sigma(s))\zeta_2(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(\theta(s))}\\
&-[\frac{r(\theta(s))\zeta_1^{\Delta}(s)+(\gamma+1)\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)[a(\sigma(s))\zeta_2(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta^{\Delta}(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}\Delta s\}\\
=&\infty. \qquad (2.31)
\end{aligned}$$

Then every solution of Eq. (1.1) *is oscillatory or tends to zero.*

Corollary 2.4. *Suppose* $-\frac{p}{a}\in\Re_+$, *and assume that* (2.1), (2.2), (2.5) *hold, and for all sufficiently large* $T\in\mathbb{T}$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{(t-t_0)^m}\{\int_{t_0}^{t}(t-s)^m\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)} - \zeta_1(s)[a(s)\zeta_2(s)]^{\Delta}\\
&+\frac{\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)\delta_2^{\gamma-1}(\theta(\sigma(s),T))a^2(\sigma(s))\zeta_2^2(\sigma(s))}{r(\theta(s))}\\
&-\frac{[r(\theta(s))\zeta_1^{\Delta}(s)+2\gamma\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)\delta_2^{\gamma-1}(\theta(\sigma(s),T))a(\sigma(s))\zeta_2(\sigma(s))]^2}{4\gamma r(\theta(s))\zeta_1(s)\theta^{\Delta}(s)\delta_1(\theta(s),T)\delta_2^{\gamma-1}(\theta(\sigma(s),T))}\}\Delta s\}\\
=&\infty. \qquad (2.32)
\end{aligned}$$

Then every solution of Eq. (1.1) *is oscillatory or tends to zero.*

Remark 2.1. The results established above improve the main results by Hassan [13, Theorems 2.1-2.4].

Remark 2.2. In Theorems 2.2-2.4, if we take $\mathbb{T}$ for some special time scales, we can obtain similar results as in Corollaries 2.1-2.2, which are omitted here.

3. Applications

In this section, we will present some applications for the established results above.

Example 3.1. Consider the following third order nonlinear delay differential equation with damping term:

$$[(tx''(t))^\gamma]' + \frac{1}{t^{\gamma+1}}(x''(t))^\gamma + \frac{1}{t^{\gamma+1}}x^\gamma(t-1)[e^{x(t-1)}+1] = 0,\ t \in [2,\infty), \tag{3.1}$$

where $\gamma \geq 1$ is a quotient of two odd positive integers.

We have in (1.1) $\mathbb{T} = \mathbb{R},\ a(t) = t^\gamma,\ p(t) = q(t) = \frac{1}{t^{\gamma+1}},\ f(x) = x^\gamma[e^x+1],\ \theta(t) = t-1,\ r(t) = 1,\ t_0 = 2$. Then $\frac{f(x)}{x^\gamma} \geq 1 = L,\ \mu(t) = \sigma(t) - t = 0$, and $-\frac{p}{a} \in \mathfrak{R}_+$. So $e_{-\frac{p}{a}}(t,t_0) = e_{-\frac{p}{a}}(t,2) = \exp(-\int_2^t \frac{p(s)}{a(s)}ds)$. Moreover, we have

$$\begin{aligned} 1 > \exp(-\int_2^t \frac{p(s)}{a(s)}ds) &\geq 1 - \int_2^t \frac{p(s)}{a(s)}ds \\ =1 - \int_2^t \frac{1}{s^{2\gamma+1}}ds &= 1 + \frac{1}{2\gamma}[t^{-2\gamma} - 2^{-2\gamma}] > \frac{1}{2}. \end{aligned}$$

Then we have

$$\int_{t_0}^\infty \frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}ds > \frac{1}{2^{\frac{1}{\gamma}}}\int_2^\infty \frac{1}{s}ds = \infty,$$

and

$$\int_{t_0}^\infty \frac{1}{r(s)}ds = \infty.$$

Furthermore,

$$\begin{aligned} &\int_{t_0}^\infty [\frac{1}{r(\xi)}\int_\xi^\infty (\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty \frac{q(s)}{e_{-\frac{p}{a}}(s,t_0)}ds)^{\frac{1}{\gamma}}d\tau]d\xi \\ =&\int_2^\infty [\int_\xi^\infty (\frac{e_{-\frac{p}{a}}(\tau,2)}{\tau^\gamma}\int_\tau^\infty \frac{1}{s^{\gamma+1}e_{-\frac{p}{a}}(s,2)}ds)^{\frac{1}{\gamma}}d\tau]d\xi \\ >&\frac{1}{2^{\frac{1}{\gamma}}}\int_2^\infty [\int_\xi^\infty (\frac{1}{\tau^\gamma}\int_\tau^\infty \frac{1}{s^{\gamma+1}}ds)^{\frac{1}{\gamma}}d\tau]d\xi \\ =&\frac{1}{(2\gamma)^{\frac{1}{\gamma}}}\int_2^\infty [\int_\xi^\infty \frac{1}{\tau^2}d\tau]d\xi = \frac{1}{(2\gamma)^{\frac{1}{\gamma}}}\int_2^\infty \frac{1}{\xi}d\xi = \infty. \end{aligned}$$

On the other hand, for a sufficiently large T, and $t \to \infty$, we have

$$\delta_1(t,T) = \int_T^t \frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}ds > \frac{1}{2^{\frac{1}{\gamma}}}\int_T^t \frac{1}{s}ds \to \infty.$$

So we can take sufficiently large $T^* > T$ such that $\delta_1(\theta(t), T) > 1$ for $t \in [T^*, \infty)$. Taking $\zeta_1(t) = t^\gamma,\ \zeta_2(t) = 0$ in (2.18), we get that

$$\begin{aligned}
&\lim_{t\to\infty}\sup\{\int_T^t\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s,t_0)}\\
&-[\frac{r(\theta(s))\zeta_1'(s)}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta'(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}ds\}\\
=&\lim_{t\to\infty}\sup\{\int_T^{T^*}\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s,t_0)}\\
&-[\frac{r(\theta(s))\zeta_1'(s)}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta'(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}ds\\
&+\int_{T^*}^t\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s,t_0)}-[\frac{r(\theta(s))\zeta_1'(s)}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta'(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}ds\}\\
>&\lim_{t\to\infty}\sup\{\int_T^{T^*}\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s,t_0)}\\
&-[\frac{r(\theta(s))\zeta_1'(s)}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta'(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}ds\\
&+\int_{T^*}^t[1-(\frac{\gamma}{\gamma+1})^{\gamma+1}]\frac{1}{s}ds=\infty.
\end{aligned}$$

So (2.15)-(2.18) all hold, and by Corollary 2.1 we deduce that every solution of Eq. (3.1) is oscillatory or tends to zero.

Example 3.2. Consider the following third order nonlinear delay difference equation:

$$\Delta[(t\Delta^2x(t))^\gamma]+\frac{1}{t^{\gamma+1}}(\Delta^2x(t))^\gamma+\frac{M}{t^{\gamma+1}}x^\gamma(\frac{t}{2})=0,\ t\in[2,\infty)_{\mathbb{Z}}, \tag{3.2}$$

where Δ denotes the difference operator, $M > 0$ is a constant, and $\gamma \geq 1$ is a quotient of two odd positive integers.

We have in (1.1) $\mathbb{T} = \mathbb{Z},\ a(t) = t^\gamma,\ p(t) = q(t) = \frac{1}{t^{\gamma+1}},\ f(x) = Mx^\gamma,\ \theta(t) = \frac{t}{2},\ r(t) = 1,\ t_0 = 2$. Then $\frac{f(t)}{x^\gamma(t)} \geq M = L,\ \mu(t) = \sigma(t) - t = 1$, and

$$1-\mu(t)\frac{p(t)}{a(t)}=1-\frac{1}{t^{2\gamma+1}}\geq 1-\frac{1}{2}>0,$$

which implies $-\frac{p}{a} \in \mathfrak{R}_+$. So by Bohner [3, Lemma 2] we obtain

$$\begin{aligned}
e_{-\frac{p}{a}}(t,t_0)=&e_{-\frac{p}{a}}(t,2)\geq 1-\int_2^t\frac{p(s)}{a(s)}\Delta s\\
=&1-\int_2^t\frac{1}{s^{2\gamma+1}}\Delta s=1-\sum_{s=2}^{t-1}\frac{1}{s^{2\gamma+1}}\\
\geq&1-\int_1^{t-1}\frac{1}{s^{2\gamma+1}}ds=1+\frac{1}{2\gamma}[(t-1)^{-2\gamma}-1]>\frac{1}{2},
\end{aligned}$$

and

$$e_{-\frac{p}{a}}(t,t_0) \leq \exp(-\int_2^t \frac{p(s)}{a(s)}\Delta s) < 1.$$

Then we have

$$\sum_{s=t_0}^{\infty} \frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} = \sum_{s=2}^{\infty} \frac{[e_{-\frac{p}{a}}(s,2)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} = \sum_{s=2}^{\infty} \frac{[e_{-\frac{p}{a}}(s,2)]^{\frac{1}{\gamma}}}{s} > \frac{1}{2^{\frac{1}{\gamma}}} \sum_{s=2}^{\infty} \frac{1}{s} = \infty,$$

and

$$\sum_{s=t_0}^{\infty} \frac{1}{r(s)} = \infty.$$

Furthermore,

$$\begin{aligned}
&\sum_{\xi=t_0}^{\infty}[\frac{1}{r(\xi)}\sum_{\tau=\xi}^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(s+1,t_0)})^{\frac{1}{\gamma}}] \\
=&\sum_{\xi=2}^{\infty}[\frac{1}{r(\xi)}\sum_{\tau=\xi}^{\infty}(\frac{e_{-\frac{p}{a}}(\tau,2)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)}{e_{-\frac{p}{a}}(s+1,2)})^{\frac{1}{\gamma}}] \\
>&\frac{1}{2^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}(\frac{1}{\tau^{\gamma}}\sum_{s=\tau}^{\infty}\frac{1}{s^{\gamma+1}})^{\frac{1}{\gamma}}] \\
>&\frac{1}{2^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}(\frac{1}{\tau^{\gamma}}\int_{s=\tau}^{\infty}\frac{1}{s^{\gamma+1}}ds)^{\frac{1}{\gamma}}] = \frac{1}{(2\gamma)^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}\frac{1}{\tau^2}] \\
>&\frac{1}{(2\gamma)^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}\sum_{\tau=\xi}^{\infty}\frac{1}{\tau(\tau+1)} = \frac{1}{(2\gamma)^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}\frac{1}{\xi} = \infty.
\end{aligned}$$

On the other hand, for a sufficiently large $T > 1$, we have

$$\delta_1(t,T) = \sum_{s=T}^{t-1} \frac{[e_{-\frac{p}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} > \frac{1}{2^{\frac{1}{\gamma}}} \sum_{s=T}^{t-1} \frac{1}{s} \to \infty.$$

So there exists $T^* > T$ such that $\delta_1(\theta(t),T) > 1$ for $t \in [T^*,\infty)_{\mathbb{Z}}$. Let $\zeta_1(t) = t^{\gamma}$, $\zeta_2(t) = 0$ in (2.22). Then by the inequality $(t+1)^{\gamma} - t^{\gamma} \leq \gamma(t+1)^{\gamma-1} < \gamma 2^{\gamma-1}t^{\gamma-1}$ for $t \geq T^*$ we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\sup\{\sum_{s=T}^{t-1}\{L\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s+1,t_0)} \\
&-[\frac{r(\theta(s))(\zeta_1(s+1)-\zeta_1(s))}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta(s+1)-\theta(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}\} \\
=&\lim_{t\to\infty}\sup\{\sum_{s=T^*}^{t-1}\{M\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s+1,t_0)} \\
&-[\frac{r(\theta(s))(\zeta_1(s+1)-\zeta_1(s))}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta(s+1)-\theta(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}
\end{aligned}$$

$$+\sum_{s=T}^{T^*}\{M\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s+1,t_0)}$$
$$-[\frac{r(\theta(s))(\zeta_1(s+1)-\zeta_1(s))}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta(s+1)-\theta(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}\}$$
$$>\sum_{s=T}^{T^*}\{M\frac{q(s)\zeta_1(s)}{e_{-\frac{p}{a}}(s+1,t_0)}$$
$$-[\frac{r(\theta(s))(\zeta_1(s+1)-\zeta_1(s))}{(\gamma+1)r^{\frac{1}{\gamma+1}}(\theta(s))\zeta_1^{\frac{\gamma}{\gamma+1}}(s)(\theta(s+1)-\theta(s))^{\frac{\gamma}{\gamma+1}}\delta_1^{\frac{\gamma}{\gamma+1}}(\theta(s),T)}]^{\gamma+1}\}$$
$$+\sum_{s=T^*}^{t-1}[M-(\frac{\gamma}{\gamma+1})^{\gamma+1}2^{\gamma^2+\gamma-1}]\frac{1}{s}\to\infty,$$

provided that $M>(\frac{\gamma}{\gamma+1})^{\gamma+1}2^{\gamma^2+\gamma-1}$. So (2.19)-(2.22) all hold, and by Corollary 2.2 we obtain that every solution of Eq. (3.2) is oscillatory or tends to zero under the condition $M>(\frac{\gamma}{\gamma+1})^{\gamma+1}2^{\gamma^2+\gamma-1}$.

Acknowledgements

The author would like to thank the reviewers very much for their valuable suggestions to improve the quality of this paper.

References

[1] R. Agarwal, M. Bohner and A. Peterson, *Inequalities on time scales: a survey*, Math. Inequal. Appl., 2001, 4(4), 535–557.

[2] R. P. Agarwal, M. Bohner and S. H. Saker, *Oscillation of second order delay dynamic equations*, Can. Appl. Math. Q., 2005, 13, 1–18.

[3] M. Bohner, *Some oscillation criteria for first order delay dynamic equations*, Far East J. Appl. Math., 2005, 18(3), 289–304.

[4] M. Bohner and A. Peterson, *Dynamic Equations On Time Scales: An Introduction With Applications*, Birkhäuser, Boston, 2001.

[5] M. Bohner and S. H. Saker, *Oscillation of second order nonlinear dynamic equations on time scales*, Rocky Mountain J. Math., 2004, 34, 1239–1254.

[6] L. Erbe and T. S. Hassan, *Oscillation of Third Order Nonlinear Functional Dynamic Equations on Time Scales*, Diff. Equ. Dynam. Sys., 2010, 18(1), 199–227.

[7] L. Erbe, T. S. Hassan and A. Peterson, *Oscillation of third-order functional dynamic equations with mixed arguments on time scales*, J. Appl. Math. Comput., 2010, 34(1-2), 353–371.

[8] Q. Feng and F. Meng, *Oscillation of solutions to nonlinear forced fractional differential equations*, Electron. J. Differ. Eq., 2013, 2013(169), 1–10.

[9] S. R. Grace, R. P. Agarwal, M. Bohner and D. O'Regan, *Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations*, Commun. Nonlinear Sci. Numer. Simul., 2009, 14, 3463–3471.

[10] S. R. Grace, J. R. Graef and M. A. El-Beltagy, *On the oscillation of third order neutral delay dynamic equations on time scales*, Comput. Math. Appl., 2012, 63(4), 775–782.

[11] L. Guo, L. Liu and Y. Wu, *Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions*, Nonlinear Anal. Model., 2016, 21(5), 635–650.

[12] Y. Guan, Z. Zhao and X. Lin, *On the existence of positive solutions and negative solutions of singular fractional differential equations via global bifurcation techniques*, Bound. Value Probl., 2016, 2016(141), 1–18.

[13] T. S. Hassan, *Oscillation of third order nonlinear delay dynamic equations on time scales*, Math. Comput. Modelling, 2009, 49, 1573–1586.

[14] T. S. Hassan, *Oscillation criteria for higher order quasilinear dynamic equations with Laplacians and a deviating argument*, J. Egypt. Math. Soc., 2016, 25(2), 178–185.

[15] S. Hilger, *Analysis on measure chains-a unified approach to continuous and discrete calculus*, Results Math., 1990, 18, 18–56.

[16] T. S. Hassan and Q. Kong, *Oscillation criteria for higher-order nonlinear dynamic equations with Laplacians and a deviating argument on time scales*, Math. Methods Appl. Sci., 2017, 40(11), 4028–4039.

[17] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities, Second edition*, Cambridge Univ. Press, Cambridge, UK, 1988.

[18] Z. Han, T. Li, S. Sun and F. Cao, *Oscillation criteria for third order nonlinear delay dynamic equations on time scales*, Ann. Polon. Math., 2010, 99(2), 143–156.

[19] Y. Huang and F. Meng, *Oscillation criteria for forced second-order nonlinear differential equations with damping*, J. Comput. Appl. Math., 2009, 224, 339–345.

[20] L. Liu and Y. Bai, *New oscillation criteria for second-order nonlinear neutral delay differential equations*, J. Comput. Appl. Math., 2009, 231, 657–663.

[21] H. Liu and F. Meng, *Oscillation criteria for second order linear matrix differential systems with damping*, J. Comput. Appl. Math., 2009, 229(1), 222–229.

[22] H. Liu and F. Meng, *Interval oscillation criteria for second-order nonlinear forced differential equations involving variable exponent*, Adv. Diff. Equ., 2016, 2016(291), 1–14.

[23] H. Liu, F. Meng and P. Liu, *Oscillation and asymptotic analysis on a new generalized Emden-Fowler equation*, Appl. Math. Comput., 2012, 219(5), 2739–2748.

[24] L. Li, F. Meng and Z. Zheng, *Some new oscillation results for linear Hamiltonian systems*, Appl. Math. Comput., 2009, 208(1), 219–224.

[25] J. Liu and Z. Zhao, *Multiple solutions for impulsive problems with nonautonomous perturbations*, Appl. Math. Lett., 2017, 64, 143–149.

[26] F. Meng and Y. Huang, *Interval oscillation criteria for a forced second-order nonlinear differential equations with damping*, Appl. Math. Comput., 2011, 218, 1857–1861.

[27] L. Ren and J. Xin, *Almost global existence for the Neumann problem of quasi-linear wave equations outside star-shaped domains in 3D*, Electron. J. Differ. Eq., 2017, 2017(312), 1–22.

[28] Y. Sahiner, *Oscillation of second-order delay differential equations on time scales*, Nonlinear Anal. TMA, 2005, 63(5), 1073–1080.

[29] S. H. Saker, *Oscillation of third-order functional dynamic equations on time scales*, Science China(Mathematics), 2011, 54(12), 2597–2614.

[30] S. H. Saker, *Oscillation of second-order nonlinear neutral delay dynamic equations on time scales*, J. Comput. Appl. Math., 2006, 187, 123–141.

[31] Y. Shi, Z. Han and Y. Sun, *Oscillation criteria for a generalized Emden-Fowler dynamic equation on time scales*, Adv. Diff. Equ., 2016, 2016(3), 1–12.

[32] Y. B. Sun, Z. Han, Y. Sun and Y. Pan, *Oscillation theorems for certain third order nonlinear delay dynamic equations on time scales*, Electron. J. Qual. Theory Differ. Equ., 2011, 75, 1–14.

[33] Y. Sun, L. Liu and Y. Wu, *The existence and uniqueness of positive monotone solutions for a class of nonlinear Schrödinger equations on infinite domains*, J. Comput. Appl. Math., 2017, 321, 478–486.

[34] J. Shao, Z. Zheng and F. Meng, *Oscillation criteria for fractional differential equations with mixed nonlinearities*, Adv. Diff. Equ., 2013, 2013(323), 1–9.

[35] F. Xu, X. Zhang, Y. Wu and L. Liu, *Global existence and temporal decay for the 3D compressible Hall-magnetohydrodynamic system*, J. Math. Anal. Appl., 2016, 438(1), 285–310.

[36] Z. Zhao, *Existence of fixed points for some convex operators and applications to multi-point boundary value problems*, Appl. Math. Comput., 2009, 215(8), 2971–2977.

[37] Z. Zheng, *Oscillation Criteria for Matrix Hamiltonian Systems via Summability Method*, Rocky Mount. J. Math., 2009, 39(5), 1751–1766.

[38] B. Zhu, L. Liu and Y. Wu, *Local and global existence of mild solutions for a class of nonlinear fractional reaction-diffusion equation with delay*, Appl. Math. Lett., 2016, 61, 73–79.

[39] Z. Zheng and F. Meng, *On Oscillation Properties for Linear Hamiltonian Systems*, Rocky Mount. J. Math., 2009, 39(1), 343–358.

[40] X. Zheng, Y. Shang and X. Peng, *Orbital stability of solitary waves of the coupled Klein-Gordon-Zakharov equations*, Math. Methods Appl. Sci., 2017, 40(7), 2623–2633.

[41] X. Zheng, Y. Shang and X. Peng, *Orbital stability of periodic traveling wave solutions to the generalized Zakharov equations*, Acta Math. Sci., 2017, 37B(4), 998–1018.

[42] Z. Zheng, X. Wang and H. Han, *Oscillation Criteria for Forced Second Order Differential Equations with Mixed Nonlinearities*, Appl. Math. Lett., 2009, 22, 1096–1101.