Journal of Applied Analysis and Computation
Volume 8, Number 6, December 2018, 1851–1862
Website:http://jaac.ijournal.cn
DOI:10.11948/2018.1851

BIFURCATIONS OF TRAVELING WAVE SOLUTIONS FOR A GENERALIZED CAMASSA-HOLM EQUATION*

Minzhi Wei[1,†], Xianbo Sun[1] and Hongying Zhu[1]

Abstract In this paper, the traveling wave solutions for a generalized Camassa-Holm equation $u_t-u_{xxt}=\frac{1}{2}(p+1)(p+2)u^pu_x-\frac{1}{2}p(p-1)u^{p-2}u_x^3-2pu^{p-1}u_xu_{xx}-u^pu_{xxx}$ are investigated. By using the bifurcation method of dynamical systems, three major results for this equation are highlighted. First, there are one or two singular straight lines in the two-dimensional system under some different conditions. Second, all the bifurcations of the generalized Camassa-Holm equation are given for p either positive or negative integer. Third, we prove that the corresponding traveling wave system of this equation possesses peakon, smooth solitary wave solution, kink and anti-kink wave solution, and periodic wave solutions.

Keywords Generalized Camassa-Holm equation, bifurcation theory, peakon, solitary wave solution, kink and anti-kink wave solutions.

MSC(2010) 35C08, 37K40, 74J35, 35Q51.

1. Introduction

During the past few decades, more and more mathematicians paid attentions to certain nonlinear evolution equations which possess solitary wave solutions, such as peaked wave solutions [13, 17–19, 22, 23]. The well-known Camassa-Holm (CH) equation [3], which describes nonlinear dispersive waves is in the following form

$$u_t-u_{xxt}=3uu_x-2uu_x-uu_{xxx}. \tag{1.1}$$

The CH equation arises from the theory of shallow water waves [3, 4] and provides a model of wave breaking for a large class of solutions [5, 6]. Particularly, the CH equation (1.1) has peakon solution with the form $u(x,t)=ce^{-|x-ct|}$. The name peakon, which means traveling wave with slope discontinuities, is used to distinguish them from general traveling wave solutions since they have a corner at the peak of height c, where c is the wave speed. However, it is necessary to point out that a peakon solution is not a classical solution, but a weak solution in the sense of satisfying a integral formulation, which is introduced in [14].

†the corresponding author. Email address:weiminzhi@21cn.com(M. Wei)
1Department of Applied mathematics, Guangxi University of Finance and Economics, Nanning, Guangxi, 530003, China
*The authors were supported by Guangxi College Enhancing Youth's Capacity Project (,KY2016YB404, KY2016LX315) and National Natural Science Foundation of China (No. 41461110,11461001).

Recently, some generalizations of the CH equation admitting breaking wave solutions have been studied, such as the b family equation [9,10]

$$u_t - u_{xxt} = -(b+1)uu_x + bu_x u_{xx} + uu_{xxx}, \quad b \neq 0 \tag{1.2}$$

has been studied for possessing peakon solutions. Furthermore, Wazwaz [20] studied nonlinear dispersive variants $CH(n,n)$ of the generalized Camassa-Holm equation in $(1+1), (2+1)$ and $(3+1)$ dimensions, respectively.

Motivated by the literature, the present paper focuses on the following generalized Camassa-Holm equation [1]:

$$u_t - u_{xxt} = \frac{1}{2}(p+1)(p+2)u^p u_x - \frac{1}{2}p(p-1)u^{p-2}u_x^3 - 2pu^{p-1}u_x u_{xx} - u^p u_{xxx}, \tag{1.3}$$

where $p \in Z$ and $p \neq 0$ is the parameter with the nonlinear terms, and derived by Anco, Recio, Gandarias and Bruzon [1], which is considered by one of the related Hamiltonian structures of the Camassa-Holm equation. (1.3) is exactly the Camassa-Holm equation (1.1) when $p = 1$. In this paper, all the bifurcations of the generalized Camassa-Holm equation (1.3) are given for p either positive or negative integer.

The present paper focuses on the bifurcation analysis (see [7,8,11,12,14,15,21, 24,25]) of equation (1.3). Let $u(x,t) = u(x-ct) = u(\xi)$, where c is the wave speed, substituting it into (1.3) and integrating once, the partial differential equation (1.3) can be transformed to the following ordinary differential equation

$$-cu + cu'' = \frac{1}{2}(p+2)u^{p+1} - \frac{1}{2}pu^{p-1}u'^2 - u^p u'' + g, \tag{1.4}$$

where $'$ is the derivative with respect to ξ, and g is an integral constant. Eq. (1.4) is equivalent to the following ODE system as follows

$$\frac{du}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{g + cu + \frac{1}{2}(p+2)u^{p+1} - \frac{1}{2}pu^{p-1}y^2}{c + u^p}, \tag{1.5}$$

with the first integral

$$H(u,y) = \frac{1}{2}[cu^2 + u^{p+2} - (c+u^p)y^2] + gu = h. \tag{1.6}$$

Eq. (1.5) is a 3-parameter planar dynamical system depending on the parameter group (c, g, p). For different c, g and a fixed p, the bifurcations of (1.5) in the phase plane (u, y) would be shown as the parameters (c, g) are changed.

Notice that when $c + u^p = 0$, the right hand side of the second equation in (1.5) is not continuous. In other words, on the straight line $c + u^p = 0$, $u''(\xi)$ has no definition, it implies that the smooth system (1.3) sometimes has non-smooth traveling wave solutions. The existence of singular straight line for a traveling wave equation would lead to traveling waves losing their smoothness.

2. Bifurcations of Phase Portraits of (1.5)

In this section, we study the phase portraits for equation (1.5). First, imposing the transformation $d\tau = (c+u^p)d\xi$ to Eq. (1.5), it yields the following nonlinear system

$$\frac{d\phi}{d\tau} = (c+u^p)y, \quad \frac{dy}{d\tau} = g + cu + \frac{1}{2}(p+2)u^{p+1} - \frac{1}{2}pu^{p-1}y^2. \tag{2.1}$$

Obviously, Eq. (2.1) has the same topological phase portraits as Eq. (1.4) except for the straight lines $c+u^p=0$. Eqs. (2.1) and (1.5) are integrable because of the same first integral (3.2). For a fixed h, Eq. (1.6) determines a set of invariant curves with different branches. As h is varied, Eq. (1.6) defines different families of orbits with different dynamical behaviors. On the following, $g=0$ and $g\neq 0$ will be considered separately.

Case 1: $g=0$. Firstly, we investigate the bifurcations of phase portraits of (2.1) when $g=0$. Let $f_1(u)=cu+\frac{1}{2}(p+2)u^{p+1}$. To investigate the critical points of the system, we need to find out all zeros of the equation $f_1(u)$=0. It is easy to be found that in the (u,y) phase plane, the abscissas of equilibrium points for Eq. (1.5) on u-axis are the zeros of $f_1(u)$. Note that $f_1'(u)=c+\frac{1}{2}(p+1)(p+2)u^p$. For an even p and $\frac{-2c}{(p+1)(p+2)}>0$, $f_1'(u)$ has two zeros $\tilde{u}_\pm=\pm(\frac{-2c}{(p+1)(p+2)})^{\frac{1}{p}}$, for an odd p and $(p+1)(p+2)\neq 0$, $f_1'(u)$ has one zero $\tilde{u}_+=(\frac{-2c}{(p+1)(p+2)})^{\frac{1}{p}}$. Using these information, we know that all zeros of $f_1(u)$ lies on u-axis. For an even p, when $\frac{-2c}{p+2}>0$, as $f_1(\tilde{u}_-)<0, f_1(\tilde{u}_+)>0$ (or $f_1(\tilde{u}_-)>0, f_1(\tilde{u}_+)<0$), there exists three equilibrium points of (2.1) at $O(0,0)$, $E_{1,2}(u_{1,2},0)$, where $u_1=-(\frac{-2c}{p+2})^{\frac{1}{p}}$,$u_2=(\frac{-2c}{p+2})^{\frac{1}{p}}$.

For an odd p, when $(p+1)(p+2)>0$, as $f_1(\tilde{u}_+)>0$ (or $f_1(\tilde{u}_+)<0$), there exists two equilibrium points of (1.6) at $O(0,0)$, $E_3(u_3,0)$, where $u_3=(\frac{-2c}{p+2})^{\frac{1}{p}}$. Otherwise, when $-c>0$, and p is even or odd, on the singular lines $u=\pm(-c)^{\frac{1}{p}}$ there are four equilibrium points $S_i(u_{s\pm},Y_\pm)$, where $u_{s\pm}=\pm(-c)^{\frac{1}{p}}, Y_\pm=\pm(-|\frac{p}{2}|)^{\frac{1}{p}}\sqrt{|\frac{1-p}{2}|}$, $i=1-4$.

Case 2: $g\neq 0$. Denote that $f_2(u)=g+cu+\frac{1}{2}(p+2)u^{p+1}$, then $f_2'(u)=c+\frac{1}{2}(p+1)(p+2)u^p$.

For $p=2m$, $u=\tilde{u}_0=\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}$ satisfies $f_2'(\pm u)=0$. Thus $f_2(\tilde{u}_0)=g+c\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\frac{1}{2}(p+2)\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{p+1}{p}}$ and $f_2(-\tilde{u}_0)=g-c\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}-\frac{1}{2}(p+2)\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{p+1}{p}}$. It implies the following relations in the (c,g)-parameter plane

$$L_a: g=\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}\frac{-pc}{p+1},\quad c<0, g>0, p>0.$$

$$L_b: g=\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}\frac{pc}{p+1},\quad c<0, g<0, p>0.$$

$$L_d: g=\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}\frac{-pc}{p+1},\quad c<0, g>0, p<0.$$

$$L_e: g=\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}\frac{pc}{p+1},\quad c<0, g<0, p<0.$$

For $p=2m+1$, $u=\tilde{u}_0=\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}$ satisfies $f_2'(u)=0$. Then we have $f_2(\tilde{u}_0)=g+c\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\frac{1}{2}(p+2)\left(\frac{-2c}{(p+1)(p+2)}\right)^{\frac{p+1}{p}}$, which implies the following relations in the (c, g)-parameter plane

$$L_c : g = \left(\frac{-2c}{(p+1)(p+2)} \right)^{\frac{1}{p}} \frac{-pc}{p+1}, \ p > 0.$$

$$L_g : g = \left(\frac{-2c}{(p+1)(p+2)} \right)^{\frac{1}{p}} \frac{-pc}{p+1}, \ p < 0.$$

Next, the bifurcations of the phase portraits of (2.1) will be studied by using the above statements. In (c, g)-parameter plane and on the line $g = 0$, the curve $c + u^p = 0$ partitions it into some regions for $p = 2m$, $p = 2m + 1$, $p = -(2m + 2)$ or $p = -(2m + 1)$ shown in Figs.1 (1-1)-(1-4), respectively.

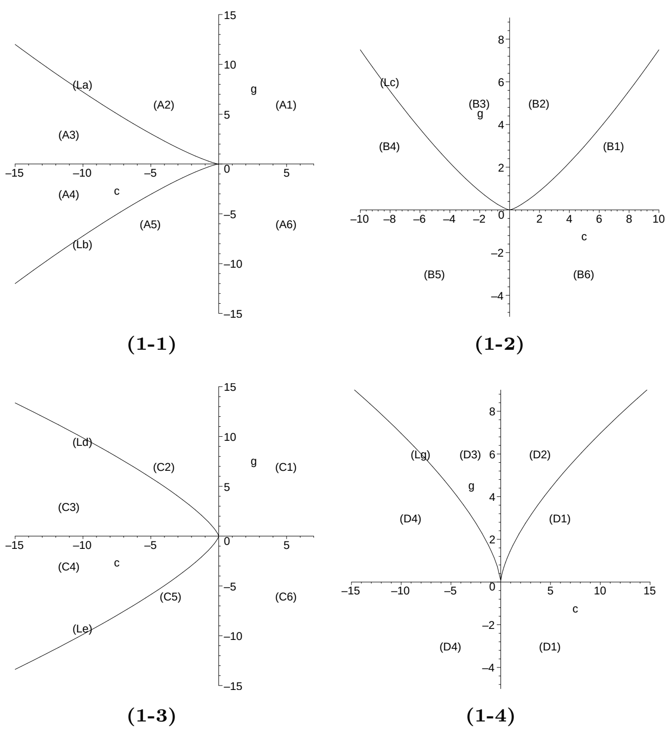


Figure 1. The bifurcations set of (1.5) in (c, g)-parameter plane, for $m \in Z^+$.

Let $M(u_i, y_i)$ be the coefficient matrix of the linearized system of (1.5) at an equilibrium point (u_i, y_i). It is given

$$M(u_i, y_i) = \begin{pmatrix} p u_i^{p-1} y_i & c + u_i^p \\ \frac{1}{2}(p+1)(p+2)u_i^p + c - \frac{1}{2}p(p-1)u_i^{p-2} y_i^2 & -p u_i^{p-1} y_i \end{pmatrix},$$

which therefore reveals so that

$$J(u_i, 0) = -(c + u_i^p) \left(\frac{1}{2}(p+1)(p+2)u_i^p + c \right).$$

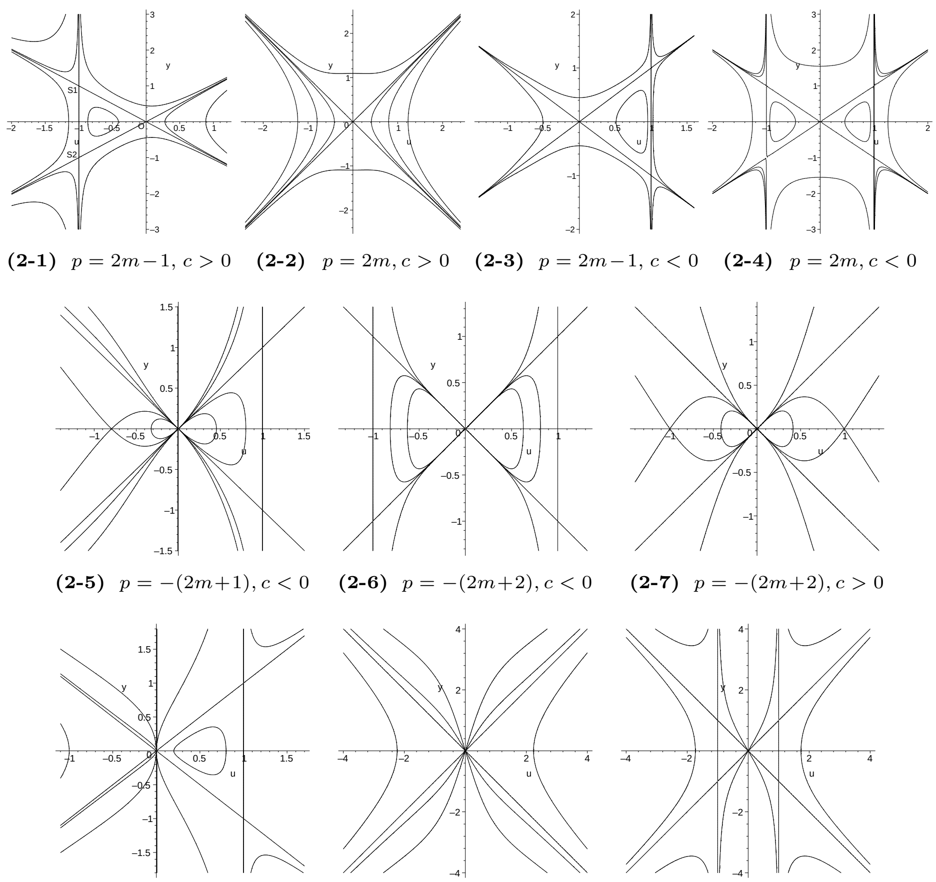
By the theory of planar dynamical systems, the equilibrium point is a saddle point if $J < 0$; and the equilibrium point is a center point if $J > 0$ and

$Trace(M(u_i, 0)) = 0$; the equilibrium point is a node if $J > 0$ and $(Trace(M(u_i, 0)))^2 - 4J(u_i, 0) > 0$, the equilibrium point is a cusp if $J = 0$ and the Poincaré index of the equilibrium point is 0.

For our convenience, denote that

$$h_i = H(u_i, 0) = gu_i + \frac{1}{2}cu^2 + \frac{1}{2}u_i^{p+2}, \qquad i = 0, \ldots, 5.$$

Using the qualitative analysis above, we can obtain the bifurcation curves and phase portraits under various parameter conditions. Figs.2-Figs.4 show the phase portraits of (2.1) in the cases $g = 0$ and $g \neq 0$, respectively.

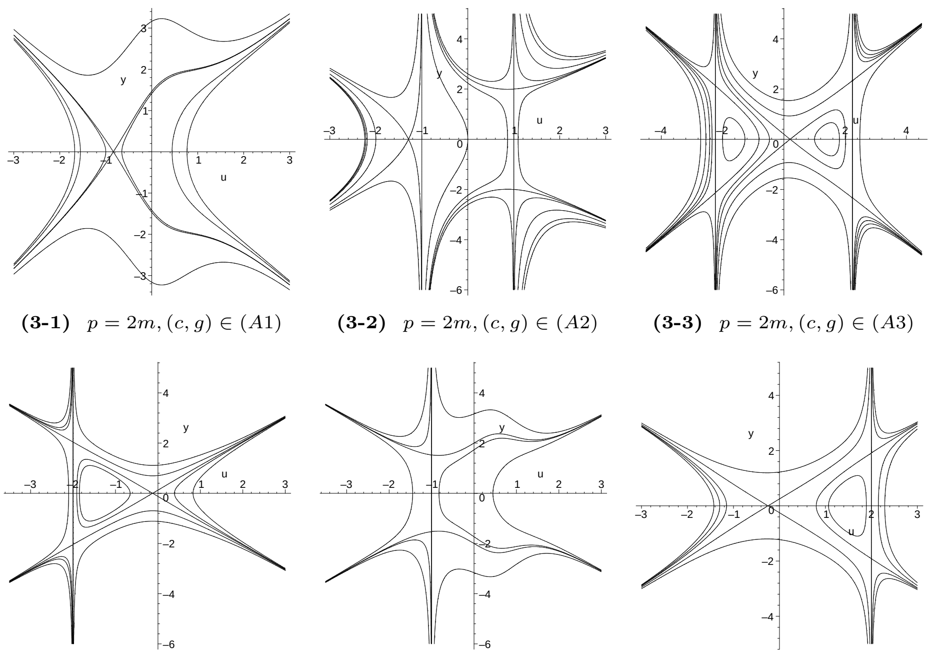


(2-1) $p = 2m-1, c > 0$ **(2-2)** $p = 2m, c > 0$ **(2-3)** $p = 2m-1, c < 0$ **(2-4)** $p = 2m, c < 0$

(2-5) $p = -(2m+1), c < 0$ **(2-6)** $p = -(2m+2), c < 0$ **(2-7)** $p = -(2m+2), c > 0$

(2-8) $p = -1, c < 0$ **(2-9)** $p = -2, c > 0$ **(2-10)** $p = -2, c < 0$

Figure 2. Phase portraits of (1.5) under the parameter conditions $g = 0$ and $m \in Z^+,\ m \geq 1$. When $p = -(2m+1), c > 0$ and $p = -1, c > 0$, the phase portraits are symmetry to Figs. (2-5) and (2-8), respectively.



(3-1) $p = 2m, (c, g) \in (A1)$ **(3-2)** $p = 2m, (c, g) \in (A2)$ **(3-3)** $p = 2m, (c, g) \in (A3)$

(3-4) $p = 2m-1, (c, g) \in (B1)$ **(3-5)** $p = 2m-1, (c, g) \in (B2)$ **(3-6)** $p = 2m-1, (c, g) \in (B5)$

Figure 3. Phase portraits of (1.5) under the parameter conditions $g \neq 0$ and $m \in Z^+$, $m \geq 1$. Similarly, for $p = 2m, (c, g) \in (A6)$, $p = 2m, (c, g) \in (A5)$, $p = 2m, (c, g) \in (A4)$, $p = 2m + 1$, $p = 1, (c, g) \in (B4)$, $p = 2m - 1$, $(c, g) \in (B3)$, $p = 2m - 1$, $(c, g) \in (B6)$, respectively, we can obtain similar bifurcations of phase portraits of (3.1). To save space, we omit them.

3. Some Exact Parametric Representations of Traveling Wave Solutions of (1.3)

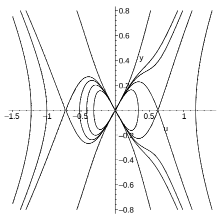
In this section, we provide some parametric representations for heteroclinic orbits, periodic orbits and homoclinic orbits defined by (1.5) in different parameter conditions besides $p = 1$. Obviously, (2.1) has the same orbits as (1.5) except for $c + u^p = 0$. The transformation of variables $d\tau = (c + u^p)d\xi$ leads to some differences between the parametric representations of orbits for (1.5) and (2.1) when $c + u^p = 0$. Notice that $H(u, y) = h$ is defined by (1.6), and in order to find the parametric representation of (1.5), (1.6) can be cast into

$$y^2 = \frac{2gu + cu^2 + u^{p+2} - 2h}{c + u^p}. \tag{3.1}$$

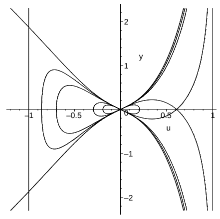
(1) Denote Γ^h is the open curve family determined by $H(u, y) = h, h \in (h_0, h_3)$. A_γ and B_γ are called turning points on the curve Γ^h, and $C(\phi_m, 0)$ stands for the intersection of Γ^h and the y-axis (see Fig. (2-1)). The detailed discussion on the existence of peakon can be seen in [14].

For Fig. (2-1), supposed that $p = 2m - 1, c > 0$, linking with Eq. (1.5), using the standard phase portrait analytical technique gives when $h = 0$ then $y^2 = u^2$. Taking the integration on both sides of equation (1.5) leads to

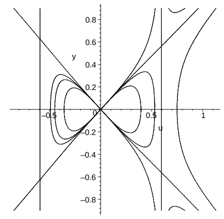
$$u = (-c)^{\frac{1}{p}} e^{-|x-ct|}, \tag{3.2}$$



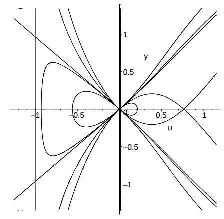
(4-1) $p = -(2m+2), (c,g) \in (C1)$



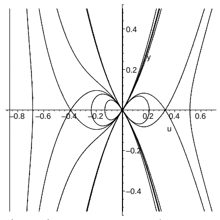
(4-2) $p = -(2m+2), (c,g) \in (C2)$



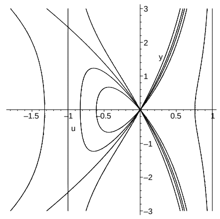
(4-3) $p = -(2m+2), (c,g) \in (C3)$



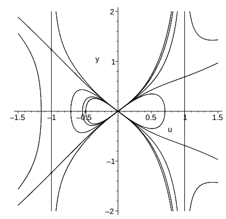
(4-4) $p = -(2m+1), (c,g) \in (D1)$



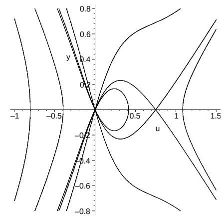
(4-5) $p = -(2m+1), (c,g) \in (D2)$



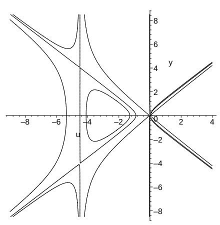
(4-6) $p = -2, (c,g) \in (C2)$



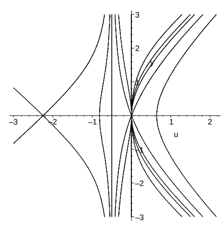
(4-7) $p = -2, (c,g) \in (C3)$



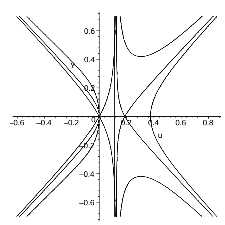
(4-8) $p = -2, (c,g) \in (C6)$



(4-9) $p = -1, (c,g) \in (D1)$



(4-10) $p = -1, (c,g) \in (D2)$



(4-11) $p = -1, (c,g) \in (D4)$

Figure 4. Phase portraits of (1.5) under the parameter conditions $g \neq 0$ and $m \in Z^+,\ m \geq 1$. Similarly, for $p = -(2m+2), (c,g) \in (C6)$, $p = -(2m+2), (c,g) \in (C5)$, $p = -(2m+2), (c,g) \in (C4)$,$p = -(2m+1), (c,g) \in (D4)$, $p = -(2m+1), (c,g) \in (D3)$, $p = -2, (c,g) \in (C5)$, $p = -2, (c,g) \in (C4)$, $p = -2, (c,g) \in (C1)$, $p = -1, (c,g) \in (D3)$, respectively, we can obtain similar bifurcations of phase portraits of (3.1). To save space, we omit them here.

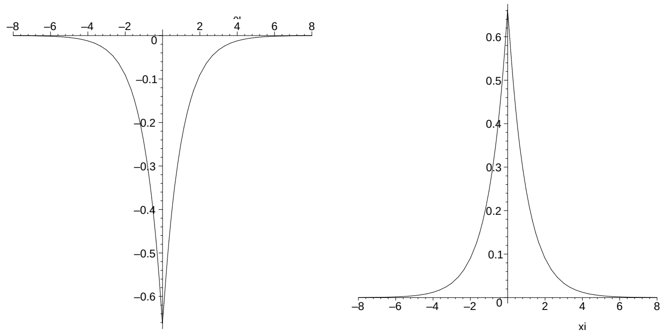
which gives rise to a peakon solution of (1.3) (see Fig. (5-1)). Similarly, (2-3) gives another peakon solution (see Fig. (5-2)), which were also proved in [1].

(2) Suppose that $p = -1, c < 0, g = 0$ (see Fig. (2-8)). Corresponding to the periodic orbit defined by $H(u,y) = h \in (0, h_3)$ enclosing the center point $(u_3, 0)$. It implies that $y^2 = \frac{cu^3+u^2-2hu}{c+u} = \frac{u(u-t_0)(u-t_1)}{u+\frac{1}{c}}$, with $0 < t_0 < t_1 < -\frac{1}{c}$. Thus, by the first equation in (1.5) and from the formulas 258.00, 258.11, 340.04 in [2], we obtain the following parametric representation of periodic orbit (see Fig. (6-1)):

$$|\xi| = \frac{-2(1+ct_1)}{\sqrt{ct_1(-1-ct_0)}}\left[\Pi(\varphi_1, \frac{-1}{ct_1}, k_1) + F(\varphi_1, k_1)\right], \tag{3.3}$$

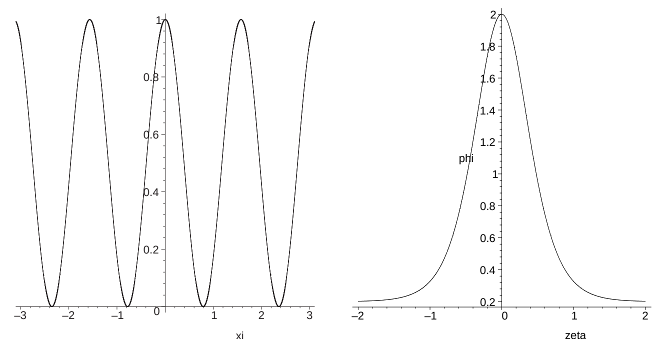
where $\varphi_1 = \arcsin\sqrt{\frac{t_1(cu+1)}{t_1-u}}$, $k_1^2 = \frac{t_0-t_1}{t_1(1+ct_0)}$, and $\Pi(\varphi_1, \frac{-1}{ct_1}, k_1), F(\varphi_1, k_1)$ are elliptic integral functions (see [2]). The profile of the periodic solutions

If $h \to 0$, the periodic orbit is closing to the triangle. This fact tells us that the wave profile of $u(\xi)$ determined by the periodic orbit is a peakon solution, that is the triangle gives rise to a peakon wave solution (see Fig. (7-2)).



(5-1) **(5-2)**

Figure 5. Peakon solutions corresponding to Figs. (2-1) and (2-3), respectively.



(6-1) **(6-2)**

Figure 6. Profiles of periodic wave solutions and smooth solitary wave solutions.

(3) Suppose that $p = 2m, (c, g) \in (A3)$. In this case, we have the phase portrait of (2.1) shown in Fig. (3-3). For concrete value, taking $p = 2$, Eq. (1.6) can be changed to

$$y^2 = \frac{u^4 + cu^2 + 2gu - 2h}{c + u^2} = \frac{(u - t_2)(u - t_3)(u - t_4)^2}{c + u^2}, \tag{3.4}$$

where $t_2 < 0 < t_4 < t_3$. Setting $G(u) = \frac{1}{(u-t_4)}\sqrt{\frac{c+u^2}{(u-t_2)(u-t_3)}}$ Thus, by the first equation of (1.5) and [2], we obtain the following parametric representation:

$$\xi = \int_u^{t_3} G(u)du = \frac{t_4 - \sqrt{-c}}{\sqrt{(t_3 - t_4)(t_4 - t_2)}}\Phi_2(u) - \Phi_3(u) + \beta, \tag{3.5}$$

where

$$\Phi_2(u) = \ln\left|\frac{(t_4-t_3)(u-t_2)+(t_4-t_2)(u-t_3)-2\sqrt{(t_4-t_3)(t_4-t_2)(u-t_3)(u-t_2)}}{u-t_4}\right|,$$

$$\Phi_3(u) = \ln\left|\sqrt{(u-t_3)(u-t_2)}+u-\frac{t_2+t_3}{2}\right|, \beta = \ln\left|\frac{t_3-t_2}{2}\right| - \frac{t_4-\sqrt{-c}}{\sqrt{(t_3-t_4)(t_4-t_2)}}\ln|t_3-t_2|\,.$$

(3.5) gives rise to a smooth solitary solution of system (1.5) (see Fig. (6-2)).

(4) Suppose that $p = -2, (c, g) \in (C2)$. In this case, we have the phase portrait of (2.1) shown in Fig. (4-6). Corresponding to the orbits defined by $H(u, y) = h \in (h_s, 0)$ and Eq. (2.1), there are a family of homoclinic orbits inside the triangle. Then Eq. (1.6) can be rewritten as

$$y^2 = \frac{u^2(cu^2 + gu + 1 - 2h)}{cu^2 + 1} = \frac{cu^2(u - t_5)(u - t_6)}{cu^2 + 1}, \tag{3.6}$$

with $t_6 < 0 < t_5$. Hence, by the first equation of (1.5), we have the following parametric representation:

$$\xi = \ln\left|\frac{-t_5 - t_6}{2} + u + \sqrt{(u - t_5)(u - t_6)}\right| - \frac{1}{\sqrt{ct_5t_6}}\Phi_1(u), \tag{3.7}$$

where $\Phi_1(u) = \ln\left|\frac{2t_5t_6 - (t_5+t_6)u + 2\sqrt{-t_5t_6}\sqrt{(u-t_5)(u-t_6)}}{u}\right|$. (3.7) gives rises to a smooth solitary solution(see Fig. (6-2)).

(5) Suppose that $p = -2, (c, g) \in (C6)$. In this case, we have the phase portrait of (2.1) shown in Fig. (4-8).

(I) Corresponding $H(u, y) = h_1$ defined by (9) and Eq. (2.1), there are two heteroclinic orbits connecting to two equilibrium points $O(0, 0)$ and $E_1(u_1, 0)$. Then Eq. (1.6) can be rewritten as

$$y^2 = \frac{u^2(cu^2 + 2gu + 1 - 2h_1)}{cu^2 + 1} = \frac{cu^2(u - t_7)^2}{cu^2 + 1}. \tag{3.8}$$

Thus, by the first equation of (1.5), we obtain the following parametric representation of a kink and anti-kink wave solutions (see Fig. (7-1)):

$$\xi = \int_u^{u_1} \sqrt{\frac{cu^2 + 1}{cu^2(u - t_7)^2}}du = -\left(1 + \frac{1}{\sqrt{c}t_7}\right)\ln\left|\frac{u - t_7}{u_1 - t_7}\right| + \frac{1}{\sqrt{c}t_7}\ln\left|\frac{u}{u_1}\right|. \tag{3.9}$$

(II) Corresponding $H(u, y) = h \in (h_1, 0)$ defined by (1.6) and (2.1) there are a family of homoclinic orbits pass through the original point $O(0, 0)$, which inside two heteroclinic orbits mentioned in (5)(I). Thus Eq. (1.6) can be rewritten as $y^2 = \frac{u^2(cu^2+2gu+1-2h)}{cu^2+1} = \frac{cu^2(u-\tilde{t_8})(u-\tilde{t_9})}{cu^2+1}$ where $0 < \tilde{t_9} < \tilde{t_8}$. It has the same parametric representation as Eq. (3.7).

(6) Suppose that $p = -1, (c, g) \in (D1)$. In this case, we have the phase portrait of (2.1) shown in Fig. (4-9).

(I) Corresponding to the curve triangle defined by $H(u, y) = h_s$, by (3.3) there are an arch heteroclinic orbits connecting to two equilibrium points $E(u_{s+}, Y_+)$ and $E(u_{s+}, Y_-)$. Eq. (1.6) can be rewritten as

$$y^2 = \frac{cu^3 + (1 + 2g)u^2 - 2h_su}{cu + 1} = u(u - t_8), \tag{3.10}$$

where $t_8 < 0$ Thus, we obtain the following parametric representation of the periodic peakon wave solution is obtained:

$$u(\xi) = t_8\left(1 - \frac{\sqrt{-t_8}}{2}\cosh(\xi)\right), \quad \xi \in \left(0, \cosh^{-1}\frac{2}{\sqrt{-t_8}}\right), \tag{3.11}$$

which gives rise to periodic peakon solution of (1.3) (see Fig. (7-2)), which is introduced in detail by Li and Zhu in Theorem A of [16].

Particularly, when $u > -c$, the direction of the vector field defined by (1.5) is opposite to the direction of the vector field defined by (2.1), they gives rise to an unbounded solution of system (1.5).

(II) Corresponding to the orbits defined by $H(u, y) = h \in (h_1, h_s)$, by Eq. (2.1) there are a family of periodic orbits enclosing the equilibrium point $E_1(u_1, 0)$. Eq. (1.6) can be rewritten as

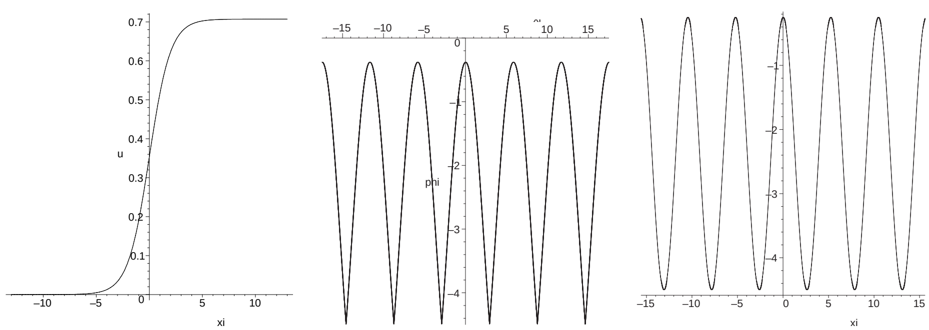
$$y^2 = \frac{u(cu^2 + (1 + 2g)u - 2h)}{cu + 1} = \frac{cu(u - t_9)(u - t_{10})}{cu + 1}, \tag{3.12}$$

with $-\frac{1}{c} < t_9 < t_{10} < 0$. Thus, by the first equation of (1.5) and from the formulas 258.00, 258.11, 340.04 in [2], we have the following parametric representation of the periodic wave solutions (see Fig. (7-3)):

$$\xi = 2\sqrt{\frac{ct_{10} + 1}{-ct_9}} \left(\frac{ct_{10}}{ct_{10} + 1} \Pi(\varphi_5, \frac{1}{ct_{10} - 1}, k_5) + F(\varphi_5, k_5) \right), \tag{3.13}$$

where $k_5^2 = \frac{t_{10} - t_9}{-t_9(ct_{10} + 1)}, \varphi_5 = \arcsin \sqrt{\frac{u(ct_{10}+1)}{(u - t_{10})}}$.

If $h \to h_s$, the periodic is going to the curve triangle. This fact indicates that the wave profile of $u(\xi)$ determined by the periodic orbit is a periodic peakon wave, which will be solved on the following, and the curve triangle gives rise to a periodic peakon wave solution such like Eq. (3.11).



(7-1) **(7-2)** **(7-3)**

Figure 7. Profiles of kink solution, periodic peakon solutions and periodic wave solutions..

4. Conclusion

In summary, the bifurcations of the generalized Camassa-Holm equation are investigated by using the bifurcation theory and phase portraits of (1.6). It shows that the system (1.3) has peakon, peakon periodic solutions, solitary wave solution, kink and anti-kink wave solutions, and periodic wave solutions under some parameter conditions. Finally, we describe the dynamical behaviors of (1.3) and explain the reason for the existence of non-smooth traveling wave. From the above analysis, it can be known that this method can be suitable for other nonlinear wave equations.

Acknowledgments

One of the authors (M. Z. Wei) is grateful to Professor Zhijun Qiao for his enthusiastic guidance and help during her visit to University of Texas-Rio Grande Valley. All the authors thank the referee for the valuable suggestions which help improve the paper.

References

[1] S. C. Anco, E. Recio, M. L. Gandarias and M. S. Bruzon, *A nonlinear generalization of the Camassa-Holm equation with peakon solutions*, Dynamical Systems. Diff. Equat. Appl. AIMS Proceedings, 2015 Special, 29–37.

[2] P. F. Byrd and M. D. Friedman, *Handbook of elliptic integrals for engineers and scientists*, New York: Springer-Verlag, 1971.

[3] R. Camassa and D. D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett., 1993, 71, 1661–1664,.

[4] R. Camassa, D. D. Holm and J. M. Hyman, *A new integrable shallow water equation*, Adv. Appl. Mech., 1994, 31, 1–33.

[5] A. Constantin, *Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, Ans. Inst. Fourier (Grenoble), 2000, 50, 321–362.

[6] A. Constantin and J. Escher, *On the blow-up rate and the blow-up set of breaking waves for a shallow water equation*, Math. Z., 2000, 33, 75–91.

[7] A. Y. Chen, J. B. Li, X. J. Deng and W.T. Huang, *Traveling wave solutions of the Fornberg-Whitham equation*, Appl. Math. Comput., 2009, 215, 3068–3075.

[8] A. Y. Chen and J. B. Li, *Single peak solitary wave solutions for the osmosis $K(2,2)$ equation under inhomogeneous boundary condition*, J. Math. Anal. Appl., 2010, 369, 758–766.

[9] A. Degasperis, D. D. Holm and A. N. W. Hone, *A new integrable equation with peakon solutions*, Theor. Math. Phys., 2002, 133, 1463–1474.

[10] D. D. Holm and A. N. W. Hone, *A class of equations with peakon and pulson solutions*, J. Non. Math. Phys., 2005, 12, 380–394.

[11] M. A. Han, *Bifurcation theory of limit cycles of planar systems*, in: A. Canada, P. Drabek, A. Fonda (Eds.), Handbook of differential equations, Ordinary differential equations, vol. 3, Elsevier, 2006.

[12] M. A. Han, J. M. Yang, A. A. Tarta and Y. Gao, *Limit cycles near homoclinic and heteroclinic loops*, J. Dyn. Diff. Equat., 2008, 20, 923–944.

[13] J. B. Li and Y. Zhang, *Exact loop solutions, cusp solutions, solitary wave solutions and periodic wave solutions for the special CH-DP equation*, Nonlinear Anal.: Real World Appl., 2009, 10, 2502–2507.

[14] J. B. Li and H. H. Dai, *On the study of singular nonlinear traveling wave equations: Dynamical System Approach*, Beijing: Science Press, 2007.

[15] J. B. Li and F. J. Chen, *Exact traveling wave solutions and bifurcations of the dual Ito equation*, Nonlinear Dyn., 2015 , 82, 1537–1550.

[16] J. B. Li, W. J. Zhu and G. R. Chen, *Understanding peakons, periodic peakons and compactons via a shallow water wave equation*, Int. J. Bifurcat. Chaos, 2016, 12, 1650207.

[17] Z. R. Liu and T. F. Qian, *Peakons of the Camassa-Holm equation*, Appl. Math. Model., 2002, 26, 473–480.

[18] Z. J. Qiao and G. P. Zhang, *On peaked and smooth solitons for the Camassa-Holm equation*, Europhys. Lett., 2006, 73, 657–663.

[19] J. W. Shen, W. Xu and W. Li, *Bifurcations of travelling wave solutions in a new integrable equation with peakon and compactons*, Chaos, Solitons & Fract., 2006, 27, 413–425.

[20] A. M. Wazwaz, *A class of nonlinear fourth order variant of a generalized Camassa-Holm equation with compact and noncompact solutions*, Appl. Math. Comput., 2005, 165, 485–501.

[21] M. Z. Wei, X. B. Sun and S. Q. Tang, *Single peak solitary wave solutions for the CH-KP(2,1) equation under boundary condition*, J. Diff. Equat., 2015, 259, 628–641.

[22] G. P. Zhang and Z. J. Qiao, *Cuspons and smooth solitons of the Degasperis-Procesi equation under inhomogeneous boundary condition*, Math. Phys. Anal. Geom., 2007, 10, 205–225.

[23] L. N. Zhang and A. Y. Chen, *Exact loop solitons, cuspons, compactons and smooth solitons for the Boussinesq-like B(2,2) equation*, Proc. Roman. Acad. A., 2014, 15, 11–17.

[24] L. N. Zhang, A. Y. Chen and J. D. Tang, *Special exact soliton solutions for the* $K(2,2)$ *equation with non-zero constant pedestal*, Appl. Math. Comput., 2011, 218, 4448–4457.

[25] W. J. Zhu and J. B. Li, *Exact traveling wave solutions and bifurcations of the Biswas-Milovic equation*, Non. Dyn., 2016, 84, 1973–1987.