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CREEPING FLOW ANALYSIS OF SLIGHTLY NON-NEWTONIAN FLUID IN A UNIFORMLY POROUS SLIT

Hameed Ullah[1], Huafei Sun[1,†], Abdul Majeed Siddiqui[2]
and Tahira Haroon[2]

Abstract This paper provides the analysis of the steady, creeping flow of a special class of slightly viscoelastic, incompressible fluid through a slit having porous walls with uniform porosity. The governing two dimensional flow equations along with non-homogeneous boundary conditions are non-dimensionalized. Recursive approach is used to solve the resulting equations. Expressions for stream function, velocity components, volumetric flow rate, pressure distribution, shear and normal stresses in general and on the walls of the slit, fractional absorption and leakage flux are derived. Points of maximum velocity components are also identified. A graphical study is carried out to show the effect of porosity and non-Newtonian parameter on above mentioned resulting expressions. It is observed that axial velocity of the fluid decreases with the increase in porosity and non-Newtonian parameter. The outcome of this theoretical study has significant importance both in industry and biosciences.

Keywords Creeping flow, non-Newtonian fluid, porous slit, recursive approach.

MSC(2010) 76A05, 76A10, 76S99, 76M55.

1. Introduction

A wide range of applications for fluid flows through permeable boundaries exists in different problems like gaseous diffusion, filtration, oil production, coalescence and in biological mechanism such as in the circulation of blood through an artificial kidney and the flow through renal tubules of nephron in kidneys. Berman [2] studied for laminar steady state incompressible viscous fluid flow in channel between porous walls and discussed the effects of wall porosity on the velocity and pressure distribution. Afterwards this problem has been further investigated by many researchers [6, 15, 20, 22], Recently Siddiqui et al. [16, 17] have studied flow in channel with different absorption patterns at permeable walls for Stokes problem. Ahmad and Naseem [1] studied the hydrodynamics of the creeping flow of a viscous, incompressible, laminar fluid flowing in a tube with permeable walls and T. Haroon et

†the corresponding author. Email address:huafeisun@bit.edu.cn(H. Sun)
1School of Mathematics and Statistics, Beijing Institute of Technology Beijing, 100081, China
2Deparment of Mathematics, Pennsylvania State University, York Campus, 1031 Edgecomb Avenue, York, PA 17403, USA

al. [5] discussed the creeping viscous fluid flow in a proximal tubule with uniform reabsorption.

Non-Newtonian fluids have their own practical, industrial and mathematical importance due to their complex and distinct features and has been a core topic for researchers and scientists from many decades. These fluids may categorize as differential type, rate type and integral type fluids. Rivlin-Ericksen [12] proposed a theory for fluids of differential type which explain several interesting features as normal stress effects, rod climbing, shear thinning and thickening effects. These fluids have a complex mathematical structure as well. Many researchers [3,4,7,13,14] have discussed many interesting and challenging issues related to differential type fluids. An important subclass of differential type fluids is a third order fluid. Kacou [8] and Ng and Saibel [11] discussed a special class of third grade fluid model for journal bearing and slider bearing, respectively and this model differs slightly from the Newtonian fluid. As per our knowledge, no attempt has been made to study the plane steady flow of special class of slightly viscoelastic fluid model in a porous slit. This model leads to a highly non linear set of partial differential equations in two dimensions along with non-homogeneous boundary conditions. In general it is very difficult to solve this type of equations either analytically or numerically. Analytical study of such type of nonlinear problem is important not only because of its technological significance but also due to the interesting mathematical features presented by these equations. An analytic technique known as recursive approach was used by Langlios [9, 10] to linearize the equations of motion for steady state, slow flows. We have generalized this approach to solve highly non-linear two-dimensional momentum equations for slow flows of incompressible slightly viscoelastic fluid model along with non-homogeneous boundary conditions. Expressions for the velocity components, flow rate, pressure field, mean pressure drop, wall shear stress, normal stresses, leakage flux and fractional reabsorption are obtained. Graphical results and discussion are also presented. We hope that this article will be useful in understanding the mechanism of flows through permeable boundaries in industry and also in biosciences, e.g., in reabsorption of blood and nutrients through renal tubule.

2. Basic equations

The basic equations that govern the steady, slow flow of isothermal, incompressible fluid in the absence of body forces are

$$\nabla \cdot \mathbf{V} = \mathbf{0}, \tag{2.1}$$

$$\nabla \cdot \mathbf{T} = \mathbf{0}, \tag{2.2}$$

where $\mathbf{V}$ is the velocity vector and $\mathbf{T}$ is the Cauchy stress tensor, which for simple fluids is given by Truesdell and Noll [21]

$$\mathbf{T} = -\mathbf{pI} + \mathbf{S}, \tag{2.3}$$

where p is the pressure, $\mathbf{I}$ is the identity tensor and

$$\mathbf{S} = \sum_{\mathbf{i=0}}^{\mathbf{i=3}} \mathbf{S_i}, \tag{2.4}$$

with

$$\mathbf{S_1} = \mu\mathbf{A_1}, \tag{2.5}$$
$$\mathbf{S_2} = \alpha_\mathbf{1}\mathbf{A_2} + \alpha_\mathbf{2}\mathbf{A_1^2}, \tag{2.6}$$
$$\mathbf{S_3} = \beta_\mathbf{1}\mathbf{A_3} + \beta_\mathbf{2}(\mathbf{A_2A_1} + \mathbf{A_1A_2}) + \beta_\mathbf{3}(\mathbf{trA_1^2})\mathbf{A_1}, \tag{2.7}$$

μ is the coefficient of dynamic viscosity, α_1, α_2, β_1, β_2 and β_3 are material constants, and $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_3}$ are kinematical tensors defined respectively as

$$\mathbf{A_1} = (\mathbf{gradV}) + (\mathbf{gradV})^\mathbf{T}, \tag{2.8}$$
$$\mathbf{A_2} = \frac{\mathbf{D}}{\mathbf{Dt}}\mathbf{A_1} + (\mathbf{A_1gradV}) + (\mathbf{A_1gradV})^\mathbf{T}, \tag{2.9}$$
$$\mathbf{A_3} = \frac{\mathbf{D}}{\mathbf{Dt}}\mathbf{A_2} + (\mathbf{A_2gradV}) + (\mathbf{A_2gradV})^\mathbf{T}, \tag{2.10}$$

where *grad* is the gradient operator and the $\frac{D}{Dt}$ is the material time derivative defined as

$$\frac{D}{Dt}(*) = \frac{\partial}{\partial t}(*) + (\mathbf{V}\cdot\mathbf{grad})(*). \tag{2.11}$$

Rajagopal and Fosdick [13] have studied the thermodynamics of fluids modeled exactly by equation (2.3) in detail. They have shown that if a fluid modeled by equation (2.3) is to be compatible with thermodynamics, that is, meet the restrictions imposed by the Clausius-Duhem inequality and the assumptions that the specific Helmholtz free energy to be minimum when the fluid is locally at rest, the material coefficients have to meet the following restrictions

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0. \tag{2.12}$$

Thus the Cauchy stress tensor $\mathbf{T}$ from equation (2.3) reduces to

$$\mathbf{T} = -\mathbf{pI} + \mu\mathbf{A_1} + \alpha_\mathbf{1}\mathbf{A_2} + \alpha_\mathbf{2}\mathbf{A_1^2} + \beta_\mathbf{3}(\mathbf{trA_1^2})\mathbf{A_1}. \tag{2.13}$$

If the material constants α_1 and α_2 vanish then the fluid modeled by equation (2.13) takes the form [8]

$$\mathbf{T} = -\mathbf{pI} + \mu\mathbf{A_1} + \beta_\mathbf{3}(\mathbf{trA_1^2})\mathbf{A_1} \tag{2.14}$$

equation (2.14) is the constitutive equation for slightly non-Newtonian fluids (fluids for which the stress deformation relation departs only slightly from that of a Newtonian fluid).

Substituting equation (2.14) into equation (2.2) yields

$$grad\, p = \mu\, div(\mathbf{A_1}) + \beta_\mathbf{3}\left[\mathbf{A_1}\,\mathbf{grad}|\mathbf{A_1}|^\mathbf{2} + |\mathbf{A_1}|^\mathbf{2}\mathbf{div}(\mathbf{A_1})\right], \tag{2.15}$$

where

$$|\mathbf{A_1}|^\mathbf{2} = \mathbf{tr}(\mathbf{A_1^2}). \tag{2.16}$$

This model has been used by some researchers [8, 11] for journal bearing as well as for slider bearing studies. It may be considered as a subclass of simple fluid theory. On the other hand, this model may be taken as a special class of Sisko fluid.

3. Problem statement

Consider steady, two dimensional, slow, isothermal flow of slightly non-Newtonian, incompressible fluid between slit walls, where walls are porous and porosity v_0 is uniformly distributed. The gap between the slit walls is assumed to be $2H$ and the breadth is taken as W. We choose rectangular coordinate system such that x axis is taken along the centre of the slit while y axis is taken normal to x axis. The volume flow rate is assumed to be Q_0 at the starting position $x = 0$.

For steady two dimensional flow, we choose velocity profile as

$$\mathbf{V} = [u(x,y),\ v(x,y)] .$$

The boundary conditions of the problem under consideration are

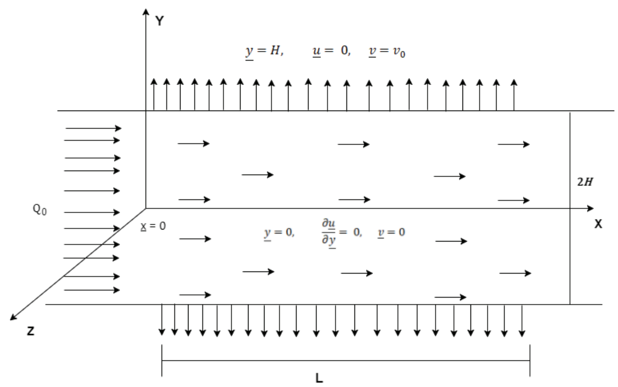


Figure 1. Geometry of the Problem

$$u = 0, \qquad v = \pm v_0, \qquad \text{at} \qquad y = \pm H, \tag{3.1}$$

$$Q_0 = W \int_{-H}^{H} u(0,y)\, dy. \tag{3.2}$$

The governing equations under the assumptions stated above are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.3}$$

$$\frac{\partial p}{\partial x} = (\mu + \beta_3 M)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \beta_3\left[2\frac{\partial u}{\partial x}\frac{\partial M}{\partial x} + \frac{\partial M}{\partial y}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right], \tag{3.4}$$

$$\frac{\partial p}{\partial y} = (\mu + \beta_3 M)\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] + \beta_3\left[2\frac{\partial v}{\partial y}\frac{\partial M}{\partial y} + \frac{\partial M}{\partial x}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right]. \tag{3.5}$$

From equation (2.14), the expression for stresses are

$$T_{xx} = -p + 2\mu\frac{\partial u}{\partial x} + 2\beta_3 M\frac{\partial u}{\partial x}, \tag{3.6}$$

$$T_{yx} = T_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \beta_3 M\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{3.7}$$

$$T_{yy} = -p + 2\mu \frac{\partial v}{\partial y} + 2\beta_3 M \frac{\partial v}{\partial y}, \tag{3.8}$$

where

$$M(x,y) = 8\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2. \tag{3.9}$$

It is noteworthy that the dissipation function for viscous incompressible fluid flow is normally defined as

$$\Phi(x,y) = 4\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2. \tag{3.10}$$

From equations (3.9) and (3.10), we have

$$M(x,y) = 2\,\Phi(x,y). \tag{3.11}$$

Introducing the dimensionless parameters as

$$x = \frac{x}{H}, \qquad y = \frac{y}{H}, \qquad u = \frac{u}{Q_0/WH}, \qquad v = \frac{v}{Q_0/WH}, \tag{3.12}$$

$$p = \frac{p}{\mu Q_0/WH^2}, \qquad T_{ij} = \frac{T_{ij}}{\mu Q_0/WH^2}, \qquad \phi = \frac{\Phi}{Q_0^2/W^2H^4}. \tag{3.13}$$

Making use of equation (3.13) into equations (3.1)−(3.10), we have dimensionless equations as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.14}$$

$$\frac{\partial p}{\partial x} = (1+2\beta\phi)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \beta\left[4\frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x} + 2\frac{\partial \phi}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right], \tag{3.15}$$

$$\frac{\partial p}{\partial y} = (1+2\beta\phi)\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] + \beta\left[4\frac{\partial v}{\partial y}\frac{\partial \phi}{\partial y} + 2\frac{\partial \phi}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right], \tag{3.16}$$

$$T_{xx} = -p + 2\frac{\partial u}{\partial x} + 4\beta\phi\frac{\partial u}{\partial x}, \tag{3.17}$$

$$T_{yx} = T_{xy} = (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + 2\beta\phi(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}), \tag{3.18}$$

$$T_{yy} = -p + 2\frac{\partial v}{\partial y} + 4\beta\phi\frac{\partial v}{\partial y}, \tag{3.19}$$

$$\phi(x,y) = 4(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2, \tag{3.20}$$

and the boundary conditions due to symmetry of center line are

$$u = 0, \qquad v = S, \qquad \text{at} \qquad y = 1, \tag{3.21}$$

$$\frac{\partial u}{\partial y} = 0, \qquad v = 0, \qquad \text{at} \qquad y = 0, \tag{3.22}$$

$$2\int_0^1 u(0,y)\,dy = 1, \tag{3.23}$$

where $\beta = \dfrac{\beta_3\, Q_0^2}{\mu\, W^2\, H^4}$ and $S = \dfrac{v_0\, W\, H}{Q_0}$ are the non-Newtonian and porosity parameters, respectively and $Q(x) = \frac{Q(x)}{Q_0}$.

4. Solution of the problem

The exact solution of the above system of partial differential equations subject to non-homogeneous boundary conditions due to non-linearity seems to be impossible. Langlios [9, 10] proposed an approach known as recursive approach to linearize the equations of motion for steady state, slow flows, so we choose

$$u(x,y) = \sum_{i=0}^{i=\infty} \epsilon^i \, u^{(i)}(x,y), \tag{4.1}$$

$$p(x,y) = Constant + \sum_{i=0}^{i=\infty} \epsilon^i \, p^{(i)}(x,y), \tag{4.2}$$

$$T(x,y) = \sum_{i=0}^{i=\infty} \epsilon^i \, T^{(i)}(x,y), \tag{4.3}$$

where ϵ is a small dimensionless number. Substituting (4.1-4.3) into equations (3.14-3.23) and collecting the coefficients of like powers of ϵ, we obtain the set of boundary value problems. We plan to solve these problems at $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^3)$.

4.1. First order problem

On equating coefficients of ϵ, we get

$$\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0, \tag{4.4}$$

$$\frac{\partial p^{(1)}}{\partial x} = \frac{\partial^2 u^{(1)}}{\partial x^2} + \frac{\partial^2 u^{(1)}}{\partial y^2}, \tag{4.5}$$

$$\frac{\partial p^{(1)}}{\partial y} = \frac{\partial^2 v^{(1)}}{\partial x^2} + \frac{\partial^2 v^{(1)}}{\partial y^2}, \tag{4.6}$$

$$T_{xx}^{(1)} = -p^{(1)} + 2\,\frac{\partial u^{(1)}}{\partial x}, \qquad T_{xy}^{(1)} = \left(\frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x}\right), \qquad T_{yy}^{(1)} = -p^{(1)} + 2\,\frac{\partial v^{(1)}}{\partial y} \tag{4.7}$$

along with the corresponding boundary conditions

$$u^{(1)} = 0,\ v^{(1)} = S, \qquad \text{at} \qquad y = 1, \tag{4.8}$$

$$\frac{\partial u^{(1)}}{\partial y} = 0,\ v^{(1)} = 0, \qquad \text{at} \qquad y = 0, \tag{4.9}$$

$$\int_0^1 u^{(1)}(0,y)\,dy = \frac{1}{2}, \qquad \text{at} \qquad x = 0. \tag{4.10}$$

On introducing stream function $\psi^{(1)}(x,y)$ as

$$u^{(1)} = \frac{\partial \psi^{(1)}}{\partial y},\ v^{(1)} = -\frac{\partial \psi^{(1)}}{\partial x}. \tag{4.11}$$

Equation (4.4) is identically satisfied and equations (4.5) and (4.6) after eliminating pressure take the form

$$\nabla^4\psi^{(1)} = 0. \tag{4.12}$$

Moreover, boundary conditions (4.8-4.10) in terms of stream functions become

$$\frac{\partial\psi^{(1)}}{\partial y} = 0, \ \frac{\partial\psi^{(1)}}{\partial x} = -S, \qquad \text{at} \qquad y = 1, \tag{4.13}$$

$$\frac{\partial^2\psi^{(1)}}{\partial y^2} = 0, \ \frac{\partial\psi^{(1)}}{\partial x} = 0, \qquad \text{at} \qquad y = 0, \tag{4.14}$$

$$\psi^{(1)}(0,1) = \frac{1}{2}, \ \psi^{(1)}(0,0) = 0. \tag{4.15}$$

To obtain the solution of equation (4.12) along with the boundary conditions (4.13-4.15), inverse method [18,19] is used, so we choose the stream function $\psi^{(1)}(x, y)$ of the form

$$\psi^{(1)}(x, y) = Sx\mathrm{R}^{(1)}(y) + \mathrm{T}^{(1)}(y), \tag{4.16}$$

where $\mathrm{R}^{(1)}(y)$ and $\mathrm{T}^{(1)}(y)$ are unknown functions to be determined. Using equation (4.16) in equations (4.12-4.15), we obtain

$$\mathrm{R}^{(1)}(y) = \frac{1}{2}\left(-3y + y^3\right), \tag{4.17}$$

$$\mathrm{T}^{(1)}(y) = \frac{1}{4}\left(3y - y^3\right). \tag{4.18}$$

Therefore, expressions for stream function, the equation (4.16) in account of equations (4.17) and (4.18) and velocity components (4.11) become

$$\psi^{(1)} = \frac{1}{4}(1 - 2Sx)\left(3y - y^3\right), \tag{4.19}$$

$$u^{(1)} = \frac{3}{4}(1 - 2Sx)\left(1 - y^2\right), \tag{4.20}$$

$$v^{(1)} = \frac{S}{2}\left(3y - y^3\right). \tag{4.21}$$

Pressure is calculated at first order problem using equations (4.20)and (4.21) into equations (4.5)and (4.6) as

$$\frac{\partial p^{(1)}}{\partial x} = -\frac{3}{2}(1 - 2Sx), \tag{4.22}$$

$$\frac{\partial p^{(1)}}{\partial y} = -3Sy. \tag{4.23}$$

Integrating equation (4.22)with respect to x, we have

$$p^{(1)} = -\frac{3}{2}x(1 - Sx) + A(y), \tag{4.24}$$

where $A(y)$ is unknown constant to be determined.

Now differentiating equation (4.24)with respect to y, and comparing with the equation (4.23), yields

$$A'(y) = -3Sy, \tag{4.25}$$

where equation (4.24) implies

$$p^{(1)} = -\frac{3}{2}x(1-Sx) - \frac{3Sy^2}{2} + p_0^{(1)}, \tag{4.26}$$

where $p(0,0) = p_0^{(1)}$ is the pressure at the entrance of the channel at $(x, y) = (0, 0)$. The dimensionlized mean pressure at any section of the slit can be obtained by using the formula

$$\begin{aligned}\bar{p}^{(1)}(x) &= \int_0^1 (p^{(1)} - p_0^{(1)})dy, \\ &= -\frac{1}{2}(3x(1-Sx)+S),\end{aligned} \tag{4.27}$$

and the pressure drop over the length L of the slit is

$$\begin{aligned}\Delta\,\bar{p}^1(L) &= \bar{p}^1(0) - \bar{p}^1(L), \\ &= \frac{3}{2}L(1-LS).\end{aligned} \tag{4.28}$$

Shear, normal stresses and normal stresses difference at first order are obtained by substituting equations (4.20), (4.21) and (4.26) in equation (4.7)

$$T_{xx}^{(1)} = -3S - \frac{3}{2}x(-1+Sx) + \frac{9Sy^2}{2}, \tag{4.29}$$

$$T_{xy}^{(1)} = -\frac{3}{2}\,(1-2Sx)y, \tag{4.30}$$

$$T_{yy}^{(1)} = 3S - \frac{3}{2}x(-1+Sx) - \frac{3Sy^2}{2}, \tag{4.31}$$

$$T_{xx}^{(1)} - T_{yy}^{(1)} = 6S\left(-1+y^2\right). \tag{4.32}$$

The normal stress difference is not zero as the walls of the channel are porous. If $S = 0$ then we see that difference of normal stresses is zero, which happened in the case of solid walls. We noticed that the result obtained for first order system is very similar to the slow flow of viscous fluid with uniform porosity [6].

4.2. Second order problem

On equating coefficients of ϵ^2, we get three equations with three unknowns along with the corresponding boundary conditions and stresses

$$\frac{\partial u^{(2)}}{\partial x} + \frac{\partial v^{(2)}}{\partial y} = 0, \tag{4.33}$$

$$\frac{\partial p^{(2)}}{\partial x} = \frac{\partial^2 u^{(2)}}{\partial x^2} + \frac{\partial^2 u^{(2)}}{\partial y^2}, \tag{4.34}$$

$$\frac{\partial p^{(2)}}{\partial x} = \frac{\partial^2 v^{(2)}}{\partial x^2} + \frac{\partial^2 v^{(2)}}{\partial y^2}, \tag{4.35}$$

$$u^{(2)} = 0, \qquad v^{(2)} = 0, \qquad \text{at} \qquad y = 1, \tag{4.36}$$

$$\frac{\partial u^{(2)}}{\partial y} = 0, \qquad v^{(2)} = 0, \qquad \text{at} \qquad y = 0, \tag{4.37}$$

$$\int_0^1 u^{(2)}(0, y)\, dy = 0, \qquad \text{at} \qquad x = 0. \tag{4.38}$$

$$T_{xx}^{(2)} = -p^{(2)} + 2\,\frac{\partial u^{(2)}}{\partial x},\ T_{xy}^{(2)} = \left(\frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x}\right),\ T_{yy}^{(2)} = -p^{(2)} + 2\,\frac{\partial v^{(2)}}{\partial y}. \tag{4.39}$$

We remark that we do not see any contribution of slightly non newtonian parameter at this order. On introducing stream function $\psi^{(2)}(x, y)$ as

$$u^{(2)} = \frac{\partial \psi^{(2)}}{\partial y}, \qquad v^{(2)} = -\frac{\partial \psi^{(2)}}{\partial x}, \tag{4.40}$$

equation (4.33) is identically satisfied and equations (4.34) and (4.35) after eliminating pressure take the form

$$\nabla^4 \psi^{(2)} = 0. \tag{4.41}$$

Moreover, equations (4.36-4.38) in terms of stream function become

$$\frac{\partial \psi^{(2)}}{\partial y} = 0, \qquad \frac{\partial \psi^{(2)}}{\partial x} = 0, \qquad \text{at} \qquad y = 1, \tag{4.42}$$

$$\frac{\partial^2 \psi^{(2)}}{\partial y^2} = 0, \qquad \frac{\partial \psi^{(2)}}{\partial x} = 0, \qquad \text{at} \qquad y = 0, \tag{4.43}$$

$$\psi^{(2)}(0, 1) = 0,\ \psi^{(2)}(0, 0) = 0. \tag{4.44}$$

The solution of equation (4.41) along with boundary conditions (4.42-4.44) for any supposed $\psi^{(2)}(x, y)$ is zero due to homogeneous boundary conditions, therefore

$$\psi^{(2)} = 0, \tag{4.45}$$

$$u^{(2)} = 0, \tag{4.46}$$

$$v^{(2)} = 0. \tag{4.47}$$

Pressure, mean pressure drop, pressure drop, shear and normal stresses at 2nd order solution becomes

$$p^{(2)} - p_0^{(2)} = 0, \tag{4.48}$$

$$\bar{p}^2(x) = 0, \tag{4.49}$$

$$\Delta \bar{p}^2(L) = 0, \tag{4.50}$$

$$T_{xx}^{(2)} = 0, \tag{4.51}$$

$$T_{xy}^{(2)} = 0, \tag{4.52}$$

$$T_{yy}^{(2)} = 0. \tag{4.53}$$

4.3. Third order problem

Coefficients of ϵ^3 provide us the system of equations:

$$\frac{\partial u^{(3)}}{\partial x} + \frac{\partial v^{(3)}}{\partial y} = 0, \tag{4.54}$$

$$\frac{\partial p^{(3)}}{\partial x} = \frac{\partial^2 u^{(3)}}{\partial x^2} + \frac{\partial^2 u^{(3)}}{\partial y^2} + 2\,\beta \left(\frac{\partial \phi^{(2)}}{\partial y}\frac{\partial u^{(1)}}{\partial y} + 2\,\frac{\partial \phi^{(2)}}{\partial x}\frac{\partial u^{(1)}}{\partial x} + \frac{\partial \phi^{(2)}}{\partial y}\frac{\partial v^{(1)}}{\partial x} + \phi^{(2)} \left(\frac{\partial^2 u^{(1)}}{\partial x^2} + \frac{\partial^2 u^{(1)}}{\partial y^2} \right) \right), \tag{4.55}$$

$$\frac{\partial p^{(3)}}{\partial y} = \frac{\partial^2 v^{(3)}}{\partial x^2} + \frac{\partial^2 v^{(3)}}{\partial y^2} + 2\,\beta \left(\frac{\partial \phi^{(2)}}{\partial x}\frac{\partial u^{(1)}}{\partial y} + 2\,\frac{\partial \phi^{(2)}}{\partial x}\frac{\partial v^{(1)}}{\partial y} + \frac{\partial \phi^{(2)}}{\partial x}\frac{\partial v^{(1)}}{\partial x} + \phi^{(2)} \left(\frac{\partial^2 v^{(1)}}{\partial x^2} + \frac{\partial^2 v^{(1)}}{\partial y^2} \right) \right), \tag{4.56}$$

$$T_{xx}^{(3)} = -p^{(3)} + 2\,\frac{\partial u^{(3)}}{\partial x} + 4\,\beta\,\phi^{(2)}\,\frac{\partial u^{(1)}}{\partial x}, \tag{4.57}$$

$$T_{xy}^{(3)} = \left(\frac{\partial u^{(3)}}{\partial y} + \frac{\partial v^{(3)}}{\partial x} \right) + 2\,\beta\,\phi^{(2)}\,\left(\frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x} \right), \tag{4.58}$$

$$T_{yy}^{(3)} = -p^{(3)} + 2\,\frac{\partial v^{(3)}}{\partial y} + 4\,\beta\,\phi^{(2)}\,\frac{\partial v^{(1)}}{\partial y}, \tag{4.59}$$

along with the corresponding boundary conditions

$$u^{(3)} = 0, \qquad v^{(3)} = 0, \qquad \text{at} \qquad y = 1, \tag{4.60}$$

$$\frac{\partial u^{(3)}}{\partial y} = 0, \qquad v^{(3)} = 0, \qquad \text{at} \qquad y = 0, \tag{4.61}$$

$$\int_0^1 u^{(3)}(0, y)\,dy = 0, \qquad \text{at} \qquad x = 0, \tag{4.62}$$

where

$$\phi^{(2)} = 4\,\left(\frac{\partial u^{(1)}}{\partial x} \right)^2 + \left(\frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x} \right)^2. \tag{4.63}$$

On introducing stream function $\psi^{(3)}(x, y)$ as

$$u^{(3)} = \frac{\partial \psi^{(3)}}{\partial y}, \qquad v^{(3)} = -\frac{\partial \psi^{(3)}}{\partial x}. \tag{4.64}$$

Equation (4.54) is identically satisfied and equations (4.55) and (4.56), after eliminating pressure takes the form

$$\nabla^4 \psi^{(3)} = (1 - 2\,S\,x) \left(\left(810\,S^2\,\beta\right) y^3 + \frac{27}{2}\left(3\,K(x) - 40\,S^2\right) \beta\,y \right), \tag{4.65}$$

where

$$K(x) = 1 - 4Sx + 4S^2x^2. \tag{4.66}$$

Moreover, boundary conditions (4.60-4.62) in terms of stream function become

$$\frac{\partial \psi^{(3)}}{\partial y}=0, \qquad \frac{\partial \psi^{(3)}}{\partial x}=0, \qquad \text{at} \qquad y=1, \tag{4.67}$$

$$\frac{\partial^2 \psi^{(3)}}{\partial y^2}=0, \qquad \frac{\partial \psi^{(3)}}{\partial x}=0, \qquad \text{at} \qquad y=0, \tag{4.68}$$

$$\psi^{(3)}(0,1)=0, \qquad \psi^{(3)}(0,0)=0. \tag{4.69}$$

To obtain the solution of equation (4.65) with boundary conditions (4.67–4.69), we choose the stream function $\psi^{(3)}(x,y)$ of the form

$$\psi^{(3)}(x,y)=(1-2Sx)\left(\mathrm{R}^{(3)}(y)+\left(\frac{27}{2}\left(3K(x)-40S^2\right)\beta\right)\mathrm{T}^{(3)}(y)\right), \tag{4.70}$$

where $\mathrm{R}^{(3)}(y)$ and $\mathrm{T}^{(3)}(y)$ are unknown functions to be determined.

Using equation (4.70) in equations (4.65-4.69), we get

$$\begin{aligned}&(1-2\,S\,x)\left(1944S^2\,\beta\,\mathrm{T}^{(3)''}(y)+\,\mathrm{R}^{(3)^{iv}}(y)+\frac{27}{2}\left(3K(x)-40S^2\right)\,\beta\,\mathrm{T}^{(3)^{iv}}(y)\right)\\ =&(1-2\,S\,x)\left(810\,S^2\,\beta\,y^3+\frac{27}{2}(3\,K(x)-40\,S^2)\beta\,y\right),\end{aligned} \tag{4.71}$$

The boundary conditions after making the use of equation (4.70)reduce to

$$\mathrm{R}^{(3)}(0)=\mathrm{R}^{(3)}(1)=\mathrm{R}^{(3)'}(1)=\mathrm{R}^{(3)''}(0)=0, \tag{4.72}$$

$$\mathrm{T}^{(3)}(0)=\mathrm{T}^{(3)}(1)=\mathrm{T}^{(3)'}(1)=\mathrm{T}^{(3)''}(0)=0. \tag{4.73}$$

The equation (4.71), gives rise to two differential equations:

$$\mathrm{R}^{(3)^{iv}}(y)+1944S^2\,\beta\,\mathrm{T}^{(3)''}(y)=810\,S^2\,\beta\,y^3, \tag{4.74}$$

$$\mathrm{T}^{(3)^{iv}}(y)=y. \tag{4.75}$$

Solving equations (4.74) and (4.75) subject to boundary conditions (4.72) and (4.73), we obtain

$$\mathrm{R}^{(3)}(y)=\frac{81}{700}S^2y\left(-1+y^2\right)^2\left(24+5y^2\right)\beta, \tag{4.76}$$

$$\mathrm{T}^{(3)}(y)=\frac{1}{120}\left(y-2y^3+y^5\right). \tag{4.77}$$

Note that equation (4.76) contributes towards non Newtonian parameter β while equation (4.77) is free of non newtonian effects.

Using equations (4.76) and (4.77) in equation (4.70), the expression for stream function and velocity components become

$$\psi^{(3)} = \frac{9\,(1-2Sx)\,\beta}{2800}\Big[\big(105K(x)-536S^2\big)\,y + \big(-210\,K(x)+1252\,S^2\big)\,y^3 + \big(105\,K(x)-896\,S^2\big)\,y^5 + 180\,S^2\,y^7\Big], \tag{4.78}$$

$$u^{(3)} = \frac{9(1-2Sx)\,\beta}{2800}\Big[105K(x)-536S^2 + \big(-630K(x)+3756S^2\big)\,y^2 + \big(525K(x)-4480S^2\big)\,y^4 + 1260S^2y^6\Big], \tag{4.79}$$

$$v^{(3)} = \frac{9S\beta}{1400}\Big[\big(315K(x)-536S^2\big)\,y + \big(-630K(x)+1252S^2\big)\,y^3 + \big(315K(x)-896S^2\big)\,y^5 + 180S^2y^7\Big]. \tag{4.80}$$

Pressure, mean pressure drop and pressure drop at 3rd order solution using equations (4.79) and (4.80) into equations (4.55) and (4.56) and following the same procedure from equations (4.22–4.29), we have

$$p^{(3)} - p_0^{(3)} = \frac{9\beta}{700}\Bigg[(3x(-1+Sx)\left(\frac{105K(x)+105}{2} - 136S^2\right) + \big(105SK(x) - 7992S^3\big)y^2 + \big(525SK(x)+7630S^3\big)\,y^4 - 2660S^3y^6\Bigg], \tag{4.81}$$

$$\bar{p}^3(x) = \frac{9\beta}{1400}\left[-3036S^3 + \big(-315+816S^2\big)\,x + \big(315S-816S^3\big)\,x^2 + 35\,\big(-9x+S\,\big(8+9x^2\big)\big)\,K(x)\right], \tag{4.82}$$

$$\Delta\bar{p}^3(L) = -\frac{9}{700}L(-1+LS)\big[315+2S(76S+315L(-1+LS))\big]\beta. \tag{4.83}$$

Shear stress, normal stresses and the normal stresses difference are obtained by substituting equations (4.79-4.81), (4.20) and (4.21) in equations (4.57–4.59), thus

$$T_{xx}^{(3)} = \frac{9\beta}{1400}\Big[-7328S^3 + x\,\big(315+1704S^2+315K(x)\big) - 630S + x^2\big(-315S-1704S^3-315SK(x)\big) + \big(33672S^3+1470SK(x)\big)\,y^2 + \big(-31500S^3-2100SK(x)\big)\,y^4 + 11200S^3y^6\Big], \tag{4.84}$$

$$T_{xy}^{(3)} = \frac{9(-1+2Sx)\beta}{700}\Big[\big(315K(x)+852S^2\big)\,y - 980S^2y^3 + 840S^2y^5\Big], \tag{4.85}$$

$$T_{yy}^{(3)} = \frac{9\beta}{1400}\Big[7328S^3 + x\,\big(315-3336S^2+315K(x)\big) + 630S + x^2\,\big(-315S+3336S^3-315SK(x)\big) + \big(-1704S^3-1890SK(x)\big)\,y^2 + 980S^3y^4 - 560S^3y^6\Big], \tag{4.86}$$

$$T_{xx}^{(3)} - T_{yy}^{(3)} = \frac{9S\beta}{350}\Big[-315K(x)-3664S^2 + \big(840K(x)+8844S^2\big)\,y^2 - \big(525K(x)+8120S^2\big)\,y^4 + 2940S^2y^6\Big]. \tag{4.87}$$

We see that the non-Newtonian parameter β contributed in velocity field, pressure field and shear and normal stresses at third order solution. If $\beta=0$ then all solutions at 3rd order become zero.

On combining first, second and third order solutions we have

$$\psi = \frac{(1-2Sx)}{2800}\big[\left(2100+9\beta\left(105K(x)-536S^2\right)\right)y-\big(700+18\beta\big(105K(x) \\ -626S^2\big)\big)y^3+9\beta\big(105K(x)-896S^2\big)y^5+1620S^2y^7\beta\big], \tag{4.88}$$

$$u = \frac{3(1-2Sx)}{2800}\big[700+3\beta\left(105K(x)-536S^2\right)-\big(700+18\beta\big(105K(x) \\ -626S^2\big)\big)y^2+15\beta\big(105K(x)-896S^2\big)y^4+3780S^2y^6\beta\big], \tag{4.89}$$

$$v = \frac{3S}{1400}\big[\left(700+3\beta\left(315K(x)-536S^2\right)\right)y-\left(\frac{700}{3}+6\beta\left(315K(x)-626S^2\right)\right)y^3 \\ +3\beta\big(315K(x)-896S^2\big)y^5+540S^2y^7\beta\big], \tag{4.90}$$

$$p-p_0 = -\frac{3}{2}x(1-Sx)-\frac{3Sy^2}{2}+\frac{9\beta}{700}\Bigg[(3x(-1+Sx)\left(\frac{105K(x)+105}{2}-136S^2\right) \\ +\left(105SK(x)-7992S^3\right)y^2+\left(525SK(x)+7630S^3\right)y^4-2660S^3y^6\Bigg], \tag{4.91}$$

$$\bar{p}(x) = -\frac{1}{2}(3x(1-Sx)+S)+\frac{9\beta}{1400}\big[-3036S^3+\left(-315+816S^2\right)x+\big(315S \\ -816S^3\big)x^2+35\left(-9x+S\left(8+9x^2\right)\right)K(x)\big], \tag{4.92}$$

$$\Delta\bar{p}(L) = -\frac{3}{700}L(-1+LS)[350+3(315+2S(76S+315L(-1+LS)))\beta], \tag{4.93}$$

$$T_{xx} = -3S-\frac{3}{2}x(-1+Sx)+\frac{9Sy^2}{2}+\frac{9\beta}{1400}\big[-7328S^3+x\big(315+1704S^2 \\ +315K(x)\big)-630S+x^2\big(-315S-1704S^3-315SK(x)\big)+\big(33672S^3 \\ +1470SK(x)\big)y^2+\left(-31500S^3-2100SK(x)\right)y^4+11200S^3y^6\big], \tag{4.94}$$

$$T_{xy} = -\frac{3}{2}(1-2Sx)y+\frac{9(-1+2Sx)\beta}{700}\big[\left(315K(x)+852S^2\right)y \\ -980S^2y^3+840S^2y^5\big], \tag{4.95}$$

$$T_{yy} = 3S-\frac{3}{2}x(-1+Sx)-\frac{3Sy^2}{2}+\frac{9\beta}{1400}\big[7328S^3+x\big(315-3336S^2 \\ +315K(x)\big)+630S+x^2\left(-315S+3336S^3-315SK(x)\right)+\big(-1704S^3 \\ -1890SK(x)\big)y^2+980S^3y^4-560S^3y^6\big], \tag{4.96}$$

$$T_{xx}-T_{yy} = 6S\left(-1+y^2\right)+\frac{9S\beta}{350}\big[-315K(x)-3664S^2+\left(840K(x)+8844S^2\right)y^2 \\ -\left(525K(x)+8120S^2\right)y^4+2940S^2y^6\big]. \tag{4.97}$$

Here we observe that pressure $p(x,y)$, mean pressure $\bar{p}(x)$, pressure drop $\Delta\bar{p}(L)$, shear and normal stresses are varying with porosity parameter and non newtonian parameter. If $\beta = 0$ then the Newtonian velocity and pressure fields can be recover [5].

The dimensionless volume flow rate is given by

$$Q(x) = 2\int_0^1 u(x,y)dy, \\ = 1-2Sx, \tag{4.98}$$

which shows the variation downstream.

The maximum axial velocity occurs at the center of the channel as

$$u_{\max} = \frac{3(1-2Sx)}{2800}(700 + 3(105K(x) - 536S^2)\beta). \tag{4.99}$$

The transverse velocity is found to be maximum at the walls, i.e.,

$$v_{\max} = S, \tag{4.100}$$

which is due to the porosity at the walls.

The wall shear stress from equation (4.95) is obtain as

$$T_w = -T_{xy}|_{y=1} = -\frac{3}{700}(-1+2Sx)\left(350 + 2136S^2\beta + 945\beta K(x)\right). \tag{4.101}$$

From equation (4.97), the normal stresses difference at the wall and at the center of the channel are given as

$$\begin{aligned} T_{xx} - T_{yy}|_{y=1} &= 0, \\ T_{xx} - T_{yy}|_{y=0} &= -6S + \frac{9}{350}S\beta\left(-3664S^2 - 315K(x)\right). \end{aligned} \tag{4.102}$$

The wall shear stress and normal stresses difference at the center of the channel are dependent on S and β.

The fractional reabsorption in a slit of length L is obtained as

$$F_a = \frac{Q(0) - Q(L)}{Q(0)} \tag{4.103}$$

$$= 2LS. \tag{4.104}$$

The leakage flux $q(x)$ is defined as

$$q(x) = -\frac{dQ}{dx} \tag{4.105}$$

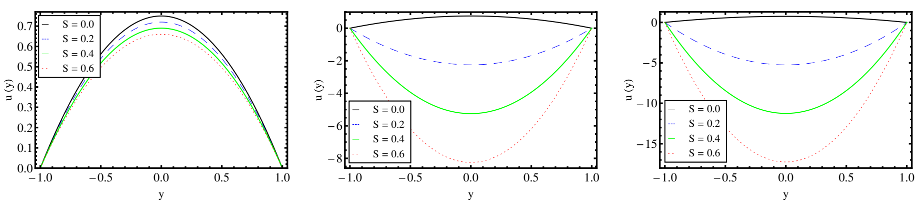
$$= 2S. \tag{4.106}$$

It is noted that fractional reabsorption F_a and leakage flux $q(x)$ has linear relationship with porosity parameter and are independent of β.

5. Results and discussion

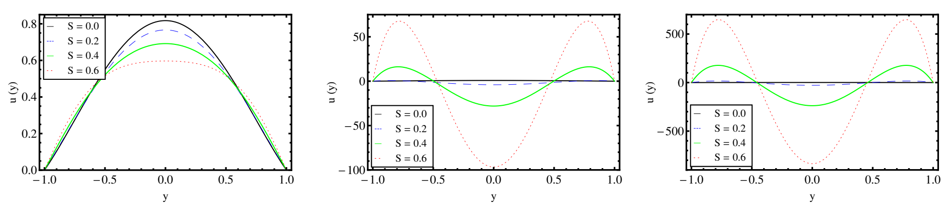
A graphical analysis is carried out to investigate the impact of porosity parameter S and non-Newtonian parameter β on the expressions obtain in previous sections for velocity components, streamlines, axial flow rate, shear and normal stresses and pressure distributions at different axial positions $x = 0.1$ (entrance), $x = 10$ and $x = 20$ of the channel. Figures 2(a-c) show the impact of S on axial velocity component u at various positions along the channel by keeping $\beta = 0$ and noticed that in absence of porosity ($S = 0$), the same parabolic profile (Poiseuille flow) for axial velocity is observed throughout the channel and magnitude of u is decreasing continuously as S increasing and the backward flow occurs in the channel. In Figures 3(a-c) the variation in axial velocity profile under the effect of S for $\beta = 0.2$

is depicted. We observed that parabolic behavior is effected throughout the channel due to the presence of non-Newtonian parameter and in comparison of figures 2(a-c) shear thickening behavior can be observe. Figures 4(a-c) is showing the effect of β on axial velocity keeping $S = 0.2$. It is seen that fluid flows slowly on increasing β at various positions of the channel which seems realistic due to shear thickening behavior. The variation of radial velocity can be seen in figures 5(a-c) on changing S when a) $\beta = 0$, b) $\beta = 0.4$ and c) $\beta = 0.8$. The thickening of the fluid with increasing β causes increase in the magnitude of v component of velocity in the channel, though $v = 0$ at the center of the channel. These figures demonstrating how v is changing its profile with the variation in S and β and also confirming the shear thickening behavior of slightly viscoelastic fluid.



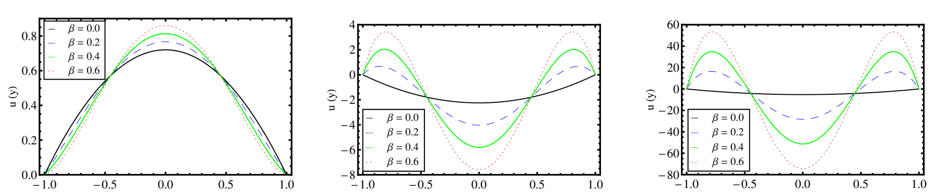
(a) (b) (c)

Figure 2. Effect of porosity parameter S on axial velocity $u(y)$ for $\beta = 0$, (a) $x = 0.1$, (b) $x = 10$, (c) $x = 20$.



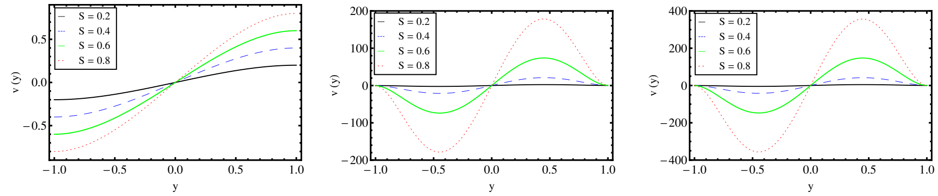
(a) (b) (c)

Figure 3. Effect of porosity parameter S on axial velocity $u(y)$ for $\beta = 0.2$,(a) $x = 0.1$, (b)$x = 10$, (c) $x = 20$.



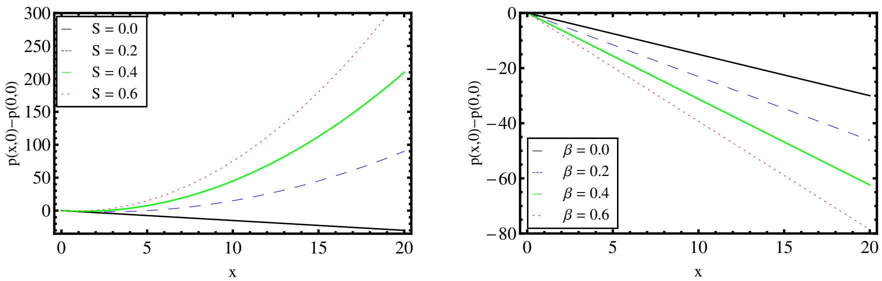
(a) (b) (c)

Figure 4. Effect of non-Newtonian parameter β on axial velocity $u(y)$ for $S = 0.2$, (a) $x = 0.1$, (b) $x = 10$, (c) $x = 20$.



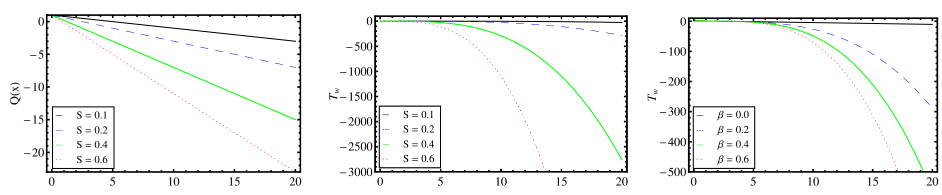
(a) (b) (c)

Figure 5. Effect of porosity parameter S on radial velocity $v(y)$ for (a) $\beta = 0$, (b) $\beta = 0.4$, (c) $\beta = 0.8$.



(a) (b)

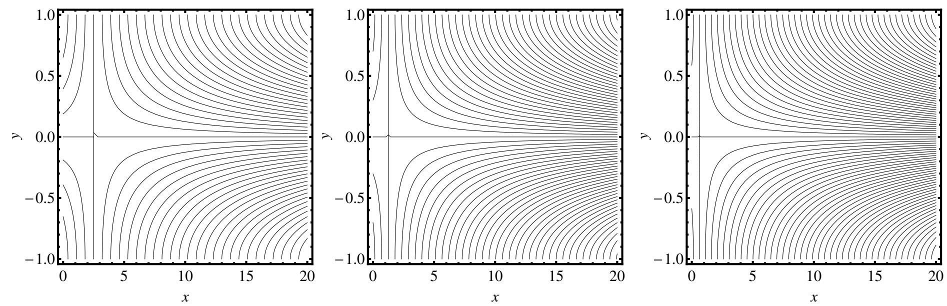
Figure 6. Effect of (a) porosity parameter S when $\beta = 0$ and (b) β when $S = 0$, on pressure difference $p(x, 0) - p(0, 0)$.



(a) (b) (c)

Figure 7. Effect of S on (a) axial flow rate and (b) wall shear stress keeping $\beta = 0.2$. Effect of β on (c) wall shear stress keeping $S = 0.2$.

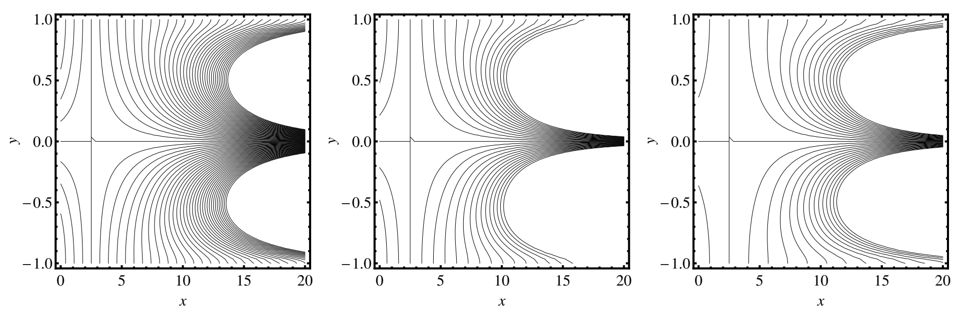
In figures 6(a-c) profile for pressure difference is plotted on varying S and β respectively. It is clear that on increasing S pressure difference increases and reduces by increasing β. Figures 7(a-c) is plotted to show the variation of porosity and non newtonian parameter on axial flow rate and shear stress at the walls of the slit. It can be seen that flow rate is decreasing on increasing S and shear stress reduces downstream in account of parameter S and β.



(a) (b) (c)

Figure 8. Stream lines for $\beta = 0$, (a) $S = 0.2$, (b) $S = 0.4$, (c) $S = 0.8$.

In figures 8(a-c) the streamlines are drawn for Newtonian fluid using different values of S. As we increase absorption back flow starts earlier in the channel. Figures 9(a-c) are showing the streamlines for different values of β when $S = 0.2$. As we increase the value of β, streamlines pattern changed.



(a) (b) (c)

Figure 9. Stream lines for $S = 0.2$ (a) $\beta = 0.4$, (b) $\beta = 0.6$, (c) $\beta = 0.8$.

6. Concluding remarks

In this paper, a mathematical study is carried out to discuss the slow flow phenomenon for a special class of third grade fluid through slit with porous walls. Recursive approach is used to linearize the system of nonlinear partial differential equations along with non-homogeneous boundary conditions and exact solutions for velocity profile, volume flow rate, pressure difference, pressure drop, leakage flux, fractional reabsorption and wall shear stress are obtained. A graphical analysis is also presented. Key outcomes of the present study can be summarized as follows:

1. The results for slow flow of the Newtonian fluid are recovered for $\beta = 0$.
2. Backward flow is observed earlier along the length of the channel with increasing porosity.
3. The maximum axial and radial velocities occur at center and at the slit walls, respectively.

4. Flow become slow due to non-Newtonian parameter β which indicates the shear thickening behavior of the fluid.
5. Pressure is increasing in account of increasing porosity parameter S and it is decreasing on increasing non-Newtonian parameter β.
6. Shear stress and axial flow rate decreases downstream.
7. It is noted from streamlines that for increasing porosity parameter S, the backward flow can be observed along the length of the channel and the large values of non newtonian parameter affected the flow pattern.

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References

[1] S. Ahmad and N. Ahmad, *On flow through renal tubule in case of periodic radial velocity component*, International Journal of Emerging Multidisciplinary Fluid Sciences, 2011, 4, 201–208.

[2] A. S. Berman, *Laminar flow in channels with porous walls*, J. Appl. Phys., 1953, 24, 1232–1235.

[3] J. E. Dunn and R. L. Fosdick, *Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade*, Archs ration. Mech. Analysis, 1974, 56, 191–252.

[4] R. L. Fosdick and K. R. Rajagopal, *Anomalous features in the model of second order fluids*, Archs ration. Mech. Analysis, 1979, 70, 145–152.

[5] T. Haroon, A. M. Siddiqui and A. Shahzad, *Creeping flow of viscous fluid through a proximal tubule with uniform reabsorption*, A mathematical study, Applied Mathematical Sciences, 2016, 10(16), 795–807.

[6] S. K. Karode, *Laminar flow in channels with porous walls, revisited*, J. Membr. Sci., 2001, 191, 237–241.

[7] A. Kacou, K. R. Rajagopal and A. Z. Szeri, *Flow of a fluid of the differential type in a journal bearing*, ASME Journal of Tribology, 1987, 109, 100–107.

[8] A. Kacou, K. R. Rajagopal and A. Z. Szeri, *A thermodynamics analysis of journal bearings lubricated by non Newtonian fluid*, ASME Journal of Tribology, 1988, 110, 414–420.

[9] W. E. Langlois, *A recursive approach to the theory of slow steady-state viscoelastic flow*, Transactions of the Society of Rheology, 1963, 7, 75–99.

[10] W. E. Langlois, *The recursive theory of slow viscoelastic flow applied to three basic problems of hydrodynamics*, Transctions of the Society of Rheology, 1964, 8, 33–60.

[11] C. W. Ng and E. Saibel, *Non-linear viscosity effects in slide bearing lubrication*, ASME Journal of Lubrication Technology, 1962, 7, 192–196.

[12] R. S. Rivlin and J. L. Ericksen, *Stress-deformation relations for istropic materials, Journal of Rational Mechanics and Analysis*, 1955, 4, 323–354.

[13] K. R. Rajagopal and R. L. Fosdick, *Thermodynamics and stability of fluids of third grade*, Proc. Roy. Soc, London, Series A, 1980, 339, 351–377.

[14] K. R. Rajagopal, *On the creeping flow of the second order fluid*, Journal of non-Newtonian Fluid Mech., 1984, 15, 239–246.

[15] H. G. Sharma and K. R. Singh, *Numerical solution of the flow of a second order fluid through a channel with porous walls under a transverse magnetic field*, Indian J. Pure and Appl. Math., 1986, 17(10), 1231-1241.

[16] A. M. Siddiqui, T. Haroon and M. Kashan, *MHD flow of Newtonian fluid in a permeable tubule, Magneto hydrodynamics*, 2015, 51(4), 655–672.

[17] A. M. Siddiqui, T. Haroon and A. Shahzad, *Hydrodynamics of viscous fluid through porous slit with linear absorption*, Appl. Math. Mech. Engl. Ed., 2016, 37(3), 361–378.

[18] A. M. Siddiqui, and P. N. Kaloni, *Certain inverse solutions of a non-Newtonian fluid*, Int. J. Non-Linear Mech. 1986, 21(6), 459–473.

[19] A. M. Siddiqui, *Some more inverse solutions of a non-Newtonian fluid*, Mech. Res. Commun., 1990, 17(3), 157–163.

[20] R. M. Terrill, *Laminar flow in a uniformly porous channel*, Aeronaut. Quart., 1964, 15, 299–310.

[21] C. Truesdell and W. Noll, *The non linear fluid theories of mechanics*, Handbuch der physick (ed. Flügge), III/3. Berlin-Heidelberg-New York: Springer 1965.

[22] S. W. Yuan, *Further investigation of laminar flow in channels with porous walls*, J. Appl. Phys., 1956, 27, 267–269.