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A BLOW-UP METHOD TO PROVE FORMAL INTEGRABILITY FOR SOME PLANAR DIFFERENTIAL SYSTEMS

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Abstract In this work we provide an effective method to prove the formal integrability of the resonant saddles. The method is based on the use of a blow-up and the resolution of a recurrence differential equation using induction. Using the method some open integrability problems for certain resonant saddles are solved.

Keywords Saddle constants, formal first integral, invariant analytic curve.

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1. Introduction

The integrability problem for systems of differential equations is one of the main problems in the qualitative theory of differential systems [5,14]. In fact, integrability, although a rare phenomenon, it is of great importance due to applications in the bifurcation theory. For a certain systems of differential equations in $\mathbb{R}^2$ it is closely connected with another problem in the theory of differential systems, i.e., with the center-focus problem [24, 26]. The center-focus problem or shorter center problem ask for the conditions under which a real system whose real part has two purely imaginary eigenvalues admits periodic solutions in a neighborhood of the singular point at the origin. Such real differential system can be embedded in the complex plane and in this way the singular point at the origin which is of center-focus type becomes a $1 : -1$ resonant saddle singular point. A natural generalization of this resonance is a $p : -q$ resonant singular point of a polynomial vector field in $\mathbb{C}^2$, see [29]. The analytic integrability of such singular points is an open problem recently studied in several works for some families of differential systems, see for instance [11, 13, 15–17, 29].

The first step to establish the analytic integrability of a $p : -q$ resonant singular point is to find the necessary conditions. For this task, one can use different methods, including the use of a series of changes of variables which brings the original system to its normal form, see for instance [1]. Other methods are based on the construction of a formal first integral, see [28, 29]. Once we obtain necessary condi-

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tions for integrability of the $p:-q$ resonant singular point we have to prove their sufficiency.

To prove the sufficiency there is no general algorithm that works for all the systems. Sometimes the sufficiency is obtained verifying that the system is Hamiltonian or time-reversible, that is, invariant by a certain symmetry. Another method is to find an explicit first integral which is well-defined in a neighborhood of the singular point. If first integral can not be found directly or explicitly as a function of elementary functions, one can try to find an integrating factor of the system and using it to construct a first integral if it is possible. Some results connect the existence of the integrating factor with the existence of an analytic first integral, see for instance [3,7,18,28] and references therein. Once in a while ad hoc methods to prove the sufficiency are used for some particular families, see [6, 7, 13, 19, 21–23, 27, 29]. However, for certain families of systems that satisfy the necessary conditions all these methods turn to be ineffective and the integrability problem for such families remains as open problems, see for instance [8, 10–12].

As we will see in the next section (see Theorem 2.1), for an isolated singularity, which is a resonant saddle, the existence of a formal first integral implies the existence of an analytic first integral. Consequently, one way to prove the sufficiency is to show the existence of a formal first integral. Hence, it is important to develop new methods for proving the existence of a formal first integral at a certain singular point.

In [2] the analytic integrability through the formal integrability and the connection of formal integrability with the existence of invariant analytic (sometimes algebraic) curves has been studied. Using the results presented in [2] some families of differential systems in $\mathbb{C}^2$ that satisfy the sufficient conditions to have a formal integrable resonant saddle are determined without previously computing the necessary conditions.

In this paper we give a method to prove formal integrability. The method can be successfully applied for showing that some necessary conditions for integrability are also sufficient. Using this method we then solve some open cases.

2. Definitions and preliminary concepts

Poincaré and Lyapunov [24, 26] proved that the elementary point at the origin of the system of differential equations

$$\dot{u} = v + P(u, v), \qquad \dot{v} = -u + Q(u, v), \tag{2.1}$$

where $P(u, v)$ and $Q(u, v)$ are real analytic functions without constant and linear terms, is a center if and only if the system admits a first integral of the form

$$\Phi(u, v) = u^2 + v^2 + \sum_{k+l\geq 3} \phi_{kl} u^k v^l. \tag{2.2}$$

This is the so-called Poincaré-Lyapunov Theorem. The theorem says that the qualitative picture of trajectories in a neighborhood of the singular point is related to local integrability of the system: the singular point is a center if and only if there exists an analytic first integral of the form (2.2). It can be also proved that there exists an analytic first integral of system (2.1) if and only if there exists a formal first integral of system (2.1) of the form (2.2) [4]. However, the Poincaré-Lyapunov

Theorem does not give an answer to the question how to establish whether for a given system of differential equations there exists a first integral of the form (2.2). The answer to this question must be found for each particular system separately, and so far we have no general method that enables us to answer this question for an arbitrary system (2.1).

One of the tools to study the integrability problem for system (2.1) is to complexify this system in the following way. First setting $x = u + iv$ system (2.1) becomes the equation

$$\dot{x} = ix + R(x, \bar{x}).$$

Adjoining to this equation its complex conjugate we have the system

$$\dot{x} = ix + R(x, \bar{x}), \quad \dot{\bar{x}} = -i\bar{x} + \bar{R}(x, \bar{x}).$$

Consider $y := \bar{x}$ as a new variable and $\bar{R}$ as a new function. Then, from the latter system we obtain a system of two complex differential equations which after the change of time $idt = dT$ and rewriting t instead of T becomes

$$\dot{x} = x + G(x, y), \quad \dot{y} = -y + H(x, y). \tag{2.3}$$

System (2.3) is called the complexification of system (2.1). For such system we can always find a function of the form

$$\Psi(x, y) = xy + \sum_{i+j>2} \psi_{ij} x^i y^j,$$

satisfying the equation

$$\dot{\Psi} = \frac{\partial \Psi}{\partial x}(x + G(x, y)) + \frac{\partial \Psi}{\partial y}(-y + H(x, y)) = v_3 (xy)^4 + v_5 (xy)^6 + \cdots,$$

where v_{2i+1} are polynomials in the coefficients of system (2.3). We see that if all the polynomials v_{2i+1} vanish then $\Psi(x, y)$ is first integral of system (2.3). And if system (2.3) has formal first integral then also real system (2.1) has analytic first integral (2.2). Complex system (2.3) has singular point at the origin which is called $1 : -1$ resonant saddle singular point. The values v_{2i+1} are called the *saddle constants* or *focal values*, see [28, 29].

A generalization of $1 : -1$ resonant saddle singular point is $p : -q$ resonant saddle singular point. Complex differential system with a $p : -q$ resonant saddle singular point at the origin is a system of the form

$$\dot{x} = p\,x + F_1(x, y), \quad \dot{y} = -q\,y + F_2(x, y), \tag{2.4}$$

where F_1 and F_2 are analytic functions without constant and linear terms with $p, q \in \mathbb{Z}$ and $p, q > 0$. The integrable resonant saddles of such systems also called *resonant centers* were introduced by Dulac [9], see also [20, 29].

Definition 2.1. A $p : -q$ resonant saddle singular point of an analytic system is a resonant center if an only if there exists a local meromorphic first integral $\Psi = x^q y^p + \sum_{i+j>p+q} \psi_{ij} x^i y^j$.

The proof of the next result is given in [25], see also [28].

Theorem 2.1. *Assume that system* (2.1) *or* (2.4) *with an isolated singularity at the origin has a formal first integral* $\Psi(x,y) \in \mathbb{C}[[x,y]]$ *in the neighborhood of the singularity. Then, there exists an analytic first integral in the neighborhood of the singularity.*

Hence, if we want to prove the existence of an analytic first integral of system (2.1) (or (2.4)) we only need to prove the existence of a formal first integral.

3. Blow-up method for resonant singular points

In this section we consider the blow-up method to detect formal integrability for a resonant singular point. This method has been already applied to some systems of ODE's [2, 16, 17, 29], but mostly not successful. The method is the following. We consider the resonant singular point at the origin of system (2.4) and we perform the blow-up $(x,y) \to (x,z) = (x, y/x)$. Then the singular point $x = y = 0$ is replaced by the line $x = 0$, which contains two singular points that correspond to the separatrices of the singular point at the origin of system (2.4). These two singular points are saddles given by: p_1 which is $(p+q) : -p$ resonant and p_2 which is $(p+q) : -q$ resonant. We now establish the following result.

Theorem 3.1. *The* $p : -q$ *resonant singular point at the origin of system (2.4) is analytically integrable if and only if either* p_1 *or* p_2 *is orbitally analytically linearizable.*

The proof of this theorem is straightforward since the fact that the $p : -q$ resonant singular point at the origin of system (2.4) has an analytic first integral $\Psi(x,y)$ means that both points p_1 or p_2 have a well-defined analytic first integral $\tilde{\Psi}(x, zx)$. The sufficiency was proven in Lemma 1 of [13] using the normal orbital form of the $p : -q$ resonant system (2.4) and the first integral of such normal orbital form.

Corollary 3.1. *The necessary integrability conditions for the* $p : -q$ *resonant singular point at the origin of system (2.4) are the same, modulo the previous ones, than the necessary integrability conditions of the singular points* p_1 *or* p_2.

In order to apply Theorem 3.1 we need to prove the existence of a formal first integral for one of the points p_1 or p_2 and this could not be done for some of the cases studied in [2, 16, 17].

4. Formal integrability for resonant singular points

In [2] the authors analyzed the formal integrability of system (2.4) so that they found the first integral as a power series in original coordinates. Here, we apply the blow-up method with $z = y/x$ and then we consider the formal integrability at the resonant saddle p_1 or p_2. In such a way system (2.4) is transformed into a system of variables (x,z) of the form

$$\dot{z} = -(p+q)z + x\mathcal{F}(x,z), \qquad \dot{x} = p\,x + x^2\mathcal{G}(x,z), \tag{4.1}$$

where $\mathcal{F}(0,0)=0$. Taking into account that $x=0$ is an invariant line of system (4.1), we propose to look for a formal first integral of the form

$$\tilde{\mathcal{H}} = \sum_{i\geq 1}^{\infty} f_i(z)x^i, \tag{4.2}$$

where $f_i(z)$ are polynomials of degree $\leq i$ (if the resonant saddle is formally integrable). The first differential equation for $f_1(z)$ is $pf_1(z)-(p+q)zf_1'(z)=0$ whose solution is $f_1(z)=c_1 z^{p/(p+q)}$. Hence, we take $c_1=0$ and, therefore, $f_1(z)=0$. The next differential equation for $f_2(z)$ is $2pf_2(z)-(p+q)zf_2'(z)=0$ and its solution is $f_2(z)=c_2 z^{(2p)/(p+q)}$. Consequently, either $(2p)/(p+q)\in\mathbb{N}$ or we take $c_2=0$. Taking into account that $p,q\in\mathbb{Z}$ with $p,q>0$ it always exists f_{k_0} such that $(k_0 p)/(p+q)\in\mathbb{N}$ (or $(k_0 q)/(p+q)\in\mathbb{N}$ for saddle point p_2). Finally, at each power of x we have the differential equation

$$k\,p\,f_k(z)-(p+q)\,z\,{f_k}'(z)+g_k(z)=0, \tag{4.3}$$

where $g_k(z)$ depends on some previous functions $f_i(z)$ for $i=k_0,\ldots,k-1$. The solution of differential equation (4.3) is given by

$$f_k(z)=c_k\,z^{\frac{kp}{p+q}}+z^{\frac{kp}{p+q}}\int^z \frac{s^{-1-\frac{kp}{p+q}}}{p+q}\,g_k(s)\,ds, \tag{4.4}$$

where c_k is an arbitrary constant. However, we will see that functions f_k in (4.4) are not necessarily polynomials. In fact, from (4.4) it is easy to see that it always exists a value k_r such that for $k\geq k_r$ the functions $f_i(z)$ for $i=k_0,\ldots,k_r-1$ can give logarithmic terms. Therefore, we are not able to apply directly the induction method to prove that the solution f_k of recursive equation (4.3) is always a polynomial.

The logarithmic terms appear in the case when in the integrating function in (4.4) there is a term αx^{-1}. This is the case when

$$-1-\frac{k_r p}{p+q}+m_k=-1,$$

where m_k is the degree of the polynomial $g_k(s)$. So, we have $k_r=m_k(p+q)/p$. Hence, if $p=1$ then $k_r=m_k(1+q)$ which can be satisfied since m_k and q are positive integers. For the case of $p\neq 1$ taking into account that the value of m_k increases with increasing of k, it can also exist a value of m_k such that m_k is divisible by p and it gives the value of k_r that can give logarithmic terms. In fact, the coefficients of these logarithmic terms are the necessary conditions for integrability at singular point p_1 and this gives by Corollary 3.1 an alternative method for computing the saddle constants of the original system (2.4).

Thus using the blow-up transformation $z=y/x$ and the formal series (4.2) not necessarily means the existence of a formal first integral. We have to find a condition under which logarithmic terms can not appear in any solution f_k. In the following sections we solve some open examples of recent papers finding the condition under which we avoid the logarithmic term in the power series.

5. Open resonant cubic system of [8]

In [8] the integrability of the complex cubic system of the form

$$
\begin{aligned}
\dot{x} &= x(1 - a_{10}x - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2),\\
\dot{y} &= -y(1 - b_{10}x - b_{01}y - b_{20}x^2 - b_{11}xy - b_{02}y^2),
\end{aligned}
\tag{5.1}
$$

where a_{ij}, $b_{ij} \in \mathbb{C}$ is studied. The study was splitted into three different cases: (a) $a_{01} = b_{10} = 1$, (b) $a_{01} = 1$ and $b_{10} = 0$, and (c) $a_{01} = b_{10} = 0$. In Theorem 1 of [8] 13 integrable cases corresponding to the case (a) are given. However, the sufficiency of case (10) of this theorem remains open problem. Here, we apply our method to solve this case.

The system associated to condition (10) of Theorem 1 in [8] with $b_{10}b_{01} \neq 0$ after a scaling of the variables x and y is given by

$$
\begin{aligned}
\dot{x} &= x - \frac{1}{4}x^2y - \frac{1}{8}xy^2,\\
\dot{y} &= -y + xy - x^2y + y^2 - \frac{3}{4}xy^2 - \frac{1}{4}y^3.
\end{aligned}
\tag{5.2}
$$

Applying the blow-up transformation

$$
(x, y) \to (z, y) = (x/y, y) \tag{5.3}
$$

we obtain the system

$$
\begin{aligned}
\dot{z} &= 2z - yz + \frac{1}{8}zy^2 - yz^2 + \frac{1}{2}y^2z^2 + y^2z^3 = \mathcal{F}(z, y)\\
\dot{y} &= -y + y^2 - \frac{1}{4}y^3 + y^2z - \frac{3}{4}y^3z - y^3z^2 = \mathcal{G}(z, y).
\end{aligned}
\tag{5.4}
$$

Now, we look for a first integral of the form

$$
\mathcal{H}(z, y) = \sum_{k=2}^{\infty} f_k(z)y^k. \tag{5.5}
$$

Computing $\dot{\mathcal{H}} = (\partial\mathcal{H}/\partial z)\mathcal{F} + (\partial\mathcal{H}/\partial y)\mathcal{G}$ and equating to zero each coefficient of power of y we obtain the following recursive differential equation for f_k

$$
\begin{aligned}
&(2-k)(\frac{1}{4} + \frac{3}{4}z + z^2)f_{k-2} + (k-1)(1+z)f_{k-1}\\
&- kf_k + (\frac{z}{8} + \frac{z^2}{2} + z^3)f'_{k-2} - (z + z^2)f'_{k-1} + 2zf'_k = 0.
\end{aligned}
$$

For $k = 2, 3, \ldots, 10$ we find $f_2 = z$, $f_3 = z(1-z)$, $f_4 = z(\frac{13}{16} + z^2)$,

$$
f_5 = z(\frac{29}{48} + \frac{19}{16}z - \frac{1}{4}z^2 - \frac{2}{3}z^3), \quad f_6 = z(\frac{655}{1536} + \frac{379}{192}z + \frac{4}{3}z^3 + \frac{5}{12}z^4),
$$

$$
f_7 = \frac{z}{7680}(-2231 - 17605z - 18680z^2 + 1280z^3 + 12160z^4 + 1664z^5),
$$

Next, $f_8 = zQ_6(z)$, where $Q_6(z)$ is a polynomial of degree 6 without the term with z^3; $f_9 = zQ_7(z)$, where $Q_7(z)$ is a polynomial of degree 7; $f_{10} = zQ_8(z)$, where

$Q_8(z)$ is a polynomial of degree 8 without the term with z^4. So, we claim for k odd that f_k is of the form

$$f_k(z) = z(C_0 + C_1 z + C_2 z^2 + \cdots + C_{k-2} z^{k-2}),$$

and for k even that f_k is of the form $f_k = zQ_{k-2}$, where Q_{k-2} is a polynomial of degree $k-2$ without the monomial term $z^{\frac{k}{2}-1}$, i.e.,

$$f_k(z) = z(C_0 + C_1 z + C_2 z^2 + \cdots + C_{\frac{k}{2}-2} z^{\frac{k}{2}-2} + C_{\frac{k}{2}} z^{\frac{k}{2}} + \cdots + C_{k-2} z^{k-2}).$$

Now, using the induction we prove that for each k we obtain a polynomial f_k. Suppose that the assumption is true for $k = 1, \ldots, n-1$ and we compute f_k for $k = n$ solving the differential equation

$$\begin{aligned} f_n' - \frac{n}{2z} f_n =& \frac{1}{2z}\Big[(n-2)(\frac{1}{4} + \frac{3}{4}z + z^2) f_{n-2} - (n-1)(1+z) f_{n-1} \\ & - (\frac{z}{8} + \frac{z^2}{2} + z^3) f_{n-2}' + (z + z^2) f_{n-1}'\Big]. \end{aligned} \tag{5.6}$$

Let first n be odd and we want to prove that $f_n(z) = zQ_{n-2}$, where Q_{n-2} is a polynomial of degree at most $n-2$. If n is odd then also $n-2$ is odd and $n-1$ is even. The expression on the righthand side of differential equation (5.6) that we call $R_{n-2}(z)$ is a polynomial of degree at most $n-2$, and the differential equation (5.6) becomes $f_n' - \frac{n}{2z} f_n = R_{n-2}(z)$. We know that the general solution of a linear differential equation of the form

$$f'(z) + g(z) f(z) = h(z) \tag{5.7}$$

is

$$f(z) = e^{-\int g(z)dz}\Big(C + \int e^{\int g(z)dz} h(z) dz\Big). \tag{5.8}$$

In our case we have $g(z) = -\frac{n}{2z}$ and $h(z) = R_{n-2}(z) = A_0 + A_1 z + \cdots + A_{n-2} z^{n-2}$. Thus, the solution is

$$\begin{aligned} f_n(z) &= e^{\int \frac{n}{2z} dz}\big[C + \int e^{-\int \frac{n}{2z} dz}(A_0 + A_1 z + \cdots + A_{n-2} z^{n-2}) dz\big] \\ &= e^{\frac{n}{2}\ln z}\big[C + \int e^{-\frac{n}{2}\ln z}(A_0 + A_1 z + \cdots + A_{n-2} z^{n-2}) dz\big] \\ &= Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int z^{-\frac{n}{2}}(A_0 + A_1 z + \cdots + A_{n-2} z^{n-2}) dz \\ &= Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int (A_0 z^{-\frac{n}{2}} + A_1 z^{1-\frac{n}{2}} + \cdots + A_{n-2} z^{\frac{n}{2}-2}) dz \\ &= Cz^{\frac{n}{2}} + z^{\frac{n}{2}}(\tilde{A}_0 z^{1-\frac{n}{2}} + \tilde{A}_1 z^{2-\frac{n}{2}} + \cdots + \tilde{A}_{n-2} z^{\frac{n}{2}-1}) \\ &= Cz^{\frac{n}{2}} + \tilde{A}_0 z + \tilde{A}_1 z^2 + \cdots + \tilde{A}_{n-2} z^{n-1}) \\ &= Cz^{\frac{n}{2}} + z(\tilde{A}_0 + \tilde{A}_1 z + \cdots + \tilde{A}_{n-2} z^{n-2}) = Cz^{\frac{n}{2}} + zQ_{n-2}(z). \end{aligned}$$

Taking $C = 0$ it yields $f_n(z) = zQ_{n-2}(z)$ as we have assumed. We note that for n odd in the calculation no logarithmic term can appear. This is not the case for n even. Therefore, a bit more detailed analysis is needed. Let n be even. Then also

$n-2$ is even and $n-1$ is odd and functions f_{n-2}, f'_{n-2}, f_{n-1}, and f'_{n-1} are of the form

$$\begin{aligned}
f_{n-2}(z) =& z\ (C_0 + C_1 z + C_2 z^2 + \cdots + C_{\frac{n}{2}-3} z^{\frac{n}{2}-3} + C_{\frac{n}{2}-1} z^{\frac{n}{2}-1} \\
& + \cdots + C_{n-4} z^{n-4}), \\
f'_{n-2}(z) =& C_0 + 2C_1 z + 3C_2 z^2 + \cdots + (\frac{n}{2}-2) C_{\frac{n}{2}-3} z^{\frac{n}{2}-3} + \frac{n}{2} C_{\frac{n}{2}-1} z^{\frac{n}{2}-1} \\
& + \cdots + (n-3) C_{n-4} z^{n-4}, \\
f_{n-1}(z) =& z(A_0 + A_1 z + A_2 z^2 + \cdots + A_{n-3} z^{n-3}), \\
f'_{n-1}(z) =& A_0 + 2A_1 z + 3A_2 z^2 + \cdots + (n-2) A_{n-3} z^{n-3},
\end{aligned}$$

as we have assumed. Moreover, we assume that the coefficients of polynomials f_{n-2} and f_{n-1} satisfy the condition

$$\frac{n}{2} A_{\frac{n}{2}-2} + \frac{n-2}{2} A_{\frac{n}{2}-1} - \frac{n}{2} C_{\frac{n}{2}-3} + \frac{8-3n}{16} C_{\frac{n}{2}-2} = 0. \tag{5.9}$$

Now, we insert these expressions in differential equation (5.6) and condition (5.9) assures that f_n with n even does not have a term $z^{n/2}$, i.e., is of the required form. Next, we compute f_{n+1} which has no problem because $n+1$ is odd. The last step is to see what happens for f_{n+2} with $n+2$ even. We compute the expression of f_{n+2} and we see that the coefficient of $z^{\frac{n}{2}+1}$ is given by

$$\frac{n+2}{2} A_{\frac{n}{2}-1} + \frac{n}{2} A_{\frac{n}{2}} - \frac{n+2}{2} C_{\frac{n}{2}-2} + \frac{2-3n}{16} C_{\frac{n}{2}-1}. \tag{5.10}$$

If condition (5.10) is zero then after integration equation no logarithmic term appears in $f_{n+2}(z)$ because the coefficient in front of monomial $z^{\frac{n}{2}+1}$ vanish. However, condition (5.10) is condition (5.9) for $n+2$. Hence, the induction is proved. Consequently, if condition (5.9) is satisfied for the coefficients of the polynomials f_{n-2} and f_{n-1}, whenever n is even we have not logarithmic terms. Then, we obtain that $f_n = zQ_{n-2}$ which is of the form

$$f_n = z(B_0 + B_1 z + B_2 z^2 + \cdots + B_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + B_{\frac{n}{2}} z^{\frac{n}{2}} + \cdots + B_{n-2} z^{n-2}).$$

Moreover, the first integral of system (5.4) is

$$\mathcal{H}(z,y) = \sum_{k=2}^{\infty} zQ_{k-2}(z) y^k = zy^2 + \sum_{k=3}^{\infty} zQ_{k-2}(z) y^k$$

and the first integral of system (5.2) is

$$\begin{aligned}
\Psi(x,y) = \mathcal{H}(\frac{x}{y}, y) &= (\frac{x}{y}) y^2 + \sum_{k=3}^{\infty} \frac{x}{y} \cdot \tilde{Q}_{k-2}(\frac{x}{y}) \cdot y^k \\
&= xy + \sum_{k=3}^{\infty} xy^{k-1} (B_0 + B_1 \frac{x}{y} + B_2 (\frac{x}{y})^2 + \cdots + B_{k-2} (\frac{x}{y})^{k-2})) \\
&= xy + \sum_{k=3}^{\infty} (B_0 x y^{k-1} + B_1 x^2 y^{k-2} + B_2 x^3 y^{k-3} + \cdots + B_{k-2} x^{k-1} y),
\end{aligned}$$

which is a formal first integral of the required form.

6. Open resonant quartic system of [11]

In [11] the integrability of the complex quartic system of the form

$$\begin{aligned} \dot{x} &= x(1 - a_{30}x^3 - a_{21}x^2y - a_{12}xy^2 - a_{03}y^3), \\ \dot{y} &= -y(1 - b_{30}x^3 - b_{21}x^2y - b_{12}xy^2 - b_{03}y^3), \end{aligned} \tag{6.1}$$

where $a_{ij},\ b_{ij} \in \mathbb{C}$ is studied. Note, that if for system (6.1) we have $a_{12} \neq 0$ and $b_{21} \neq 0$ then by a linear change of variables we can take in system (6.1) $a_{12} = b_{21} = 1$. In view of this remark the study of system (6.1) in [11] was splitted into three different cases: (a) $a_{12} = b_{21} = 1$, (b) $a_{12} = 1$ and $b_{21} = 0$, and (c) $a_{12} = b_{21} = 0$. In Theorem 2 of [11] 13 integrable cases corresponding to the case (a) are stated. However, the sufficiency of case (10) remains open. We are going to apply our method to solve this case.

The system associated to statement (10) of Theorem 2 in [11] with $a_{12} = b_{21} = 1$ is given by

$$\begin{aligned} \dot{x} &= x - \frac{3}{16}x^4 + \frac{1}{2}x^3y - x^2y^2 + \frac{8}{9}xy^3, \\ \dot{y} &= -y - \frac{9}{16}x^3y + x^2y^2 - \frac{5}{3}xy^3 - \frac{16}{9}y^4. \end{aligned} \tag{6.2}$$

We proceed in a similar way as in the case of previous section. After transformation (5.3) we obtain system

$$\begin{aligned} \dot{z} &= 2z + \frac{8}{3}y^3z + \frac{2}{3}y^3z^2 - \frac{1}{2}y^3z^3 + \frac{3}{8}y^3z^4, \\ \dot{y} &= -y - \frac{16}{9}y^4 - \frac{5}{3}y^4z + y^4z^2 - \frac{9}{16}y^4z^3. \end{aligned} \tag{6.3}$$

Next, we look for a first integral of the form (5.5). We compute $\dot{\mathcal{H}} = \dot{z}\partial\mathcal{H}/\partial z + \dot{y}\partial\mathcal{H}/\partial y$ for system (6.3) and equating to zero the coefficients of the same power of y yields the following recurrence differential equation

$$\begin{aligned} &(k-3)(-\frac{16}{3} - 5z + 3z^2 - \frac{27}{16}z^3)f_{k-3}(z) \\ &+ \frac{1}{24}z(64 + 16z - 12z^2 - 9z^3)f'_{k-3}(z) - kf_k(z) + 2zf'_k(z) = 0. \end{aligned}$$

We now compute the first several f_k using the recurrence equations and find $f_2(z) = z$, $f_3(z) = 0$, $f_4(z) = z^2$, $f_5(z) = z(4+3z)(-8-66z+9z^2)/108$, $f_6(z) = z^3$, $f_7(z) = z(-32z - 288z^2 - 162z^3 + 27z^4)/54$,

$$\begin{aligned} f_8(z) &= z(\frac{224}{729} + \frac{238}{81}z + \frac{28}{3}z^2 + \frac{7}{12}z^4 - \frac{105}{128}z^5 + \frac{7}{128}z^6), \\ f_9(z) &= z(-\frac{8}{9}z^2 - 8z^3 - \frac{9}{2}z^4 + \frac{3}{4}z^5), \\ f_{10}(z) &= z(\frac{512}{729}z + \frac{604}{81}z^2 + \frac{80}{3}z^3 + \frac{25}{12}z^5 - \frac{153}{64}z^6 + \frac{11}{64}z^7), \end{aligned}$$

$f_{11}(z) = zQ_9(z)$, where $Q_9(z)$ is a polynomial of degree 9, $f_{12}(z) = zQ_8(z)$, where $Q_8(z)$ is a polynomial of degree 8 without the term with z^5, $f_{13}(z) = zQ_{10}(z)$, where $Q_{10}(z)$ is a polynomial of degree 10, $f_{14}(z) = zQ_{12}(z)$, where $Q_{12}(z)$ is a polynomial of degree 12 without the term with z^6.

Hence, we assume for k being odd $f_k(z)$ is of the form

$$f_k(z) = z(C_0 + C_1 z + \cdots + C_{k-2} z^{k-2}),$$

and for k being even $f_k(z)$ is of the form

$$f_k(z) = z(C_0 + C_1 z + \cdots + C_{\frac{k}{2}-2} z^{\frac{k}{2}-2} + C_{\frac{k}{2}} z^{\frac{k}{2}} + \cdots + C_{k-2} z^{k-2}),$$

i.e., a polynomial without the term containing monomial $z^{\frac{k}{2}}$.

We prove this by induction. Suppose that this is the case for $k = n-1$ and we will prove that it is true also for $k = n$. In order to do that we solve the differential equation

$$\begin{aligned} f_n'(z) - \frac{n}{2z} f_n(z) = & \frac{1}{2z}\big[(3-k)(-\frac{16}{3} - 5z + 3z^2 - \frac{27}{16} z^3) f_{k-3}(z) \\ & - \frac{1}{24} z(64 + 16z - 12z^2 - 9z^3) f_{k-3}'(z)\big]. \end{aligned} \tag{6.4}$$

Let first n be odd. Then $n-3$ is even and the expression on the right hand side of (6.4) is a polynomial of degree at most $n-2$ and we call it $P_{n-2}(z)$. Therefore the differential equation (6.4) becomes

$$f_n'(z) - \frac{n}{2z} f_n(z) = P_{n-2}(z).$$

In the sense of (5.7) and (5.8) $g(z) = -\frac{n}{2z}$ and $h(z) = P_{n-2}(z) = A_0 + A_1 z + \cdots + A_{n-2} z^{n-2}$ and the solution takes the form

$$f_n(z) = z^{\frac{n}{2}}(C + \int z^{-\frac{n}{2}}(A_0 + A_1 z + \cdots + A_{n-2} z^{n-2}) dz,$$

which after the same procedure as integration in previous section yields

$$f_n(z) = C z^{\frac{n}{2}} + z(C_0 + C_1 z + \cdots + C_{n-2} z^{n-2}).$$

Taking $C = 0$ yields $f_n(z) = z(C_0 + C_1 z + \cdots + C_{n-2} z^{n-2})$ as we assumed.

Let n be an even number, then $n-3$ is odd and the expression on the right hand side of (6.4) is polynomial of degree at most $n-2$ and we call it $\tilde{P}_{n-2}(z)$. Differential equation (6.4) becomes

$$f_n'(z) - \frac{n}{2z} f_n(z) = \tilde{P}_{n-2}(z).$$

Since n is even number after integration we could obtain a logarithmic term in $f_n(z)$. To avoid this the coefficients C_i in $f_{n-3}(z)$ must satisfy the condition

$$\begin{aligned} & -\frac{3}{16}(2n-3) C_{\frac{n}{2}-4} + (\frac{3n}{4} - 2) C_{\frac{n}{2}-3} \\ & + \frac{1}{3}(13 - 4n) C_{\frac{n}{2}-2} - \frac{4}{9}(n - 12) C_{\frac{n}{2}-1} = 0. \end{aligned} \tag{6.5}$$

Under these conditions for n even $n-3$ is odd and f_{n-3} is of the form

$$f_{n-3}(z) = z(C_0 + C_1 z + \cdots + C_{n-5} z^{n-5}).$$

We insert this expression in differential equation (6.4) and no logarithmic term appear in function $f_n(z)$ since the coefficient in front of monomial $z^{\frac{n}{2}}$ vanish due to the coefficients of polynomial f_{n-3} satisfy the condition (6.5). Now, we will take a look what happens in the next step when we compute f_{n+3} and f_{n+6} where $n+6$ is also even. The computation of f_{n+3} is carried out without any problem since $n+3$ is odd. However, when we compute f_{n+6} for $n+6$ even we see that the coefficient of $z^{\frac{n}{2}+3}$ is given by

$$\begin{aligned} &-\frac{3}{16}(2n+9)C_{\frac{n}{2}-1}+(\frac{3n}{4}+\frac{10}{4})C_{\frac{n}{2}} \\ &-\frac{1}{3}(4n+11)C_{\frac{n}{2}+1}-\frac{4}{9}(n-6)C_{\frac{n}{2}+2}=0. \end{aligned} \tag{6.6}$$

If condition (6.6) is satisfied then after integration no logarithmic term appears in $f_{n+6}(z)$ since the coefficient in front of monomial $z^{\frac{n}{2}+3}$ vanishes. However, condition (6.6) is condition (6.5) for $n+6$. Hence, the induction is proved. Consequently, as long as condition (6.5) is satisfied for the coefficients of the polynomials f_{n-3} with n being even we have no logarithmic term.

Hence, whenever n is even we obtain

$$f_n = z(B_0 + B_1 z + B_2 z^2 + \cdots + B_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + B_{\frac{n}{2}} z^{\frac{n}{2}} + \cdots + B_{n-2} z^{n-2}).$$

So, first integral of system (6.3) is

$$\mathcal{H}(z,y) = \sum_{k=2}^{\infty} zQ_{k-2}(z)y^k = zy^2 + \sum_{k=3}^{\infty} zQ_{k-2}(z)y^k,$$

and the first integral of system (6.2) is

$$\Psi(x,y) = xy + \sum_{k=3}^{\infty}(B_0 xy^{k-1} + B_1 x^2 y^{k-2} + B_2 x^3 y^{k-3} + \cdots + B_{k-2} x^{k-1} y).$$

7. Open resonant quintic systems of [12]

In [12] the integrable resonant saddles of the following system with quintic homogeneous nonlinearities

$$\begin{aligned} \dot{x} &= x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4 - a_{-15}y^5, \\ \dot{y} &= -y + b_{5,-1}x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{40}y^5, \end{aligned}$$

where $x, y, a_{ij}, b_{ij} \in \mathbb{C}$ with $a_{-15} = 0$ were studied. The study was split in four different cases: (C_1) $a_{31} = b_{13} = 1$, (C_2) $a_{31} = 1$, $b_{13} = 0$, (C_3) $a_{31} = 0$, $b_{13} = 1$ and (C_4) $a_{31} = b_{13} = 0$. In Theorem 1.2 of [12] 17 integrable cases corresponding to the case (C_1) are given. However, the sufficiency of case (8) and (17) remains open. We prove the sufficiency of these two cases using the method proposed in this paper.

The system associated to statement (8) of Theorem 1.2 in [12] is written as

$$\begin{aligned} \dot{x} &= x - \frac{5}{3}b_{22}x^5 - x^4y - b_{22}x^3y^2 + \frac{1}{5}x^2y^3 + \frac{3}{5}b_{22}xy^4, \\ \dot{y} &= -y - \frac{25}{3}b_{22}x^4y - x^3y^2 + b_{22}x^2y^3 + xy^4 - \frac{3}{2}b_{22}y^5, \end{aligned}$$

with $b_{22} = \pm\sqrt{2/15}$. Here, we consider only the case $b_{22} = \sqrt{2/15}$ but the case of negative b_{22} can be solved in a similar way. Hence, we begin with the system

$$\begin{aligned}\dot{x} &= x - \frac{1}{3}\sqrt{\frac{10}{3}}x^5 - x^4y - \sqrt{\frac{2}{15}}x^3y^2 + \frac{1}{5}x^2y^3 + \frac{1}{5}\sqrt{\frac{6}{5}}xy^4\\ \dot{y} &= -y - \frac{5}{3}\sqrt{\frac{10}{3}}x^4y - x^3y^2 + \sqrt{\frac{2}{15}}x^2y^3 + xy^4 - \frac{1}{5}\sqrt{\frac{6}{5}}y^5.\end{aligned} \tag{7.1}$$

After applying the blow-up (5.3) system (7.1) takes the form

$$\begin{aligned}\dot{z} &= 2z + \frac{2}{5}\sqrt{\frac{6}{5}}y^4z - \frac{4}{5}y^4z^2 - 2\sqrt{\frac{2}{15}}y^4z^3 + \frac{4}{3}\sqrt{\frac{10}{3}}y^4z^5\\ \dot{y} &= -y - \frac{1}{5}\sqrt{\frac{6}{5}}y^5 + y^5z + \sqrt{\frac{2}{15}}y^5z^2 - y^5z^3 - \frac{5}{3}\sqrt{\frac{10}{3}}y^5z^4.\end{aligned} \tag{7.2}$$

Now, we look for a power series of the form (5.5). As in the previous section we compute $\dot{\mathcal{H}} = (\partial\mathcal{H}/\partial z)\dot{z} + (\partial\mathcal{H}/\partial y)\dot{y}$ for system (7.2) and equate to zero each coefficient of different powers of y. So, we obtain the following recursive differential equation

$$\begin{aligned}&\frac{k-4}{225}(-9\sqrt{30} + 225z + 15\sqrt{30}z^2 - 225z^3 - 125\sqrt{30}z^4)f_{k-4}\\ &+ \frac{2}{225}(9\sqrt{30}z - 90z^2 - 15\sqrt{30}z^3 + 50\sqrt{30}z^5)f'_{k-4}\\ &- kf_k + 2zf'_k = 0.\end{aligned} \tag{7.3}$$

Computing f_k for the first several k we find $f_2 = z$, $f_4 = z^2$, $f_6 = \frac{3}{5}z^2 + z^4 + \sqrt{\frac{5}{6}}z^5$, $f_8 = \frac{6}{5}z^3 + 2z^5 + \sqrt{\frac{10}{3}}z^6$, $f_{10} = -\frac{1}{25}\sqrt{\frac{6}{5}}z^2 + \frac{33}{50}z^3 + \frac{2}{5}\sqrt{\frac{6}{5}}z^4 + \frac{19}{\sqrt{30}}z^6 + \frac{11}{6}z^7 + \frac{23}{9}\sqrt{\frac{5}{6}}z^8 + \frac{25}{36}z^9$, $f_{12} = -\frac{2}{25}\sqrt{\frac{6}{5}}z^3 + \frac{42}{25}z^4 + \frac{4}{5}\sqrt{\frac{6}{5}}z^5 + 22\sqrt{\frac{2}{15}}z^7 + \frac{14}{3}z^8 + \frac{32}{9}\sqrt{\frac{10}{3}}z^9 + \frac{20}{9}z^{10}$, and $f_k = 0$ for $k = 3, 5, 7, 9, 11$. Therefore, we assume that $f_k = 0$ if k is odd and

$$f_k = z(C_0 + C_1z + \cdots + C_{\frac{k}{2}-2}z^{\frac{k}{2}-2} + C_{\frac{k}{2}}z^{\frac{k}{2}} + \cdots + C_{k-2}z^{k-2}), \tag{7.4}$$

if k is even. We prove this using induction. Suppose that the assumption holds for $k = 1, \ldots, n-4$ and we compute f_k for $k = n$ solving the differential equation (7.3).

If n is odd, then also $n-4$ is odd and $f_{n-4} = f'_{n-4} = 0$. The recurrence formula yields homogeneous differential equation

$$2zf'_n - nf_n = 0,$$

which has solution $f_n = C_{n/2}z^{\frac{n}{2}}$. Choosing at each step $C_{n/2} = 0$ we obtain $f_n = 0$ for each n being odd.

Now, assuming that n is even then also $n-4$ is even. Since n is even number after integration we could obtain a logarithmic term in $f_n(z)$. To avoid this the coefficients C_i of $f_{n-4}(z)$ must satisfy the condition

$$\begin{aligned}&-\frac{1}{3}\frac{\sqrt{10}}{3}(3n-4)C_{\frac{n}{2}-5} + (4-n)C_{\frac{n}{2}-4}\\ &+ \frac{3n-16}{5}C_{\frac{n}{2}-2} + \frac{4\sqrt{30}}{25}C_{\frac{n}{2}-1} = 0.\end{aligned} \tag{7.5}$$

Under this condition f_{n-4} and f'_{n-4} are of the form

$$\begin{aligned} f_{n-4} &= C_0 z + C_1 z^2 + \cdots + C_{\frac{n}{2}-4} z^{\frac{n}{2}-3} + C_{\frac{n}{2}-2} z^{\frac{n}{2}-1} + \cdots + C_{n-6} z^{n-5}, \\ f'_{n-4} &= C_0 + 2C_1 z + \cdots + (\frac{n}{2}-3) C_{\frac{n}{2}-4} z^{\frac{n}{2}-4} + (\frac{n}{2}-1) C_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + \cdots \\ &\quad + (n-5) C_{n-6} z^{n-6}. \end{aligned}$$

We insert these expressions in differential equation (7.3) and no logarithmic term appear in function $f_n(z)$ since the coefficient in front of monomial $z^{\frac{n}{2}}$ vanish due to the coefficients of polynomial f_{n-4} satisfy the condition (7.5). Now, we compute f_{n+4} where $n+4$ is even and we see that the coefficient in front of $z^{\frac{n}{2}+2}$ is given by

$$-\frac{1}{3}\sqrt{\frac{10}{3}}(3n+8) C_{\frac{n}{2}-3} - n C_{\frac{n}{2}-2} + \frac{3n-4}{5} C_{\frac{n}{2}} + \frac{4\sqrt{30}}{25} C_{\frac{n}{2}+1}. \tag{7.6}$$

If coefficient (7.6) is zero then after integration no logarithmic term appears in $f_{n+4}(z)$ since the coefficient in front of monomial $z^{\frac{n}{2}+2}$ vanishes. However, condition (7.5) is condition (7.6) for $n+4$. Hence, the induction is proven. Consequently, if condition (7.5) is satisfied for the coefficients of the polynomials f_{n-4}, whenever n is even there is no logarithmic term.

Finally, we insert the expressions of f_{n-4} and f'_{n-4} in differential equation (7.3). Then differential equation (7.3) becomes

$$\begin{aligned} f'_n - \frac{n}{2z} f_n &= \frac{z(B_0 + B_1 z + \cdots + B_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + B_{\frac{n}{2}} z^{\frac{n}{2}} + \cdots + B_{n-2} z^{n-2})}{2z} \\ &= \overline{B}_0 + \overline{B}_1 z + \cdots + \overline{B}_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + \overline{B}_{\frac{n}{2}} z^{\frac{n}{2}} + \cdots + \overline{B}_{n-2} z^{n-2}, \end{aligned}$$

which regarding to (5.7) and (5.8) has the solution

$$\begin{aligned} f_n(z) &= C z^{\frac{n}{2}} + z^{\frac{n}{2}} \int z^{-\frac{n}{2}} (\overline{B}_0 + \overline{B}_1 z + \cdots + \overline{B}_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + \overline{B}_{\frac{n}{2}} z^{\frac{n}{2}} \\ &\quad + \cdots + \overline{B}_{n-2} z^{n-2}) dz \\ &= C z^{\frac{n}{2}} + z^{\frac{n}{2}} \int (\overline{B}_0 z^{-\frac{n}{2}} + \overline{B}_1 z^{1-\frac{n}{2}} + \cdots + \overline{B}_{\frac{n}{2}-2} z^{-2} + \overline{B}_{\frac{n}{2}} \\ &\quad + \cdots + \overline{B}_{n-2} z^{\frac{n}{2}-2}) dz \\ &= C z^{\frac{n}{2}} + z^{\frac{n}{2}} (D_0 z^{1-\frac{n}{2}} + D_1 z^{2-\frac{n}{2}} + \cdots + D_{\frac{n}{2}-2} z^{-1} + D_{\frac{n}{2}} z \\ &\quad + \cdots + D_{n-2} z^{\frac{n}{2}-1}) \\ &= C z^{\frac{n}{2}} + z (D_0 + D_1 z + \cdots + D_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + D_{\frac{n}{2}} z^{\frac{n}{2}} \\ &\quad + \cdots + D_{n-2} z^{n-2}). \end{aligned}$$

Setting $C = 0$ we finally obtain

$$f_n(z) = z(D_0 + D_1 z + \cdots + D_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + D_{\frac{n}{2}} z^{\frac{n}{2}} + \cdots + D_{n-2} z^{n-2}),$$

which has the claimed form (7.4). The first integral of system (7.2) is

$$\mathcal{H}(z,y) = zy^2 + \sum_{k=3}^{\infty} z(D_0 + D_1 z + \cdots + D_{\frac{k}{2}-2} z^{\frac{k}{2}-2} + D_{\frac{k}{2}} z^{\frac{k}{2}} + \cdots + D_{k-2} z^{k-2}) y^k$$

and the first integral of system (7.1) is

$$\begin{aligned}\Psi(x,y) = \mathcal{H}(\frac{x}{y},y) &= (\frac{x}{y})y^2 + \sum_{k=3}^{\infty}\frac{x}{y}\cdot(D_0 + D_1\frac{x}{y} + D_2(\frac{x}{y})^2 + \cdots + D_{k-2}(\frac{x}{y})^{k-2})\cdot y^k \\ &= xy + \sum_{k=3}^{\infty} xy^{k-1}(D_0 + D_1\frac{x}{y} + D_2(\frac{x}{y})^2 + \cdots + D_{k-2}(\frac{x}{y})^{k-2})) \\ &= xy + \sum_{k=3}^{\infty}(D_0 xy^{k-1} + D_1 x^2 y^{k-2} + D_2 x^3 y^{k-3} + \cdots + D_{k-2}x^{k-1}y).\end{aligned}$$

The system corresponding to open case (17) of Theorem 1.2 in [12] is of the form

$$\begin{aligned}\dot{x} &= x - x^4 y + \frac{7}{6}b_{04}x^3y^2 + \frac{1}{7}x^2y^3, \\ \dot{y} &= -y + 28x^5 + \frac{245}{3}b_{04}x^4y - 3x^3y^2 - \frac{7}{6}b_{04}x^2y^3 + xy^4 + b_{04}y^5,\end{aligned} \tag{7.7}$$

where $b_{04} = \pm\frac{2\sqrt{3}}{7\sqrt{7}}$. As in the case above we take the positive root but the proof is similar for the negative one. Hence, we consider the system

$$\begin{aligned}\dot{x} &= x - x^4 y + \frac{1}{\sqrt{21}}x^3y^2 + \frac{1}{7}x^2y^3, \\ \dot{y} &= -y + 28x^5 + 10\sqrt{\frac{7}{3}}x^4y - 3x^3y^2 - \frac{x^2y^3}{\sqrt{21}} + xy^4 + \frac{2}{7}\sqrt{\frac{3}{7}}y^5.\end{aligned} \tag{7.8}$$

After applying (5.3) we obtain the system

$$\begin{aligned}\dot{z} &= 2z - \frac{2}{7}\sqrt{\frac{3}{7}}y^4z - \frac{6}{7}y^4z^2 + \frac{2}{\sqrt{21}}y^4z^3 + 2y^4z^4 - 10\sqrt{\frac{7}{3}}y^4z^5 - 28y^4z^6, \\ \dot{y} &= -y + \frac{2}{7}\sqrt{\frac{3}{7}}y^5 + y^5z - \frac{1}{\sqrt{21}}y^5z^2 - 3y^5z^3 + 10\sqrt{\frac{7}{3}}y^5z^4 + 28y^5z^5.\end{aligned} \tag{7.9}$$

The proof of integrability of system (7.8) is done in a similar way as before. We look for a power series of the form (5.5). We compute $\dot{\mathcal{H}} = (\partial\mathcal{H}/\partial z)\dot{z} + (\partial\mathcal{H}/\partial y)\dot{y}$ for system (7.9) and equate to zero each coefficient of different powers of y. We obtain the following recurrence differential equation

$$\begin{aligned}&(k-4)\Big(\frac{2\sqrt{3}}{7\sqrt{7}} + z - \frac{1}{\sqrt{21}}z^2 - 3z^3 + 10\sqrt{\frac{7}{3}}z^4 + 28z^5\Big)f_{k-4} \\ &- \frac{2}{147}\Big(3\sqrt{21}z + 63z^2 - 7\sqrt{21}z^3 - 147z^4 + 245\sqrt{21}z^5 + 2058z^6\Big)f'_{k-4} \\ &- kf_k + 2zf'_k = 0.\end{aligned} \tag{7.10}$$

For $k = 2, 3, \ldots, 14$ we find $f_2 = z$, $f_4 = z^2$, and

$$\begin{aligned}f_6 &= \frac{1}{14}\sqrt{\frac{3}{7}}z + \frac{4}{7}z^2 + 2z^4 - \frac{5}{2}\sqrt{\frac{7}{3}}z^5 - \frac{14}{3}z^6, \\ f_8 &= -\frac{1}{147}z(-3\sqrt{21}z - 168z^2 - 588z^4 + 245\sqrt{21}z^5 + 1375z^6),\end{aligned}$$

$f_{10} = zp_9(y)$, where $p_9(y)$ is a polynomial of degree 9 without the term with the monomial z^4, $f_{12} = zp_{11}$, where $p_{11}(y)$ is a polynomial of degree 11 without the term with the monomial z^5, $f_{14} = zp_{13}$, where $p_{13}(y)$ is a polynomial of degree 13 without the term with the monomial z^6, and $f_3 = f_5 = f_7 = f_9 = f_{11} = f_{13} = 0$. Hence, we assume that $f_k = 0$ if k is odd and

$$f_k = z(C_0 + C_1 z + \cdots + C_{\frac{k}{2}-2} z^{\frac{k}{2}-2} + C_{\frac{k}{2}} z^{\frac{k}{2}} + \cdots + C_{k-1} z^{k-1}) \tag{7.11}$$

if k is even. We prove this by induction. Suppose that the assumption holds for $k = 1, \ldots, n-4$ and we compute f_k for $k = n$ solving the recursive differential equation (7.10).

If n is odd, then also $n-4$ is odd and consequently $f_{n-4} = f'_{n-4} = 0$. The recurrence formula yields the homogeneous differential equation

$$2zf'_n - nf_n = 0,$$

which has solution $f_n = C_{n/2} z^{\frac{n}{2}}$. Choosing at each step $C_{n/2} = 0$ we obtain $f_n = 0$ for each n being odd.

Now, assuming that n is even yields that also $n-4$ is even and after integration a logarithmic term could appear in $f_n(z)$. To avoid this the coefficients C_i in $f_{n-4}(z)$ must satisfy the condition

$$\begin{aligned} &14(n+2)C_{\frac{n}{2}-6} + 5\sqrt{\frac{7}{3}}nC_{\frac{n}{2}-5} + 2(3-n)C_{\frac{n}{2}-4} \\ &+ \frac{2}{7}(2n-11)C_{\frac{n}{2}-2} + \frac{\sqrt{21}}{49}(n-8)C_{\frac{n}{2}-1} = 0. \end{aligned} \tag{7.12}$$

Under this condition and taking into account that f_{n-4} and f'_{n-4} are of the form

$$\begin{aligned} f_{n-4} &= C_0 z + C_1 z^2 + \cdots + C_{\frac{n}{2}-4} z^{\frac{n}{2}-3} + C_{\frac{n}{2}-2} z^{\frac{n}{2}-1} + \cdots + C_{n-5} z^{n-4}, \\ f'_{n-4} &= C_0 + 2C_1 z + \cdots + (\frac{n}{2}-3)C_{\frac{n}{2}-4} z^{\frac{n}{2}-4} + (\frac{n}{2}-1)C_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + \cdots \\ &\quad + (n-4)C_{n-5} z^{n-5}, \end{aligned}$$

we insert these expression in differential equation (7.10) and no logarithmic term appears in function $f_n(z)$ since the coefficient in front of monomial $z^{\frac{n}{2}}$ vanishes due to the coefficients of polynomial f_{n-4} satisfy the condition (7.12). Computing f_{n+4} for $n+4$ even we see that the coefficient of $z^{\frac{n}{2}+2}$ is given by

$$\begin{aligned} &14(n+6)C_{\frac{n}{2}-4} + 5\sqrt{\frac{7}{3}}(n+4)C_{\frac{n}{2}-3} - 2(1+n)C_{\frac{n}{2}-2} \\ &+ \frac{2}{7}(2n-3)C_{\frac{n}{2}} + \frac{\sqrt{21}}{49}(n-4)C_{\frac{n}{2}+1}. \end{aligned} \tag{7.13}$$

If coefficient (7.13) vanishes then after integration no logarithmic term appears in $f_{n+4}(z)$ since the coefficient in front of monomial $z^{\frac{n}{2}+2}$ is zero. However, condition (7.13) is condition (7.12) for $n+4$. Hence the induction is proved. Therefore, if condition (7.12) is satisfied in the coefficients of the polynomial f_{n-4}, where n is even there is no logarithmic term.

Now, we insert the expressions of f_{n-4} and f'_{n-4} in differential equation (7.10) and differential equation (7.10) becomes

$$
\begin{aligned}
f'_n - \frac{n}{2z} f_n &= \frac{z(B_0 + B_1 z + \cdots + B_{\frac{k}{2}-2} z^{\frac{k}{2}-2} + B_{\frac{k}{2}} z^{\frac{k}{2}} + \cdots + B_{k-2} z^{k-2}}{2z} \\
&= \overline{B}_0 + \overline{B}_1 z + \cdots + \overline{B}_{\frac{k}{2}-2} z^{\frac{k}{2}-2} + \overline{B}_{\frac{k}{2}} z^{\frac{k}{2}} + \cdots + \overline{B}_{k-1} z^{k-1},
\end{aligned}
$$

which regarding to (5.7) and (5.8) has the solution

$$
\begin{aligned}
f_n(z) &= C z^{\frac{n}{2}} + z^{\frac{n}{2}} \int z^{-\frac{n}{2}} (\overline{B}_0 + \overline{B}_1 z + \cdots + \overline{B}_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + \overline{B}_{\frac{n}{2}} z^{\frac{n}{2}} \\
&\quad + \cdots + \overline{B}_{n-1} z^{n-1}) dz \\
&= C z^{\frac{n}{2}} + z^{\frac{n}{2}} \int (\overline{B}_0 z^{-\frac{n}{2}} + \overline{B}_1 z^{1-\frac{n}{2}} + \cdots + \overline{B}_{\frac{n}{2}-2} z^{-2} + \overline{B}_{\frac{n}{2}} \\
&\quad + \cdots + \overline{B}_{n-1} z^{\frac{n}{2}-1}) dz \\
&= C z^{\frac{n}{2}} + z^{\frac{n}{2}} (D_0 z^{1-\frac{n}{2}} + D_1 z^{2-\frac{n}{2}} + \cdots + D_{\frac{n}{2}-2} z^{-1} + D_{\frac{n}{2}} z \\
&\quad + \cdots + \overline{D}_{n-1} z^{\frac{n}{2}}) \\
&= C z^{\frac{n}{2}} + z(D_0 + D_1 z + \cdots + D_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + D_{\frac{n}{2}} z^{\frac{n}{2}} \\
&\quad + \cdots + D_{n-1} z^{n-1}).
\end{aligned}
$$

Now setting $C = 0$ we obtain

$$
f_n(z) = z(D_0 + D_1 z + \cdots + D_{\frac{n}{2}-2} z^{\frac{n}{2}-2} + D_{\frac{n}{2}} z^{\frac{n}{2}} + \cdots + D_{n-1} z^{n-1}).
$$

Then, first integral of system (7.9) is

$$
\mathcal{H}(z, y) = zy^2 + \sum_{k=3}^{\infty} z(D_0 + D_1 z + \cdots + D_{\frac{k}{2}-2} z^{\frac{k}{2}-2} + D_{\frac{k}{2}} z^{\frac{k}{2}} + \cdots + D_{k-1} z^{k-1}) y^k,
$$

which yields a first integral of system (7.8)

$$
\begin{aligned}
\Psi(x, y) = \mathcal{H}(\frac{x}{y}, y) &= (\frac{x}{y}) y^2 + \sum_{k=3}^{\infty} \frac{x}{y} \cdot (D_0 + D_1 \frac{x}{y} + D_2 (\frac{x}{y})^2 + \cdots + D_{k-1} (\frac{x}{y})^{k-1}) \cdot y^k \\
&= xy + \sum_{k=3}^{\infty} x y^{k-1} (D_0 + D_1 \frac{x}{y} + D_2 (\frac{x}{y})^2 + \cdots + D_{k-1} (\frac{x}{y})^{k-1})) \\
&= xy + \sum_{k=3}^{\infty} (D_0 x y^{k-1} + D_1 x^2 y^{k-2} + D_2 x^3 y^{k-3} + \cdots + D_{k-1} x^k),
\end{aligned}
$$

which is a formal first integral of the required form.

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