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BOUNDEDNESS OF SOLUTIONS FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH INTEGRAL JUMP CONDITIONS*

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Abstract The boundedness of solutions for certain nonlinear impulsive differential equations are obtained, the jumping conditions at discontinuous points are related to the integral of the past states, rather than a left hand limit at the discontinuous points. These results are obtained by new built impulsive integral inequalities with integral jumping conditions using the method of successive iteration.

Keywords Impulsive differential equation, boundedness, integral inequalities, integral jump condition, discontinuous function.

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1. Introduction

Impulsive differential equations are differential equations involving impulse effect, which appear as a natural description of observed evolution phenomena of several real world problems. The theories of impulsive differential equations are firstly researched by V.D. Milman and A.D. Myshkis in 1960s, and they take on a blooming research scene in recent years. They are widely used in many areas such as biological mathematics (they are used in characterizing the heart beats, blood flows, population dynamics), theoretical physics, pharmacokinetics, mathematical economy, chemical technology, electric technology and so on (see the monographs [1, 2, 9] and [15] for details).

In spite of their importance of impulsive differential equations, the development of the theory of impulsive differential equations is quite slow due to the special features possessed by impulsive differential equations in general. Among these results, differential inequalities and integral inequalities with impulsive effects play increasingly important roles in the study of quantitative and qualitative properties of solutions of impulsive differential systems. However, most of these results involved the impulsive effects are of point-discontinuous, i.e., jumping conditions at a sequences

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of discontinuous points depend on the left hand limits (see [3, 4, 6–8, 10–14, 16–23] for details). For example, D. S. Borysenko [3] considered the following integral inequality with impulsive effect

$$u(t) \leq a(t) + \int_{t_0}^{t} f(s)u(s)ds + \sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0),$$

and gave an estimate of unknown function; in [8], G. Iovane studied the following integral inequalities

$$u(t) \leq a(t) + \int_{t_0}^{t} f(s)u(\lambda(s))ds + \sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0),$$

$$u(t) \leq a(t) + q(t)\left[\int_{t_0}^{t} f(s)u(\alpha(s))ds + \int_{t_0}^{t} f(s)\int_{t_0}^{s} g(t)u(\tau(t))dtds + \sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0)\right];$$

in [20], Wusheng Wang gave the upper bound for the nonlinear inequality of the form

$$v^p(t) \leq A_0(t) + \frac{p}{p-q}\int_{t_0}^{t} f(s)v^q(\tau(s))ds + \sum_{t_0<t_i<t} \alpha_i v^q(t_i - 0).$$

As we know, most of phenomena occur in natural world are not sudden changed, thus the impulsive differential equations with integral jump conditions are more accurate than impulsive differential equations with stationary discontinuous points in characterizing the nature.

In paper [18], the authors investigated the following integral inequality

$$m(t) \leq c + \int_{t_0}^{t} p(s)m(s)ds + \sum_{t_0<t_k<t} \beta_k m(t_k) + \sum_{t_0<t_k<t} \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds, \quad (1.1)$$

and obtained the estimation of $m(t)$:

$$m(t) \leq c \prod_{t_0<t_k<t}\left[(1+\beta_k)e^{\int_{t_{k-1}}^{t_k} p(\tau)d\tau} + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s} p(\tau)d\tau} ds\right] e^{\int_{t_{k-1}}^{t} p(\tau)d\tau}. \quad (1.2)$$

Using the method of successive iteration, the authors in paper [16] researched some new nonlinear impulsive differential inequalities and integral inequalities with integral jump conditions.

Motivated by the above mentioned papers, we will investigate some more generalized integral inequalities with integral jump conditions, and give the upper-bound estimate of unknown functions firstly, then we use the new built inequalities to investigate the boundedness of solutions for impulsive differential equations with integral jump conditions at the discontinuous points.

2. New integral inequalities with integral jump conditions

Firstly, we give some auxiliary lemmas which are important in the proofs of the main results.

Lemma 2.1. *If $z(t) \in C[t_0, b)$ such that*

$$\begin{cases} z'(t) \le p(t)z(t) + q(t), \\ z(t_0) \le z_0, \end{cases}$$

where $p(t),\ q(t) \in C[t_0, b)$, then

$$z(t) \le e^{\int_{t_0}^{t} p(s)ds} \left[z_0 + \int_{t_0}^{t} q(s) e^{-\int_{t_0}^{s} p(\tau)d\tau} ds \right].$$

Particularly, $z(t_0) = 0$ implies that

$$z(t) \le \int_{t_0}^{t} q(s) e^{\int_{s}^{t} p(\tau)d\tau} ds.$$

By the Young inequality (see [5] for details), we have the following lemma.

Lemma 2.2. *Let $C(t)$ and $x(t)$ be nonnegative functions, let $0 < \lambda < 1$ be a real number. Then for any positive function $K(t)$,*

$$Cx^{\lambda} \le \lambda K^{\lambda - 1} C^{\alpha} x + (1 - \lambda) K^{\lambda} C^{\beta},$$

where α and β are nonnegative constants satisfying

$$\lambda\alpha + (1 - \lambda)\beta = 1.$$

Throughout this paper, we always assume that $0 \le t_0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = +\infty$, $\mathbb{R}_+ = [0, +\infty)$, $I \subset \mathbb{R}$. $x(t) : [t_0, \infty) \to \mathbb{R}_+$ is a continuous function for $t \ne t_k$, $x(0^+)$, $x(t_k^-)$ and $x(t_k^+)$ exist, $x(t_k^-) = x(t_k)$, $k = 1, 2, \cdots$, i.e., $x(t)$ is left-continuous at t_k, $k = 1, 2, \cdots$.

Theorem 2.1. *Suppose that $x(t)$ is left-continuous at t_k and satisfies*

$$x(t) \le a(t) + b(t) \int_{t_0}^{t} p(s)x(s)ds + \sum_{t_0 < t_k < t} \alpha_k x(t_k) + \sum_{t_0 < t_k < t} C_k \int_{t_k - \tau_k}^{t_k - \sigma_k} x(s)ds, \tag{2.1}$$

where $a(t),\ b(t),\ p(t) \in C([t_0, \infty), \mathbb{R}_+)$, $\alpha_k \ge 0$, $C_k \ge 0$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$ are constants. Then for $t \in (t_k, t_{k+1}]$,

$$x(t) \le a_k(t) + b(t) \int_{t_k}^{t} p(s) a_k(s) e^{\int_{s}^{t} p(\tau)b(\tau)d\tau} ds, \tag{2.2}$$

where

$$\begin{aligned} a_k(t) = a(t) &+ \sum_{t_0 < t_k < t} \alpha_k a_{k-1}(t_k) \\ &+ \sum_{t_0 < t_k < t} (b(t) + \alpha_k b(t_k)) \int_{t_{k-1}}^{t_k} p(s) a_{k-1}(s) e^{\int_{s}^{t_k} p(\tau)b(\tau)d\tau} ds \\ &+ \sum_{t_0 < t_k < t} C_k \int_{t_k - \tau_k}^{t_k - \sigma_k} \left[a_{k-1}(s) + b(s) \int_{t_{k-1}}^{s} p(v) a_{k-1}(v) e^{\int_{v}^{s} p(\tau)b(\tau)d\tau} dv \right] ds, \end{aligned} \tag{2.3}$$

with $a_0(t) \equiv a(t)$.

Proof. For $t \in [t_0, t_1]$, we have that

$$x(t) \le a(t) + b(t) \int_{t_0}^{t} p(s)x(s)ds. \tag{2.4}$$

Let $y_1(t) = \int_{t_0}^{t} p(s)x(s)ds$, then $y_1(t_0) = 0$, $x(t) \le a(t) + b(t)y_1(t)$, $y_1^{'}(t) = p(t)x(t) \le p(t)b(t)y_1(t) + p(t)a(t)$. Hence

$$y_1(t) \le \int_{t_0}^{t} p(s)a(s)e^{\int_s^t p(\tau)b(\tau)d\tau}ds. \tag{2.5}$$

When $t \in (t_1, t_2]$, we have

$$\begin{aligned}
x(t) \le & a(t) + b(t) \int_{t_0}^{t} p(s)x(s)ds + \alpha_1 x(t_1) + C_1 \int_{t_1-\tau_1}^{t_1-\sigma_1} x(s)ds \\
= & [a(t) + \alpha_1 a(t_1)] + [b(t) + \alpha_1 b(t_1)] \int_{t_0}^{t_1} p(s)a(s)e^{\int_s^{t_1} p(\tau)b(\tau)d\tau}ds \\
& + C_1 \int_{t_1-\tau_1}^{t_1-\sigma_1} \left[a(s) + b(s) \int_{t_0}^{s} p(\nu)a(\nu)e^{\int_\nu^s p(\tau)b(\tau)d\tau}d\nu\right] ds \\
& + b(t) \int_{t_1}^{t} p(s)x_1(s)ds \\
\triangleq & a_1(t) + b(t) \int_{t_1}^{t} p(s)x(s)ds.
\end{aligned}$$

Let $y_2(t) = \int_{t_1}^{t} p(s)x(s)ds$, we have by Lemma 2.2 that

$$x(t) \le a_1(t) + b(t) \int_{t_1}^{t} p(s)a_1(s)e^{\int_s^t p(\tau)b(\tau)d\tau}ds, \tag{2.6}$$

this shows that (2.2) holds for $k = 1$.

Now we assume that (2.2) holds for $t \in [t_0, t_k]$. This implies that for $t \in (t_{k-1}, t_k]$,

$$x(t) \le a_{k-1}(t) + b(t) \int_{t_{k-1}}^{t} p(s)a_{k-1}(s)e^{\int_s^t p(\tau)b(\tau)d\tau}ds,$$

where

$$\begin{aligned}
a_{k-1}(t) = a(t) &+ \sum_{i=1}^{k-1} \alpha_i a_{i-1}(t_i) \\
&+ \sum_{i=1}^{k-1} [b(t) + \alpha_i b(t_i)] \int_{t_{i-1}}^{t_i} p(s)a_{i-1}(s)e^{\int_s^{t_i} p(\tau)b(\tau)d\tau}ds \\
&+ \sum_{i=1}^{k-1} C_i \int_{t_i-\tau_i}^{t_i-\sigma_i} \left[a_{i-1}(s) + b(s) \int_{t_{i-1}}^{s} p(\nu)a_{i-1}(\nu)e^{\int_\nu^s p(\tau)b(\tau)d\tau}d\nu\right] ds.
\end{aligned}$$

Then for $t \in (t_k, t_{k+1}]$, we get

$$x(t) \le a(t) + b(t) \int_{t_0}^{t} p(s)x(s)ds + \sum_{i=1}^{k} \alpha_i x(t_i) + \sum_{i=1}^{k} C_i \int_{t_i-\tau_i}^{t_i-\sigma_i} x(s)ds$$

$$
\begin{aligned}
=& a(t) + b(t)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} p(s)x(s)ds + \sum_{i=1}^{k}\alpha_i x(t_i) \\
&+ \sum_{i=1}^{k} C_i \int_{t_i-\tau_i}^{t_i-\sigma_i} x(s)ds + b(t)\int_{t_k}^{t} p(s)x(s)ds \\
\leq& a(t) + b(t)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} p(s)a_{i-1}(s)e^{\int_s^{t_i} p(\tau)b(\tau)d\tau}ds \\
&+ \sum_{i=1}^{k}\alpha_i\left(a_{i-1}(t_i) + b(t_i)\int_{t_{i-1}}^{t_i} p(s)a_{i-1}(s)e^{\int_s^{t_i} p(\tau)b(\tau)d\tau}ds\right) \\
&+ \sum_{i=1}^{k} C_i \int_{t_i-\tau_i}^{t_i-\sigma_i} \Big[a_{i-1}(s) \\
&+ b(s)\int_{t_{i-1}}^{s} p(\nu)a_{i-1}(\nu)e^{\int_\nu^{s} p(\tau)b(\tau)d\tau}d\nu\Big] ds + b(t)\int_{t_k}^{t} p(s)x(s)ds \\
=& a_k(t) + b(t)\int_{t_k}^{t} p(s)x(s)ds.
\end{aligned}
$$

Hence we have for $t \in (t_k, t_{k+1}]$,

$$x(t) \leq a_k(t) + b(t)\int_{t_k}^{t} p(s)a_k(s)e^{\int_s^{t} p(\tau)b(\tau)d\tau}ds,$$

where

$$
\begin{aligned}
a_k(t) = a(t) &+ \sum_{t_0<t_k<t}\alpha_k a_{k-1}(t_k) \\
&+ \sum_{t_0<t_k<t}[b(t) + \alpha_k b(t_k)]\int_{t_{k-1}}^{t_k} p(s)a_{k-1}(s)e^{\int_s^{t_k} p(\tau)b(\tau)d\tau}ds \\
&+ \sum_{t_0<t_k<t} C_k \int_{t_k-\tau_k}^{t_k-\sigma_k}\left[a_{k-1}(s) + b(s)\int_{t_{k-1}}^{s} p(\nu)a_{k-1}(\nu)e^{\int_\nu^{s} p(\tau)b(\tau)d\tau}d\nu\right]ds.
\end{aligned}
$$

This completes the proof of Theorem 2.1 by the mathematical induction. □

Corollary 2.1. *Suppose that $\alpha_k \equiv 0$ in (2.1), i.e.,*

$$x(t) \leq a(t) + b(t)\int_{t_0}^{t} p(s)x(s)ds + \sum_{t_0<t_k<t} C_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds,$$

then for $t \in (t_k, t_{k+1}]$,

$$x(t) \leq \widetilde{a}_k(t) + b(t)\int_{t_k}^{t} p(s)\widetilde{a}_k(t)e^{\int_s^{t} p(\tau)b(\tau)d\tau}ds,$$

where

$$\widetilde{a}_k(t) = a(t) + b(t)\sum_{t_0<t_k<t}\int_{t_{k-1}}^{t_k} p(s)a_{k-1}(s)e^{\int_s^{t_k} p(\tau)b(\tau)d\tau}ds$$

$$+\sum_{t_0<t_k<t} C_k \int_{t_k-\tau_k}^{t_k-\sigma_k} [\widetilde{a}_{k-1}(s) + b(s) \int_{t_{k-1}}^{s} p(\nu)\widetilde{a}_{k-1}(\nu)e^{\int_\nu^s p(\tau)b(\tau)d\tau} d\nu]ds$$

with $\widetilde{a_0}(t) \equiv a(t)$.

Next, we will give some nonlinear integral inequalities with integral jump conditions.

Theorem 2.2. *We suppose that* $x(t)$ *is left-continuous at* t_k *and satisfies*

$$\begin{aligned} x(t) \le a(t) + b(t)\int_{t_0}^{t} \left[g(s)x(s) + h(s)x^{\lambda}(s)\right] ds + \sum_{t_0<t_k<t} \alpha_k x(t_k) \\ + \sum_{t_0<t_k<t} \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds, \end{aligned} \tag{2.7}$$

where $a(t)$, $b(t)$, $g(t)$, $h(t) \in C([t_0,\infty),\mathbb{R}_+)$, $\alpha_k \ge 0$, $C_k \ge 0$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$ *are constants. Then for* $t \in (t_k, t_{k+1}]$, *we have*

$$x(t) \le a_k(t) + b(t)\int_{t_k}^{t} Q(s)e^{\int_s^t P(\tau)d\tau} ds, \tag{2.8}$$

where

$$\begin{aligned} a_k(t) =& a(t) + \sum_{t_0<t_k<t} \alpha_k a_{k-1}(t_k) \\ &+ \sum_{t_0<t_k<t} (b(t) + \alpha_{k-1}b(t_k)) \int_{t_{k-1}}^{t_k} Q(s)e^{\int_s^{t_k} P(\tau)b(\tau)d\tau} ds \\ &+ \sum_{t_0<t_k<t} C_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \left[a_{k-1}(s) + b(s)\int_{t_{k-1}}^{s} Q(v)e^{\int_\nu^s P(\tau)d\tau} d\nu\right] ds, \end{aligned} \tag{2.9}$$

$$P(t) = b(t)[g(t) + \lambda K^{\lambda-1}(t)h^{\alpha}(t)], \tag{2.10}$$

and

$$Q(t) = a(t)[g(t) + \lambda K^{\lambda-1}(t)h^{\alpha}(t)] + (1-\lambda)K^{\lambda}(t)h^{\beta}(t). \tag{2.11}$$

Proof. For $t \in [t_0, t_1]$, we have that

$$x(t) \le a(t) + b(t)\int_{t_0}^{t} \left[g(s)x(s) + h(s)x^{\lambda}(s)\right] ds. \tag{2.12}$$

Let $y_1(t) = \int_{t_0}^{t} \left[g(s)x(s) + h(s)x^{\lambda}(s)\right] ds$, then $y_1(t_0) = 0$, $x(t) \le a(t) + b(t)y_1(t)$. Hence we get

$$\begin{aligned} y_1^{'}(t) &= g(t)x(t) + h(t)x^{\lambda}(t) \\ &\le g(t)x(t) + \lambda K^{\lambda-1}(t)h^{\alpha}(t)x(t) + (1-\lambda)K^{\lambda}(t)h^{\beta}(t) \\ &= \left[g(t) + \lambda K^{\lambda-1}(t)h^{\alpha}(t)\right][a(t) + b(t)y_1(t)] + (1-\lambda)K^{\lambda}(t)h^{\beta}(t) \\ &= P_1(t)y_1(t) + Q(t). \end{aligned} \tag{2.13}$$

By Lemma 2.2, we have

$$y_1(t) \le \int_{t_0}^{t} Q(s) e^{\int_s^t p(\tau)d\tau} ds. \tag{2.14}$$

Hence for $t \in (t_0, t_1]$, we obtain that

$$x(t) \le a(t) + b(t) \int_{t_0}^{t} Q(s) e^{\int_s^t p(\tau)d\tau} ds.$$

Now for $t \in (t_1, t_2]$,

$$\begin{aligned} x(t) \le & a(t) + b(t) \int_{t_0}^{t} \left[g(s)x(s) + h(s)x^{\lambda}(s) \right] ds + \alpha_1 x(t_1) + \int_{t_1-\tau_1}^{t_1-\sigma_1} x(s) ds \\ = & [a(t) + \alpha_1 a(t_1)] + [b(t) + \alpha_1 b(t_1)] \int_{t_0}^{t_1} Q(s) e^{\int_s^{t_1} p(\tau)d\tau} ds \\ & + \int_{t_1-\tau_1}^{t_1-\sigma_1} \left[a(s) + b(s) \int_{t_0}^{s} Q(\nu) e^{\int_\nu^s p(\tau)d\tau} d\nu \right] ds \\ & + b(t) \int_{t_1}^{t} \left[g(s)x(s) + h(s)x^{\lambda}(s) \right] \\ \triangleq & a_1(t) + b(t) \int_{t_1}^{t} \left[g(s)x(s) + h(s)x^{\lambda}(s) \right] ds. \end{aligned}$$

Similar calculation as (2.14) implies that

$$x(t) \le a_1(t) + b(t) \int_{t_1}^{t} Q(s) e^{\int_s^t p(\tau)d\tau} ds, t \in (t_1, t_2].$$

This means that (2.7) holds for $k = 1$. Suppose for $t \in [t_0, t_k]$, (2.7) holds, that is

$$x(t) \le a_{k-1}(t) + b(t) \int_{t_{k-1}}^{t} Q(s) e^{\int_s^t p(\tau)d\tau} ds,$$

where

$$\begin{aligned} a_{k-1}(t) = & a(t) + \sum_{i=1}^{k-1} \alpha_i a_{i-1}(t_i) \\ & + \sum_{t_0 < t_k < t} (b(t) + \alpha_{k-1} b(t_k)) \int_{t_{k-1}}^{t_k} Q(s) e^{\int_s^{t_k} p(\tau)b(\tau)d\tau} ds \\ & + \sum_{t_0 < t_k < t} C_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \left[a_{k-1}(s) + b(s) \int_{t_{k-1}}^{s} Q(v) e^{\int_\nu^s p(\tau)d\tau} d\nu \right] ds, \end{aligned}$$

Hence we have for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} x(t) \le & a(t) + b(t) \left[\int_{t_0}^{t_1} Q(s) e^{\int_s^{t_1} p(\tau)d\tau} + \int_{t_1}^{t_2} Q(s) e^{\int_s^{t_1} p(\tau)d\tau} + \cdots \right. \\ & \left. + \int_{t_{k-1}}^{t_k} Q(s) e^{\int_s^{t_1} p(\tau)d\tau} \right] \end{aligned}$$

$$
\begin{aligned}
&+ b(t)\int_{t_k}^{t}[g(s)x(s)+h(s)x^{\lambda}(s)]ds + (\alpha_1 x(t_1)+\cdots+\alpha_k x(t_k))\\
&+ C_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds\\
=&a_k(t)+b(t)\int_{t_k}^{t}\left[g(s)x(s)+h(s)x^{\lambda}(s)\right]ds.
\end{aligned}
$$

Therefore, we have by Lemma 2.1 that

$$x(t)\le a_k(t)+b(t)\int_{t_k}^{t}Q(s)e^{\int_s^t p(\tau)d\tau}ds.$$

This completes the proof of Theorem 2.2. □

Corollary 2.2. *If $a(t)\equiv a$ and $b(t)\equiv 1$ in Theorem 2.2, we have the estimate*

$$x(t)\le \hat{a}_k+\int_{t_k}^{t}\hat{Q}(s)e^{\int_s^t \hat{P}(\tau)d\tau}ds \text{ for } t\in(t_k,t_{k+1}], \tag{2.15}$$

$$
\begin{aligned}
\hat{a}_k =&a+\sum_{t_0<t_k<t}(\alpha_k+C_k(\tau_k-\sigma_k))\\
&+\sum_{t_0<t_k<t}(1+\alpha_{k-1})\int_{t_{k-1}}^{t_k}\hat{Q}(s)e^{\int_s^{t_k}\hat{P}(\tau)d\tau}ds\\
&+\sum_{t_0<t_k<t}C_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{t_{k-1}}^{s}\hat{Q}(v)e^{\int_\nu^s \hat{P}(\tau)d\tau}d\nu ds,
\end{aligned}
$$

and

$$\hat{P}(t)=g(t)+\lambda K^{\lambda-1}(t)h^{\alpha}(t), \hat{Q}(t)=a[g(t)+\lambda K^{\lambda-1}(t)h^{\alpha}(t)]+(1-\lambda)K^{\lambda}(t)h^{\beta}(t).$$

3. Boundedness of solutions for impulsive differential equations

Consider the following impulsive differential equation

$$\begin{cases} x'(t)=H(t,x(t),x^{\lambda}(t)), x\neq t_k,\\ x(t_k^+)=\alpha_k x(t_k)+C_k\int_{t_k-\tau_k}^{t_k-\sigma_k}x(s)ds, \end{cases}\tag{3.1}$$

where $\alpha_k\ge 0$, $C_k\ge 0$, $0\le\sigma_k\le\tau_k\le t_k-t_{k-1}$, $0<\lambda<1$ are constants.

Theorem 3.1. *If there exists nonnegative continuous functions $g(t)$ and $h(t)$ such that*

$$|H(t,u,v)|\le g(t)|u|+h(t)|\nu|^{\lambda} \text{ for } t\in[t_0,\infty), \tag{3.2}$$

and

$$\begin{aligned}
&\sum_k [\alpha_k + C_k(\tau_k - \sigma_k)] < \infty, \\
&\sum_k (1+\alpha_k) \int_{t_{k-1}}^{t_k} Q(\hat{s}) e^{\int_s^{t_k} \hat{P}(\tau)d\tau} ds < \infty, \\
&\sum_k C_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_{k-1}}^{s} \hat{Q}(\nu) e^{\int_\nu^s \hat{P}(\tau)d\tau} d\nu ds < \infty, \\
&\limsup_{t\to\infty} \int_{t_0}^{t} \hat{Q}(s) e^{\int_s^t \hat{P}(\tau)d\tau} ds < \infty,
\end{aligned} \tag{3.3}$$

where $\hat{P}(t)$ and $\hat{Q}(t)$ are defined as in Corollary 2.2, then all solutions of (3.1) *are bounded.*

Proof. Given arbitrarily initial value x_0, we consider the following initial value problem

$$\begin{cases} x'(t) = H(t, x(t), x^{\lambda}(t)), x \neq t_k, \\ x(t_0) = x_0, \\ x(t_k^+) = \alpha_k x(t_k) + C_k \displaystyle\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds. \end{cases}$$

Since the initial value problem is equivalent to integral equation

$$x(t) = x_0 + \int_{t_0}^{t} H(s, x(s), x^{\lambda}(s))ds + \sum_{t_0<t_k<t} \alpha_k x(t_k) + \sum_{t_0<t_k<t} C_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds, \tag{3.4}$$

we can obtain the following estimate easily:

$$\begin{aligned}
|x(t)| \leq & |x_0| + \int_{t_0}^{t} |H(s, x(s), x^{\lambda}(s))| ds + \sum_{t_0<t_k<t} \alpha_k |x(t_k)| \\
& + \sum_{t_0<t_k<t} \int_{t_k-\tau_k}^{t_k-\sigma_k} |x(s)| ds \\
\leq & |x_0| + \int_{t_0}^{t} \left[g(s)|x(s)| + h(s)|x^{\lambda}(s)|\right] ds + \sum_{t_0<t_k<t} \alpha_k |x(t_k)| \\
& + \sum_{t_0<t_k<t} \int_{t_k-\tau_k}^{t_k-\sigma_k} |x(s)| ds.
\end{aligned} \tag{3.5}$$

Now an application of Corollary 2.2 implies the estimate

$$|x(t)| \leq \hat{a}_k + \int_{t_k}^{t} \hat{Q}(s) e^{\int_s^t \hat{P}(\tau)d\tau} ds \text{ for } t \in (t_k, t_{k+1}],$$

here $\hat{a}_k$ is defined by

$$\begin{aligned}
\hat{a}_k = & |x_0| + \sum_{t_0<t_k<t} (\alpha_k + C_k(\tau_k - \sigma_k)) \\
& + \sum_{t_0<t_k<t} (1+\alpha_{k-1}) \int_{t_{k-1}}^{t_k} \hat{Q}(s) e^{\int_s^{t_k} \hat{P}(\tau)d\tau} ds
\end{aligned}$$

$$+\sum_{t_0<t_k<t} C_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_{k-1}}^{s} \hat{Q}(v) e^{\int_{\nu}^{s} \hat{P}(\tau)d\tau} d\nu ds,$$

since the series are convergent, and $\limsup_{t\to\infty} \int_{t_0}^{t} \hat{Q}(s) e^{\int_s^t \hat{P}(\tau)d\tau} ds < \infty$, we obtain that $x(t)$ is bounded, by the arbitrarily of initial value x_0, we see that all solutions are bounded. This completes the proof. □

4. Conclusions

In this content, we give some new integral inequalities with integral jump conditions and obtain the boundedness for impulsive differential equations (The impulsive effects of these differential equations are of interval jump type, rather than at some points). New impulsive integral inequalities with weak singular kernels and their applications to fractional differential equations are still open and we will focus on the open problem in the future study.

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