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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF TRAVELING WAVE SOLUTION FOR KORTEWEG-DE VRIES-BURGERS EQUATION WITH DISTRIBUTED DELAY*

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Abstract In this paper, we investigate the existence and asymptotic behavior of traveling wave solution for delayed Korteweg-de Vries-Burgers (KdV-Burgers) equation. Using geometric singular perturbation theory and Fredholm alternative, we establish the existence of traveling wave solution for this equation. Employing the standard asymptotic theory, we obtain asymptotic behavior of traveling wave solution of the equation.

Keywords Delayed Korteweg-de Vries-Burgers equation, traveling wave solution, geometric singular perturbation theory, asymptotic behavior.

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1. Introduction

During the past two decades, traveling wave solution of nonlinear evolutionary equations with delay is one of fastest developing areas of modern mathematics due to their significant nature, chemical and biological phenomena [1, 12, 13, 26, 33, 43] etc. Recently, Zhao and Xu [44] studied time-delayed KdV equation of the two forms

$$\begin{aligned}&\alpha u_t(x,t)+(1-\alpha)u_t(x,t-\tau)+u(x,t)u_x(x,t)\\&+\tau u_{xx}(x,t-\tau)-u_{xxx}(x,t)=0\end{aligned}\tag{1.1}$$

and

$$u_t(x,t)+u(x,t-\tau)u_x(x,t)+\tau u_{xx}(x,t-\tau)-u_{xxx}(x,t)=0,\tag{1.2}$$

where the time delay $\tau\in[0,b]$, for some $b\geq 0$ and $\alpha\in[0,1]$. By the inertial manifold theory and differential manifold geometric theory, the authors obtained the existence of solitary wave solutions of (1.1) and (1.2) when τ is small enough.

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Zhao [43] investigated the generalized KdV equation with distributed delay

$$u_t + (f * u)u^n u_x + \tau u_{xx} + u_{xxx} = 0, \tag{1.3}$$

where

$$f * u(x,t) = \int_{-\infty}^{t} f(t-s)u(x,s)ds, \tag{1.4}$$

and the kernel

$$f(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}, \quad \tau > 0. \tag{1.5}$$

Eq. (1.5) is sometimes called the weak generic delay kernel because it reflects the idea the importance of events in the past decreases exponentially the further one looks into the past [12]. The average delay for the distributed delay kernel $f(t)$ is defined as

$$\tau = \int_0^\infty t f(t)dt. \tag{1.6}$$

He proved that the solitary wave solutions persist when the average delay is suitably small.

It is known that the simplest form of wave equation in which nonlinearity uu_x, dispersion u_{xxx} and dissipation u_{xx} all occur is the KdV-Burgers equation

$$u_t + \alpha u_{xx} + \beta u u_x + s u_{xxx} = 0, \quad t \geq 0, \ \ x \in \mathbb{R}, \tag{1.7}$$

where α, β and s are real constants with $\alpha\beta s \neq 0$. During a study of the propagation of waves on liquid-filled elastic tubes, it was found by Johnson [17] that a particular limit of the problem led to (1.7), where $u(x,t)$ is proportional to the radial perturbation of the tube wall, and x and t are the characteristic and time variables, respectively. Eq. (1.7) was valid in the far field of an initially linear (s-mall amplitude) near-field solution. It is non-integrable in the sense that its spectral problem is non-existent. Feudel and Steudel [10] first pointed this out by showing that the equation has no prolongation structure. Eq. (1.7) is usually considered as a combination of the KdV equation and the Burgers equation, since $\alpha = 0$ in (1.7), it reduces to the famous KdV equation

$$u_t + \beta u u_x + s u_{xxx} = 0, \quad t \geq 0, \ \ x \in \mathbb{R}, \tag{1.8}$$

which was first suggested by Korteweg and de Vries who used it as a nonlinear model to study the change in the form of long waves advancing in a rectangular channel [19]. While taking $s = 0$ in (1.7), it become the well-known Burgers equation

$$u_t + \alpha u_{xx} + \beta u u_x = 0, \quad t \geq 0, \ \ x \in \mathbb{R}, \tag{1.9}$$

which is named after its use by Burgers for studying turbulence in 1939 [3]. It is known that both (1.8) and (1.9) are exactly solvable and have traveling wave solutions as follows, respectively,

$$u(x,t) = \frac{12sk^2}{\alpha} sech^2 k(x - 4sk^2 t)$$

and

$$u(x,t) = \frac{2k}{\alpha} + \frac{2\beta k}{\alpha}\tanh(x - 2kt).$$

A great number of theoretical issues concerning the KdV-Burgers equation have received considerable attention. In particular, the traveling wave solution to different types of KdV equations has been studied extensively, and many powerful methods have been established. Among these methods, we may cite, for instance, first integral method [8], the bifurcation method of dynamical systems [21], sub-ODE method [22], amplitude ansatz method [35], $(\frac{G'}{G})$-expansion method [23], numerical methods [5], Lie group theoretical methods [37] and so on. On the other hand, there has been widely argued and accepted [15, 40, 43, 44] that for various reasons, time delay should be considered in many models. However, the issue concerns the existence and asymptotic behavior of traveling wave solutions for KdV-Burgers equation with time-delay in the nonlinear term seems not to be exploited yet.

Inspired by [43, 44], we concern with KdV-Burgers equation with distributed delay

$$u_t + \alpha u_{xx} + \beta(f * u)u_x + su_{xxx} = 0, \quad t > 0, \ x \in \mathbb{R}, \tag{1.10}$$

where α, β and s are real constants with $\alpha\beta s \neq 0$. The convolution $f * u$ is defined in (1.4) and the kernel is in (1.5). Mathematically, the term $f * u$ can modify the nature of solutions of the PDE (1.7) dramatically. The existence of traveling wave solutions will be established by geometric singular perturbation theory. We note that geometric singular perturbation theory was introduced by Fenichel [9]. This is a geometric approach to problems with a clear separation in timescales. It uses invariant manifolds in phase space in order to understand the global structure of the phase space or to construct orbits with desired properties [16]. Since then, the method has evolved and found that way toward applications, which has received a great deal of interest and has been used by many researchers to obtain the existence of traveling waves for generalized KdV equations [18, 41–44], KdV-Burgers-Kuramoto equation [11], nonlinear dispersive-dissipative equation [28–30], reaction-diffusion equations [12,13,26,32,33,38,39], stochastic differential equations [2], slow-fast dynamic systems [25,27,36], Lienard equations [4,6,7] and biological models [14, 20, 24], etc.

In the present paper, our aim is to discuss the existence and asymptotic behavior of traveling wave solutions of equation (1.10) with a typical kernel (1.4). The rest of this paper is organized as follows. In Section 2, we first establish the existence of traveling wave solution for equation (1.10). In Section 3, we obtain the asymptotic behavior of the traveling wave solution which are obtained in Section 2 for the first time. In Section 4, we will give a brief conclusion.

2. Existence of traveling wave solution

In this section, we will establish a proposition which will be employed in the main proof of the existence of traveling wave solutions of equation (1.10) with (1.4). We first introduce the following result on invariant manifolds which is due to Fenichel [9]. For convenience, we use a version of this theorem due to Jones [18].

Lemma 2.1 (Geometric Singular Perturbation Theorem)**.** *For the system*

$$\begin{cases} x'(t) = f(x, y, \varepsilon), \\ y'(t) = \varepsilon g(x, y, \varepsilon), \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$ and ε is a real parameter, f, g are C^∞ on the set $V \times I$ where $V \in \mathbb{R}^{n+l}$ and I is an open interval, containing 0. If $\varepsilon = 0$, the system has a compact, normally hyperbolic manifold of critical points M_0 which is contained in the set $\{f(x,y,0) = 0\}$. Then for any $0 < r < +\infty$, if $\varepsilon > 0$, but sufficiently small, there exists a manifold M_ε:

- *(I) which is locally invariant under the flow of system (2.1);*
- *(II) which is a manifold of class C^r in x, y and ε;*
- *(III) $M_\varepsilon = \{(x,y) : x = h^\varepsilon(y)\}$ for some C^r function $h^\varepsilon(y)$ and y in some compact set K;*
- *(IV) there exist locally invariant stable and unstable manifolds $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ that lie within $O(\varepsilon)$ of, and are diffeomorphic to, $W^s(M_0)$ and $W^u(M_0)$.*

Then we give a useful result for the nondelay case (1.7). Assume that Eq. (1.7) has traveling wave solution in the form $u(x,t) = \varphi(\xi)$, $\xi = x - ct$, with wave speed $c > 0$. Substituting it into (1.7) and integrating once yields

$$-c\varphi + \alpha\varphi' + \frac{\beta}{2}\varphi^2 + s\varphi'' = 0, \tag{2.2}$$

where prime denotes the derivative by ξ and the integration constant is taken to be zero. Let $\varphi' = \phi$, then Eq. (2.2) can be written as the following system,

$$\begin{cases} \varphi' = \phi, \\ \phi' = \dfrac{c}{s}\varphi - \dfrac{\beta}{2s}\varphi^2 - \dfrac{\alpha}{s}\phi. \end{cases} \tag{2.3}$$

The following lemma leads to the existence of a traveling wave solution of nondelay equation.

Lemma 2.2. *Assume that $\alpha > 0, \beta > 0, s < 0$ and $0 < c < -\frac{\alpha^2}{4s}$. Then in the (φ, ϕ) phase plane satisfies system (2.3), there is a heteroclinic orbit connection its two equilbria $(0,0)$ and $(\frac{2c}{\beta}, 0)$. This connection is confined to $\phi > 0$.*

Proof. It is easy to verify that system (2.3) has two equilibria $E_0 = (0,0)$ and $E_1 = (\frac{2c}{\beta}, 0)$. Linearising (2.3) at E_1 shows that this positive equilibrium point is always a saddle point. However, the origin E_0 is always an unstable node for $0 < c < -\frac{\alpha^2}{4s}$. To confirm the existence of a heteroclinic connection between the two, and confined to $\phi > 0$, we shall show that for a suitable value of $\lambda > 0$, the triangular set

$$\Omega = \left\{(\varphi, \phi) : 0 \le \varphi \le \frac{2c}{\beta},\ \ 0 \le \phi \le \lambda\varphi\right\}$$

is negative invariant. This will be so if $\vec{m} \cdot \vec{n} \le 0$ everywhere on the boundary of the set where $\vec{n}$ is an inward pointing normal and $\vec{m}$ is the vector whose components are the right-hand side of (2.3). Two sides of the triangle are trivial to deal with. On the third, $\phi = \lambda\varphi$, we can take $\vec{n} = (\lambda, -1)$, then

$$\begin{aligned} \vec{m} \cdot \vec{n} &= \lambda\phi - \frac{c}{s}\varphi + \frac{\alpha}{s}\phi + \frac{\beta}{2s}\varphi^2 \bigg|_{(\varphi, \lambda\varphi)} \\ &= \varphi\left(\lambda^2 + \frac{\alpha}{s}\lambda - \frac{c}{s} + \frac{\beta}{2s}\varphi\right) \end{aligned}$$

$$\leq \varphi\left(\lambda^2 + \frac{\alpha}{s}\lambda - \frac{c}{s}\right) \leq 0.$$

Since $0 < c < -\frac{\alpha^2}{4s}$, $\lambda^2 + \frac{\alpha}{s}\lambda - \frac{c}{s} = 0$ has two real positive roots λ_1 and λ_2 satisfying $0 < \lambda_1 \leq \lambda_2$. This implies that $\vec{F} \cdot \vec{n} \leq 0$ provided $\lambda_1 \leq \lambda \leq \lambda_2$.

From the above discussion, it is easy to get that one branch of the unstable manifold at $(\frac{2c}{\beta}, 0)$ emanates into the invariant set Ω defined above. Once in this set, it has nowhere to go except into the origin. Thus the desired heteroclinic connection exists. This completes the proof. $\square$

Next we will seek the existence of the traveling wave solutions for Eq. (1.10) with the kernel

$$f(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}, \tag{2.4}$$

where $\tau > 0$, which measures the delay. For this kernel, if we define $v = f * u$, i.e.,

$$v(\xi) = \int_0^{+\infty} \frac{1}{\tau}e^{-\frac{t}{\tau}}u(\xi + ct)dt, \tag{2.5}$$

then by direct computation, we get

$$\frac{dv}{d\xi} = \frac{1}{c\tau}(v - u). \tag{2.6}$$

Assume that equation (1.10) with (1.4) has traveling wave solution in the form $u(x,t) = \varphi(\xi)$, $\xi = x - ct$, with wave speed $c > 0$, and integrating once meanwhile taking the integration constant to be zero, we have

$$s\varphi'' + \alpha\varphi' - c\varphi + \beta G = 0, \tag{2.7}$$

where prime also denotes the derivative by ξ and

$$G(\xi) = \int_{-\infty}^{\xi} \varphi' v d\delta. \tag{2.8}$$

Thus it is possible to reformulate the integro-differential equation (1.10) as the system

$$\begin{cases} s\varphi'' + \alpha\varphi' - c\varphi + \beta G = 0, \\ c\tau v' = v - \varphi, \end{cases} \tag{2.9}$$

under the boundary value conditions

$$\begin{cases} \lim\limits_{\xi \to -\infty} (\varphi(\xi), v(\xi)) = (0, 0), \\ \lim\limits_{\xi \to +\infty} (\varphi(\xi), v(\xi)) = \left(\frac{2c}{\beta}, \frac{2c}{\beta}\right). \end{cases} \tag{2.10}$$

Obviously, system (2.9) is not a delay differential system. The delay in the original problem now plays its role through the parameter τ. Thus we can deal with the question of traveling wave solutions in the delay Eq. (1.10) with (1.5) by seeking the existence of the traveling waves in system (2.9). Note that Eq. (1.10) with (1.5) is an infinite-dimensional dynamical system, but the phase space for the traveling wave equations associated with system (2.9) is finite dimensional. Hence the geometric singular perturbation theory for the ordinary differential equations can be applied to the system (2.9) to tackle existence of the traveling wave solutions. And our task is equivalent to establish the existence of traveling wave solutions of (1.10) connecting the two uniform steady-states $(0,0)$ and $(\frac{2c}{\beta}, \frac{2c}{\beta})$, for sufficiently small delay.

If further denote $\varphi' = \phi$, the system (2.9) can be cast into standard form for a singular perturbed problem

$$\begin{cases} \varphi' = \phi, \\ \phi' = \dfrac{c}{s}\varphi - \dfrac{\alpha}{s}\phi - \dfrac{\beta}{s}G, \\ c\tau v' = v - \varphi, \end{cases} \tag{2.11}$$

which can be called a slow system. When $\tau = 0$, the system becomes

$$\begin{cases} \varphi' = \phi, \\ \phi' = \dfrac{c}{s}\varphi - \dfrac{\alpha}{s}\phi - \dfrac{\beta}{2s}\varphi^2. \end{cases} \tag{2.12}$$

System (1.5) has a heteroclinic orbit connecting its two equilibria $(0,0)$ and $(\frac{2c}{\beta}, 0)$, which has been guaranteed by Lemma 2.2. What we will prove is that the system (2.11) has a traveling wave solution connecting $(0,0,0)$ and $(\frac{2c}{\beta}, 0, \frac{2c}{\beta})$ for $\tau > 0$ sufficiently small. Note that when $\tau = 0$, system (2.11) does not define a dynamic system in $\mathbb{R}^3$. This problem can be overcome through the transformation $\xi = \tau\eta$, under which the system (2.11) becomes

$$\begin{cases} \dot{\varphi} = \tau\phi, \\ \dot{\phi} = \dfrac{c\tau}{s}\varphi - \dfrac{\alpha\tau}{s}\phi - \dfrac{\beta\tau}{s}G, \\ \dot{v} = \dfrac{1}{c}(v - \varphi), \end{cases} \tag{2.13}$$

where dot denotes the derivative with respect to η. The system (2.13) is called a fast system. The slow system (2.11) and fast system (2.13) are equivalent when $\tau > 0$. Let $\tau \to 0$ in system (2.11), then the flow of system (2.11) is confined to the set

$$M_0 = \left\{ (\varphi, \phi, v) \in \mathbb{R}^3 : v = \varphi \right\},$$

which is a two-dimensional invariant manifold for system (2.11). If this manifold is normally hyperbolic, then the geometric singular perturbation theory of Fenichel [9] will be applied, and provides us with a two-dimensional invariant manifold M_τ, for the flow when $\tau > 0$. The idea is to study the flow of (2.11) restricted to this manifold, and the resulting system will be two-dimensional. This does not in itself establish the existence of a traveling wave solution; we still have to study the system reduced to M_τ, and show that it possesses a heteroclinic connection.

To check that M_0 is a normally hyperbolic manifold, we need to check that the linearization of the fast system (2.13), restricted to M_0, has exactly $\dim M_0$ eigenvalues on the imaginary axis with the remainder of the spectrum being hyperbolic.

The linearization of the fast system, restricted to M_0, is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{c} & 0 & \frac{1}{c} \end{pmatrix},$$

the eigenvalues of which are $0, 0, \frac{1}{c}$. Thus M_0 is normally hyperbolic. We now know that there exists a two-dimensional manifold M_τ with the properties described above. Next we will determine M_τ explicitly. M_τ can be written in the form

$$M_\tau = \Big\{(\varphi, \phi, v) \in \mathbb{R}^3 : v = \varphi + g(\varphi, \phi, \tau)\Big\},$$

where the function g depends smoothly on τ and satisfies

$$g(\varphi, \phi, 0) = 0.$$

Plugging $v = \varphi + g(\varphi, \phi, \tau)$ into slow system (2.11) yields

$$c\tau\left(\phi + \frac{\partial g}{\partial \varphi}\phi + \frac{\partial g}{\partial \phi}\left(\frac{c}{s}\varphi - \frac{a}{s}\phi - \frac{\beta}{s}G\right)\right) = g. \tag{2.14}$$

Since g is zero when $\tau = 0$, let g be expressed as

$$g(\varphi, \phi, \tau) = \tau g_1(\varphi, \phi) + \tau^2 g_2(\varphi, \phi) + \cdots\,. \tag{2.15}$$

Substituting (2.15) into (2.14) and comparing powers of τ give

$$g_1 = c\phi, \quad g_2 = c^2\left(\frac{c}{s}\varphi - \frac{a}{s}\phi - \frac{\beta}{s}G\right).$$

Hence we obtain

$$g = (c\phi)\tau + \left[c^2\left(\frac{c}{s}\varphi - \frac{a}{s}\phi - \frac{\beta}{s}G\right)\right]\tau^2 + o(\tau^2).$$

The slow system (2.11) restricted to M_τ is therefore given by

$$\begin{cases} \varphi' = \phi, \\ \phi' = \dfrac{c}{s}\varphi - \dfrac{\alpha}{s}\phi - \dfrac{\beta}{2s}\varphi^2 - \dfrac{\tau\beta c}{s}\displaystyle\int_{-\infty}^{\xi}\phi^2 d\delta + O(\tau^2). \end{cases} \tag{2.16}$$

Obviously, the system (2.16) is simplified to (2.12) when $\tau = 0$. It is easy to see system (2.16) has two equilibria points $(\varphi, \phi) = (0, 0)$ and $(\frac{2c}{\beta}, 0)$ for $\tau > 0$ sufficiently small. We wish now to establish the existence of a heteroclinic connection between these two critical points. From Lemma 2.2, we know that such a connection exists when $\tau = 0$. For $\tau > 0$ sufficiently small, we shall seek such a solution of system (2.16), that is a perturbation of traveling waves of system (2.3). Let $(\varphi_{10}, \phi_{10})$ be the solution of (2.16) when $\tau = 0$. To solve the system for $\tau > 0$ sufficiently small, we set

$$\varphi = \varphi_{10} + \tau\psi_1 + \cdots\,, \quad \phi = \phi_{10} + \tau\psi_2 + \cdots\,.$$

Substituting, we find that, to the lowest order, the differential equation system determining ψ_1 and ψ_2 is

$$\frac{d}{d\xi}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}+\begin{pmatrix}0 & -1\\ \frac{\beta}{s}\varphi_1-\frac{c}{s} & \frac{\alpha}{s}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}0\\ -\frac{\beta c}{s}\int_{-\infty}^{\xi}\phi_1 d\delta\end{pmatrix}, \tag{2.17}$$

and we seek to prove that this system has a solution satisfying $\psi_1(\pm\infty)=0$ and $\psi_2(\pm\infty)=0$.

Working in the space L^2 of square integrable functions, with inner product

$$\int_{-\infty}^{+\infty}\Big(x(\xi),y(\xi)\Big)d\xi,$$

$(\cdot,\cdot)$ being the Euclidean inner product on $\mathbb{R}^2$. Fredholm theory states that (2.17) have a solution if and only if

$$\int_{-\infty}^{\infty}\left(x(\xi),\begin{pmatrix}0\\ -\frac{\beta c}{s}\int_{-\infty}^{\xi}\phi_1 d\delta\end{pmatrix}\right)d\xi=0,$$

for all function $x(\xi)$ in the kernel of the adjoint of the operator L defined by the left-hand side of (2.17). It is easy to verify that the adjoint operator L^* is given by

$$L^*=-\frac{d}{d\xi}+\begin{pmatrix}0 & \frac{\beta}{s}\varphi_1-\frac{c}{s}\\ -1 & \frac{\alpha}{s}\end{pmatrix},$$

and thus compute KerL^*, we have to find all $x(\xi)$ satisfying

$$\frac{dx}{d\xi}=\begin{pmatrix}0 & \frac{\beta}{s}\varphi_1-\frac{c}{s}\\ -1 & \frac{\alpha}{s}\end{pmatrix}x. \tag{2.18}$$

The general solution of equation (2.18) is difficult to find because the matrix is non-constant. However, we are only looking for solutions satisfying $x(\pm\infty)=0$, in fact the only such solution is the zero solution. Recall that $\varphi_{10}(\xi)$ is the solution of the unperturbed problem and we do know it tends to zero as $\xi\to-\infty$ (see [8]). Letting $\xi\to-\infty$ in (2.18), the matrix becomes a constant matrix, with eigenvalues λ satisfying

$$\lambda^2-\frac{\alpha}{s}\lambda-\frac{c}{s}=0,$$

since $0<c<-\frac{\alpha^2}{4s}$, the eigenvalues are therefore both real and negative. As $\xi\to-\infty$, any solution of (2.18) other than the zero solution must be growing exponentially for lager ξ. So the only solution satisfying $x(\pm\infty)=0$ is the zero solution. This means that the Fredholm orthogonality condition trivially holds and so solutions of (2.18) exist. This completes the proof that a heteroclinic connection exists between the two equilibrum points $(0,0,0)$ and $(\frac{2c}{\beta},0,\frac{2c}{\beta})$ of (2.16).

From the above discussions, we obtain the first main theorem.

Theorem 2.1. *For any fixed $0<c<-\frac{\alpha^2}{4s}$, equation (1.10) with the weak generic kernel (2.4) has a traveling wave solution $u(x,t)=\varphi(x-ct)$ satisfying $\varphi(-\infty)=0$ and $\varphi(+\infty)=\frac{2c}{\beta}$, provided the delay $\tau>0$ is sufficiently small.*

3. Asymptotic behavior of traveling wave solution

Let $\Phi(\xi) = (\varphi_0(\xi), v_0(\xi))$ be the traveling wave solution of system (2.9). Differentiating (2.9) with respect to ξ and denoting $\Phi'(\xi) = (\varphi_1(\xi), v_1(\xi))$, we get

$$\begin{cases} s\varphi_1'' + \alpha\varphi_1' - c\varphi_1 + \beta\varphi_1 v_0 = 0, \\ c\tau v_1' = v_1 - \varphi_1. \end{cases} \tag{3.1}$$

Due to $\varphi(-\infty) = v(-\infty) = 0$, the limiting system for (3.1) as $\xi \to -\infty$ is

$$\begin{cases} s\varphi_{1-}'' + \alpha\varphi_{1-}' - c\varphi_{1-} = 0, \\ c\tau v_{1-}' = v_{1-} - \varphi_{1-}, \end{cases} \tag{3.2}$$

where (φ_{1-}, v_{1-}) is the traveling wave solution of system (3.2) as $\xi \to -\infty$. Setting $\varphi_{1-}' = \varphi_{2-}$, we can write system (3.2) as a first order system of ordinary differential equations

$$Z' = AZ, \quad Z = (\varphi_{1-}, \varphi_{2-}, v_{1-})^T, \tag{3.3}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{c}{s} & -\frac{\alpha}{s} & 0 \\ -\frac{1}{c\tau} & 0 & \frac{1}{c\tau} \end{pmatrix}.$$

Solving the system (3.3), we have

$$Z = (\varphi_{1-}, \varphi_{2-}, v_{1-})^T = \sum_{i=1}^{3} c_i h_i e^{\lambda_i \xi}, \tag{3.4}$$

where

$$\lambda_1 = \frac{1}{c\tau}, \quad \lambda_2 = \frac{-\alpha + \sqrt{\alpha^2 + 4cs}}{2s}, \quad \lambda_3 = -\frac{\alpha + \sqrt{\alpha^2 + 4cs}}{2s},$$

$h_i (i = 1, 2, 3)$ are eigenvectors of the matrix A with λ_i $(i = 1, 2, 3)$ as corresponding eigenvalues, and c_i are arbitrary constants. For $s < 0$, $0 < c < -\frac{\alpha^2}{4s}$, it is easy to check that $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$ and

$$(\varphi_{1-}, \varphi_{2-}, v_{1-})^T = c_1 h_1 e^{\lambda_1 \xi} + c_2 h_2 e^{\lambda_2 \xi} + c_3 h_3 e^{\lambda_3 \xi}.$$

We deduce the following asymptotic behavior as $\xi \to -\infty$

$$\begin{pmatrix} \varphi_1(\xi) \\ v_1(\xi) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{3} \alpha_i (m_i + o(1)) e^{\lambda_i \xi} \\ \sum_{i=1}^{3} \alpha_i (n_i + o(1)) e^{\lambda_i \xi} \end{pmatrix}, \tag{3.5}$$

where m_i, n_i $(i = 1, 2, 3)$ are constants, n_i and α_i $(i = 1, 2, 3)$ cannot be zero simultaneously. Then we claim that $n_i \neq 0$ $(i = 1, 2, 3)$ in (3.5). Consider the solution $h_i e^{\lambda_i \xi}$ of the system (3.3). If the third components of eigenvector h_i $(i = 1, 2, 3)$ is zero, the matrix A implies that the other components are zero. So we conclude that $n_i \neq 0$ $(i = 1, 2, 3)$.

Similarly, $\varphi(+\infty) = v(+\infty) = \frac{2c}{\beta}$ and the limiting system for (3.1) as $\xi \to +\infty$ is

$$\begin{cases} s\varphi_{1+}'' + \alpha\varphi_{1+}' + c\varphi_{1+} = 0, \\ c\tau v_{1+}' = v_{1+} - \varphi_{1+}, \end{cases} \tag{3.6}$$

where (φ_{1+}, v_{1+}) is the traveling wave solution of system (3.6) as $\xi \to +\infty$. Setting $\varphi_{1+}' = \varphi_{2+}$, we can write system (3.6) as a first order system of ordinary differential equations

$$Z' = BZ, \quad Z = (\varphi_{1+}, \varphi_{2+}, v_{1+})^T, \tag{3.7}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{c}{s} & -\frac{\alpha}{s} & 0 \\ -\frac{1}{c\tau} & 0 & \frac{1}{c\tau} \end{pmatrix}.$$

One can obtain that the general solution of the system (3.7) has the following form

$$Z = (\varphi_{1+}, \varphi_{2+}, v_{1+})^T = \sum_{i=1}^{3} d_i f_i e^{\Lambda_i \xi},$$

where

$$\Lambda_1 = \frac{1}{c\tau}, \quad \Lambda_2 = \frac{-\alpha + \sqrt{\alpha^2 - 4cs}}{2s}, \quad \Lambda_3 = -\frac{\alpha + \sqrt{\alpha^2 - 4cs}}{2s},$$

where $f_i (i = 1, 2, 3)$ are eigenvectors of the matrix B with Λ_i $(i = 1, 2, 3)$ as the corresponding eigenvalues, and D_i are arbitrary constants. For $s < 0$, $0 < c < -\frac{\alpha^2}{4s}$, it is easy to check that $\Lambda_1 > 0, \Lambda_2 < 0, \Lambda_3 > 0$. Since $(\varphi_{1+}, \varphi_{2+}, v_{1+})^T \to (0, 0, 0)^T$ as $\xi \to +\infty$, it is easy to verify that $d_1 = d_3 = 0$ and

$$(\varphi_{1+}, \varphi_{2+}, v_{1+})^T = d_3 f_2 e^{\Lambda_2 \xi}. \tag{3.8}$$

Similarly to the case $\xi \to -\infty$, we have the following asymptotic behavior of traveling wave solutions as $\xi \to +\infty$

$$\begin{pmatrix} \varphi_1(\xi) \\ v_1(\xi) \end{pmatrix} = \begin{pmatrix} \beta_1 (p_1 + o(1)) e^{\Lambda_2 \xi} \\ \beta_1 (q_1 + o(1)) e^{\Lambda_2 \xi} \end{pmatrix}, \tag{3.9}$$

where p_1, q_1 are constants, q_1 and β_1 cannot be zero, simultaneously. Then we claim that $q_1 \neq 0$ in (3.9). Consider the solution $f_2 e^{\Lambda_2 \xi}$ of the system (3.8). If the third components of eigenvector f_2 is zero, the matrix B implies that the other components are zero. So we conclude that $q_1 \neq 0$.

From the above discussions, we obtain the second main theorem.

Theorem 3.1. *Under the assumptions of Theorem 2.1, there exist positive constants A_i and B_i $(i = 1, 2)$ such that system (2.9) has a traveling wave solution $\Phi(\xi)$ satisfying the following asymptotic properties*

$$\Phi(\xi) = \begin{pmatrix} (A_1 + o(1)) e^{\lambda \xi} \\ (A_2 + o(1)) e^{\lambda \xi} \end{pmatrix}, \qquad \xi \to -\infty$$

and

$$\Phi(\xi) = \begin{pmatrix} \frac{2c}{\beta} - (B_1 + o(1))e^{\Lambda_2 \xi} \\ \frac{2c}{\beta} - (B_2 + o(1))e^{\Lambda_2 \xi} \end{pmatrix}, \qquad \xi \to +\infty,$$

where λ *may be one of* λ_1, λ_2 *and* λ_3.

Remark 3.1. If the kernel is taken to be one of the following kernels

$$f(x,t) = \delta(x)\delta(t-\tau), \qquad f(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}}, \qquad f(x,t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} \frac{1}{\sqrt{4\pi st}} e^{-\frac{x^2}{4st}},$$

by using the above method we can get similar results, respectively.

4. Conclusion

This paper is concerned with the existence and asymptotic behavior of the traveling wave solutions of the KdV-Burgers equation with a particular kernel commonly called the weak generic kernel. This choice enables the equation, in particular the traveling wave equation, to be recast as a system which is of higher order but is not a delay differential system. In this system, the parameter in the kernel which measures the delay appears as a coefficient. By exploiting this fact and the supposed smallness of the delay, we have shown how the theory of dynamical systems can be used to construct a two-dimensional invariant manifold within $\mathbb{R}^3$ for the system. By recasting the equation as this manifold, we are then able to show the existence of a heteroclinic connection in the manifold and this establishes the existence of traveling wave solutions for the original problem when the delay is sufficiently small. Furthermore, by employing the standard asymptotic theory, we obtain the asymptotic behavior of traveling wave solutions accordingly.

In the near future, we are going to develop and extend the techniques to other nonlinear evolutionary equations. We believe that this method must be advantageous for a rather diverse group of scientists.

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