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UPPER BOUNDS FOR THE ASSOCIATED NUMBER OF ZEROS OF ABELIAN INTEGRALS FOR TWO CLASSES OF QUADRATIC REVERSIBLE CENTERS OF GENUS ONE*

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Abstract In this paper, by using the method of Picard-Fuchs equation and Riccati equation, we study the upper bounds for the associated number of zeros of Abelian integrals for two classes of quadratic reversible centers of genus one under any polynomial perturbations of degree n, and obtain that their upper bounds are $3n-3$ $(n \geq 2)$ and $18\left[\frac{n}{2}\right]+3\left[\frac{n-1}{2}\right]$ $(n \geq 4)$ respectively, both of the two upper bounds linearly depend on n.

Keywords Abelian integral, quadratic reversible center, weakened Hilbert's 16th problem.

MSC(2010) 34C07, 34C08, 37G15.

1. Introduction

Consider a planar polynomial system of the form

$$\frac{dx}{dt}=\frac{H_y(x,y)}{N(x,y)}+\mu p(x,y),\quad \frac{dy}{dt}=-\frac{H_x(x,y)}{N(x,y)}+\mu q(x,y), \tag{1.1}$$

where μ $(0<\mu\ll 1)$ is a real parameter, $\frac{H_y(x,y)}{N(x,y)}$, $\frac{H_x(x,y)}{N(x,y)}$, $p(x,y)$, $q(x,y)$ are any polynomials of x, y, $\max\left\{\deg\left(\frac{H_y(x,y)}{N(x,y)}\right),\deg\left(\frac{H_x(x,y)}{N(x,y)}\right)\right\}=m$, $\max\{\deg(p(x,y)),\}$ $\{\deg(q(x,y))\}=n$. System (1.1) is an integrable system when $\mu=0$, where $H(x,y)$ is a first integral, and $N(x,y)$ is an integrating factor. Further, we suppose that system (1.1) has at least a center when $\mu=0$. i.e., we can define a continuous family of ovals

$$\Gamma_h\subset\left\{(x,y)\in\mathbb{R}^2: H(x,y)=h, h\in\Omega\right\},$$

which are defined on a maximal open interval $\Omega=(h_1,h_2)$. The question of this paper is: for any small μ, how many limit cycles in system (1.1) can be bifurcated from the period annulus Γ_h? It is well known that in any compact region of the

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period annulus, the number of limit cycles of system (1.1) is no more than isolated zeros of the following Abelian integral $A(h)$,

$$A(h) = \oint_{\Gamma_h} N(x,y)\left[q(x,y)\,dx - p(x,y)\,dy\right], h \in \Omega. \tag{1.2}$$

A) If $\mu = 0$, and (1.1) is a Hamiltonian system, that is, $N(x,y)$ is a real number, and $N(x,y) \neq 0$, $\deg(H(x,y)) = m+1$. Finding an upper bound $Z(m,n)$ for the number of the isolated zeros of the Abelian integral $A(h)$ is an important and significant problem, where the upper bound $Z(m,n)$ only depends on m, n and does not depend on the concrete forms of $H(x,y)$, $p(x,y)$, and $q(x,y)$. It is called the weakened Hilbert's 16th problem by Arnold in [1]. This problem has been studied widely, such as, researchers obtain plentiful important results for Liénard systems [9, 10], and more details can be found in the books [2, 4], and the review article [7].

B) If $\mu = 0$, however (1.1) is an integrable non-Hamiltonian system, that is, $N(x,y)$ is not a real number. Maybe $N(x,y)$ and $H(x,y)$ are not polynomials, so the study of the associated Abelian integral become much more difficult. Thus, researchers consider this problem by starting from the simplest case: the degrees of the unperturbed systems are low, specific, $m = 2$, and conjecture that the upper bound linearly depends on n, nevertheless, this conjecture is still far from being solved.

For quadratic reversible centers of genus one, there are essentially 22 cases in the classification in [3], namely $(r1)$–$(r22)$. The linear dependance of case $(r1)$ was studied in [11]; cases $(r3)$–$(r6)$ were studied in [8]; cases $(r9)$, $(r13)$, $(r17)$, and $(r19)$ were studied in [6]; cases $(r11)$, $(r16)$, $(r18)$, and $(r20)$ were studied in [5], all of these upper bounds linearly depend on n. In order to thoroughly study these problems, whereas we consider other two cases $(r21)$ and $(r12)$, we obtain that their upper bounds are $3n-3$ $(n \geq 2)$ and $18\left[\frac{n}{2}\right] + 3\left[\frac{n-1}{2}\right]$ $(n \geq 4)$, respectively, both of the two upper bounds linearly depend on n.

The forms of cases $(r21)$ and $(r12)$ are as follows

$$(r21) \quad \begin{array}{l} \dot{x} = -xy, \quad \dot{y} = -y^2 + \frac{1}{2^4}x - \frac{1}{2^4}. \\ H(x,y) = x^{-2}\left(\frac{1}{2}y^2 - \frac{1}{2^4}x + \frac{1}{2^5}\right) = h, \quad h \in \left(-\frac{1}{2^5}, 0\right), \end{array} \tag{1.3}$$

with an integrating factor $N(x,y) = x^{-3}$;

$$(r12) \quad \begin{array}{l} \dot{x} = -xy, \quad \dot{y} = -\frac{1}{6}y^2 + \frac{1}{3^2 \cdot 2^6}x - \frac{1}{3^2 \cdot 2^6}. \\ H(x,y) = x^{-\frac{1}{3}}\left(\frac{1}{2}y^2 + \frac{1}{3 \cdot 2^7}x + \frac{1}{3 \cdot 2^6}\right) = h, \quad h \in \left(\frac{1}{2^7}, +\infty\right), \end{array} \tag{1.4}$$

with an integrating factor $N(x,y) = x^{-\frac{4}{3}}$.

In this paper, our main results include the following theorem.

Theorem 1.1. *For any polynomials $p(x,y)$ and $q(x,y)$ of degree n, the upper bounds for the corresponding number of zeros of the Abelian integral*

$$A(h) = \oint_{\Gamma_h} N(x,y)\left[q(x,y)\,dx - p(x,y)\,dy\right], \ h \in \Omega,$$

where Γ_h are orbits of the corresponding system given above, which linearly depend on n. Concretely, for $(r21)$, the upper bound is $3n-3$ when $n \geq 2$, 0 when $n = 1, 0$;

for (r12), the upper bound is $18\left[\frac{n}{2}\right]+3\left[\frac{n-1}{2}\right]$ *when* $n\geq 4$, 33 *when* $1\leq n\leq 3$, 0 *when* $n=0$.

2. Abelian Integral

Suppose polynomials $p(x,y)=\sum\limits_{0\leq r+s\leq n} a_{r,s}x^r y^s$, $q(x,y)=\sum\limits_{0\leq r+s\leq n} b_{r,s}x^r y^s$, so, the Abelian integral $A(h)$ in Theorem 1.1 has the form

$$A(h)=\oint_{\Gamma_h} x^{-\nu}\left(\sum_{0\leq r+s\leq n} b_{r,s}x^r y^s dx-\sum_{0\leq r+s\leq n} a_{r,s}x^r y^s dy\right),$$

where $x^{-\nu}$ is an integrating factor.

For concision, Abelian integral is denoted by $I_{r,s}(h)$ as follows:

$$I_{r,s}(h)=\oint_{\Gamma_h} x^{r-\nu}y^s dx,$$

where $r=-1,0,1,\cdots,n-1,n$; $s=0,1,2,\cdots,n,n+1$. When $s=1$, we use $J_r(h)$ instead of $I_{r,1}(h)$.

Note that

$$\oint_{\Gamma_h} x^{r-\nu}y^s dy=\frac{\oint_{\Gamma_h} x^{r-\nu}dy^{s+1}}{s+1}=\frac{\nu-r}{s+1}\oint_{\Gamma_h} x^{r-\nu-1}y^{s+1}dx=\frac{\nu-r}{s+1}I_{r-1,s+1}(h).$$

Thus, $A(h)$ can be written as

$$A(h)=\sum_{\substack{0\leq r\leq n,\\ 0\leq s\leq n,\\ 0\leq r+s\leq n}} b_{r,s}I_{r,s}(h)+\sum_{\substack{0\leq r\leq n,\\ 0\leq s\leq n,\\ 0\leq r+s\leq n}} a_{r,s}\tfrac{r-\nu}{s+1}I_{r-1,s+1}(h)=\sum_{\substack{-1\leq r\leq n,\\ 0\leq s\leq n+1,\\ 0\leq r+s\leq n}} \tilde{b}_{r,s}I_{r,s}(h). \tag{2.1}$$

For the Abelian integral $A(h)$, we have the following proposition.

Proposition 2.1. *1) For (r21), the Abelian integral* $A(h)$ *can be expressed as*

$$A(h)=\begin{cases}\frac{1}{h^{n-2}}K(h), & K(h)=\alpha(h)J_0(h)+\beta(h)J_1(h), \quad (n\geq 2),\\ \alpha(h)J_0(h), & (n=0,1),\end{cases} \tag{2.2}$$

where $\deg(\alpha(h))\leq n-2$, $\deg(\beta(h))\leq n-1$, *when* $n\geq 2$*; and* $\deg(\alpha(h))=0$, *when* $n=0,1$.

2) For (r12), the Abelian integral $A(h)$ *can be expressed as*

$$A(h)=\begin{cases}\alpha(h)J_0(h)+\beta(h)J_{\frac{1}{3}}(h)+\gamma(h)J_{\frac{2}{3}}(h), & (n\geq 1),\\ \delta(h)J_{-1}(h), & (n=0),\end{cases} \tag{2.3}$$

where $\deg(\alpha(h))\leq 3\left[\frac{n}{2}\right]-3$, $\deg(\beta(h))\leq 3\left[\frac{n}{2}\right]-2$, $\deg(\gamma(h))\leq 3\left[\frac{n-1}{2}\right]-1$, *when* $n\geq 4$*;* $\deg(\alpha(h))\leq 3$, $\deg(\beta(h))\leq 1$, $\deg(\gamma(h))\leq 2$, *when* $n=1,2,3$*; and* $\deg(\delta(h))=0$, *when* $n=0$.

Proof. For $(r21)$ and $(r12)$, because $I_{i,s}(h)=0$ when s is an even number, we only need to consider odd values for s.

1) For $(r21)$, $\nu=3$, using C replace $\frac{1}{2^5}$, and it follows from (1.3) that

$$-x^{-3}y^2+x^{-2}y\frac{\partial y}{\partial x}+2Cx^{-2}-2Cx^{-3}=0. \tag{2.4}$$

Multiplied (2.4) by x^iy^{s-2} and integrating over Γ_h, we obtain

$$\frac{i+s-2}{s}I_{i,s}(h)=2C\left[I_{i+1,s-2}(h)-I_{i,s-2}(h)\right], \tag{2.5}$$

where s is a positive odd number, i.e. $s=1,3,5,\cdots,2[\frac{n}{2}]+1$, for system $(r21)$, we restrict $i=-1,0,1,2,3,\cdots,n-1$, and $0\le i+s\le n$.

(i) If $i+s-2=0$, i.e., $(i,s)=(1,1)$, or $(i,s)=(-1,3)$, $A(h)$ includes $I_{1,1}(h)$ and $I_{-1,3}(h)$. Because $I_{1,1}(h)$ is $J_1(h)$, we mainly consider $I_{-1,3}(h)$. Let $i=-1$, and $s=3$, from equality (2.5), we obtain

$$J_{-1}(h)=J_0(h). \tag{2.6}$$

(ii) If $i+s-2\neq 0$, i.e., $(i,s)\neq(1,1)$, and $(i,s)\neq(-1,3)$, we rewrite (2.5) as

$$I_{i,s}(h)=\frac{2Cs}{i+s-2}\left[I_{i+1,s-2}(h)-I_{i,s-2}(h)\right], \tag{2.7}$$

which indicates $I_{i,s}(h)$ can be expressed in terms of $I_{i+1,s-2}(h)$ and $I_{i,s-2}(h)$, and then step by step, thus, $I_{i,s}(h)$ can be written as a linear combination of $J_i(h)(i=-1,0,\cdots)$ and $I_{-1,3}(h)$ with the form

$$I_{i,s}(h)=\begin{cases} J_i(h), \quad (i\neq 1, s=1),\\ \sum\limits_{k=0}^{\frac{s-1}{2}} c_{(i,s),\,k}J_{i+k}(h), \quad (i\ge 0, s\ge 3),\\ \sum\limits_{k=0}^{\frac{s-3}{2}} c_{(-1,s),\,k}J_k(h)+d_{-1,s}I_{-1,3}(h), \quad (i=-1, s\ge 5).\end{cases} \tag{2.8}$$

From (2.1) and (2.8), we get

$$A(h)=A_1(h)+A_2(h)+A_3(h)+A_4(h),$$

where $A_1(h)=\sum\limits_{k=-1}^{n-1}\tilde{b}_{k,1}J_k(h)$, $A_3(h)=\sum\limits_{\substack{5\le s\le 2[\frac{n}{2}]+1,\\ s\equiv 1\, mod\, 2}}\tilde{b}_{-1,s}\sum\limits_{k=0}^{\frac{s-3}{2}}c_{(-1,s),\,k}J_k(h)$, $A_2(h)$ $=\sum\limits_{\substack{0\le r\le n-3,\\ s\equiv 1\, mod\, 2,\\ 3\le r+s\le n}}\tilde{b}_{r,s}\sum\limits_{k=0}^{\frac{s-1}{2}}c_{(r,s),\,k}J_{r+k}(h)$, $A_4(h)=\tilde{b}_{-1,3}I_{-1,3}(h)+\sum\limits_{\substack{5\le s\le 2[\frac{n}{2}]+1,\\ s\equiv 1\, mod\, 2}}\tilde{b}_{-1,s}d_{-1,s}\times$ $I_{-1,3}(h)=\tilde{d}_{-1,3}I_{-1,3}(h)$. For $A_2(h)$, the maximum number of $r+k$ is $r+\frac{s-1}{2}=n-3+\frac{3-1}{2}=n-2$, and the minimum number is $0+0=0$. For $A_3(h)$, the maximum number of k is $\left[\frac{n+1-3}{2}\right]=\left[\frac{n-2}{2}\right]\le n-3$ $(n\ge 4)$, and the minimum number is 0. If denoted $A_1(h)+A_2(h)+A_3(h)$ by $A_5(h)$, i.e., $A_5(h):=A_1(h)+A_2(h)+A_3(h)$, so $A_5(h)$ is a linear combination of $J_k(h)(k=-1,0,\cdots,n-1)$, we have that

$$A(h)=A_5(h)+A_4(h)=\sum_{k=-1}^{n-1}e_kJ_k(h)+\tilde{d}_{-1,3}I_{-1,3}(h), \tag{2.9}$$

where $e_k \in \mathbb{R}$ $(k=-1,0,\cdots,n-1)$.

Again, it follows from (1.3) that

$$\frac{1}{2}x^{-2}y^2-2Cx^{-1}+Cx^{-2}=h. \tag{2.10}$$

Multiplied (2.10) by $x^{i-1}y^{s-2}$ and integrating over Γ_h, we obtain

$$\frac{1}{2}I_{i,s}(h)=hI_{i+2,s-2}(h)+2CI_{i+1,s-2}(h)-CI_{i,s-2}(h). \tag{2.11}$$

(i) If $i+s-2=0$, i.e., $(i,s)=(1,1)$, or $(i,s)=(-1,3)$, if $\hbar:=\frac{1}{h}$, let $i=-1$, $s=3$, from equalities (2.6), (2.9) and (2.11), we obtain

$$I_{-1,3}(h)=2hJ_1(h)+4CJ_0(h)-2CJ_{-1}(h)=2hJ_1(h)+2CJ_0(h),$$

thus,

$$\hbar I_4(h)=\hbar\tilde{d}_{-1,3}I_{-1,3}(h)=2\tilde{d}_{-1,3}C\hbar J_0(h)+2\tilde{d}_{-1,3}J_1(h).$$

(ii) If $i+s-2\neq 0$, i.e., $(i,s)\neq(1,1)$, and $(i,s)\neq(-1,3)$, let $s=3$, by (2.7), equality (2.11) can be written as

$$(i-2)CJ_i(h)=(i+1)hJ_{i+2}(h)+(2i-1)CJ_{i+1}(h), \quad (i\neq -1). \tag{2.12}$$

A) If $i\geq 2$, we rewrite equality (2.12) as

$$\hbar J_i(h)=\frac{i-4}{i-1}C\hbar^2J_{i-2}(h)-\frac{2i-5}{i-1}C\hbar^2J_{i-1}(h),$$

which indicates that $\hbar J_i(h)$ can be expressed in terms of $\hbar^2J_{i-2}(h)$ and $\hbar^2J_{i-1}(h)$, and then step by step, $\hbar J_i(h)$ can be written as a linear combination of $J_0(h)$ and $J_1(h)$ with polynomial coefficients of $\hbar$:

$$\hbar J_i(h)=\alpha_i(\hbar)J_0(h)+\beta_i(\hbar)J_1(h),$$

where $2\leq \deg(\alpha_i(\hbar))\leq i$, $2\leq\deg(\beta_i(\hbar))\leq i$.

B) If $i=0,1$, $\hbar J_i(h)$ can also be expressed by $J_0(h)$ and $J_1(h)$ as $\hbar J_0(h)=\hbar J_0(h)$, $\hbar J_1(h)=\hbar J_1(h)$.

C) If $i=-1$, from equality (2.6), we get

$$\hbar J_{-1}(h)=\hbar J_0(h).$$

As a consequence, all $\hbar J_k(h)(k=-1,0,1,\cdots,n-1)$ can be expressed in terms of $J_0(h)$ and $J_1(h)$, so $\hbar A_5(h)$ and $\hbar A_4(h)$ can be expressed by $J_0(h)$ and $J_1(h)$.

If $n\geq 2$, $J(h):=\hbar A(h)$, from (2.9), using formulae above, we obtain

$$J(h)=\hbar A(h)=\hbar A_5(h)+\hbar A_4(h)=\alpha(\hbar)J_0(h)+\beta(\hbar)J_1(h), \tag{2.13}$$

where $1\leq\deg(\alpha(\hbar))\leq n-1$, $0\leq\deg(\beta(\hbar))\leq n-1$. Multiplied (2.13) by h^{n-1} and if $K(h):=h^{n-1}J(h)$, we obtain

$$K(h)=h^{n-1}J(h)=\alpha(h)J_0(h)+\beta(h)J_1(h),$$

where $\deg(\alpha(h))\leq n-2$, $\deg(\beta(h))\leq n-1$.

If $n = 0, 1$, from (2.1) and (2.6), we obtain

$$A(h) = \tilde{b}_{-1,1} J_{-1}(h) + \tilde{b}_{0,1} J_0(h) = \left(\tilde{b}_{-1,1} + \tilde{b}_{0,1}\right) J_0(h) = \alpha(h) J_0(h),$$

where $\alpha(h) := \tilde{b}_{-1,1} + \tilde{b}_{0,1}$ and $\deg(\alpha(h)) = 0$.

2) For $(r12)$, $\nu = \frac{4}{3}$, using C replace $\frac{1}{3 \cdot 2^7}$, and it follows from (1.4) that

$$-\frac{1}{6} x^{-\frac{4}{3}} y^2 + x^{-\frac{1}{3}} y \frac{\partial y}{\partial x} + \frac{2}{3} C x^{-\frac{1}{3}} - \frac{2}{3} C x^{-\frac{4}{3}} = 0. \tag{2.14}$$

Multiplied (2.14) by $x^i y^{s-2}$ and integrating over Γ_h, we obtain

$$\frac{6i + s - 2}{s} I_{i,s}(h) = 4C \left[I_{i+1,s-2}(h) - I_{i,s-2}(h) \right], \tag{2.15}$$

where s is an odd number, i.e. $s = 1, 3, 5, \cdots, 2[\frac{n}{2}] + 1$, for system $(r12)$, we restrict $i = \frac{1}{3} t$ ($t = -3, -2, -1, 0, 1, \cdots, 3n - 3$) and $0 \le i + s \le n$. So $6i + s - 2 \ne 0$, equality (2.15) can be written as

$$I_{i,s}(h) = \frac{4Cs}{6i + s - 2} \left[I_{i+1,s-2}(h) - I_{i,s-2}(h) \right], \tag{2.16}$$

which indicates $I_{i,s}(h)$ can be expressed in terms of $I_{i+1,s-2}(h)$ and $I_{i,s-2}(h)$, and then step by step, $I_{i,s}(h)$ can be written as a linear combination of $J_i(h)(i = -1, 0, \cdots)$ with the form

$$I_{i,s}(h) = \sum_{k=0}^{\frac{s-1}{2}} c_{(i,s),\,k} J_{i+k}(h).$$

From equality (2.1), we get

$$A(h) = \sum_{\substack{0 \le r+s \le n, \\ -1 \le r \le n, \\ s \equiv 1 \, mod \, 2}} \tilde{b}_{r,s} \sum_{k=0}^{\frac{s-1}{2}} c_{(r,s),\,k} J_{r+k}(h).$$

The maximum number of $r + k$ is $n - 1 + \frac{1-1}{2} = n - 1$, and the minimum number is $-1 + \frac{1-1}{2} = -1$, so $A(h)$ is a linear combination of $J_k(h)(k = -1, 0, \cdots, n - 1)$, we have that

$$A(h) = \sum_{k=-1}^{n-1} e_k J_k(h), \tag{2.17}$$

where $e_k \in \mathbb{R}$ ($k = -1, 0, \cdots, n - 1$).

Again, it follows from (1.4) that

$$\frac{1}{2} x^{-\frac{1}{3}} y^2 + C x^{\frac{2}{3}} + 2C x^{-\frac{1}{3}} = h. \tag{2.18}$$

Multiplied (2.18) by $x^{i-1} y^{s-2}$ and integrating over Γ_h, we obtain

$$\frac{1}{2} I_{i,s}(h) = h I_{i+\frac{1}{3},s-2}(h) - C I_{i+1,s-2}(h) - 2C I_{i,s-2}(h). \tag{2.19}$$

Let $s = 3$, using (2.16), equality (2.19) can be written as

$$4C(3i-1)J_i(h) = (6i+1)hJ_{i+\frac{1}{3}}(h) - (6i+7)CJ_{i+1}(h). \tag{2.20}$$

A) If $i \geq 1$, we rewrite equality (2.20) as

$$J_i(h) = \frac{6i-5}{C(6i+1)}hJ_{i-\frac{2}{3}}(h) - \frac{4(3i-4)}{6i+1}J_{i-1}(h),$$

which indicates that $J_i(h)$ can be expressed in terms of $hJ_{i-\frac{2}{3}}(h)$ and $J_{i-1}(h)$, and then step by step, $J_i(h)$ can be written as a linear combination of $J_0(h)$, $J_{\frac{1}{3}}(h)$ and $J_{\frac{2}{3}}(h)$ with polynomial coefficients of h:

$$J_i(h) = \alpha_i(h)J_0(h) + \beta_i(h)J_{\frac{1}{3}}(h) + \gamma_i(h)J_{\frac{2}{3}}(h),$$

where $\deg(\alpha_i(h)) \leq 3\left[\frac{i+1}{2}\right] - 3$, $\deg(\beta_i(h)) \leq 3\left[\frac{i+1}{2}\right] - 2$, $\deg(\gamma_i(h)) \leq 3\left[\frac{i}{2}\right] - 1$, when $i \geq 2$; and $\deg(\alpha_i(h)) = 0$, $\deg(\gamma_i(h)) = 1$, $\beta_i(h) = 0$, when $i = 1$.

B) If $i = 0$, $J_i(h)$ can also be expressed by $J_0(h)$, $J_{\frac{1}{3}}(h)$, $J_{\frac{2}{3}}(h)$ as $J_0(h) = J_0(h)$.

C) If $i < 0$, we rewrite equality (2.20) as

$$J_i(h) = \frac{6i+1}{4C(3i-1)}hJ_{i+\frac{1}{3}}(h) - \frac{6i+7}{4(3i-1)}J_{i+1}(h),$$

which indicates that $J_{-1}(h)$ can be expressed in terms of $J_0(h)$, $J_{\frac{1}{3}}(h)$ and $J_{\frac{2}{3}}(h)$:

$$J_{-1}(h) = \left(\tfrac{5}{2^9C^3}h^3 + \tfrac{1}{16}\right)J_0(h) + \tfrac{5}{2^6C}hJ_{\frac{1}{3}}(h) + \tfrac{25}{2^9C^2}h^2J_{\frac{2}{3}}(h).$$

As a consequence, all $J_k(h)(k = -1, 0, 1, \cdots, n-1)$ can be expressed in terms of $J_0(h)$, $J_{\frac{1}{3}}(h)$ and $J_{\frac{2}{3}}(h)$. By substituting these formulae into (2.17), we obtain

$$A(h) = \alpha(h)J_0(h) + \beta(h)J_{\frac{1}{3}}(h) + \gamma(h)J_{\frac{2}{3}}(h),$$

where $\deg(\alpha(h)) \leq 3\left[\frac{n}{2}\right] - 3$, $\deg(\beta(h)) \leq 3\left[\frac{n}{2}\right] - 2$, $\deg(\gamma(h)) \leq 3\left[\frac{n-1}{2}\right] - 1$, when $n \geq 4$; and $\deg(\alpha(h)) \leq 3$, $\deg(\beta(h)) \leq 1$, $\deg(\gamma(h)) \leq 2$, when $n = 1, 2, 3$.

If $n = 0$, from equality (2.1), we obtain $A(h) = -\frac{4}{3}a_{0,0}J_{-1}(h) = \delta(h)J_{-1}(h)$, where $\delta(h) := -\frac{4}{3}a_{0,0}$, and $\deg(\delta(h)) = 0$. □

3. Picard-Fuchs Equation and Riccati Equation

Lemma 3.1. *1) For (r21), the Abelian integrals $J_i(h)$ $(i = 0, 1)$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J_0(h) \\ J_1(h) \end{pmatrix} = \begin{pmatrix} h+C & 0 \\ 2C & 2h \end{pmatrix} \begin{pmatrix} J_0'(h) \\ J_1'(h) \end{pmatrix}. \tag{3.1}$$

2) For (r12), the Abelian integrals $J_i(h)$ $\left(i = 0, \frac{1}{3}, \frac{2}{3}\right)$ satisfy the following Picard-Fuchs equation

$$\begin{pmatrix} J_0(h) \\ J_{\frac{1}{3}}(h) \\ J_{\frac{2}{3}}(h) \end{pmatrix} = \begin{pmatrix} h & 0 & -3C \\ -4C & \frac{4}{3}h & 0 \\ 0 & -\frac{12}{5}C & \frac{4}{5}h \end{pmatrix} \begin{pmatrix} J_0'(h) \\ J_{\frac{1}{3}}'(h) \\ J_{\frac{2}{3}}'(h) \end{pmatrix}. \tag{3.2}$$

Proof. 1) For $(r21)$, by (1.3), we obtain $y^2 = 2hx^2+4Cx-2C$, $\frac{\partial y}{\partial h} = \frac{x^2}{y}$, and $ydy = x^2\left(x^{-3}y^2 - 2Cx^{-2} + 2Cx^{-3}\right)dx$. Since $J_i(h) = \oint_{\Gamma_h} x^{i-3}ydx$, $J_i'(h) = \oint_{\Gamma_h} \frac{x^{i-1}}{y}dx$, we get

$$\begin{aligned}(i-2)J_i(h) = \oint_{\Gamma_h} ydx^{i-2} &= -\oint_{\Gamma_h} x^{i-2}x^2\frac{x^{-3}y^2-2Cx^{-2}+2Cx^{-3}}{y}dx \\ &= -J_i + 2CJ_{i-1}'(h) - 2CJ_{i-2}'(h).\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}J_i(h) = \oint_{\Gamma_h} \frac{x^{i-3}y^2}{y}dx &= \oint_{\Gamma_h} \frac{x^{i-3}\left(2hx^2+4Cx-2C\right)}{y}dx \\ &= 2hJ_i'(h) + 4CJ_{i-1}'(h) - 2CJ_{i-2}'(h).\end{aligned} \tag{3.4}$$

From (3.3), it follows that

$$(i-1)J_i(h) = 2CJ_{i-1}'(h) - 2CJ_{i-2}'(h). \tag{3.5}$$

From (3.4) and (3.5), we get

$$(i-2)J_i(h) = -2hJ_i'(h) - 2CJ_{i-1}'(h). \tag{3.6}$$

From (3.6), let $i = 0, 1$, we obtain

$$J_0(h) = hJ_0'(h) + CJ_{-1}'(h), \tag{3.7}$$

$$J_1(h) = 2hJ_1'(h) + 2CJ_0'(h). \tag{3.8}$$

By equality (2.6), we get

$$J_{-1}'(h) = J_0'(h). \tag{3.9}$$

By (3.7)–(3.9), we obtain equation (3.1).

2) For $(r12)$, by (1.4), we obtain $\frac{1}{2}x^{-\frac{1}{3}}y^2 + Cx^{\frac{2}{3}} + 2Cx^{-\frac{1}{3}} = h$, $\frac{\partial y}{\partial h} = \frac{x^{\frac{1}{3}}}{y}$, and $ydy = \left(\frac{1}{3}hx^{-\frac{2}{3}} - C\right)dx$. Since $J_i(h) = \oint_{\Gamma_h} x^{i-\frac{4}{3}}ydx$, $J_i'(h) = \oint_{\Gamma_h} \frac{x^{i-1}}{y}dx$, we get

$$\begin{aligned}J_i(h) = \oint_{\Gamma_h} \frac{x^{i-\frac{4}{3}}y^2}{y}dx &= \oint_{\Gamma_h} \frac{x^{i-\frac{4}{3}}\left(2hx^{\frac{1}{3}}-2Cx-4C\right)}{y}dx \\ &= 2hJ_i'(h) - 2CJ_{i+\frac{2}{3}}'(h) - 4CJ_{i-\frac{1}{3}}'(h),\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\left(i-\tfrac{1}{3}\right)J_i(h) &= \oint_{\Gamma_h}\left(i-\tfrac{1}{3}\right)x^{i-\frac{4}{3}}ydx = \oint_{\Gamma_h} ydx^{i-\frac{1}{3}} \\ &= -\oint_{\Gamma_h} \frac{x^{i-\frac{1}{3}}\left(\frac{1}{3}hx^{-\frac{2}{3}}-C\right)}{y}dx = -\tfrac{1}{3}hJ_i'(h) + CJ_{i+\frac{2}{3}}'(h).\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), it follows that

$$(3i-1)J_i(h) = -hJ_i'(h) + 3CJ_{i+\frac{2}{3}}'(h), \tag{3.12}$$

$$(6i+1)J_i(h) = 4hJ_i'(h) - 12CJ_{i-\frac{1}{3}}'(h). \tag{3.13}$$

From (3.12) and (3.13), let $i = 0, \frac{1}{3}, \frac{2}{3}$, we get

$$J_0(h) = hJ_0'(h) - 3CJ_{\frac{2}{3}}'(h), \tag{3.14}$$

$$3J_{\frac{1}{3}}(h) = 4hJ_{\frac{1}{3}}'(h) - 12CJ_0'(h), \tag{3.15}$$

$$5J_{\frac{2}{3}}(h) = 4hJ_{\frac{2}{3}}'(h) - 12CJ_{\frac{1}{3}}'(h). \tag{3.16}$$

By (3.14)–(3.16), we obtain equation (3.2). □

Lemma 3.2. *1) For (r21), the Abelian integrals $J_i(h)$ $(i=0,1)$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J_0'(h) \\ J_1'(h) \end{pmatrix} = \frac{1}{B(h)} \begin{pmatrix} 2h & 0 \\ -2C & h+C \end{pmatrix} \begin{pmatrix} J_0(h) \\ J_1(h) \end{pmatrix}, \tag{3.17}$$

where $B(h) = 2h(h+\frac{1}{2^5})$.

2) For (r12), the Abelian integrals $J_i'(h)$ $\left(i=0,\frac{1}{3},\frac{2}{3}\right)$ satisfy the following Picard-Fuchs equation

$$\begin{pmatrix} J_0''(h) \\ J_{\frac{1}{3}}''(h) \\ J_{\frac{2}{3}}''(h) \end{pmatrix} = \frac{1}{B(h)} \begin{pmatrix} -9C^2 & 3Ch \\ -h^2 & 9C^2 \\ -3Ch & h^2 \end{pmatrix} \begin{pmatrix} J_{\frac{1}{3}}'(h) \\ J_{\frac{2}{3}}'(h) \end{pmatrix}, \tag{3.18}$$

where $B(h) = 4\left(h-\frac{1}{2^7}\right)\left(h^2+\frac{1}{2^7}h+\frac{1}{2^{14}}\right)$.

Proof. 1) For $(r21)$, it can be calculated directly from equation (3.1).

2) For $(r12)$, by differentiating both sides of equation (3.2) with respect to h, we obtain

$$0\cdot J_0'(h) = -hJ_0''(h) + 3CJ_{\frac{2}{3}}''(h), \tag{3.19}$$

$$J_{\frac{1}{3}}'(h) = 12CJ_0''(h) - 4hJ_{\frac{1}{3}}''(h), \tag{3.20}$$

$$J_{\frac{2}{3}}'(h) = -12CJ_{\frac{1}{3}}''(h) + 4hJ_{\frac{2}{3}}''(h). \tag{3.21}$$

By (3.19)−(3.21), we obtain equation (3.18). □

Lemma 3.3. *1) For (r21), $J_i\left(-\frac{1}{2^5}\right)=0$ $(i=0,1)$; $J_i(h)<0$ $(i=0,1)$, when $h\in\left(-\frac{1}{2^5},0\right)$.*

2) For (r12), $J_i\left(\frac{1}{2^7}\right)=0$ $\left(i=0,\frac{1}{3},\frac{2}{3}\right)$; $J_{-1}(h)<0$, $J_i'(h)>0$ $\left(i=0,\frac{1}{3},\frac{2}{3}\right)$, when $h\in\left(\frac{1}{2^7},+\infty\right)$.

For $(r21)$, since $J_i(h)=\oint_{\Gamma_h} x^{i-3}ydx$, for $(r12)$, since $J_i(h)=\oint_{\Gamma_h} x^{i-\frac{4}{3}}ydx$ and $J_i'(h)=\oint_{\Gamma_h}\frac{x^{i-1}}{y}dx$. The proof only requires some simple calculation, so it is omitted.

Corollary 3.1. *1) For (r21), if $D(h):=\frac{J_0(h)}{J_1(h)}$, $D(h)$ satisfies the following Riccati equation*

$$B(h)D'(h) = 2CD^2(h) + (h-C)D(h), \tag{3.22}$$

where $B(h) = 2h(h+\frac{1}{2^5})$.

2) For (r12), if $D(h):=\frac{J_{\frac{1}{3}}'(h)}{J_{\frac{2}{3}}'(h)}$, $D(h)$ satisfies the following Riccati equation

$$B(h)D'(h) = 3ChD^2(h) - 2h^2D(h) + 9C^2, \tag{3.23}$$

where $B(h) = 4\left(h-\frac{1}{2^7}\right)\left(h^2+\frac{1}{2^7}h+\frac{1}{2^{14}}\right)$.

Proof. For $(r21)$ and $(r12)$, using equations (3.17) and (3.18), and by differentiating both sides of $D(h)$ with respect to h, respectively, we obtain equations (3.22) and (3.23). □

4. Upper Bound for the Associated Number

For $(r12)$, if $n \geq 1$, from equality (2.3), using equation (3.2), we obtain

$$A(h) = \alpha_1(h)J_0'(h) + \beta_1(h)J_{\frac{1}{3}}'(h) + \gamma_1(h)J_{\frac{2}{3}}'(h), \tag{4.1}$$

where $\alpha_1(h) = h\alpha(h) - 4C\beta(h)$, $\beta_1(h) = \frac{4}{3}h\beta(h) - \frac{12}{5}C\gamma(h)$, $\gamma_1(h) = \frac{4}{5}h\gamma(h) - 3C\alpha(h)$, thus, when $n \geq 4$, $\deg(\alpha_1(h)) \leq 3\left[\frac{n}{2}\right] - 2$, $\deg(\beta_1(h)) \leq 3\left[\frac{n}{2}\right] - 1$, $\deg(\gamma_1(h)) \leq 3\left[\frac{n-1}{2}\right]$; when $n = 1, 2, 3$, $\deg(\alpha_1(h)) \leq 4$, $\deg(\beta_1(h)) \leq 2$, $\deg(\gamma_1(h)) \leq 3$.

By differentiating both sides of equality (4.1) with respect to h and using equation (3.18), we obtain

$$B(h)A'(h) = B(h)\alpha_1'(h)J_0'(h) + \beta_2(h)J_{\frac{1}{3}}'(h) + \gamma_2(h)J_{\frac{2}{3}}'(h), \tag{4.2}$$

where $\beta_2(h) = B(h)\beta_1'(h) - h^2\beta_1(h) - 9C^2\alpha_1(h) - 3Ch\gamma_1(h)$, $\gamma_2(h) = B(h)\gamma_1'(h) + h^2\gamma_1(h) + 3Ch\alpha_1(h) + 9C^2\beta_1(h)$, thus, when $n \geq 4$, $\deg(\beta_2(h)) \leq 3\left[\frac{n}{2}\right] + 1$, $\deg(\gamma_2(h)) \leq 3\left[\frac{n-1}{2}\right] + 2$; when $n = 1, 2, 3$, $\deg(\beta_2(h)) \leq 4$, $\deg(\gamma_2(h)) \leq 5$.

By equalities (4.1) and (4.2), we get

$$B(h)\alpha_1(h)A'(h) = B(h)\alpha_1'(h)A(h) + I(h), \tag{4.3}$$

$$I(h) = E(h)J_{\frac{1}{3}}'(h) + F(h)J_{\frac{2}{3}}'(h), \tag{4.4}$$

where $E(h) = \alpha_1(h)\beta_2(h) - B(h)\alpha_1'(h)\beta_1(h)$, $F(h) = \alpha_1(h)\gamma_2(h) - B(h)\alpha_1'(h)\gamma_1(h)$, thus, when $n \geq 4$, $\deg(E(h)) \leq 6\left[\frac{n}{2}\right] - 1$, $\deg(F(h)) \leq 3\left[\frac{n}{2}\right] + 3\left[\frac{n-1}{2}\right]$; when $n = 1, 2, 3$, $\deg(E(h)) \leq 8$, $\deg(F(h)) \leq 9$.

Using Proposition 2.1, Corollary 3.1 and equality (4.4), after some simple calculations, we obtain the following lemma.

Lemma 4.1. *1) For $(r21)$, if $n \geq 2$, $W(h) := \frac{K(h)}{J_1(h)}$, $W(h)$ satisfies the following Riccati equation*

$$B(h)\alpha(h)W'(h) = 2CW^2(h) + M(h)W(h) + G(h),$$

where $M(h) = B(h)\alpha'(h) + (h - C)\alpha(h) - 4C\beta(h)$, $G(h) = B(h)\alpha(h)\beta'(h) - B(h)\alpha'(h)\beta(h) - (h-C)\alpha(h)\beta(h) + 2C\beta^2(h)$, thus, $\deg(M(h)) \leq n-1$, $\deg(G(h)) \leq 2n - 2$, for $n \geq 2$.

2) For $(r12)$, if $n \geq 1$, $W(h) := \frac{I(h)}{J_{\frac{2}{3}}'(h)}$, $W(h)$ satisfies the following Riccati equation

$$B(h)E(h)W'(h) = 3ChW^2(h) + M(h)W(h) + G(h),$$

where $M(h) = B(h)E'(h) - 2h^2E(h) - 6ChF(h)$, $G(h) = 9C^2E^2(h) + 3ChF^2(h) + B(h)E(h)F'(h) - B(h)E'(h)F(h) + 2h^2E(h)F(h)$, thus, when $n \geq 4$, $\deg(M(h)) \leq 6\left[\frac{n}{2}\right] + 1$, $\deg(G(h)) \leq 9\left[\frac{n}{2}\right] + 3\left[\frac{n-1}{2}\right] + 1$; when $n = 1, 2, 3$, $\deg(M(h)) \leq 10$, $\deg(G(h)) \leq 19$.

In the following, we use $\natural A(h)$ to denote the number of zeros of $A(h)$ in Ω, and we need the following lemma.

Lemma 4.2 ([8]). *The smooth functions* $V(h), \phi(h), \psi(h), \xi(h)$, *and* $\eta(h)$ *satisfy the following Riccati equation*

$$\eta(h)V'(h) = \phi(h)V^2(h) + \psi(h)V(h) + \xi(h),$$

then

$$\natural V(h) \le \natural\eta(h) + \natural\xi(h) + 1.$$

Lemma 4.2 is the Lemma 5.3 in [8], and the proof can be found in [8], so it is omitted.

Finally, we finish the proof of Theorem 1.1.

Proof. 1) For $(r21)$, using Proposition 2.1, Lemmas 4.1 and 4.2, we get

$$\natural A(h) = \natural K(h) = \natural W(h) \le \natural B(h) + \natural\alpha(h) + \natural G(h) + 1.$$

When $n \ge 2$, since $\deg(\alpha(h)) \le n-2$ and $\deg(G(h)) \le 2n-2$, noticing that $B(h) = 2h(h+\frac{1}{2^5})$ and there is no zero in $\left(-\frac{1}{2^5}, 0\right)$, we obtain

$$\natural A(h) = \natural K(h) = \natural W(h) \le (n-2) + (2n-2) + 1 = 3n-3.$$

When $n = 0, 1$, since $A(h) = \alpha(h)J_0(h)$, where $\deg(\alpha(h)) = 0$, $J_0(h) \ne 0$, we get $\natural A(h) = 0$.

2) For $(r12)$, using Lemma 4.2, from (4.3), we get

$$\natural A(h) \le \natural B(h) + \natural\alpha_1(h) + \natural I(h) + 1. \tag{4.5}$$

Using Lemmas 4.1, we get

$$\natural I(h) = \natural W(h) \le \natural B(h) + \natural E(h) + \natural G(h) + 1. \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$\natural A(h) \le 2\natural B(h) + \natural\alpha_1(h) + \natural E(h) + \natural G(h) + 2.$$

When $n \ge 4$, since $\deg(\alpha_1(h)) \le 3\left[\frac{n}{2}\right] - 2$, $\deg(E(h)) \le 6\left[\frac{n}{2}\right] - 1$, and $\deg(G(h)) \le 9\left[\frac{n}{2}\right] + 3\left[\frac{n-1}{2}\right] + 1$, noticing that $B(h) = 4\left(h - \frac{1}{2^7}\right)\left(h^2 + \frac{1}{2^7}h + \frac{1}{2^{14}}\right)$ and there is no zero in $\left(-\frac{1}{3\cdot 2^6}, 0\right)$, we obtain

$$\natural A(h) \le \left(3\left[\tfrac{n}{2}\right] - 2\right) + \left(6\left[\tfrac{n}{2}\right] - 1\right) + \left(9\left[\tfrac{n}{2}\right] + 3\left[\tfrac{n-1}{2}\right] + 1\right) + 2 = 18\left[\tfrac{n}{2}\right] + 3\left[\tfrac{n-1}{2}\right].$$

When $n = 1, 2, 3$, some similar discussions show that $\natural A(h) \le 4+8+19+2 = 33$.

When $n = 0$, since $A(h) = \delta(h)J_{-1}(h)$, where $\deg(\delta(h)) = 0$, $J_{-1}(h) \ne 0$, we get $\natural A(h) = 0$. □

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