

BOUNDARY CONTROLLABILITY FOR THE TIME-FRACTIONAL NONLINEAR KORTEWEG-DE VRIES (KDV) EQUATION*

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Abstract In this paper, we study the time-fractional nonlinear Korteweg-de Vries (KdV) equation. By using the theory of semigroups, we prove the well-posedness of the time-fractional nonlinear KdV equation. Moreover, we present the boundary controllability result for the problem.

Keywords Nonlinear time-fractional Korteweg-de Vries (KdV) equation, well-posedness, boundary controllability.

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1. Introduction and main results

In the present paper, we are concerned with the well-posedness and boundary controllability for the following time-fractional nonlinear KdV equation posed on a finite domain $(0, L)$ with nonhomogeneous boundary conditions

$$\begin{cases} D_t^\alpha u + 6auu_x + au_{xxx} = 0, u(x, 0) = u_0, & (x, t) \in (0, L) \times (0, T), \\ u(0, t) = h_1(t), u_x(L, t) = h_2(t), u_{xx}(L, t) = h_3(t), t \in (0, T), \end{cases} \quad (1.1)$$

where $a > 0$, $L > 0$, $T > 0$, $\alpha \in (0, 1)$, $D_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (u(\xi) - u(0)) d\xi$ and Γ represents the Euler Gamma function. We call it “time-fractional” as the derivative order $\alpha \in (0, 1)$ on the time space.

The Korteweg-de Vries equation was first introduced by Korteweg and de Vries [24] in 1895 as a model for propagation of some surface water waves along a channel. In recent years, it attracted much attention and appeared in several areas such as the models for some water waves, the unidirectional propagation of small-amplitude long waves, the blood pressure waves in large arteries and acoustic-gravity waves in a compressible heavy fluid, see e.g. [3, 5–7, 11, 11, 13–16, 20, 21, 26, 33].

In [9], Cerpa and Crépeau considered the locally controllable for nonlinear KdV

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equation with only one control input $u_x(L, t) = g_3(t)$

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, u(x, 0) = u_0, & x \in (0, L), \\ u(0, t) = 0, u(L, t) = 0, u_x(L, t) = g_3(t), & t \in \mathbb{R}^+, \end{cases} \quad (1.2)$$

via performing a power series expansion of the solution around the origin in these critical cases. Li and Liu [28] studied the problem

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, u(x, 0) = u_0, & x \in (0, L), \\ u(0, t) = h_1(t), u(1, t) = h_2(t), u_x(1, t) = h_3(t), & t \in \mathbb{R}^+, \end{cases} \quad (1.3)$$

and proved the well-posedness for the above problem (1.3).

Very recently, Caicedo and Zhang [8] studied the initial-boundary-value problem of the Korteweg-de Vries in the space $H^s(0, L)$, for any $s \geq 0$

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, u(x, 0) = u_0, & x \in (0, L), \\ u_{xx}(0, t) + u(0, t) - \frac{1}{6}u^2(0, t) = h(t), u(L, t) = 0, u_x(L, t) = 0, & \text{for } t \geq 0, t \in \mathbb{R}^+, \end{cases} \quad (1.4)$$

where they addressed a question left by Rosier in [32] by using Lagrangian coordinates. Moreover, they given its controllability. In the last decades, results involving the initial-boundary-value problems of the KdV equation posed on the finite domain had studied by many researchers. The interested readers are referred to [18, 22, 25] and other importance references, see, e.g. [2, 4, 12, 17, 18, 20]. While the time-fractional nonlinear KdV equation is rarely studied, the recent study about the time-fractional nonlinear KdV equation only can be found in [1].

In the present paper, we present other results (the well-posedness and boundary controllability) for problem (1.1). In the process of our study, we face three main difficulties. Firstly, as we all know that, applying the theory of semigroups is a crucial method to investigate the linear estimates and properties of solution for the partial differential equations. But the method only applies to the integer order equations. To solve the time-fractional nonlinear problem (1.1), we resort to the so-called the Laplace transform such that time-fractional nonlinear problem (1.1) has good linear estimates and properties. Secondly, due to the presence of the nonlinear boundary condition, the Kato smoothing property is not strong enough to enable us to establish the controllability for the time-fractional nonlinear problem (1.1) via the contraction mapping principle. Instead, we consider the sharp Kato smoothing property of the backward adjoint system of the linear system associated to the time-fractional nonlinear problem (1.1)

$$\begin{cases} D_t^\alpha \phi + a\phi_{xxx} = 0, & (x, t) \in (0, L) \times (0, T), \\ \phi(x, T) = \phi_T, \\ \phi(0, t) = 0, \phi_x(0, t) = 0, \phi_{xx}(L, t) = 0, & t \in (0, T). \end{cases} \quad (1.5)$$

But it is very difficulty to show that the solution of system (1.5) possesses the hidden regularities for any $\phi_T \in L^2(0, L)$. To get around, we will invoke some harmonic analysis tools (see [8]) to solve our difficulty. Finally, we encounter some difficulties

that how to treat the extra term which derived from the process of investing the control of the linear system of problem (1.1) by the usual multiplier method and compactness arguments. To achieve our purpose, we resort to the hidden regularity, again. Finding the observability inequality for the adjoint system (1.5) to overcome the problem.

This paper is organized as follows. In Section 2, we give some linear estimates. In Section 3, we present the linear result. In Section 4, we prove the nonlinear results. In the final section, we give some conclusions.

2. Linear estimates

In this section, we first consider the linear problem of the system (1.1) with nonhomogeneous boundary datas of the form

$$\begin{cases} D_t^\alpha u + au_{xxx} = 0, u(x, 0) = u_0, & (x, t) \in (0, L) \times (0, T), \\ u(0, t) = h_1(t), u_x(L, t) = h_2(t), u_{xx}(L, t) = h_3(t), t \in (0, T). \end{cases} \quad (2.1)$$

Applying the Laplace transform with respect to t in both sides of in (2.1), we have

$$\begin{cases} s^\alpha \hat{u} + a\hat{u}_{xxx} = 0, \\ \hat{u}(0, t) = \hat{h}_1(t), \quad \hat{u}_x(L, t) = \hat{h}_2(t), \quad \hat{u}_{xx}(L, t) = \hat{h}_3(t), \end{cases} \quad (2.2)$$

where

$$\begin{aligned} \hat{u}(x, s) &= \int_0^\infty e^{-st} u(x, t) dt, \\ \hat{h}_j(s) &= \int_0^\infty e^{-st} h_j(t) dt, \quad j = 1, 2, 3. \end{aligned}$$

As we all known that, when $\hat{h}_1(t) = \hat{h}_2(t) = \hat{h}_3(t) = 0$, the solution u of (2.2) can be denoted by

$$\hat{u}(t) = W_0(t)\hat{u}_0,$$

where $W_0(t)$ is the C_0 -semigroup in the space $L^2(0, L)$ (see [29]) generated by the dissipative linear operator

$$A\hat{u} = -\frac{a}{s^\alpha} \hat{u}'''.$$

And its definition domain is

$$D(A) = \hat{u} \in H^3(0, L); \hat{u}(0) = \hat{u}'(L) = \hat{u}''(L) = 0.$$

If $u_0 = 0$, the solution u of (2.2) can be written as

$$\hat{u}(t) = W_{bdr}(t) \vec{\hat{h}}, \quad \vec{\hat{h}} = (\hat{h}_1, \hat{h}_2, \hat{h}_3).$$

The operator $W_{bdr}(t)$ is the boundary integral operator of Eq.(2.2). In the following, we first look for the explicit representation formula of $W_{bdr}(t)$.

The characteristic equation of (2.2) is

$$s^\alpha + a\lambda^3 = 0,$$

and its solutions are

$$\lambda_1 = -\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}, \quad \lambda_2 = \frac{1 + \sqrt{3}i}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}, \quad \lambda_3 = \frac{1 - \sqrt{3}i}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}.$$

Then, the solutions $\widehat{u}(x, s)$ of (2.2) can be written as

$$\widehat{u}(x, s) = c_1(s)e^{\lambda_1(s)x} + c_2(s)e^{\lambda_2(s)x} + c_3(s)e^{\lambda_3(s)x},$$

and $c_j = c_j(s), j = 1, 2, 3$ solve the linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \\ \lambda_1^2 e^{\lambda_1 L} & \lambda_2^2 e^{\lambda_2 L} & \lambda_3^2 e^{\lambda_3 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \widehat{h}_1 \\ \widehat{h}_2 \\ \widehat{h}_3 \end{pmatrix}.$$

Using the inverse Laplace transform of \widehat{u} for any $c > 0$, we obtain that

$$u(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \widehat{u}(x, s) ds = \sum_{j=1}^3 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \frac{\Delta_j(s)}{\Delta(s)} e^{\lambda_j(s)x} ds,$$

where

$$c_j = \frac{\Delta_j}{\Delta}, \quad j = 1, 2, 3,$$

$$\Delta = \Delta(s) = \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \\ \lambda_1^2 e^{\lambda_1 L} & \lambda_2^2 e^{\lambda_2 L} & \lambda_3^2 e^{\lambda_3 L} \end{vmatrix},$$

$\Delta(s)$ is the determinant of the coefficient matrix and $\Delta_j(s), j = 1, 2, 3$ are the determinants of the matrix by replacing the j th-column of $\Delta(s)$ by the column vector $(\widehat{h}_1, \widehat{h}_2, \widehat{h}_3)^T$. Taking the same arguments as those in [30], we infer that the solution u can be written as the following representation

$$u(x, t) = \sum_{m=1}^3 u_m(x, t), \quad (2.3)$$

where

$$u_m(x, t) = \sum_{j=1}^3 u_{j,m}(x, t) \quad \text{and} \quad u_{j,m}(x, t) = u_{j,m}^+(x, t) + u_{j,m}^-(x, t), \quad (2.4)$$

with

$$u_{j,m}^+(x, t) = \frac{1}{2\pi i} \int_0^{i\infty} e^{st} \frac{\Delta_{j,m}(s)}{\Delta(s)} \widehat{h}_m(s) e^{\lambda_j(s)x} ds, \quad (m, j = 1, 2, 3),$$

$$u_{j,m}^-(x, t) = \frac{1}{2\pi i} \int_{-i\infty}^0 e^{st} \frac{\Delta_{j,m}(s)}{\Delta(s)} \widehat{h}_m(s) e^{\lambda_j(s)x} ds, \quad (m, j = 1, 2, 3).$$

Here, when $k \neq m$ ($k, m = 1, 2, 3$), $\Delta_{j,m}(s)$ is obtained from $\Delta_j(s)$ by letting $\widehat{h}_m(s) = 1$ and $\widehat{h}_k(s) = 0$.

Next, let $s = i\rho^3$, we have

$$u_{j,m}^+(x, t) = \frac{1}{2\pi} \int_0^\infty e^{i\rho^3 t} \frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)} \widehat{h}_m^+(\rho) e^{\lambda_j^+(\rho)x} 3\rho^2 d\rho,$$

and

$$\widehat{h}_m^+(\rho) = \widehat{h}_m(i\rho^3), \quad \Delta^+(\rho) = \Delta(i\rho^3), \quad \Delta_{j,m}^+(\rho) = \Delta_{j,m}(i\rho^3), \quad \lambda_j^+(\rho) = \lambda_j(i\rho^3).$$

Then, we turn to estimate the solution $u(x, t)$ of Eq.(2.2). The following technical Lemmas due to Bona, Sun and Zhang [2, 3] are needed which play a similar role as the Plancherel theorem in estimating $u(x, t)$.

Lemma 2.1 (see [8]). *For any $f \in L^2(\mathbb{R}^+)$, let Kf be the function defined by*

$$Kf(x) = \int_0^\infty e^{\gamma(\mu)x} f(\mu) d\mu$$

where $\gamma(\mu)$ is a continuous complex-valued function defined on $(0, \infty)$ satisfying the following two conditions:

(1) *There exists $\delta > 0$ and $b > 0$ such that*

$$\sup_{0 < \mu < \delta} \frac{|Re\gamma(\mu)|}{\mu} \geq b;$$

(2) *There exist a complex number $\alpha + i\beta$ such that*

$$\lim_{\mu \rightarrow \infty} \frac{\gamma(\mu)}{\mu} = \alpha + i\beta.$$

Then there exists a constant C such that for all $f \in L^2(0, \infty)$,

$$\|Kf\|_{L^2(0,1)} \leq C(\|e^{Re\gamma(\mu)} f(\cdot)\|_{L^2(\mathbb{R})} + \|f(\cdot)\|_{L^2(\mathbb{R})}).$$

Lemma 2.2. *Let $T > 0$ be given and $0 \leq s \leq 3$. For any given $\vec{h} = (h_1, h_2, h_3) \in \kappa_T$, $\kappa_T = H_0^{\frac{s+1}{3}}(\mathbb{R}^+) \times H_0^{\frac{s}{3}}(\mathbb{R}^+) \times H_0^{\frac{s-1}{3}}(\mathbb{R}^+)$, problem (1.1) has a unique solution $u \in X_T$, $X_T = C([0, T]; H^s(0, L)) \cap L^2([0, T]; H^{s+1}(0, L))$. Moreover, there exists a constant $C > 0$ such that $\|u\|_{X_T} + \sum_{j=0}^2 \|\partial_x^j u\|_{L_x^\infty(0, L; H^{\frac{(1-j)}{3}}(0, T))} \leq C \|\vec{h}\|_{\kappa_T}$.*

Proof. Note that as stated above, the solution u can be written as

$$u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t).$$

Now, we only prove the result for u_1 , the proofs for u_2, u_3 are similar. Note that

$$\begin{aligned} \lambda_1^+(\rho) &= i\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho, \lambda_2^+(\rho) = \frac{1}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho(\sqrt{3}-i), \lambda_3^+(\rho) = \frac{1}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho(-\sqrt{3}-i), \\ \Delta^+(\rho) &= \sqrt{3}\frac{s^\alpha}{a}\rho^3 e^{-\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho iL} \\ &\quad + \sqrt{3}\frac{s^\alpha}{a}\rho^3 e^{\frac{1}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho(\sqrt{3}+i)L} \\ &\quad + \sqrt{3}\frac{s^\alpha}{a}\rho^3 e^{\frac{1}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho(-\sqrt{3}+i)L}, \\ \Delta_{1,1}^+(\rho) &= \sqrt{3}\frac{s^\alpha}{a}\rho^3 e^{-\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho iL}, \\ \Delta_{2,1}^+(\rho) &= \frac{1+\sqrt{3}i}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{2}{3}}\rho^2 e^{\frac{1}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho(-\sqrt{3}-i)L} + \left(\frac{s^\alpha}{a}\right)^{\frac{2}{3}}\rho^2 e^{i\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho L}, \\ \Delta_{3,1}^+(\rho) &= \frac{1}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{2}}\rho e^{\frac{1}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho(\sqrt{3}-i)L} - i\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho e^{i\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho L}. \end{aligned}$$

When $\rho \rightarrow \infty$,

$$\frac{\Delta_{1,1}^+(\rho)}{\Delta^+(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho L}, \frac{\Delta_{2,1}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1} e^{-\frac{\sqrt{3}}{2}\left(\frac{s^\alpha}{a}\right)^{\frac{1}{3}}\rho L}, \frac{\Delta_{3,1}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}.$$

From (2.3) and (2.4), we have

$$\begin{aligned} u_1^+(x, t) &= \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty e^{i\rho^3 t} \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} \widehat{h}_1^+(\rho) e^{\lambda_j^+(\rho)x} 3\rho^2 d\rho, \\ \partial_x^k u_1^+(x, t) &= \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty e^{i\rho^3 t} (\lambda_j^+(\rho))^k \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} \widehat{h}_1^+(\rho) e^{\lambda_j^+(\rho)x} 3\rho^2 d\rho, \end{aligned}$$

where $k = 0, 1, 2, 3$.

According to Lemma (2.1), we infer that there exists a constant $K > 0$ such that

$$\begin{aligned} \|u_1^+(x, t)\|_{L^2(0,L)}^2 &\leq K \sum_{j=1}^3 \int_0^\infty (e^{Re\lambda_j^+(\rho)} + 1)^2 |3\rho^2 \widehat{h}_1^+(\rho)|^2 d\rho \\ &\leq K \int_0^\infty |\widehat{h}_1^+(\rho)|^2 d\rho \leq K \int_0^\infty \left| \int_0^\infty e^{-i\rho^3 \tau} h_1(\tau) d\tau \right|^2 d\rho. \end{aligned}$$

Now, setting $\mu = \rho^3, \theta(\mu) = \rho^{\frac{1}{3}}$, one has

$$\|u_1^+(x, t)\|_{L^2(0,L)}^2 \leq K \int_0^\infty \mu^{-\frac{2}{3}} \left| \int_0^\infty e^{-i\mu\tau} h_1(\tau) d\tau \right|^2 d\mu \leq K \|h_1\|_{H^{-\frac{1}{3}}(\mathbb{R}^+)}^2.$$

Similarly, when $k = 3$, we have

$$\begin{aligned} \|\partial_x^3 u_1^+(x, t)\|_{L^2(0,L)}^2 &\leq K \sum_{j=1}^3 \int_0^\infty (e^{Re\lambda_j^+(\rho)} + 1)^2 |\lambda_j^+(\rho)|^6 |3\rho^2 \widehat{h}_1^+(\rho)|^2 d\rho \\ &\leq K \int_0^\infty |\lambda_j^+(\rho)|^6 |\widehat{h}_1^+(\rho)|^2 d\rho \leq K \int_0^\infty |\lambda_j^+(\rho)|^6 \left| \int_0^\infty e^{-i\mu\tau} h_1(\tau) d\tau \right|^2 d\tau \\ &\leq K \int_0^\infty \mu^{\frac{4}{3}} \left| \int_0^\infty e^{-i\mu\tau} h_1(\tau) d\tau \right|^2 d\mu \leq K \|h_1\|_{H^{\frac{2}{3}}(\mathbb{R}^+)}^2. \end{aligned}$$

Furthermore, when $k = 0, 1, 2$, using Plancherel's Theorem in time t , for any $x \in (0, L)$, we infer that

$$\begin{aligned}
& \|\partial_x^k u_1^+(x, t)\|_{H^{\frac{s+1-k}{3}}(0, T)}^2 \\
& \leq \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty |\mu|^{\frac{2(s+1-k)}{3}} |(\lambda_j^+(\theta(\mu)))^k e^{\lambda_j^+(\theta(\mu))x} \frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}|^2 |\widehat{h}_1^+(i\mu)|^2 d\mu \\
& \leq K \int_0^\infty |(\lambda_j^+(\rho))^k e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)}|^2 |\widehat{h}_1^+(\rho)|^2 \rho^{2s+4-2k} d\rho \\
& \leq K \int_0^\infty \rho^{2s} |\widehat{h}_1^+(\rho)|^2 d\rho \\
& \leq K \int_0^\infty \mu^{\frac{2s-2}{3}} \left| \int_0^\infty e^{-i\mu\tau} h_1(\tau) d\tau \right|^2 d\mu \\
& \leq K \|h_1\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)}^2.
\end{aligned}$$

Therefore, we have

$$\sup_{0 \leq t \leq T} \|u_1^+(x, t)\|_{L^2(0, L)} \leq K \|h_1\|_{H^{\frac{-1}{3}}(\mathbb{R}^+)}, \quad \sup_{0 \leq t \leq T} \|u_1^+(x, t)\|_{H^3(0, L)} \leq K \|h_1\|_{H^{\frac{2}{3}}(\mathbb{R}^+)}.$$

By interpolation, for $0 \leq s \leq 3$,

$$\sup_{0 \leq t \leq T} \|u_1^+(x, t)\|_{H^s(0, L)} \leq K \|h_1\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)},$$

$$\sup_{0 < x < L} \|\partial_x^k u_1^+(x, t)\|_{H^{\frac{s+1-k}{3}}(0, T)} \leq K \|h_1\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)}.$$

Which ends the proof of Lemma (2.2) for u_1 . \square

In the following, we prove the continuity of $\partial_x^k u_1^+(k = 0, 1, 2)$ from the space $(0, L)$ to the space $H^{\frac{s+1-k}{3}}(0, T)$, for any $x, x_0 \in (0, L)$, we have

$$\begin{aligned}
\partial_x^k u_1^+(x, t) - \partial_x^k u_1^+(x_0, t) &= \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty e^{i\mu t} (\lambda_j^+(\theta(\mu)))^k (e^{\lambda_j^+(\theta(\mu))x} \\
&\quad - e^{\lambda_j^+(\theta(\mu))x_0}) \frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))} \widehat{h}_1^+(i\mu) d\mu.
\end{aligned}$$

Taking the Plancherel's Theorem in time t yields

$$\begin{aligned}
& \|\partial_x^k u_1^+(x, t) - \partial_x^k u_1^+(x_0, t)\|_{H^{\frac{s+1-k}{3}}(0, T)}^2 \\
& \leq \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty |\mu|^{\frac{2(s+1-k)}{3}} |(\lambda_j^+(\theta(\mu)))^k (e^{\lambda_j^+(\theta(\mu))x} \\
& \quad - e^{\lambda_j^+(\theta(\mu))x_0}) \frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}|^2 |\widehat{h}_1^+(i\mu)|^2 d\mu.
\end{aligned}$$

In views of Fatou's Lemma,

$$\begin{aligned}
& \lim_{x \rightarrow x_0} \|\partial_x^k u_1^+(x, t) - \partial_x^k u_1^+(x_0, t)\|_{H^{\frac{s+1-k}{3}}(0, T)}^2 \\
& \leq \lim_{x \rightarrow x_0} \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty |\mu|^{\frac{2(s+1-k)}{3}} |(\lambda_j^+(\theta(\mu)))^k (e^{\lambda_j^+(\theta(\mu))x} \\
& \quad - e^{\lambda_j^+(\theta(\mu))x_0}) \frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}|^2 |\widehat{h}_1^+(i\mu)|^2 d\mu \\
& = 0.
\end{aligned}$$

This leads to the continuity of $\partial_x^k u_1^+(x, t)$.

3. Linear control results

In this section, we study the boundary controllability of the following linear problem

$$\begin{cases} D_t^\alpha u + au_{xxx} = f, & (x, t) \in (0, L) \times (0, T), \\ u(x, 0) = u_0, \\ u(0, t) = h_1(t), u_x(L, t) = h_2(t), u_{xx}(L, t) = h_3(t), t \in (0, T), \end{cases} \quad (3.1)$$

where $f \in L^1(0, T; L^2(0, L))$.

Arguing as before, the solution of Eq.(3.1) can be written as

$$u(x, t) = W_0(t)u_0 + W_{bdr} \vec{h} + \int_0^t W_0(t - \tau)f(\tau)d\tau.$$

According to Lemma (2.2), we have

$$\|u\|_{X_T} + \sum_{j=0}^2 \|\partial_x^j u\|_{L_x^\infty(0, L; H^{\frac{(1-j)}{3}}(0, T))} \leq C \|\vec{h}\|_{\kappa_T} + \|f\|_{L^1(0, T; L^2(0, L))}.$$

The backward adjoint system of Eq.(3.1) is

$$\begin{cases} D_t^\alpha \phi + a\phi_{xxx} = 0, & (x, t) \in (0, L) \times (0, T), \\ \phi(x, T) = \phi_T, \\ \phi(0, t) = 0, \phi_x(0, t) = 0, \phi_{xx}(L, t) = 0 \quad t \in (0, T). \end{cases} \quad (3.2)$$

Let $x' = L - x, t' = T - t$, Eq.(3.2) becomes

$$\begin{cases} D_t^\alpha \phi + a\phi_{xxx} = 0, & (x, t) \in (0, L) \times (0, T), \\ \phi(x, 0) = \phi_0, \\ \phi(L, t) = 0, \phi_x(L, t) = 0, \phi_{xx}(0, t) = 0, \quad t \in (0, T). \end{cases} \quad (3.3)$$

In order to the convenience of the readers, we first introduce the following Lemmas.

Lemma 3.1. *Assume that $T > 0$, if we define $N_T = \{\phi_T \in L^2(0, L) : \phi_T \in X_T \text{ is the mild solution of Eq.(3.2) with } \phi_x(L, t) = 0 \text{ in } L^2(0, T)\}$, we can obtain $N_T = \{0\}$ if and only if L does not belong to $F = \{L \in \mathbb{R}^+ : L^2 = -(d^2 + db + b^2)\}$ with $d, b \in C^2$ satisfying $\frac{d^2}{e^b - e^c} = \frac{c^2}{e^b - e^d} = \frac{b^2}{e^c - e^d}$.*

Proof. Suppose that $N_T \neq \{0\}$, there exist $\lambda \in C$ and $\phi_0 \in H^3(0, L) \setminus \{0\}$ such that

$$\begin{cases} \lambda \phi_0 = -\frac{a}{s^\alpha} \phi_0''', \\ \phi_0(0) = 0, \phi_0'(0) = 0, \phi_0'''(L) = 0. \end{cases} \quad (3.4)$$

Note that the characteristic equation of Eq.(3.4) can be written as

$$\lambda + \frac{a}{s^\alpha} \mu^3 = 0. \quad (3.5)$$

Now, let μ_1, μ_2, μ_3 be three roots of Eq.(3.5), we yield that

$$\begin{cases} \mu_1 + \mu_2 + \mu_3 = 0, \\ \mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3 = 0, \\ \mu_1\mu_2\mu_3 = -\frac{s^\alpha}{a} \lambda. \end{cases} \quad (3.6)$$

Suppose that there exist double roots. Without loss of generality, we assume that

$$\mu_1 = \mu_2 = \pm\sigma, \quad \mu_3 = \mp\sigma.$$

From Eq.(3.6), we have $\sigma = 0$. This contradicts with our assumption.

If $\mu_1 \neq \mu_2 \neq \mu_3$, the solution of Eq.(3.4) is

$$\phi_0(x) = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x} + c_3 e^{\mu_3 x},$$

where $c_i (i = 1, 2, 3)$ are the solutions of the system

$$\begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_1^2 & \mu_2^2 & \mu_3^2 \\ \mu_1^2 e^{\mu_1 L} & \mu_2^2 e^{\mu_2 L} & \mu_3^2 e^{\mu_3 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.7)$$

Set $\mu_1 = \frac{d}{L}, \mu_2 = \frac{b}{L}, \mu_3 = \frac{c}{L}$. In views of Eq.(3.6), one has

$$c = -(d + b), \quad L^2 = -(db + bc + dc).$$

By reducing the rows of the matrix Eq.(3.7), we obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{c-d}{b-d} \\ 0 & 0 & M \\ 0 & 0 & N \end{pmatrix},$$

with $M = c^2 - d^2 - \frac{c-d}{b-d}(b^2 - d^2)$, $N = c^2e^c - d^2e^d - \frac{c-d}{b-d}(b^2e^b - d^2e^d)$.

Obviously, the system (3.7) has nonzero solutions if and only if

$$M = 0, N = 0. \quad (3.8)$$

Therefore, when $c \neq 0$, the system (3.8) has a unique solution

$$d^2 = \frac{c^2(e^b - e^c)}{e^b - e^d}, \quad b^2 = \frac{c^2(e^c - e^d)}{e^b - e^d},$$

which is contradicts with the inequality $N_T \neq \{0\}$ if and only if L does not belong to F , this implies $N_T = \{0\}$. \square

Lemma 3.2. *For $T > 0$, $L \notin F$, there exists a constant $C > 0$ such that for any $\phi_T \in L^2(0, L)$, the corresponding solution ϕ of Eq.(3.2) satisfies*

$$\|\phi_T\|_{L^2(0,L)} \leq C \int_0^T (|\Delta_t^{-\frac{1}{3}}\phi_{xx}(0,t)|^2 + |\phi_x(L,t)|^2 + |\Delta_t^{\frac{1}{3}}\phi(L,t)|^2) dt, \quad (3.9)$$

where $\Delta_t := (I - \partial_t^2)^{\frac{1}{2}}$.

Proof. If inequality (3.9) is false, there is a sequence $\{\phi_T^n\}_{n \in \mathbb{N}} \in L^2(0, L)$ with $\|\phi_T^n\|_{L^2(0,L)} = 1$ such that the corresponding solutions $\{\phi^n\}_{n \in \mathbb{N}}$ of Eq.(3.2) satisfy

$$n \int_0^T |\Delta_t^{-\frac{1}{3}}\phi_{xx}^n(0,t)|^2 dt \leq \|\phi_T^n\|_{L^2(0,L)} = 1,$$

for any n . Thus, we have $\|\phi_{xx}^n(0,t)\|_{L^2(0,L)} \rightarrow 0$ as $n \rightarrow \infty$. From Lemma (2.2), the sequence solutions $\{\phi^n\}_{n \in \mathbb{N}}$ and $\{\phi_x^n(L,t)\}_{n \in \mathbb{N}}$ are bounded. Next, by multiplying both sides of Eq.(3.3) by $(T-t)\phi$ and integrating by parts over $(0, L) \times (0, T)$, we have

$$\int_0^L (T-t)S^\alpha \phi^2|_0^T dx + \int_0^T \int_0^L S^\alpha \phi^2 dt dx - \int_0^T a(T-t)\phi_x^2|_0^L = 0.$$

That is,

$$\int_0^L \phi_0^2 dx \leq \frac{1}{T} \int_0^T \int_0^L \phi^2 dt dx + a \int_0^T \phi_x^2(0,t) dt,$$

which implies

$$\|\phi_T\|_{L^2(0,L)}^2 \leq \frac{1}{T} \|\phi\|_{L^2((0,L) \times (0,T))}^2 + a \|\phi_x(L, \cdot)\|_{L^2(0,L)}^2. \quad (3.10)$$

Therefore,

$$\|\phi_T^n\|_{L^2(0,L)}^2 \leq \frac{1}{T} \|\phi^n\|_{L^2((0,L) \times (0,T))}^2 + a \|\phi_x^n(L, \cdot)\|_{L^2(0,L)}^2.$$

Because of $D_t^\alpha \phi^n = -a\phi_{xxx}^n$ is bounded in $L^2(0, T; H^1(0, L))$, by the embedding theorem, the sequence $\{\phi^n\}_{n \in \mathbb{N}}$ is relatively compact in $L^2(0, T; L^2(0, L))$ (see [34]). Moreover, the sequence $\{\phi_T^n\}_{n \in \mathbb{N}}$ is an $L^2(0, L) - Cauchy$ sequence. Denote $\phi_T = \lim_{n \rightarrow \infty} \phi_T^n$ and let ϕ be the corresponding solution of Eq.(3.2). When $n \rightarrow \infty$, $\phi_x^n(L, t) \rightarrow \phi_x(L, t)$, for any n , we have $\|\phi_T\|_{L^2(0,L)} = 1$, $\phi_{xx}(0, t) = 0$ in $L^2(0, T)$

and $\|\phi_T^n\|_{L^2(0,L)} = 1$. From Lemma 3.1, one can conclude that $\phi_T(x) = 0$, which contradicts with $\|\phi_T\|_{L^2(0,L)} = 1$. This proves Lemma 3.2. \square

In the following, we consider the linear system of Eq.(1.1)

$$\begin{cases} D_t^\alpha u + au_{xxx} = 0, u(x, 0) = u_0, & (x, t) \in (0, L) \times (0, T), \\ u(0, t) = h_1(t), u_x(L, t) = h_2(t), u_{xx}(L, t) = h_3(t), & t \in (0, T), \end{cases} \quad (3.11)$$

we can get the following linear control result.

Theorem 3.1. *Suppose that $T > 0$, $L \notin F$ hold, there exists a linear and bounded operator $\Psi : L^2(0, L) \times L^2(0, L) \times L^2(0, L) \rightarrow H^{\frac{1}{3}}(0, T) \times L^2(0, T) \times H^{-\frac{1}{3}}(0, T)$ such that problem (3.11) possesses a solution $u \in C(0, T; L^2(0, L)) \times L^2(0, T; H^1(0, L))$ satisfying*

$$u|_{t=0} = u_0, \quad u|_{t=T} = u_T,$$

for any $u_0, u_T \in L^2(0, L)$ in the case of $h_1(t) = \frac{1}{a}\Delta_t^{-\frac{2}{3}}\phi_{xx}(0, t)$, $h_2(t) = \frac{1}{a}\phi_x(L, t)$, $h_3(t) = \frac{1}{a}\Delta_t^{\frac{2}{3}}\phi(L, t)$.

Proof. Without loss of generality, let $u_0 = 0$, ϕ be the solution of Eq.(3.2), then multiplying both sides of Eq.(3.11) by ϕ and integrating by parts over $(0, L) \times (0, T)$, we obtain

$$\int_0^L s^\alpha \phi(T, x)u(T, x)dx + \int_0^T a(\phi_{xx}(0, t)h_1(t) + \phi_x(L, t)h_2(t) - \phi(L, t)h_3(t))dt = 0.$$

In the following, set Ψ be the linear bounded map from $L^2(0, L) \rightarrow L^2(0, L)$,

$$\Psi : \phi_T(\cdot) \rightarrow \frac{1}{s^\alpha}u(\cdot, T),$$

and let u be the corresponding solution of Eq.(3.11). From Lemma 3.2, we infer that

$$\begin{aligned} (\Psi(\phi_T), s^\alpha \phi_T)L^2(0, L) &= (\frac{1}{a}\Delta_t^{-\frac{2}{3}}\phi_{xx}(0, t), a\phi_{xx}(0, t))_{L^2(0, T)} \\ &+ (\frac{1}{a}\phi_x(L, t), a\phi_x(L, t))_{L^2(0, T)} + (\frac{1}{a}\Delta_t^{-\frac{2}{3}}\phi(L, t), a\Delta_t^{-\frac{2}{3}}\phi(L, t))_{L^2(0, T)} \\ &\geq C^{-2}\|\phi_T\|_{L^2(0, T)}. \end{aligned}$$

Using the Lax-Milgram theorem, we obtain that Ψ is invertible. For given $u_T \in L^2(0, L)$, one can define $\phi_T = \Psi^{-1}\frac{1}{s^\alpha}u(\cdot, T)$ such that the solution $u_T \in X_T$ of Eq.(3.11) satisfies

$$u|_{t=0} = 0, \quad u|_{t=T} = u_T.$$

\square

4. Nonlinear results

In this section, we first introduce $S_T = H^s(0, L) \times H^{s+1}(0, T) \times H^{\frac{s}{3}}(0, T) \times H^{\frac{s-1}{3}}(0, T)$ and set $Y_{s,T}$ be the space consisting of all functions u in the space $C(0, T; H^s(0, L)) \cap$

$L^2(0, T; H^{s+1}(0, L))$ with $\partial_x^j u \in L_x^\infty(0, L; H^{\frac{s+1-j}{3}}(0, T))$ ($j = 0, 1, 2$), where its norm defined as

$$\|u\|_{Y_{s,T}} := \|u\|_{C(0,T;H^s(0,L)) \cap L^2(0,T;H^{s+1}(0,L))} + \sum_{j=0}^2 \|\partial_x^j u\|_{L_x^\infty(0,L;H^{\frac{s+1-j}{3}}(0,T))}.$$

And the following Lemma.

Lemma 4.1 (see [8]). *Let $0 \leq s \leq 3$, $T > 0$ be satisfied. Then there exists a constant C such that for any $u, v \in Y_{s,T}$,*

$$\int_0^T \|u(\cdot, t)v(\cdot, t)_x\|_{H^s(0,L)} dt \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}}) \|u\|_{Y_{s,T}} \|v\|_{Y_{s,T}}.$$

Then, we consider the following time-fraction nonlinear system

$$\begin{cases} D_t^\alpha u + 6auu_x + au_{xxx} = 0, u(x, 0) = u_0, & (x, t) \in (0, L) \times (0, T), \\ u(0, t) = h_1(t), u_x(L, t) = h_2(t), u_{xx}(L, t) = h_3(t), & t \in (0, T), \end{cases} \quad (4.1)$$

and get the results.

Theorem 4.1. *Assume that $T > 0$ is satisfied, then there exists $T^* \in (0, T)$ such that system (4.1) has unique solution*

$$u \in C([0, T^*]; H^s(0, L)) \cap L^2([0, T^*]; H^{s+1}(0, L)),$$

with $u_0 \in H^s(0, L)$, $h_1 \in H^{\frac{s+1}{3}}(0, T)$, $h_2 \in H^{\frac{s}{3}}(0, T)$ and $h_3 \in H^{\frac{s-1}{3}}(0, T)$, for $0 \leq s \leq 3$. Moreover, the corresponding solution map is Lipschitz continuous and satisfies the sharp Kato smoothing properties

$$\partial_x^k u \in L_x^\infty(0, L; H^{\frac{s+1-k}{3}}(0, T^*)) \text{ for } k = 0, 1, 2.$$

Proof. For given $(u_0, h_1, h_2, h_3) \in S_T$, we define a set

$$S_{\theta,k}^s := \{u \in Y_{s,\theta}, \|u\|_{Y_{s,\theta}} \leq k\},$$

where $k > 0$, $\theta > 0$. It is clear that the set $S_{\theta,k}^s$ is closed, convex and bounded. In the following, set a map Ψ_1 on $S_{\theta,k}^s$ by $u(x, t) = \Psi_1(v(x, t))$. For $v(x, t) \in S_{\theta,k}^s$ and $u(x, t)$ is the unique solution of

$$\begin{cases} D_t^\alpha u + 6avv_x + au_{xxx} = 0, u(x, 0) = u_0 & (x, t) \in (0, L) \times (0, T), \\ u(0, t) = h_1(t), u_x(L, t) = h_2(t), u_{xx}(L, t) = h_3(t), & t \in (0, T). \end{cases}$$

By Lemma 2.1 and Lemma 4.1, there exist constants c_1, c_2 such that

$$\begin{aligned} \|\Psi_1(v)\|_{Y_{s,\theta}} &\leq c_1 \|(u_0, h_1, h_2, h_3)\|_{S_T} + c_2 \int_0^\theta \|6avv_x\|_{H^s(0,L)} dt \\ &\leq c_1 \|(u_0, h_1, h_2, h_3)\|_{S_T} + c_2 (\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}) \|v\|_{Y_{s,\theta}}^2. \end{aligned}$$

Setting $k > 0$ and $\theta > 0$ such that

$$\begin{cases} k = 2c_1 \|(u_0, h_1, h_2, h_3)\|_{S_T}, \\ c_2(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}})k \leq \frac{1}{2}, \end{cases}$$

then, we infer that

$$\|\Psi_1(v)\|_{Y_{s,\theta}} \leq k.$$

Moreover, for any $v_1, v_2 \in S_{\theta,k}^s$, we obtain that $\omega(x, t) = \Psi_1(v_1) - \Psi_1(v_2)$ solves

$$\begin{cases} D_t^\alpha \omega + 3a(v_1^2 - v_2^2)_x + a\omega_{xxx} = 0, \omega(x, 0) = 0, & (x, t) \in (0, L) \times (0, T), \\ \omega(0, t) = 0, \omega_x(L, t) = 0, \omega_{xx}(L, t) = 0, & t \in (0, T). \end{cases}$$

Applying Lemma 2.1 again yields,

$$\|\Psi_1(v_1) - \Psi_1(v_2)\|_{Y_{s,\theta}} \leq C\|v_1 - v_2\|_{Y_{s,\theta}}.$$

When $v_1 \rightarrow v_2$, one has

$$\|\Psi_1(v_1) - \Psi_1(v_2)\|_{Y_{s,\theta}} \rightarrow 0.$$

Therefore, Ψ_1 is a contraction mapping of $S_{\theta,k}^s$ and its fixed point $\Psi_1(u) = u$ is the unique solution of Eq.(4.1). \square

Theorem 4.2. *Let $T > 0$ and $L > 0$ is satisfied, then there exists $\delta > 0$, for any $u_0, u_T \in L^2(0, L)$, if*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \leq \delta,$$

we can find $h_1 \in H^{\frac{1}{3}}(0, T), h_2 \in L^2(0, T), h_3 \in H^{-\frac{1}{3}}(0, T)$ such that the system (4.1) has a unique solution

$$u \in C(0, T; L^2(0, L)) \times L^2(0, T; H^1(0, L))$$

satisfying

$$u|_{t=0} = u_0, \quad u|_{t=T} = u_T.$$

Proof. As before, the solution of Eq.(4.1) can be written as

$$u(x, t) = W_0(t)u_0 + W_{bdr} \vec{h} - \int_0^t W_0(t - \tau)(6auu_x)(\tau)d\tau,$$

where $W_{bdr} \vec{h} = W_{bdr}(h_1, h_2, h_3)$.

Arguing as in the proof of Theorem 4.1, for $v \in Y_{s,\theta}$, we have

$$\Psi_2(T, v) := \int_0^T W_0(t - \tau)(6avv_x)(\tau)d\tau.$$

According to Theorem 3.1, for any $u_0, u_T \in L^2(0, L)$, we obtain that $(h_1, h_2, h_3) = F(u_0, u_T)$ satisfies

$$v(t) = W_0(t)u_0 + W_{bdr}F(u_0, u_T) - \int_0^t W_0(t - \tau)(6avv_x)(\tau)d\tau,$$

with

$$v|_{t=0} = u_0, \quad v|_{t=T} = u_T.$$

In the following, we consider the map Φ as form

$$\Phi(v) = W_0(t)u_0 + W_{bdr}F(u_0, u_T) - \int_0^t W_0(t-\tau)(6avv_x)(\tau)d\tau.$$

Similar to the discussion of Theorem 4.1, we infer that the map Φ is the contraction mapping. Moreover, it is continuous. Hence, its fixed point $\Phi(u) = u$ is a solution of Eq.(4.1) with $(h_1, h_2, h_3) = F(u_0, u_T)$, and satisfying

$$u|_{t=0} = 0, \quad u|_{t=T} = u_T.$$

□

5. Conclusion

As we know, the results of boundary controllability seem to be considered by few authors. In particular, Caicedo and Zhang (2017) dealt with the boundary controllability of the Korteweg-de Vries (KdV) equation on a bounded domain. Motivated by the works described above, in this paper, we study the following time-fractional nonlinear Korteweg-de Vries (KdV) equation

$$\begin{cases} D_t^\alpha u + 6auu_x + au_{xxx} = 0, u(x, 0) = u_0, & (x, t) \in (0, L) \times (0, T), \\ u(0, t) = h_1(t), u_x(L, t) = h_2(t), u_{xx}(L, t) = h_3(t), & t \in (0, T), \end{cases} \quad (\text{SP})$$

where $a > 0$, $L > 0$, $T > 0$, $\alpha \in (0, 1)$, $D_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (u(\xi) - u(0)) d\xi$ and Γ represents the Euler Gamma function. Firstly, for the fractional order, we give a proper treatment. Then, we obtain that the control result for the linear system of problem (SP). Finally, we prove the well-posedness and boundary controllability of problem (SP) posed on a finite domain $(0, L)$ with nonhomogeneous boundary conditions.

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References

- [1] A. Atangana and A. Secer, *The time-fractional coupled-Korteweg-de-Vries equations*. Hindawi Publishing Corporation, Abstr. Appl. Anal., 2013, 2013, 947986.
- [2] J. L. Bona, S. M. Sun and B. Y. Zhang, *A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane*, T. Am. Math. Soc., 2002, 354, 427–490.

- [3] J. L. Bona, S. M. Sun and B. Y. Zhang, *A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain*, Commun. Partial Differential Eq., 2003, 28, 1391–1436.
- [4] J. L. Bona, S. M. Sun and B. Y. Zhang, *Non-homogeneous boundary value problems for the Korteweg-de Vries and the Korteweg-de Vries-Burgers equations in a quarter plane*, Annales de l'IHP Analyse non linéaire, 2008, 25, 1145–1185.
- [5] J. L. Bona, S. M. Sun and B. Y. Zhang, *A non-homogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain II*, J. Differential Equations, 2009, 247, 2558–2596.
- [6] B. A. Bubnov, *General boundary value problems for the Korteweg-de Vries equation in a bounded domain*, Differentsial'nye Uravneniya, 1979, 15, 26–31.
- [7] B. A. Bubnov, *Solvability in the large of nonlinear boundary value problems for the Korteweg-de Vries equation in a bounded domain*, Differentsial'nye Uravneniya, 1980, 16, 34–41.
- [8] M. A. Caicedo and B. Y. Zhang, *Well-posedness of a nonlinear boundary value problem for the Korteweg-de Vries equation on a bounded domain*, J. Math. Anal. Appl., 2017, 448, 797–814.
- [9] E. Cerpa and E. Crššpeau, *Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain*, Annales de l'Institut Henri Poincaré. Non Linear Analysis. Elsevier Masson, 2009, 26, 457–475.
- [10] E. Cerpa, I. Rivas and B. Y. Zhang, *Boundary Controllability of the Korteweg-de Vries Equation on a Bounded Domain*, SIAM J. Control. Optim., 2013, 51, 2976–3010.
- [11] T. Colin and J. M. Ghidaglia, *An initial-boundary value problem for the Korteweg-de Vries equation posed on a finite interval*, Adv. Differential Equ., 2001, 6, 1463–1492.
- [12] J. E. Colliander and C. E. Kenig, *The generalized Korteweg-de Vries equation on the half line*, Commun. Partial Differential Eq., 2002, 27, 2187–2266.
- [13] E. Crépeau and M. Sorine, *A reduced model of pulsatile flow in an arterial compartment*, Chaos Soliton Fract., 2007, 34, 594–605.
- [14] E. Crépeau and M. Sorine, *Identifiability of a reduced model of pulsatile flow in an arterial compartment*, In: Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, Seville, Spain, 2005, 12, 12–15.
- [15] H. Demiray, *Nonlinear waves in a thick-walled viscoelastic tube filled with in-viscid fluid*, Int. J. Eng. Sci., 1998, 36, 345–357.
- [16] W. S. Duan, B. R. Wang and R. J. Wei, *Nonlinear waves propagating in an inhomogeneous blood vessel*, Acta Phys. Sin., Overseas Ed., 1997, 6, 801–823.
- [17] A. V. Faminskii, *Mixed problems for the Korteweg-de Vries equation*, Sbornik: Mathematics, 1999, 190, 903.
- [18] A. V. Faminskii, *On an initial boundary value problem in a bounded domain for the generalized Korteweg-de Vries equation*, Funct. Differ. Equ., 2001, 8, 183–194.

- [19] A. V. Faminskii, *An initial boundary-value problem in a half-strip for the Korteweg-de Vries equation in fractional-order Sobolev spaces*, Commun. Partial Differential Eq., 2005, 29, 1653–1695.
- [20] A. V. Faminskii, *Global well-posedness of two initial-boundary-value problems for the Korteweg-de Vries equation*, Differ. Integral Equ., 2007, 20, 601–642.
- [21] J. M. Ghidaglia, *Weakly damped forced Korteweg-de Vries equations behave as a finite dimensional dynamical system in the long time*, J. Differential Equations, 1988, 74, 369–390.
- [22] J. Holmer, *The initial-boundary value problem for the Korteweg-de Vries equation*, Commun. Partial Differential Eq., 2006, 31, 1151–1190.
- [23] G. Jumarie, *Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results*, Comput. Math. Appl., 2006, 51, 1367–1376.
- [24] D. J. Korteweg and G. De Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 1895, 39, 422–443.
- [25] E. F. Kramer and B. Y. Zhang, *Nonhomogeneous boundary value problems for the Korteweg-de Vries equation on a bounded domain*, J. Syst. Sci. Complex., 2010, 23, 499–526.
- [26] T. M. Laleg, E. Crépeau, Y. Papelier and M. Sorin, *Arterial blood pressure analysis based on scattering transform I*, In: Engineering in Medicine and Biology Society, 29th Annual International Conference of the IEEE, 2007.
- [27] C. Laurent, L. Rosier and B. Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation on a periodic domain*, Commun. Partial Differential Eq., 2010, 35, 707–744.
- [28] J. Li and K. S. Liu, *Well-posedness of Korteweg-de Vries-Burgers Equation on a finite domain*, India J. Pure Appl. Math, 2017, 48, 91–116.
- [29] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci., 44, Springer-Verlag, New York, 1983.
- [30] I. Rivas, M. Usman and B. Y. Zhang, *Global well-posedness and asymptotic behavior of a class of initial-boundary-value problem of the Korteweg-de Vries equation on a finite domain*, arXiv preprint arXiv:1010.4658, 2010
- [31] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Contr. Optim. Ca., 1997, 2, 33–55.
- [32] L. Rosier, *Control of the surface of a fluid by a wavemaker*, ESAIM Contr. Optim. Ca., 2004, 10, 346–380.
- [33] D. L. Russell and B. Y. Zhang, *Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain*, SIAM J. Control. Optim., 1993, 31, 659–676.
- [34] J. Simon, *Compact sets in the space $L^p(O, T; B)$* , Ann. Mat. Pur. Appl., 1986, 146, 65–96.