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BIFURCATION OF TRAVELING WAVE SOLUTIONS OF THE $K(M,N)$ EQUATION WITH GENERALIZED EVOLUTION TERM

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Abstract In this paper, by using bifurcation theory and methods of plane dynamic system, we investigate the bifurcations of the traveling wave system corresponding to the $K(m, n)$ equation with generalized evolution term. Under different parameter conditions, some exact explicit parametric representations of traveling wave solution are obtained.

Keywords Bifurcation curve, singular traveling wave equation, solitary wave, smooth periodic wave, periodic cusp wave.

MSC(2010) 34C23, 34C25, 35C07, 74J35.

1. Introduction

In 1993, to understand the role of nonlinear dispersion in the formation of patterns in liquid drops, P. Rosenau and J.M. Hyman [19] introduced and studied the nonlinear dispersive equations $K(m, n)$

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3. \tag{1.1}$$

They found that (1.1) has a class of solitary waves with compact support which is called compactons. P. Rosenau [18] also studied the $K(m, n)$ equation

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \tag{1.2}$$

where a is a constant. He found a number of dispersive effect and obtained kinks, solitons, dark solitons with cusp all being manifestations of nonlinear dispersion in action. In 2008, A. Biswas [1] proposed the following $K(m,\ n)$ equation with generalized evolution term

$$(u^l)_t + au^m u_x + b(u^n)_{xxx} = 0, \tag{1.3}$$

where a, b are constants, $l, m, n \in Z^+$. He obtained 1-soliton solution and used the solitary wave ansatz to get the exact solution. Especially, the case $l = m = n = 1$ leads to the KdV equation.

In this paper, we consider the following traveling wave system of (1.3) when $l = 2$, $m = 3$, $n = 2$

$$(u^2)_t + au^3 u_x + b(u^2)_{xxx} = 0. \tag{1.4}$$

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Recently, in the book [11] written by Li and Dai, the authors have found and studied theoretically that a lot of nonlinear wave equations have nonanalytic solitary wave solutions (which was called the peakon or valleykon) and periodic cusp wave solutions. Moreover, they gave a more systematic account for the bifurcation theory method of dynamical systems to find traveling wave solutions with an emphasis on singular waves. By using the bifurcation method of dynamical systems, many authors [3–10,12–17,20–26] also investigated traveling wave solutions of some partial differential equations. Therefore, in this paper, to consider traveling wave solutions of the partial differential equation (1.4), we will investigate the dynamical behavior of the corresponding ordinary differential equation (traveling wave equation).

2. The bifurcation of phase portraits of (1.4)

Substituting $u(x,t) = \varphi(\xi)$ and $\xi = x - ct$ (c is the wave speed) into (1.4), one gets the following ODE

$$-c(\varphi^2)_\xi + \frac{a}{4}(\varphi^4)_\xi + b(\varphi^2)_{\xi\xi\xi} = 0,$$

integrating the above equation once, we have

$$-c\varphi^2 + \frac{a}{4}\varphi^4 + 2b\varphi\varphi_{\xi\xi} + 2b(\varphi_\xi)^2 = g, \tag{2.1}$$

where g is an integral constant. (2.1) is equivalent to the system

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{c\varphi^2 - \frac{a}{4}\varphi^4 - 2by^2 + g}{2b\varphi}, \end{cases} \tag{2.2}$$

which has the first integral

$$H(\varphi, y) = \varphi^2(by^2 - \frac{c}{4}\varphi^2 + \frac{a}{24}\varphi^4 - \frac{g}{2}) = h. \tag{2.3}$$

Since the traveling wave solutions of Equation (1.4) are determined by the phase portraits of system (2.2), we only need to study system (2.2).

Let $d\xi = 2b\varphi d\zeta$. Then, except on the singular straight line $\varphi = 0$, the system (2.2) has the same topological phase portraits and the first integral (2.3) as the following system

$$\begin{cases} \frac{d\varphi}{d\zeta} = 2b\varphi y, \\ \frac{dy}{d\zeta} = -2by^2 - \frac{a}{4}\varphi^4 + c\varphi^2 + g. \end{cases} \tag{2.4}$$

Therefore, we can obtain the topological phase portraits of system (2.2) from those of system (2.4).

Assume $b > 0$ without loss of generality. When $g > 0$, system (2.4) has two equilibrium points $(0, \pm\sqrt{\frac{g}{2b}})$ in the singular straight line $\varphi = 0$.

Let $f(\varphi) = c\varphi^2 - \frac{a}{4}\varphi^4 + g$, $f'(\varphi) = 2c\varphi - a\varphi^3$. Assume that $a > 0$, $c > 0$, then it is easy to show that the following facts hold.

(1) For $g > 0$, $f(\varphi)$ has two simple zeros at $\varphi_{1\pm}$, and $|\varphi_{1\pm}| > \sqrt{\frac{2c}{a}}$.

(2) For $g = 0$, $f(\varphi)$ has three simple zeros at $\varphi_{1\pm}$, φ_0, and $|\varphi_{1\pm}| > \sqrt{\frac{2c}{a}}$, $\varphi_0 = 0$.

(3) For $g<0$, when $-\frac{c^2}{a}<g<0$, $f(\varphi)$ has four simple zeros at $\varphi_{1\pm}$, $\varphi_{2\pm}$, and $|\varphi_{1\pm}|>\sqrt{\frac{2c}{a}}$, $0<|\varphi_{2\pm}|<\sqrt{\frac{2c}{a}}$. When $g=-\frac{c^2}{a}$, $f(\varphi)$ has two simple zeros at $\varphi_{e\pm}$, and $|\varphi_{e\pm}|=\sqrt{\frac{2c}{a}}$. When $g<-\frac{c^2}{a}$, $f(\varphi)$ has no zero.

Let $M(\phi,0)$ be the coefficient matrix of the linearized system of (2.4) at the equilibrium point $(\phi,0)$, then

$$\begin{aligned} Tr(M(\phi,0)) &= 0, \\ detM(\phi,0) &= 2b\phi^2(a\phi^2-2c). \end{aligned}$$

By the theory of planar dynamical systems, for an equilibrium point of a planar integral system, since $Tr(M(\phi,0))=0$, if $detM(\phi,0)>0$, then it is a center point. If $detM(\phi,0)<0$, then it is a saddle point. If $detM(\phi,0)=0$ and the Poincaré index of the equilibrium point is 0, then it is a cusp point.

Case I When $a>0$, according to the theory of planar dynamical systems, one can see the following facts.

Assume that $c>0$.

(1) For $g>0$, system (2.4) has two equilibrium points at $(\varphi_{1\pm},0)$ which are center, and there exist two equilibrium points of system (2.4) at $(0,y_\pm)$ in the singular straight line $\varphi=0$. Here, $y_\pm=\pm\sqrt{\frac{g}{2b}}$.

(2) For $g=0$, system (2.4) has three equilibrium points at $(\varphi_{1\pm},0)$, $(\varphi_0,0)$. $(\varphi_{1\pm},0)$ are center and $(\varphi_0,0)$ is a cusp point.

(3) For $g<0$, when $-\frac{c^2}{a}<g<0$, system (2.4) has two center at $(\varphi_{1\pm},0)$ and two saddle points at $(\varphi_{2\pm},0)$. When $g=-\frac{c^2}{a}$, system (2.4) has two cusp points at $(\varphi_{e\pm},0)$. When $g<-\frac{c^2}{a}$, system (2.4) has no critical point.

Assume that $c<0$.

(1) For $g>0$, system (2.4) has two equilibrium points at $(\varphi_\pm,0)$ which are center, and there exist two equilibrium points of system (2.4) at $(0,y_\pm)$ in the singular straight line $\varphi=0$. Here, $y_\pm=\pm\sqrt{\frac{g}{2b}}$.

(2) For $g=0$, system (2.4) has an equilibrium point at $(\varphi_0,0)$ which is a cusp point.

(3) For $g<0$, system (2.4) has no critical point.

For a given $a>0$, there is a bifurcation curve in the (g,c)-parameter plane as following. These curves partition the (g,c)-parameter plane into the following six regions (see Figure 1).

$$\begin{aligned} &(I_a)=\{(g,c)|g>0\}, && (I_b)=L_1^+=\{(g,c)|c>0,\ g=0\}, \\ &(I_c)=\{(g,c)|c>\sqrt{-ag},\ g<0\}, && (I_d)=L_2^+=\{(g,c)|c=\sqrt{-ag},\ g<0\}, \\ &(I_e)=\{(g,c)|c<\sqrt{-ag},\ g<0\}, && (I_f)=L_1^-=\{(g,c)|c<0,\ g=0\}. \end{aligned}$$

According to the qualitative theory of dynamical systems, we draw the bifurcation of phase portraits of system (2.4) as Figure 2.

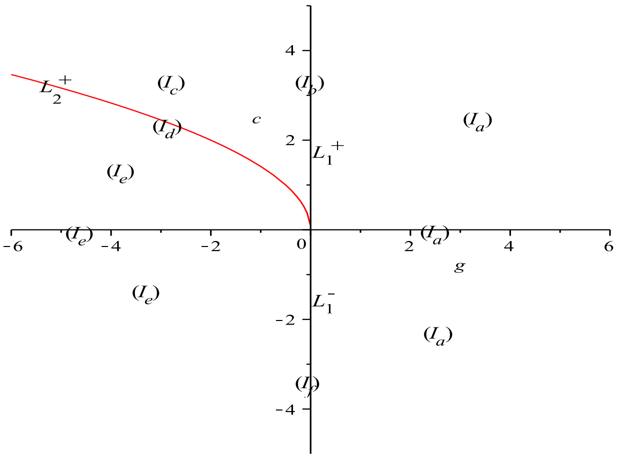

Figure 1. (g, c)-plane, where $L_1 = L_1^+ \cup L_1^- : g = 0$, $L_2^+ : c = \sqrt{-ag}$ $(g < 0)$.

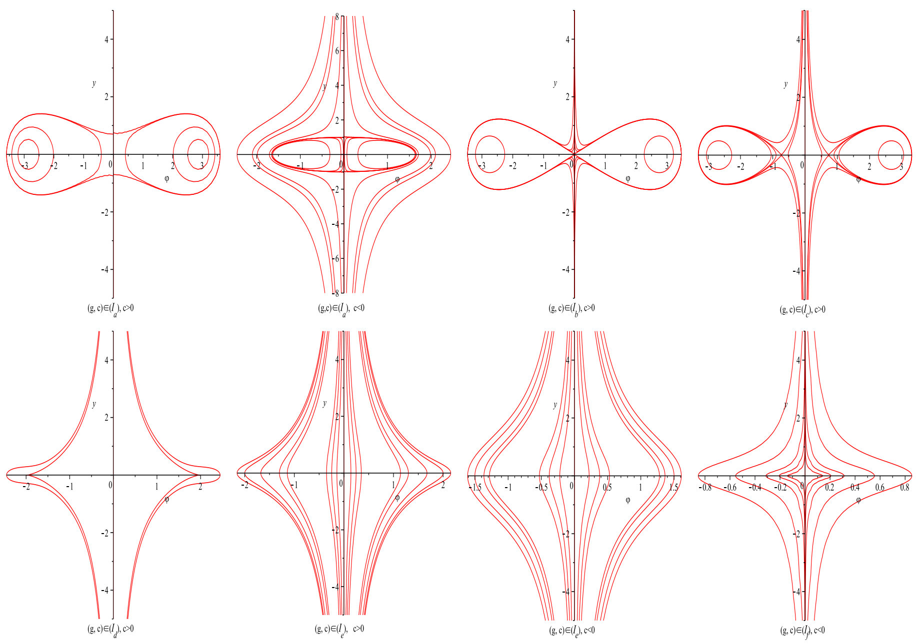

Figure 2. The phase portrait bifurcation of system (2.4) when $a > 0$.

From the above analysis, we can get the following results.

Theorem 2.1. *If $a > 0$, system (2.4) has two center at $(\varphi_{1\pm}, 0)$ and two equilibrium points in the singular straight line $\varphi = 0$ when $(g, c) \in (I_a)$. System (2.4) has two center at $(\varphi_{1\pm}, 0)$ and a cusp point at $(\varphi_0, 0)$ when $(g, c) \in (I_b)$. System (2.4) has two center at $(\varphi_{1\pm}, 0)$ and two saddle points at $(\varphi_{2\pm}, 0)$ when $(g, c) \in (I_c)$. System (2.4) has two cusp points at $(\varphi_{e\pm}, 0)$ when $(g, c) \in (I_d)$. System (2.4) has no critical* *points when* $(g,c) \in (I_e)$*. System* (2.4) *has a cusp point at* $(\varphi_0, 0)$ *when* $(g,c) \in (I_f)$*.*

Case II When $a < 0$, one can see the following facts.

Assume that $c < 0$.

(1) For $g < 0$, system (2.4) has two equilibrium points at $(\varphi_{1\pm}, 0)$ which are saddle points, and there exist two equilibrium points of system (2.4) at $(0, y_\pm)$ in the singular straight line $\varphi = 0$. Here, $|\varphi_{1\pm}| > \sqrt{\frac{2c}{a}}$, and $y_\pm = \pm\sqrt{\frac{g}{2b}}$.

(2) For $g = 0$, system (2.4) has three equilibrium points at $(\varphi_{1\pm}, 0)$ and $(\varphi_0, 0)$. Furthermore, $(\varphi_{1\pm}, 0)$ are saddles point and $(\varphi_0, 0)$ is a cusp point. Here, $|\varphi_{1\pm}| > \sqrt{\frac{2c}{a}}$, $\varphi_0 = 0$.

(3) For $g > 0$, when $0 < g < -\frac{c^2}{a}$, system (2.4) has four equilibrium points at $(\varphi_{1\pm}, 0)$ and $(\varphi_{2\pm}, 0)$. $(\varphi_{1\pm}, 0)$ are saddle points and $(\varphi_{2\pm}, 0)$ are center. When $g = -\frac{c^2}{a}$, system (2.4) has two equilibrium points $(\varphi_{e\pm}, 0)$ which are cusp points. When $g > -\frac{c^2}{a}$, system (2.4) has no critical point. Here, $|\varphi_{1\pm}| > \sqrt{\frac{2c}{a}}$, $0 < |\varphi_{2\pm}| < \sqrt{\frac{2c}{a}}$ and $|\varphi_{e\pm}| = \sqrt{\frac{2c}{a}}$.

Assume that $c > 0$.

(1) For $g < 0$, system (2.4) has two equilibrium points at $(\varphi_\pm, 0)$ which are saddle points.

(2) For $g = 0$, system (2.4) has an equilibrium point at $(\varphi_0, 0)$ which is a cusp point.

(3) For $g > 0$, system (2.4) has no critical point.

For a given $a < 0$, there is a bifurcation curve in the (g,c)-parameter plane as following.

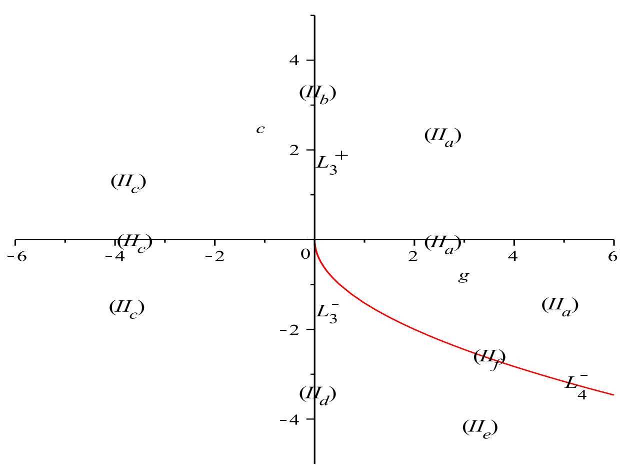

Figure 3. (g,c)-plane, where $L_1 = L_3^+ \cup L_3^- : g = 0$, $L_4^- : c = -\sqrt{-ag}$ $(g < 0)$.

These curves partition the (g,c)-parameter plane into the following six regions (see Figure 3).

$$(II_a) = \{(g,c)|c > -\sqrt{-ag},\ g > 0\},\quad (II_b) = L_3^+ = \{(g,c)|c > 0,\ g = 0\},$$
$$(II_c) = \{(g,c)|g < 0\},\quad (II_d) = L_3^- = \{(g,c)|c < 0,\ g = 0\},$$
$$(II_e) = \{(g,c)|c < -\sqrt{-ag},\ g > 0\},\quad (II_f) = L_4^- = \{(g,c)|c = -\sqrt{-ag},\ g > 0\}.$$

According to the qualitative theory of dynamical systems, we draw the bifurcation of phase portraits of system (2.4) as Figure 4.

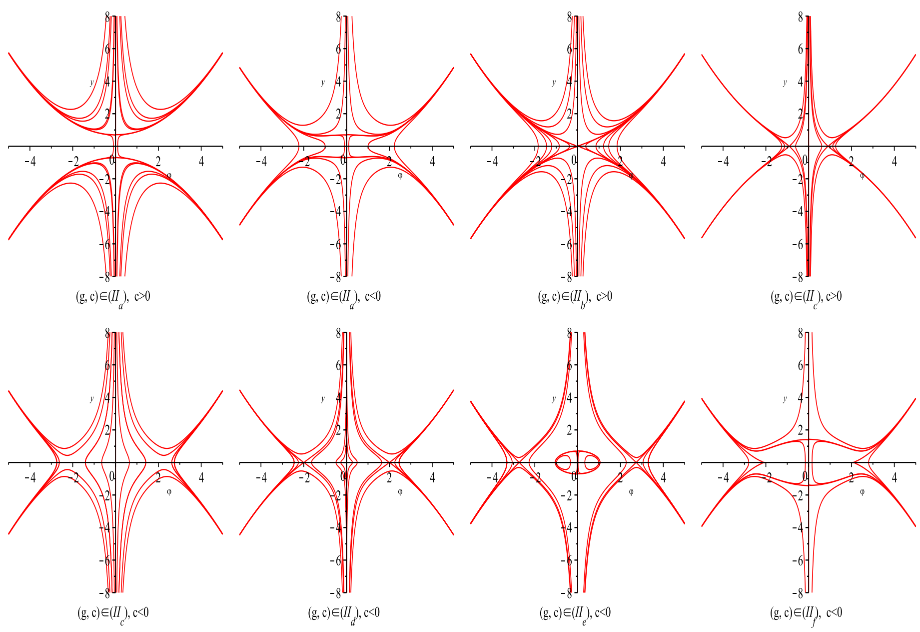

Figure 4. The phase portrait bifurcation of system (2.4) when $a < 0$.

From the above analysis, we can also obtain the following results.

Theorem 2.2. *If $a < 0$, system (2.4) has no equilibrium point except in the singular straight line $\varphi = 0$ when $(g,c) \in (II_a)$. System (2.4) has a cusp point at $(\varphi_0, 0)$ when $(g,c) \in (II_b)$. System (2.4) has two saddle points at $(\varphi_{1\pm}, 0)$ when $(g,c) \in (II_c)$. System (2.4) has two saddle points at $(\varphi_{1\pm}, 0)$ and a cusp point at $(\varphi_0, 0)$ when $(g,c) \in (II_d)$. System (2.4) has two saddle points at $(\varphi_{1\pm}, 0)$ and two center at $(\varphi_{2\pm}, 0)$ when $(g,c) \in (II_e)$. System (2.4) has two cusp points at $(\varphi_{e\pm}, 0)$ when $(g,c) \in (II_f)$.*

3. Some traveling wave solutions determined by phase portraits of (2.2) and their exact explicit parametric representations

Definition 3.1 ([11]). If the "time interval" of existence of a traveling wave solution $\phi(\xi)$ with respect to ξ in the positive direction or (and) negative direction of ξ is finite, then the profile of wave defined by $\phi(\xi)$ is called a breaking wave on one (two) side(s).

Let $H(0,\pm\sqrt{\frac{g}{2b}})=h_0, H(\varphi_{e\pm},0)=h_{e\pm}, H(\varphi_{1\pm},0)=h_{1\pm}, H(\varphi_{2\pm},0)=h_{2\pm}$.

Based on the results of Section 2 and Theorem 2.5 in [11], we can get the following proposition.

Proposition 3.1. *The following results hold.*

(i) If $a>0$, when $(g,c)\in(I_a)$, the curve defined by $H(\varphi,y)=h$, $h\in(h_{1-},h_0)$(or $h\in(h_0,h_{1+})$) gives a family of smooth periodic wave solutions of (1.4). *If $h\longrightarrow h_0^-$(or $h\longrightarrow h_0^+$), this family of smooth periodic wave solutions will become a periodic cusp wave solution of* (1.4).

(ii) If $a>0$, when $(g,c)\in(I_c)$, the curves defined by $H(\varphi,y)=h_{2+}$ and $H(\varphi,y)=h_{2-}$ give peak-type solitary wave solution and valley-type solitary wave solution of (1.4) *respectively. The curves defined by $H(\varphi,y)=h$, $h\in(h_{1+},h_{2+})$ and $H(\varphi,y)=h$, $h\in(h_{2-},h_{1-})$ give two families of smooth periodic traveling wave solutions of* (1.4). *If $h\longrightarrow h_2$, the two families of smooth periodic wave solutions will become two smooth solitary wave solutions of* (1.4).

(iii) If $a<0$, when $(g,c)\in(II_e)$ or $(g,c)\in(II_c)$, the curves defined by $H(\varphi,y)=h$, $h\in(h_0,h_{1+})$ and $H(\varphi,y)=h$, $h\in(h_{1-},h_0)$ give two families of open orbits of (2.4). *For every orbit if $\xi\longrightarrow\pm\infty$, then $\varphi(\xi)\longrightarrow 0$, $y=\varphi'(\xi)\longrightarrow\infty$. Then they give two families of breaking wave solutions of* (1.4).

Next we consider the exact explicit traveling solutions of (1.4) which cross $(0,y_\pm)$ when $\varphi>0$.

Let $H(\varphi,y)=\varphi^2(by^2+\frac{a}{24}\varphi^4-\frac{c}{4}\varphi^2-\frac{g}{2})=h_0=0$, then

$$by^2+\frac{a}{24}\varphi^4-\frac{c}{4}\varphi^2-\frac{g}{2}=0.$$

When $a>0$, $(g,c)\in(I_a)$, and if $c\neq 0$, then we have

$$\begin{aligned}(\frac{d\varphi}{d\xi})^2&=-\frac{a}{24b}(\varphi^4-\frac{6c}{a}\varphi^2-\frac{12g}{a})\\&=\frac{a}{24b}(r_1^2-\varphi^2)(\varphi^2+r_2^2).\end{aligned}$$

Here, $r_{1\pm}$ and $r_{2\pm}$ are four zeros of $g(\varphi)=\varphi^4-\frac{6c}{a}\varphi^2-\frac{12g}{a}$, $r_1^2=r_{1\pm}^2=\frac{3(c+\sqrt{c^2+\frac{4ag}{3}})}{a}$, $r_2^2=r_{2\pm}^2=-\frac{3(c-\sqrt{c^2+\frac{4ag}{3}})}{a}$. Suppose $k=\frac{a}{24b}$, then

$$\frac{d\varphi}{d\xi}=-\sqrt{k(r_1^2-\varphi^2)(\varphi^2+r_2^2)}. \tag{3.1}$$

Integrating on both sides of (3.1) over the interval $[0,\xi]$, we obtain (see [2])

$$\sqrt{k}\int_0^{\xi} d\xi = \int_{\varphi}^{r_{1+}} \frac{d\varphi}{\sqrt{(r_1^2-\varphi^2)(\varphi^2+r_2^2)}} = qcn^{-1}(\cos\psi, p),$$

where $q = \frac{1}{\sqrt{r_1^2+r_2^2}}$, $p^2 = \frac{r_1^2}{r_1^2+r_2^2}$, $\psi = \cos^{-1}(\frac{\varphi}{r_{1+}})$, i.e.

$$\sqrt{k}\xi = \frac{1}{\sqrt{r_1^2+r_2^2}} cn^{-1}(\frac{\varphi}{r_{1+}}, \frac{r_{1+}}{\sqrt{r_1^2+r_2^2}}).$$

Therefore,

$$\varphi(\xi) = r_{1+}cn(\sqrt{\frac{a(r_1^2+r_2^2)}{25b}}\xi, \frac{r_{1+}}{\sqrt{r_1^2+r_2^2}}). \tag{3.2}$$

Thus, we have a family of smooth periodic wave solutions of (1.4)

$$u(x,t) = r_{1+}cn(\sqrt{\frac{a(r_1^2+r_2^2)}{25b}}(x-ct), \frac{r_{1+}}{\sqrt{r_1^2+r_2^2}}), \quad c \neq 0. \tag{3.3}$$

The profiles of (3.2) and (3.3) are shown in Figure 5 and Figure 6, respectively.

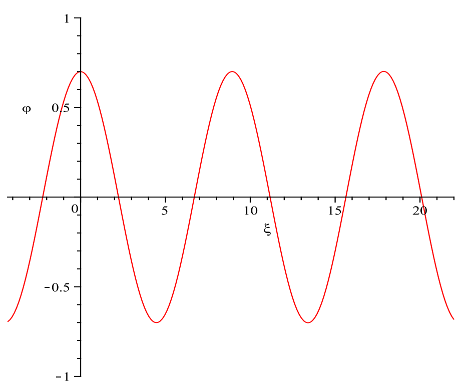

Figure 5. The smooth periodic wave solution of the equation (3.2) with a=1, b=2, c=-4, g=1.

If $c = 0$, then

$$(\frac{d\varphi}{d\xi})^2 = \frac{-a}{24b}(\varphi^4 - \frac{12g}{a}) = \frac{a}{24b}(r^2-\varphi^2)(\varphi^2+r^2),$$

where $r_{\pm} = \pm(\frac{12g}{a})^{\frac{1}{4}}$. Suppose $k = \frac{a}{24b}$, it then follows that

$$\frac{d\varphi}{d\xi} = -\sqrt{k(r^2-\varphi^2)(\varphi^2+r^2)}. \tag{3.4}$$

Integrating on both sides of (3.4) over the interval $[0,\xi]$, we obtain

$$\sqrt{k}\int_0^{\xi} d\xi = \int_{\varphi}^{r_+} \frac{d\varphi}{\sqrt{(r^2-\varphi^2)(\varphi^2+r^2)}} = qcn^{-1}(\cos\psi, p),$$

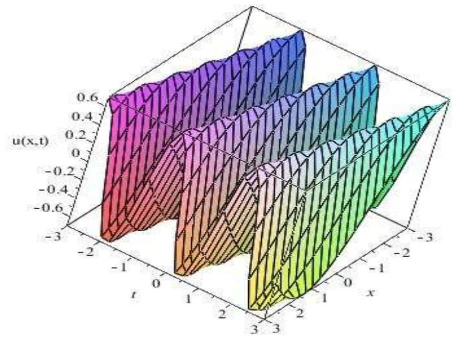

Figure 6. The smooth periodic wave solution of the equation (3.3) when a=1, b=2, c=-4, g=1.

where $q = \frac{\sqrt{2}}{2r}$, $p^2 = \frac{1}{2}, \psi = \cos^{-1}(\frac{\varphi}{r_+})$, i.e.

$$\sqrt{k}\xi = \frac{\sqrt{2}}{2r} cn^{-1}(\frac{\varphi}{r_+}, \frac{\sqrt{2}}{2}),$$

then we have

$$\varphi(\xi) = r_+ cn(\sqrt{2k} r_+ \xi, \frac{\sqrt{2}}{2}) = r_+ cn(\sqrt{\frac{a}{12b}} r_+ \xi, \frac{\sqrt{2}}{2}).$$

Thus, we have a family of smooth periodic wave solutions of (1.4)

$$u(x,t) = r_+ cn(\sqrt{\frac{a}{12b}} r_+ (x - ct), \frac{\sqrt{2}}{2}), \quad c = 0. \tag{3.5}$$

Similarly, when $a < 0$, $(g, c) \in (II_e)$, $\varphi > 0$, the exact explicit traveling solution which cross $(0, y_\pm)$ of (1.4) is as following.

$$u(x,t) = r_{3+} cn(\sqrt{\frac{a(r_3^2 + r_4^2)}{-25b}} (x - ct), \frac{r_{3+}}{\sqrt{r_3^2 + r_4^2}}), \tag{3.6}$$

where $r_{3\pm}^2 = \frac{3(c+\sqrt{c^2+\frac{4ag}{3}})}{a}$, $r_4^2 = -\frac{3(c-\sqrt{c^2+\frac{4ag}{3}})}{a}$.

With the above analysis, we present the following proposition.

Proposition 3.2. *The following results hold.*

(i) If $a > 0$, when $(g, c) \in (I_a)$, system (1.4) has two families of smooth periodic wave solutions and two periodic cusp wave solutions. When $(g, c) \in (I_c)$, system (1.4) has two smooth periodic wave solutions. If $a < 0$, when $(g, c) \in (II_e)$, system (1.4) has two families of periodic cusp wave solutions and two families of breaking wave solutions. When $(g, c) \in (II_c)$, system (1.4) has two families of breaking wave solutions.

(ii) If $a > 0$, on the (g, c)-parametric plane, when (g, c) goes from (I_a) to (I_c) by the bifurcation curve L_1^+, system (1.4) has two families of smooth periodic wave solutions and two smooth solitary wave solutions. When (g, c) goes to (I_e) by the bifurcation curve L_2^+, system (1.4) has no periodic solution.

(iii) If $a < 0$, on the (g, c)-parametric plane, when (g, c) goes from (II_a) to (II_c) by the bifurcation curve L_3^+, system (1.4) *has two families of breaking wave solutions. When (g, c) goes to (II_e) by the bifurcation curve L_3^-, system* (1.4) *has two periodic cusp wave solutions and two families of breaking wave solutions. When (g, c) goes from (II_e) to (II_a) by the bifurcation curve L_4^-, system* (1.4) *has no periodic solution and breaking wave solution.*

4. Conclusion

In this paper, by using the bifurcation theory and the method of phase portrait analysis, we investigated the bifurcations of Equation (1.4) and obtained some traveling wave solutions determined by phase portraits of (2.2) and their exact explicit parametric representations. Based on the method given in this paper, we can also obtain new exact traveling wave solutions for the $K(m, n)$ equation with generalized evolution term, i.e. $(u^3)_t + au^3u_x + b(u^2)_{xxx} = 0$. We would like to study the $K(m, n)$ equation further.

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