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CODIMENSION-TWO BIFURCATION ANALYSIS OF THE CONTINUOUS STIRRED TANK REACTOR MODEL WITH DELAY*

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Abstract The aim of this paper is to research the dynamical behaviors of the continuous stirred tank reactor (CSTR) model with delay. Firstly, we discuss the situation that its related characteristic equation has a simple zero root and a pair of purely imaginary roots. Secondly, the center manifold method and the normal form method are used to reduce the equation of CSTR model. Finally, some characteristics about the CSTR model can be obtained. We analyze three different topological structure and give entire bifurcation diagrams and phase portraits, which are innovative phenomenon. At the end, we obtain the stable and unstable periodic solutions by numerical simulation.

Keywords Delay, CSTR model, normal form, Zero-Hopf bifurcation.

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1. Introduction

There has been great concern in high codimensional bifurcation analysis for some differential equations with delay, including the Zero-Hopf bifurcations [4, 16, 21, 23, 26], Bogdanov-Takens bifurcations [11, 12, 25, 28], and bifurcation analysis has been widely applied in chemical engineering field [9, 15, 18, 20, 29]. There is a class of saddle-node-Hopf bifurcation also being studied (see [10, 27]).

Within the framework of Faria and Magalhaes [5, 6], many scholars sum up approaches which are detailed and accessible, see e.g. [11, 23]. And He Xing, et al. study the Zero-Hopf bifurcation about the bidirectional ring network model with delay [8]. In the case of CSTR model, considering about the exponential term, we obtain the linear part by Taylor expansion, and in the next calculation, in order to prevent the lack of items in normal forms, two perturbation parameters are not equal to zero.

CSTR as a kind of reactor tank of chemical experiments, it has many characteristics, for example, low cost, strong heat exchange capacity and great product quality. So it becomes a main equipment of producing polymer. It has been used widely in the production process of chemical industry, oil production and other industrial production process.

In [17], a high-performance small continuous stirred-tank reactor with non-

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contact hypnotic coupling providing intensive disturbance inside the sealed miniaturized chamber is presented. D.Kastsian and M.Mönnigmann [13] exhibit that the normal vector method for robust optimization of nonlinear systems can be continued to delayed systems. S.Pushpavanam and A.Kienle [19] discussed the balance of the constant states of a reactor-separator system, where a first-order exothermic irreversible reaction is performed in a continuous stirred tank reactor, and as for the reaction in a nonisothermal CSTR, small delays could have a stabilizing effect (see [1, 7]). Disjoint bifurcations and isola behavior were found in [15], which make the reactor to be easier to control problems. In [2], the traditional Van der Pol Oscillator with a forcing dependent on a delay is considered, and researchers study the Zero-Hopf bifurcation and provide a physical understanding of the oscillator which is very useful.

In order to treat the reactor tank as a whole when constructing a consequent CSTR model, this section makes the following assumptions:

i Materials in the reactor tank are fully mixed.

ii The volume of materials which flow into reactor tank is equal to the volume of materials which flow out reactor tank.

iii The chemical reaction is a first order irreversible chemical reaction in the reaction process.

According to the principle of material and energy balance [3], the original equation is following:

$$\begin{cases} \dot{x_1}(t) = f_1(x) + (\frac{1}{\lambda_0} - 1)x_1(t-\tau), \\ \dot{x_2}(t) = f_2(x) + (\frac{1}{\lambda_0} - 1)x_2(t-\tau) + \beta x_1(t-\tau), \end{cases} \tag{1.1}$$

where $x(t) = (x_1(t), x_2(t))^T, t \in [-\tau, 0]$, and

$$\begin{cases} f_1(x) = -\frac{1}{\lambda_0}x_1(t) + D_\alpha(1 - x_1(t))e^{\frac{x_2}{1+x_2/\gamma_0}}, \\ f_2(x) = -(\frac{1}{\lambda_0} + \beta)x_2(t) + HD_\alpha(1 - x_1(t))e^{\frac{x_2}{1+x_2/\gamma_0}}. \end{cases} \tag{1.2}$$

The variable $x_1(t)$ is the transformation rate of the reaction, and $0 \leq x_1(t) \leq 1$, and $x_2(t)$ represents temperature. H, D_α, γ_0 and τ are all positive constants. In order to simplify the system, we make the control term vanish, so $\beta x_1(t-\tau) = 0$, namely, $\beta = 0$.

The objective of the paper is to study the Zero-Hopf bifurcation by regrading H and τ as bifurcation parameters. In section 2, the existence conditions of Zero-Hopf bifurcation are given in Lemma 2.1. In section 3, center manifold theory and normal form method [5, 6] are used to research Zero-Hopf bifurcation, and we get the normal form for Zero-Hopf bifurcation with parameters. A primary difficulty for figuring out the third order terms with parameters is that we have to deal with the linear system whose coefficient matrices are singular. This problem is solved by adopting the approach in [24]. Then in section 4, we select some parameter values, and obtain some numerical simulations to support our theoretical results.

2. The existence of codimension-two bifurcation

In this section, we give the necessary and sufficient conditions for existence of Zero-Hopf bifurcation, which can guarantee the characteristic equation has a simple root

0 and a simple pair of purely imaginary roots $\pm i\omega_0$ and all other roots of the characteristic equation have negative real parts, then the Zero-Hopf bifurcation will occur.

Assuming (x_1^*, x_2^*) is the equilibrium point of equation (1.1), let $\widetilde{x_1} = x_1 - x_1^*, \widetilde{x_2} = x_2 - x_2^*, -\frac{1}{\lambda_0} = a, g(x_2) = e^{\frac{x_2+x_2^*}{1+\frac{x_2+x_2^*}{\gamma_0}}}$, then system (1.1) becomes

$$\begin{cases} \dot{\widetilde{x_1}} = a\widetilde{x_1} + D_\alpha(1-\widetilde{x_1}-x_1^*)g(\widetilde{x_2}) + (-a-1)\widetilde{x_1}(t-\tau) - D_\alpha(1-x_1^*)g(0), \\ \dot{\widetilde{x_2}} = a\widetilde{x_2} + HD_\alpha(1-\widetilde{x_1}-x_1^*)g(\widetilde{x_2}) + (-a-1)\widetilde{x_2}(t-\tau) - HD_\alpha(1-x_1^*)g(0). \end{cases}$$

Omitting the " $\sim$ ", and use the Taylor expansion at the origin and the system above becomes

$$\begin{cases} \dot{x_1} = ax_1 + D_\alpha(1-x_1-x_1^*)(\alpha_1+\alpha_2x_2+\frac{\alpha_3}{2}x_2^2+\frac{\alpha_4}{6}x_2^3) \\ \qquad + (-a-1)x_1(t-\tau) - D_\alpha(1-x_1^*)\alpha_1, \\ \dot{x_2} = ax_2 + HD_\alpha(1-x_1-x_1^*)(\alpha_1+\alpha_2x_2+\frac{\alpha_3}{2}x_2^2+\frac{\alpha_4}{6}x_2^3) \\ \qquad + (-a-1)x_2(t-\tau) - HD_\alpha(1-x_1^*)\alpha_1. \end{cases} \tag{2.1}$$

We assume $g(0) = \alpha_1, g'(0) = \alpha_2, g''(0) = \alpha_3, g'''(0) = \alpha_4$. The linearization of the system above is

$$\begin{cases} \dot{x_1} = ax_1 + D_\alpha(1-x_1^*)\alpha_2x_2 - D_\alpha x_1\alpha_1 - (a+1)x_1(t-\tau), \\ \dot{x_2} = ax_2 + HD_\alpha(1-x_1^*)\alpha_2x_2 - HD_\alpha x_1\alpha_1 - (a+1)x_2(t-\tau). \end{cases} \tag{2.2}$$

The characteristic equation of system (2.2) is

$$\Delta(\lambda,\tau) = (\lambda-a+(a+1)e^{-\lambda\tau})(\lambda-a+(a+1)e^{-\lambda\tau}+D_\alpha\alpha_1-HD_\alpha\alpha_2(1-x_1^*)). \tag{2.3}$$

If $\lambda = 0$ is one root of the equation (2.3), we obtain $H = \frac{D_\alpha\alpha_1+1}{D_\alpha\alpha_2(1-x_1^*)}$. Let $H = \beta_1$. We obtain that if $\tau = 0$, except a single zero eigenvalue, the other root of equation (2.3) has negative real part, so the stability of system is uncertain in this case. When $\tau \neq 0$, let $i\omega(\omega > 0)$ into $\lambda - a + (a+1)e^{-\lambda\tau}$, and separate the real and imaginary parts, we have

$$\begin{cases} \frac{a}{a+1} = \cos(\omega\tau), \\ \frac{w}{a+1} = \sin(\omega\tau). \end{cases} \tag{2.4}$$

Eliminating τ from (2.4), one has

$$\omega^2 = 2a+1.$$

In order to assure the existence of a simple pair of purely imaginary roots, let $a > -\frac{1}{2}$, because $a = -\frac{1}{\lambda_0} < 0$, so $-\frac{1}{2} < a < 0$. We obtain

$$\omega_0 = \sqrt{2a+1},$$

$$\tau_k = \frac{1}{\omega_0}\arccos\frac{\omega_0^2-1}{\omega_0^2+1} + 2k\pi, k = 0,1,2,\ldots.$$

And we have

$$\frac{d\tau}{d\lambda} = \frac{\tau(a+1)e^{-\lambda\tau} - 1}{(a+1)e^{-\lambda\tau}(-\lambda)} = \frac{\tau(a+1) - e^{\lambda\tau}}{(a+1)(-\lambda)} = -\frac{\tau}{\lambda} + \frac{e^{\lambda\tau}}{(a+1)\lambda},$$

then

$$Re[\frac{d\tau}{d\lambda}] = \frac{\sin(\omega\tau)}{(a+1)\omega} = \frac{1}{(a+1)^2} > 0,$$

so we have the following lemma.

Lemma 2.1. *If* $H = \frac{D_\alpha\alpha_1+1}{D_\alpha\alpha_2(1-x_1^*)}$ *and* $-\frac{1}{2} < a < 0$ *hold, when* $\tau = \tau_k (k = 0, 1, 2, ...)$, *the system* (1.1) *undergoes a Zero-Hopf bifurcation at equilibrium* (x_1^*, x_2^*).

3. Normal form with parameters for Zero-Hopf bifurcation

In this section, normal form is obtained by performing a center manifold reduction and by applying the normal form method. First, Let $H = \mu_1 + \beta_1, \tau = \mu_2 + \tau_0$, then μ_1, μ_2 are bifurcation parameters. After scaling the time by $t \to t/\tau$, the system (2.1) can be written as

$$\begin{cases} \dot{x_1} =(\mu_2+\tau_0)ax_1 + (\mu_2+\tau_0)D_\alpha(1-x_1-x_1^*)(\alpha_1+\alpha_2x_2+\frac{\alpha_3}{2}x_2^2+\frac{\alpha_4}{6}x_2^3) \\ \qquad -(\mu_2+\tau_0)D_\alpha(1-x_1^*)\alpha_1 - (\mu_2+\tau_0)(a+1)x_1(t-1), \\ \dot{x_2} =(\mu_2+\tau_0)ax_2+(\mu_1+\beta_1)(\mu_2+\tau_0)D_\alpha(1-x_1-x_1^*)(\alpha_1+\alpha_2x_2+\frac{\alpha_3}{2}x_2^2+\frac{\alpha_4}{6}x_2^3) \\ \qquad -(\mu_2+\tau_0)(\mu_1+\beta_1)D_\alpha(1-x_1^*)\alpha_1 - (\mu_2+\tau_0)(a+1)x_2(t-1). \end{cases} \tag{3.1}$$

The linearization of system (3.1) at $(0, 0)$ is

$$\begin{cases} \dot{x_1} = \tau_0ax_1 + \tau_0D_\alpha(1-x_1^*)\alpha_2x_2 - \tau_0D_\alpha x_1\alpha_1 - \tau_0(a+1)x_1(t-1), \\ \dot{x_2} = \tau_0ax_2 + \tau_0\beta_1D_\alpha(1-x_1^*)\alpha_2x_2 - \tau_0\beta_1D_\alpha x_1\alpha_1 - \tau_0(a+1)x_2(t-1). \end{cases}$$

Let

$$\eta(\theta) = \mathbf{A}\delta(\theta) + \mathbf{B}\delta(\theta+1),$$

where

$$A = \begin{pmatrix} \tau_0(a - D_\alpha\alpha_1) & \tau_0D_\alpha(1-x_1^*)\alpha_2 \\ -\tau_0\beta_1D_\alpha\alpha_1 & \tau_0(a+\beta_1D_\alpha(1-x_1^*)\alpha_2) \end{pmatrix}, B = \begin{pmatrix} -\tau_0(a+1) & 0 \\ 0 & -\tau_0(a+1) \end{pmatrix}.$$

Define

$$L\psi = \int_{-1}^{0} d\eta(\theta)\psi(\theta), \forall\psi \in C = C([-1,0], C^2).$$

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and let $F(X_t, \mu) = \begin{pmatrix} F^1 \\ F^2 \end{pmatrix}$. Choosing the phase space $C = C([-1, 0], C^2)$, then system (1.1) can be transformed into

$$\dot{X}(t) = L(\mu)X_t + F(X_t, \mu) \tag{3.2}$$

and the bilinear form on $C^* \times C$ is

$$\langle \psi, \varphi \rangle = \psi(0)\varphi(0) + \int_{-1}^{0} \psi(\xi+1)\mathbf{B}\varphi(\xi)d\xi,$$

where $\varphi(\theta) = (\varphi_1(\theta), \bar{\varphi}_1(\theta), \varphi_2(\theta)) \in C,\ \psi(s) = \begin{pmatrix} \psi_1(s) \\ \bar{\psi}_1(s) \\ \psi_2(s) \end{pmatrix} \in C^*$. Then the space C can be decomposed by $\Lambda = \{0, \pm i\omega_0\tau_0\}$ as $C = P + Q$, where $Q = \{\varphi \in C : (\psi, \varphi) = 0, \forall \psi \in P^*\}$. Choosing the bases for P and the adjoint P^* are

$$\Phi(\theta) = \begin{pmatrix} \frac{\alpha_2}{\alpha_1}(1-x_1^*)e^{i\omega_0\tau_0\theta} & \frac{\alpha_2}{\alpha_1}(1-x_1^*)e^{-i\omega_0\tau_0\theta} & \frac{1}{\beta_1} \\ e^{i\omega_0\tau_0\theta} & e^{-i\omega_0\tau_0\theta} & 1 \end{pmatrix}$$

and

$$\Psi(s) = \begin{pmatrix} -D_1\beta_1 e^{-i\omega_0\tau_0 s} & D_1 e^{-i\omega_0\tau_0 s} \\ -\bar{D}_1\beta_1 e^{i\omega_0\tau_0 s} & -\bar{D}_1 e^{i\omega_0\tau_0 s} \\ -\frac{D_2\alpha_1}{\alpha_2(1-x_1^*)} & D_2 \end{pmatrix},$$

where

$$(\Psi(s), \Phi(\theta)) = 1, 0 < s < 1.\ D_1 = \frac{D_\alpha \alpha_1}{\tau_0(a - i\omega_0) - 1},\ D_2 = \frac{D_\alpha\alpha_1 + 1}{1 - \tau_0(a+1)}.$$

Thus, the dual bases satisfy $\dot{\Phi} = \Phi J, -\dot{\Psi} = J\Psi$ with

$$J = \begin{pmatrix} i\omega_0\tau_0 & 0 & 0 \\ 0 & -i\omega_0\tau_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To consider system (3.2), we need the enlarged phase space BC of function from $[-1, 0]$ to C^2:

$$BC = \{\alpha : [-1, 0] \to C^2 : \alpha \text{ is continuous on } [-1, 0), \exists \lim_{\theta \to 0^-} \alpha(\theta) \in C^2\}$$

The items of BC can be represented by $\psi = \varphi + T_0\alpha$ with $\varphi \in C, \alpha \in C^2$, and

$$T_0(\theta) = \begin{cases} 0, & -\tau \le \theta < 0, \\ I, & \theta = 0. \end{cases}$$

Define the continuous projection $\pi : BC \to P$ by $\pi(\varphi + T_0\alpha) = \Phi[(\Psi, \varphi) + \Psi(0)\alpha]$. Then we can decompose the enlarge phase space as $BC = P \oplus ker\pi$.

Define $\mathcal{A} : C^1 \to BC$ is

$$\mathcal{A}\varphi = \dot{\varphi} + T_0[L\varphi - \dot{\varphi}(0)] = \begin{cases} \dot{\varphi}, & if\ \ -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(t)\varphi(t), & if\ \theta = 0. \end{cases}$$

In BC, equation (3.2) can be written as an abstract ODE:

$$\frac{d}{dt}u = \mathcal{A}u + T_0 F(u,\mu). \tag{3.3}$$

Let $X_t = \Phi z(t) + y(\theta), where\ z(t) = (z_1, z_2, z_3)^T$, namely

$$\begin{cases} x_1(\theta) = \dfrac{\alpha_2}{\alpha_1}(1-x_1^*)e^{i\omega_0\tau_0\theta}z_1 + \dfrac{\alpha_2}{\alpha_1}(1-x_1^*)e^{-i\omega_0\tau_0\theta}z_2 + \dfrac{1}{\beta_1}z_3 + y_1(\theta), \\ x_2(\theta) = e^{i\omega_0\tau_0\theta}z_1 + e^{-i\omega_0\tau_0\theta}z_2 + z_3 + y_2(\theta). \end{cases}$$

Equation (3.3) can be decomposed into the equation

$$\begin{cases} \dot{z} = Jz + \Psi(0)F(\Phi z + y(\theta),\mu), \\ \dot{y} = \mathcal{A}_{Q_1}y + (I-\pi)T_0F(\Phi z + y(\theta),\mu), \end{cases} \tag{3.4}$$

where $y(\theta) \in Q^1 := Q\bigcap C^1 \subset ker\pi$, $\mathcal{A}_{Q_1}$ is an operator from Q_1 to the Banach space $ker\pi$. And equation (3.4) can be written as

$$\begin{cases} \dot{z} = Jz + \dfrac{1}{2!}f_2^1(z,y,\mu) + \dfrac{1}{3!}f_3^1(z,y,\mu) + h.o.t., \\ \dot{y} = \mathcal{A}_{Q_1}y + \dfrac{1}{2!}f_2^2(z,y,\mu) + \dfrac{1}{3!}f_3^2(z,y,\mu) + h.o.t.. \end{cases} \tag{3.5}$$

On the center manifold, (3.5) can be written as the following

$$\dot{z} = Jz + \frac{1}{2}g_2^1(z,y,\mu) + \frac{1}{6}g_3^1(z,y,\mu) + h.o.t..$$

We have

$$\frac{1}{2}f_2^1(z,y,\mu) = \begin{pmatrix} \psi_{11}F_2^1(z,y,\mu) + \psi_{12}F_2^2(z,y,\mu) \\ \psi_{21}F_2^1(z,y,\mu) + \psi_{22}F_2^2(z,y,\mu) \\ \psi_{31}F_2^1(z,y,\mu) + \psi_{32}F_2^2(z,y,\mu) \end{pmatrix} =$$

$$\begin{pmatrix} \frac{-i\omega_0 D_1}{D_\alpha\alpha_1}\mu_2 z_1 + \frac{i\omega_0 D_1}{D_\alpha\alpha_1}\mu_2 z_2 + \frac{D_1\tau_0}{\beta_1}\mu_1 z_3 + D_1\beta_1((a+1)\mu_2 y_1(-1) - a\mu_2 y_1(0)) \\ \quad + D_1(a\mu_2 y_2(0) - (a+1)\mu_2 y_2(-1)) + D_1\tau_0(\frac{D_\alpha\alpha_1+1}{\beta_1}\mu_1 y_2(0) - D_\alpha\alpha_1\mu_1 y_1(0)) \\ \frac{-i\omega_0 \bar{D}_1}{D_\alpha\alpha_1}\mu_2 z_1 + \frac{i\omega_0 \bar{D}_1}{D_\alpha\alpha_1}\mu_2 z_2 + \frac{\bar{D}_1\tau_0}{\beta_1}\mu_1 z_3 + \bar{D}_1\beta_1((a+1)\mu_2 y_1(-1) - a\mu_2 y_1(0)) \\ \quad + \bar{D}_1(a\mu_2 y_2(0) - (a+1)\mu_2 y_2(-1)) + \bar{D}_1\tau_0(\frac{D_\alpha\alpha_1+1}{\beta_1}\mu_1 y_2(0) - D_\alpha\alpha_1\mu_1 y_1(0)) \\ \frac{D_2\tau_0}{\beta_1}\mu_1 z_3 + D_2\tau_0[(\frac{\alpha_3}{2\alpha_2} - \frac{\alpha_2}{\alpha_1})(z_1^2+z_2^2) + (\frac{\alpha_3}{2\alpha_2} - \frac{D_\alpha\alpha_2}{D_\alpha\alpha_1+1})z_3^2] + D_2\tau_0[(\frac{\alpha_3}{\alpha_2} - \frac{2\alpha_2}{\alpha_1})z_1 z_2 \\ \quad + (\frac{\alpha_3}{\alpha_2} - \frac{\alpha_2}{\alpha_1} - \frac{D_\alpha\alpha_2}{D_\alpha\alpha_1+1})(z_1 z_3 + z_2 z_3)] + [(D_\alpha\alpha_1 - a)\frac{\mu_2 D_2\alpha_1}{\alpha_2(1-x_1^*)} \\ \quad - D_2 D_\alpha\alpha_1(\mu_1\tau_0 + \beta_1\mu_2)]y_1(0) \\ \quad + \tau_0 D_\alpha\alpha_2 D_2(\frac{\alpha_1}{\alpha_2(1-x_1^*)} - \beta_1)(z_1+z_2+z_3)y_1(0) + \frac{D_2\mu_2\alpha_1(a+1)}{\alpha_2(1-x_1^*)}y_1(-1) \\ \quad + \frac{D_2\tau_0\alpha_3}{2\alpha_2}y_2^2(0) + (\frac{D_2\tau_0\alpha_3}{\alpha_2} - \frac{D_2\tau_0\alpha_2}{\alpha_1})(z_1+z_2)y_2(0) + (\frac{D_2\tau_0\alpha_3}{\alpha_2} - \frac{D_2\tau_0\alpha_2 D_\alpha}{D_\alpha\alpha_1+1})z_3 y_2(0) \\ \quad + D_2[\mu_2(a+1) + \mu_1\tau_0\frac{D_\alpha\alpha_1+1}{\beta_1}]y_2(0) - D_2\mu_2(a+1)y_2(-1) - \frac{D_2\tau_0}{1-x_1^*}y_1(0)y_2(0) \end{pmatrix}$$

and

$$\frac{1}{2}g_2^1(z,0,\mu) = \frac{1}{2}Proj_{s_1}f_2^1(z,0,\mu) + \mathcal{O}(|\mu|^2)$$
$$= \begin{pmatrix} (a_{11}\mu_1 + a_{12}\mu_2)z_1 + a_{13}z_1z_3 \\ (\bar{a}_{11}\mu_1 + \bar{a}_{12}\mu_2)z_2 + \bar{a}_{13}z_2z_3 \\ (a_{21}\mu_1 + a_{22}\mu_2)z_3 + a_{23}z_1z_2 + a_{24}z_3^2 \end{pmatrix} + \mathcal{O}(|\mu|^2)$$

where s_1 is spanned by

$$\{\mu_i z_1 e_1, z_1z_3e_1, \mu_i z_2 e_2, z_2z_3e_2, z_1z_2e_3, \mu_i z_3 e_3, z_3^2 e_3\},\ i = 1, 2,$$
$$a_{11} = a_{13} = a_{22} = 0$$
$$a_{12} = -\frac{i\omega_0 D_1}{D_\alpha \alpha_1} = \frac{i\omega_0}{1 - \tau_0(a - i\omega_0)},$$
$$a_{21} = \frac{\tau_0 D_2}{\beta_1},$$
$$a_{23} = D_2\tau_0(\frac{\alpha_3}{\alpha_2} - \frac{2\alpha_2}{\alpha_1}),$$
$$a_{24} = D_2\tau_0(\frac{\alpha_3}{2\alpha_2} - \frac{D_\alpha\alpha_2}{D_\alpha\alpha_1 + 1}).$$

Next we calculate $g_3^1(z,0,\mu)$. Note that, from paper [23], we have

$$\tfrac{1}{6}g_3^1(z,0,\mu) = \tfrac{1}{6}Proj_{ker(M_2^1)}\tilde{f}_3^1(z,0,\mu)$$
$$= \tfrac{1}{6}Proj_{s_2}\tilde{f}_3^1(z,0,\mu) + \mathcal{O}(|z||\mu|^2 + |z|^2|\mu|)$$
$$= \tfrac{1}{6}Proj_{s_2}f_3^1(z,0,\mu) + \tfrac{1}{4}Proj_{s_2}[(D_z f_2^1)(z,0,\mu)U_2^1(z,\mu)$$
$$+ (D_y f_2^1)(z,0,\mu)U_2^2(z,\mu)] + \mathcal{O}(|z||\mu|^2 + |z|^2|\mu|).$$

First let us calculate $Proj_{s_2}f_3^1(z,0,\mu)$. Since

$$\frac{1}{6}f_3^1(z,0,\mu) = \frac{1}{6}\begin{pmatrix} \psi_{11}F_3^1(z,0,\mu) + \psi_{12}F_3^2(z,0,\mu) \\ \psi_{21}F_3^1(z,0,\mu) + \psi_{22}F_3^2(z,0,\mu) \\ \psi_{31}F_3^1(z,0,\mu) + \psi_{32}F_3^2(z,0,\mu) \end{pmatrix}.$$

Then we have

$$\frac{1}{6}Proj_{s_2}f_3^1(z,0,\mu) = \begin{pmatrix} b_{11}\mu_1z_1z_3 \\ \bar{b}_{11}\mu_1z_2z_3 \\ b_{21}\mu_1z_1z_2 + b_{22}\mu_2z_1z_2 + b_{23}\mu_1z_3^2 + b_{24}\mu_2z_3^2 \\ +b_{25}\mu_1\mu_2z_3 + b_{26}z_1z_2z_3 + b_{27}z_3^3 \end{pmatrix}$$

where s_2 is spanned by

$$\{\mu_1\mu_2z_1e_1, \mu_i^2z_1e_1, \mu_iz_1z_3e_1, z_1^2z_2e_1, z_1z_3^2e_1, \mu_1\mu_2z_2e_2, \mu_i^2z_2e_2, \mu_iz_2z_3e_2, z_1z_2^2e_2,$$
$$z_2z_3^2e_2, \mu_iz_1z_2e_3, \mu_iz_3^2e_3, \mu_1\mu_2z_3e_3, z_1z_2z_3e_3, z_3^3e_3, \mu_i^2z_3e_3\},\ i = 1, 2,$$

$$
\begin{aligned}
b_{11} &= D_1\tau_0 D_\alpha[(1-x_1^*)\alpha_3 - (1-x_1^*)\tfrac{\alpha_2^2}{\alpha_1} - \tfrac{\alpha_2}{\beta_1}],\\
b_{21} &= D_2\tau_0 D_\alpha(1-x_1^*)(\alpha_3 - \tfrac{2\alpha_2^2}{\alpha_1}),\\
b_{22} &= D_2(\tfrac{\alpha_3}{\alpha_2} - \tfrac{2\alpha_2}{\alpha_1}),\\
b_{23} &= D_2\tau_0 D_\alpha[\tfrac{\alpha_3(1-x_1^*)}{2} - \tfrac{\alpha_2}{\beta_1}],\\
b_{24} &= D_2[\tfrac{\alpha_3}{2\alpha_2} - \tfrac{1}{(1-x_1^*)\beta_1}],\\
b_{25} &= D_2 D_\alpha[(1-x_1^*)\alpha_2 - \tfrac{\alpha_1}{\beta_1}],\\
b_{26} &= D_2\tau_0[\tfrac{\alpha_4}{\alpha_2} - \tfrac{2\alpha_3}{\alpha_1} - \tfrac{D_\alpha\alpha_3}{D_\alpha\alpha_1+1}],\\
b_{27} &= D_2\tau_0[\tfrac{\alpha_4}{6\alpha_2} - \tfrac{D_\alpha\alpha_3}{2(D_\alpha\alpha_1+1)}].
\end{aligned}
$$

Next let us calculate $Proj_{s_2}[(D_z f_2^1(z,0,\mu))U_2^1(z,\mu)]$. Since

$$
\begin{aligned}
&U_2^1(z,\mu) = (M_2^1)^{-1} Proj_{Im(M_2^1)} f_2^1(z,0,\mu)\\
&= \frac{2}{i\omega_0}\begin{pmatrix} -\frac{1}{2}\frac{i\omega_0 D_1}{D_\alpha\alpha_1}\mu_2 z_2 - \frac{D_1\tau_0}{\beta_1}\mu_1 z_3 \\ -\frac{1}{2}\frac{i\omega_0 \bar{D}_1}{D_\alpha\alpha_1}\mu_2 z_1 + \frac{\bar{D}_1\tau_0}{\beta_1}\mu_1 z_3 \\ D_2\tau_0(\frac{\alpha_3}{\alpha_2} - \frac{\alpha_2}{\alpha_1} - \frac{D_\alpha\alpha_2}{D_\alpha\alpha_1+1})(z_1 z_3 - z_2 z_3) + \frac{1}{2}D_2\tau_0(\frac{\alpha_3}{2\alpha_2} - \frac{\alpha_2}{\alpha_1})(z_1^2 - z_2^2)\end{pmatrix}
\end{aligned}
$$

we have

$$
\frac{1}{4}Proj_{s_2}[(D_z f_2^1(z,0,\mu))U_2^1(z,\mu)] = \begin{pmatrix} c_{11}\mu_2^2 z_1 + c_{12}\mu_1 z_1 z_3 \\ \bar{c}_{11}\mu_2^2 z_2 + \bar{c}_{12}\mu_1 z_2 z_3 \\ c_{21}\mu_2 z_1 z_2 + c_{22}\mu_1 z_3^2 \end{pmatrix}
$$

where

$$
\begin{aligned}
c_{11} &= -\frac{i\omega_0 D_1 \bar{D}_1}{2(D_\alpha\alpha_1)^2},\\
c_{12} &= \frac{D_1 D_2 \tau_0^2}{i\omega_0\beta_1}(\frac{\alpha_3}{\alpha_2} - \frac{\alpha_2}{\alpha_1} - \frac{D_\alpha\alpha_2}{D_\alpha\alpha_1+1}),\\
c_{21} &= -\frac{D_2\tau_0}{D_\alpha\alpha_1}(\frac{\alpha_3}{2\alpha_2} - \frac{\alpha_2}{\alpha_1})(D_1 + \bar{D}_1),\\
c_{22} &= \frac{D_2\tau_0^2}{i\omega_0\beta_1}(\frac{\alpha_3}{\alpha_2} - \frac{\alpha_2}{\alpha_1} - \frac{D_\alpha\alpha_2}{D_\alpha\alpha_1+1})(\bar{D}_1 - D_1).
\end{aligned}
$$

Now let us calculate $Proj_{s_2}[(D_y f_2^1)(z,0,\mu)U_2^2(z,\mu)]$. Define $h = h(z)(\theta) = U_2^2(z,\mu)$, and let

$$
\begin{aligned}
h(\theta) = &\begin{pmatrix} h^{(1)}(\theta) \\ h^{(2)}(\theta) \end{pmatrix}\\
&= h_{11}z_1^2 + h_{12}z_2^2 + h_{13}z_3^2 + h_{14}\mu_1^2 + h_{15}\mu_2^2 + h_{21}z_1 z_2 + h_{22}z_1 z_3 + h_{23}z_1\mu_1\\
&\quad + h_{24}z_1\mu_2 + h_{31}z_2 z_3 + h_{32}z_2\mu_1 + h_{33}z_2\mu_2 + h_{41}z_3\mu_1 + h_{42}z_3\mu_2 + h_{51}\mu_1\mu_2
\end{aligned}
$$

where $h_{ij} \in Q^1$. The coefficients of h are resolved by $(M_2^2 h)(z) = f_2^2(z, 0, \mu)$, which is equal to

$$D_z hJz - \mathcal{A}_{Q^1}(h) = (I - \pi)X_0 F_2(\Phi z, \mu)$$

where $\dot{h}$ stands for the derivative of $h(\theta)$ related to θ. Let

$$\begin{aligned} F_2(\Phi z, \mu) &= A_{11}z_1^2 + A_{12}z_2^2 + A_{13}z_3^2 + A_{14}\mu_1^2 + A_{15}\mu_2^2 + A_{21}z_1z_2 + A_{22}z_1z_3 \\ &\quad + A_{23}z_1\mu_1 + A_{24}z_1\mu_2 + A_{31}z_2z_3 + A_{32}z_2\mu_1 + A_{33}z_2\mu_2 + A_{41}z_3\mu_1 \\ &\quad + A_{42}z_3\mu_2 + A_{51}\mu_1\mu_2 \\ &= \begin{pmatrix} \frac{2i\omega_0\mu_2 a\alpha_2}{\alpha_1}(1 - x_1^*)(z_1 - z_2) + \tau_0 D_\alpha(1 - x_1^*)\alpha_3(z_1 + z_2 + z_3)^2 \\ -2\tau_0 D_\alpha \alpha_2[\frac{\alpha_2}{\alpha_1}(1 - x_1^*)(z_1 + z_2) + \frac{1}{\beta_1}z_3](z_1 + z_2 + z_3) \\ 2i\omega_0\mu_2(z_1 + z_2) + \frac{2\mu_1\tau_0}{\beta_1}z_3 + \tau_0[(D_\alpha\alpha_1 + 1)(\frac{\alpha_3}{\alpha_2} - \frac{2\alpha_2}{\alpha_1})(z_1 + z_2) \\ + \frac{D_\alpha\alpha_1\alpha_3 + \alpha_3 - 2D_\alpha\alpha_2^2}{\alpha_2}z_3](z_1 + z_2 + z_3) \end{pmatrix} \end{aligned}$$

where $A_{ij} \in \mathcal{C}^2$. Comparing the coefficients of all terms, we have that

$$\bar{h}_{23} = h_{32} = 0, \bar{h}_{24} = h_{33}, \bar{h}_{22} = h_{31}, h_{51} = h_{14} = h_{15} = h_{42} = 0$$

and that $h_{24}, h_{22}, h_{41}, h_{21}, h_{13}$ satisfy the following differential equations. Respectively,

$$\begin{cases} \dot{h}_{24} - i\omega_0\tau_0 h_{24} = \Phi(\theta)\Psi(0)A_{24}, \\ \dot{h}_{24}(0) - L(h_{24}) = A_{24}, \end{cases} \tag{3.6}$$

$$\begin{cases} \dot{h}_{22} - i\omega_0\tau_0 h_{22} = \Phi(\theta)\Psi(0)A_{22}, \\ \dot{h}_{22}(0) - L(h_{22}) = A_{22}, \end{cases} \tag{3.7}$$

$$\begin{cases} \dot{h}_{41} = \Phi(\theta)\Psi(0)A_{41}, \\ \dot{h}_{41}(0) - L(h_{41}) = A_{41}, \end{cases} \tag{3.8}$$

$$\begin{cases} \dot{h}_{21} = \Phi(\theta)\Psi(0)A_{21}, \\ \dot{h}_{21}(0) - L(h_{21}) = A_{21}, \end{cases} \tag{3.9}$$

$$\begin{cases} \dot{h}_{13} = \Phi(\theta)\Psi(0)A_{13}, \\ \dot{h}_{13}(0) - L(h_{13}) = A_{13}. \end{cases} \tag{3.10}$$

Then we have

$$\frac{1}{4}Proj_{s_2}[(D_y f_2^1(z, 0, 0))U_2^2(z, 0)] = \begin{pmatrix} d_{11}\mu_1\mu_2 z_1 + d_{12}\mu_2^2 z_1 + d_{13}\mu_1 z_1 z_3 + d_{14}\mu_2 z_1 z_3 \\ \bar{d}_{11}\mu_1\mu_2 z_2 + \bar{d}_{12}\mu_2^2 z_2 + \bar{d}_{13}\mu_1 z_2 z_3 + \bar{d}_{14}\mu_2 z_2 z_3 \\ d_{21}\mu_1 z_1 z_2 + d_{22}\mu_2 z_1 z_2 + d_{23}z_1 z_2 z_3 + d_{24}\mu_1 z_3^2 \\ + d_{25}z_3^3 + d_{26}\mu_1^2 z_3 \end{pmatrix}$$

where

$$
\begin{aligned}
d_{11} =& D_1\tau_0[\frac{D_\alpha\alpha_1+1}{\beta_1}h_{24}^{(2)}(0)-D_\alpha\alpha_1h_{24}^{(1)}(0)],\\
d_{12} =& D_1\beta_1h_{24}^{(1)}(0)-D_1h_{24}^{(2)}(0),\\
d_{13} =& D_1\tau_0[\frac{D_\alpha\alpha_1+1}{\beta_1}h_{22}^{(2)}(0)-D_\alpha\alpha_1h_{22}^{(1)}(0)],\\
d_{14} =& D_1\beta_1h_{22}^{(1)}(0)-D_1h_{22}^{(2)}(0),\\
d_{21} =& D_2\tau_0D_\alpha[(1-x_1^*)\alpha_2h_{21}^{(2)}(0)-\alpha_1h_{21}^{(1)}(0)],\\
d_{22} =& -\frac{D_2\tau_0}{1-x_1^*}[h_{33}^{(1)}(0)+h_{24}^{(1)}(0)]+D_2\tau_0(\frac{\alpha_3}{\alpha_2}-\frac{\alpha_2}{\alpha_1})[h_{33}^{(2)}(0)+h_{24}^{(2)}(0)],\\
d_{23} =& -\frac{D_2\tau_0}{1-x_1^*}(h_{31}^{(1)}(0)+h_{22}^{(1)}(0)+h_{21}^{(1)}(0))+D_2\tau_0[(\frac{\alpha_3}{\alpha_2}-\frac{\alpha_2}{\alpha_1})(h_{31}^{(2)}(0)+h_{22}^{(2)}(0))\\
&+(\frac{\alpha_3-\alpha_2}{\alpha_2}-\frac{D_\alpha\alpha_1}{D_\alpha\alpha_1+1})h_{21}^{(2)}(0)],\\
d_{24} =& D_2\tau_0[(\frac{\alpha_3-\alpha_2}{\alpha_2}-\frac{D_\alpha\alpha_1}{D_\alpha\alpha_1+1})h_{41}^{(2)}(0)-\frac{h_{41}^{(1)}(0)}{1-x_1^*}]\\
&+D_2\tau_0D_\alpha[(1-x_1^*)\alpha_2h_{13}^{(2)}(0)-\alpha_1h_{13}^{(1)}(0)],\\
d_{25} =& D_2\tau_0[(\frac{\alpha_3-\alpha_2}{\alpha_2}-\frac{D_\alpha\alpha_1}{D_\alpha\alpha_1+1})h_{13}^{(2)}(0)-\frac{h_{13}^{(1)}(0)}{1-x_1^*}],\\
d_{26} =& D_2\tau_0D_\alpha[(1-x_1^*)\alpha_2h_{41}^{(2)}(0)-\alpha_1h_{41}^{(1)}(0)]
\end{aligned}
$$

and $h_{ij}^{(1)}$, $h_{ij}^{(2)}$ will be calculated by the method in [24]. Here are the formulas of h_{ij} which we need.

$$
\begin{aligned}
h_{24}(0) =& (i\omega_0\tau_0I-L(e^{i\omega_0\tau_0\theta}))^{INV}[(I-\Phi(0)\Psi(0))A_{24}\\
&-B\int_{-1}^{0}e^{-i\omega_0\tau_0(t+1)}\Phi(t)\Psi(0)A_{24}dt],\\
h_{22}(0) =& (i\omega_0\tau_0I-L(e^{i\omega_0\tau_0\theta}))^{INV}[(I-\Phi(0)\Psi(0))A_{22}\\
&-B\int_{-1}^{0}e^{-i\omega_0\tau_0(t+1)}\Phi(t)\Psi(0)A_{22}dt],\\
h_{41}(0) =& (L(1))^{INV}[(\Phi(0)\Psi(0)-I)A_{41}+B\int_{-1}^{0}\Phi(t)\Psi(0)A_{41}dt],\\
h_{21}(0) =& (L(1))^{INV}[(\Phi(0)\Psi(0)-I)A_{21}+B\int_{-1}^{0}\Phi(t)\Psi(0)A_{21}dt],\\
h_{13}(0) =& (L(1))^{INV}[(\Phi(0)\Psi(0)-I)A_{13}+B\int_{-1}^{0}\Phi(t)\Psi(0)A_{13}dt].
\end{aligned}
$$

After simplified calculation, we have the expression as followed:

$$
\begin{aligned}
h_{24}(0) &= (M_1)^{INV}(\kappa_{24}^1+\kappa_{24}^2),\\
h_{22}(0) &= (M_1)^{INV}(\kappa_{22}^1+\kappa_{22}^2),\\
h_{41}(0) &= (M_2)^{INV}\kappa_{41},
\end{aligned}
$$

$$h_{21}(0) = (M_2)^{INV}(\kappa_{21}^1 + \kappa_{21}^2),$$
$$h_{13}(0) = (M_2)^{INV}(\kappa_{13}^1 + \kappa_{13}^2),$$

where

$$
\begin{aligned}
M_1 &= i\omega_0\tau_0 I - L(e^{i\omega_0\tau_0\theta}),\\
M_2 &= L(1),\\
\kappa_{24}^1 &= A_{24}^{(1)}\begin{pmatrix} 1+\gamma_1\beta_1 D_1(1-a\tau_0+i\omega_0\tau_0)+\frac{iD_2}{\omega_0\gamma_1\beta_1} \\ \beta_1 D_1(1-a\tau_0+i\omega_0\tau_0)+\frac{iD_2}{\omega_0\gamma_1} \end{pmatrix},\\
\kappa_{24}^2 &= A_{24}^{(2)}\begin{pmatrix} D_1\gamma_1(a\tau_0-i\omega_0\tau_0-1)-\frac{iD_2}{\omega_0\beta_1} \\ 1+D_1(a\tau_0-i\omega_0\tau_0-1)-\frac{iD_2}{\omega_0} \end{pmatrix},\\
\kappa_{22}^1 &= A_{22}^{(1)}\begin{pmatrix} 1+\gamma_1\beta_1 D_1(1-a\tau_0+i\omega_0\tau_0)+\frac{iD_2}{\omega_0\gamma_1\beta_1} \\ \beta_1 D_1(1-a\tau_0+i\omega_0\tau_0)+\frac{iD_2}{\omega_0\gamma_1} \end{pmatrix},\\
\kappa_{22}^2 &= A_{22}^{(2)}\begin{pmatrix} D_1\gamma_1(a\tau_0-i\omega_0\tau_0-1)-\frac{iD_2}{\omega_0\beta_1} \\ 1+D_1(a\tau_0-i\omega_0\tau_0-1)-\frac{iD_2}{\omega_0} \end{pmatrix},\\
\kappa_{41} &= A_{41}^{(2)}\begin{pmatrix} -\frac{2}{\omega_0}\gamma_1 Im[D_1]+\frac{D_2}{\beta_1}(1-a\tau_0-\tau_0) \\ -\frac{2}{\omega_0}\gamma_1 Im[D_1]+D_2(1-a\tau_0-\tau_0)-1 \end{pmatrix},\\
\kappa_{21}^1 &= A_{21}^{(1)}\begin{pmatrix} \frac{2}{\omega_0}\gamma_1\beta_1 Im[D_1]+\frac{D_2}{\beta_1\gamma_1}(a\tau_0+\tau_0-1)-1 \\ \frac{2}{\omega_0}\beta_1 Im[D_1]+\frac{D_2}{\gamma_1}(a\tau_0+\tau_0-1) \end{pmatrix},\\
\kappa_{21}^2 &= A_{21}^{(2)}\begin{pmatrix} -\frac{2}{\omega_0}\gamma_1 Im[D_1]+\frac{D_2}{\beta_1}(1-a\tau_0-\tau_0) \\ -\frac{2}{\omega_0}\gamma_1 Im[D_1]+D_2(1-a\tau_0-\tau_0)-1 \end{pmatrix},\\
\kappa_{13}^1 &= A_{13}^{(1)}\begin{pmatrix} \frac{2}{\omega_0}\gamma_1\beta_1 Im[D_1]+\frac{D_2}{\beta_1\gamma_1}(a\tau_0+\tau_0-1)-1 \\ \frac{2}{\omega_0}\beta_1 Im[D_1]+\frac{D_2}{\gamma_1}(a\tau_0+\tau_0-1) \end{pmatrix},\\
\kappa_{13}^2 &= A_{13}^{(2)}\begin{pmatrix} -\frac{2}{\omega_0}\gamma_1 Im[D_1]+\frac{D_2}{\beta_1}(1-a\tau_0-\tau_0) \\ -\frac{2}{\omega_0}\gamma_1 Im[D_1]+D_2(1-a\tau_0-\tau_0)-1 \end{pmatrix},\\
\gamma_1 &= \frac{\alpha_2(1-x_1^*)}{\alpha_1},\\
A_{24} &= \begin{pmatrix} 2i\omega_0 a\gamma_1 \\ 2i\omega_0 \end{pmatrix}, A_{22} = \begin{pmatrix} 2\tau_0 D_\alpha[\alpha_3(1-x_1^*)-\alpha_2\gamma_1-\frac{\alpha_2}{\beta_1}] \\ 2\tau_0[(D_\alpha\alpha_1+1)(\frac{\alpha_3}{\alpha_2}-\frac{\alpha_2}{\alpha_1})-D_\alpha\alpha_2] \end{pmatrix},\\
A_{41} &= \begin{pmatrix} 0 \\ \frac{2\tau_0}{\beta_1} \end{pmatrix}, A_{21} = \begin{pmatrix} 2\tau_0 D_\alpha[\alpha_3(1-x_1^*)-2\alpha_2\gamma_1] \\ 2\tau_0(D_\alpha\alpha_1+1)(\frac{\alpha_3}{\alpha_2}-\frac{2\alpha_2}{\alpha_1}) \end{pmatrix},\\
A_{13} &= \begin{pmatrix} \tau_0 D_\alpha[\alpha_3(1-x_1^*)-\frac{2\alpha_2}{\beta_1}] \\ \tau_0[\frac{\alpha_3}{\alpha_2}(D_\alpha\alpha_1+1)-2D_\alpha\alpha_2] \end{pmatrix}.
\end{aligned}
$$

Finally, we can get

$$\frac{1}{6}g_3^1(z,0,\mu)$$
$$=\begin{pmatrix} (b_{11}+c_{12}+d_{13})\mu_1z_1z_3+(c_{11}+d_{12})\mu_2^2z_1+d_{11}\mu_1\mu_2z_1+d_{14}\mu_2z_1z_3 \\ (\bar{b}_{11}+\bar{c}_{12}+\bar{d}_{13})\mu_1z_2z_3+(\bar{c}_{11}+\bar{d}_{12})\mu_2^2z_2+\bar{d}_{11}\mu_1\mu_2z_2+\bar{d}_{14}\mu_2z_2z_3 \\ (b_{21}+d_{21})\mu_1z_1z_2+(b_{22}++c_{21}+d_{22})\mu_2z_1z_2+(b_{23}++c_{22}+d_{24})\mu_1z_3^2 \\ +b_{24}\mu_2z_3^2+b_{25}\mu_1\mu_2z_3+(b_{26}+d_{23})z_1z_2z_3+(b_{27}+d_{25})z_3^3+d_{26}\mu_1^2z_3 \end{pmatrix}$$
$$+\mathcal{O}(|z||\mu|^2+|z|^2|\mu|).$$

On the center manifold, system (3.5) can be written as the following form

$$\begin{cases} \dot{z_1}=i\omega_0\tau_0z_1+(m_1v_1+m_2v_2)z_1+p_1z_1z_3+h.o.t., \\ \dot{z_2}=-i\omega_0\tau_0z_2+(\bar{m}_1v_1+\bar{m}_2v_2)z_2+\bar{p}_1z_2z_3+h.o.t., \\ \dot{z_3}=(n_1v_1+n_2v_2)z_3+p_2z_1z_2+p_3z_3^2+p_4z_1z_2z_3+p_5z_3^3+h.o.t., \end{cases} \tag{3.11}$$

where $v_i=v_i(\mu)(i=1,2)$, and the quadratic terms or higher order terms of μ are ignored. Then we have

$$\begin{aligned} &m_1v_1+m_2v_2=a_{12}\mu_2,\ n_1v_1+n_2v_2=a_{21}\mu_1, \\ &p_1=(b_{11}+c_{12}+d_{13})\mu_1+d_{14}\mu_2,\ p_2=(b_{21}+d_{21})\mu_1+(b_{22}+c_{21}+d_{22})\mu_2, \\ &p_3=(b_{23}+c_{22}+d_{24})\mu_1+b_{24}\mu_2,\ p_4=b_{26}+d_{23},\ p_5=b_{27}+d_{25}. \end{aligned}$$

Through the change of variables $z_1=r\cos\theta+ir\sin\theta,\ z_2=r\cos\theta-ir\sin\theta$, $z_3=\xi$, the system (3.11) becomes

$$\begin{cases} \dot{r}=Re[m_1v_1+m_2v_2]r+Re[p_1]r\xi+h.o.t., \\ \dot{\theta}=\omega_0\tau_0+Im[m_1v_1+m_2v_2]+h.o.t., \\ \dot{\xi}=(n_1v_1+n_2v_2)\xi+p_2r^2+p_3\xi^2+p_4r^2\xi+p_5\xi^3+h.o.t.. \end{cases} \tag{3.12}$$

Let $Re[m_1v_1+m_2v_2]=\varepsilon_1,\ Re[p_1]=a_1,\ n_1v_1+n_2v_2=\varepsilon_2$. As the second equation depicts a rotation around the $\xi-axis$, it is unrelated to our analysis and we will ignore it. Thus we can have a system in the plane (r,ξ),

$$\begin{cases} \dot{r}=\varepsilon_1r+a_1r\xi, \\ \dot{\xi}=\varepsilon_2\xi+p_2r^2+p_3\xi^2+p_4r^2\xi+p_5\xi^3. \end{cases} \tag{3.13}$$

4. Bifurcation diagrams

Let us introduce the transformation $\sqrt{|p_2p_3|}r\to r,\ p_3\xi\to\xi$, and we consider about the cubic terms of system(3.13)

$$\begin{cases} \dot{r}=\varepsilon_1r+br\xi, \\ \dot{\xi}=\varepsilon_2\xi+cr^2+\xi^2+er^2\xi+f\xi^3, \end{cases} \tag{4.1}$$

where $b = \frac{a_1}{p_3},\ c = \frac{p_2 p_3}{|p_2 p_3|} = \pm 1,\ e = \frac{p_4}{|p_2 p_3|},\ f = \frac{p_5}{p_3^2}$.

Truncating (4.1) at quadratic terms, then we can obtain system (4.2) with lower order terms , and in [14], a categorization of the probable mode interactions for a similar bifurcation is given.

$$\begin{cases} \dot{r} = \varepsilon_1 r + br\xi, \\ \dot{\xi} = \varepsilon_2 \xi + cr^2 + \xi^2. \end{cases} \tag{4.2}$$

There are four different topological structure, which are depending on the signs of b and c :
case I : $c = 1,\ b > 0$; case II : $c = 1,\ b < 0$;
case III : $c = -1,\ b > 0$; case IV : $c = -1,\ b < 0$.

In [22], similarly, we can obtain system (4.2) has a trivial equilibrium $O = (0, 0)$, a semi-trivial equilibrium $M_1 = (0, -\varepsilon_2)$ and a nontrivial equilibrium $M_2 = (\sqrt{\frac{\varepsilon_1(b\varepsilon_2 - \varepsilon_1)}{b^2 c}}, -\frac{\varepsilon_1}{b})$ for $c\varepsilon_1(\varepsilon_1 - b\varepsilon_2) < 0$. On the line $T_1 : \varepsilon_2 = 0$, the trivial equilibrium shows a transcritical bifurcation which means that two equilibria O and M_1 collide and exchange stability. On the line $T_2 : \varepsilon_1 = 0$, a pitchfork bifurcation occurs at O. On the line $T_3 : \varepsilon_1 = b\varepsilon_2$, M_1 shows a pitchfork bifurcation, bringing about a new equilibrium point M_2 for $c\varepsilon_1(\varepsilon_1 - b\varepsilon_2) < 0$ on the positive quadrant of r.

Then we exhibit the bifurcation diagrams for case II, III, IV in Fig.1, and phase portraits for case II, III, IV in Fig.2. As for case I, it will not be discussed in this paper because of its complexity.

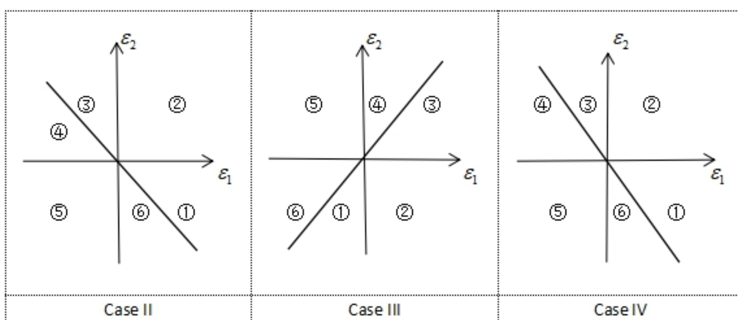

Figure 1. The bifurcation diagrams for case II, III, IV.

5. Numerical simulations

In this section, we will give some examples to illustrate our theoretical results. From Lemma 2.1, in order to satisfy the bifurcation conditions, We select $a = -0.2$, $\gamma_0 = 20$, $D_\alpha = 0.1050$ and $H = 5.0807$ into the system (1.1). There are three equilibria of original system, only choose one, such that $x_1^* = 0.5160057495$, $x_2^* = 2.621670411$, and we can also obtain $\omega_0 = 0.7745966692$, $\tau_0 = 2.354098145$, $\alpha_1 =$

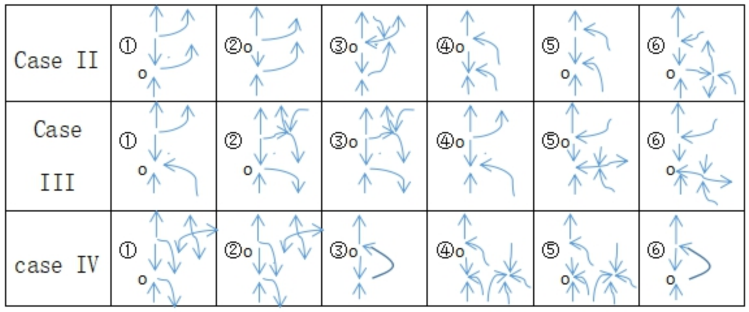

Figure 2. The phase portraits for case II, III, IV.

10.15371662, $\alpha_2 = 7.936621857$, $\alpha_3 = 6.168552293$, $D_1 = -0.2857128683 + 0.3542179573i$, $D_2 = -2.339171855$, $a_{12} = 0.2573545565 + 0.2075826676i$, $a_{21} = -1.083834929$, $b_{11} = 0.1115196025 - 0.1382585462i$, $b_{21} = 1.745869318$, $b_{22} = 1.838747001$, $b_{23} = 0.04009217394$, $b_{24} = 0.04222501805$, $c_{12} = 0.4757533068 + 0.38374350091i$, $c_{21} = 1.160010729$, $c_{22} = -0.9515066139$, $d_{13} = 2.569140234 + 0.2054337492i$, $d_{14} = -4.102625453 - 1.774723308i$, $d_{21} = -566.4596114$, $d_{22} = -10.01252480$, $d_{24} = -410.2724108$.

(1)Choose $\mu_1 = 0.001$, $\mu_2 = -0.1$, then $b = -0.9952 < 0, c = -1$. It will be considered in case IV. For $\varepsilon_1 = -0.0257, \varepsilon_2 = -0.001084$, we have Fig.3 shows the stable periodic orbit of system(1.1) in ⑤ of case IV.

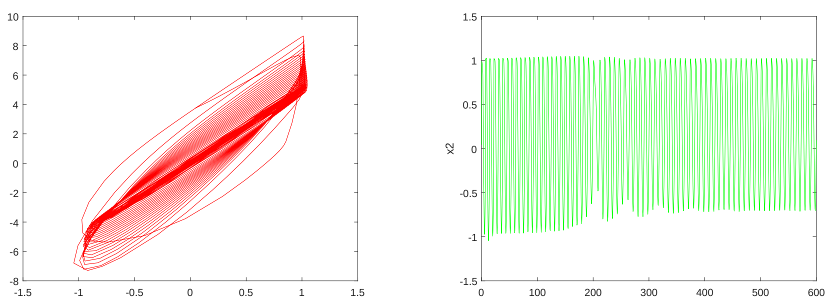

Figure 3. The periodic orbit of system(1.1) is stable in ⑤ of case IV, and the initial value is $(0.044, 3.75)$. (left)Phase plane of (x_1, x_2). (right) Wave plot of x_2.

(2)Choose $\mu_1 = -0.000009$, $\mu_2 = -0.1$, then $b = -786.1190 < 0, c = -1$. It will be considered in case IV. For $\varepsilon_1 = -0.0257, \varepsilon_2 = 0.00000975$, we have Fig.4 shows the stable periodic orbit of system(1.1) in ④ of case IV.

(3)Choose $\mu_1 = -0.001$, $\mu_2 = 0.1$, then $b = -0.9952 < 0, c = -1$. It will be considered in case IV. For $\varepsilon_1 = 0.0257, \varepsilon_2 = 0.001084$, we have Fig.5 shows the unstable periodic orbit of system(1.1) in ② of case IV.

(4)Choose $\mu_1 = 0.000009$, $\mu_2 = 0.1$, then $b = -786.1190 < 0, c = -1$. It will be considered in case IV. For $\varepsilon_1 = 0.0257, \varepsilon_2 = -0.00000975$, we have Fig.6 shows the unstable periodic orbit of system(1.1) in ① of case IV.

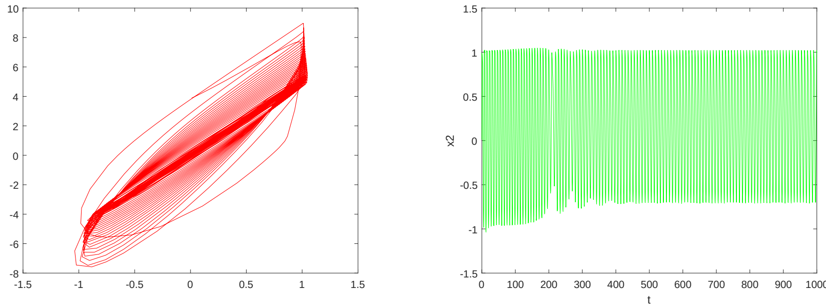

Figure 4. The periodic orbit of system(1.1) is stable in ④ of case IV, and the initial value is $(0.01, 3.88)$. (left)Phase plane of (x_1, x_2). (right) Wave plot of x_2.

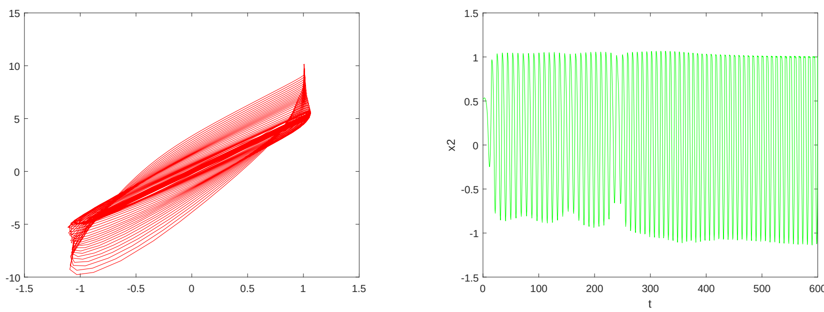

Figure 5. The periodic orbit of system(1.1) is unstable in ② of case IV, and the initial value is $(0.516, 2.650)$. (left)Phase plane of (x_1, x_2). (right) Wave plot of x_2.

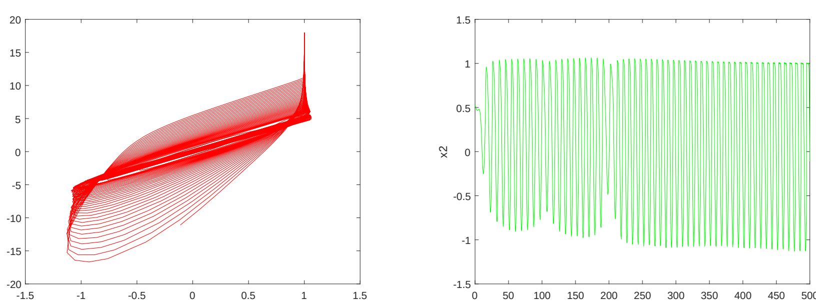

Figure 6. The periodic orbit of system(1.1) is unstable in ① of case IV, and the initial value is $(0.530, 2.622)$. (left)Phase plane of (x_1, x_2). (right) Wave plot of x_2.

6. Conclusions

In this paper, we have considered about the codimension-two bifurcation of CSTR model with delay. By applying the normal form method and the center manifold theorem, we have shown how to reduce CSTR model with parameters near the critical point of the Zero-Hopf bifurcation. According to the value of these unfolding parameters, we can make sure the existence of periodic orbits. And we found the emergence of Hopf-transcritical and pitchfork bifurcation.

When parameter values satisfy the conditions ④ and ⑤ of case IV, there is a stable period orbit nearby the equilibrium, namely, the chemical reaction in tank reactor will tend to balanced, and the stable state is described by the equilibrium. When parameter values satisfy the conditions ① and ② of case IV, there is an unstable period orbit nearby the equilibrium, namely, the chemical reaction in tank reactor is going to become the unbalanced state from balanced state.

Our work is a further investigation of CSTR model, which is helpful for the investigation about complex phenomenon caused by high codimensional bifurcation of delay differential equations.

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