Journal of Applied Analysis and Computation
Volume 8, Number 6, December 2018, 1821–1832 Website:http://jaac.ijournal.cn
DOI:10.11948/2018.1821

SIXTEEN LARGE-AMPLITUDE LIMIT CYCLES IN A SEPTIC SYSTEM*

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Abstract In this paper, bifurcation of limit cycles from the infinity of a two-dimensional septic polynomial differential system is investigated. Sufficient and necessary conditions for the infinity to be a center are derived and the fact that there exist 16 large amplitude limit cycles bifurcated from the infinity is proved as well. The study relays on making use of a recursive formula for computing the singular point quantities of the infinity. As far as we know, this is the first example of a septic system with 16 limit cycles bifurcated from the infinity.

Keywords Septic system, Infinity, Singular point quantities, Limit cycles.

MSC(2010) 34C05, 34C07.

1. Introduction and Statement of the Results

The problem of bifurcation of limit cycles for planar polynomial systems which belongs to the second part of the Hilbert 16th problem, is known as a hot but challenging issue in the qualitative theory of dynamical systems. The determination of limit cycles bifurcated from a singular point (which is of center-focus type) is strongly related with the center-focus problem. Consider a real analytic differential system in the form of a linear center perturbed by higher order terms, that is

$$\dot{x} = -y + \sum_{k=2}^{\infty} X_k(x,y), \qquad \dot{y} = x + \sum_{k=2}^{\infty} Y_k(x,y), \tag{1.1}$$

where X_k and Y_k are homogeneous polynomials of degree k. In polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, system (1.1) becomes into

$$\dot{r} = \sum_{k=1}^{\infty} \varphi_{k+2}(\theta) r^{k+1}, \qquad \dot{\theta} = 1 + \sum_{k=1}^{\infty} \psi_{k+2}(\theta) r^k, \tag{1.2}$$

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*The authors were supported by National Natural Science Foundation of China (Nos. 61473332 and 11272120), National Science Foundation of Shandong Province(No.ZR2018MA002) and the Natural Science Foundation of Zhejiang Province (No. LQ14A010009).

where $\varphi_{k+2} = \cos\theta X_k(\theta) + \sin\theta Y_k(\theta)$ and $\psi_{k+2} = \cos\theta Y_k(\theta) - \sin\theta X_k(\theta)$. For sufficiently small h, the Poincaré succession function is

$$\Delta(h) = \widetilde{r}(2\pi, h) - h = \sum_{k=2}^{\infty} \nu_k(2\pi)h^k. \tag{1.3}$$

If $\nu_2(2\pi) = \nu_3(2\pi) = \cdots = \nu_{2k}(2\pi) = 0 \neq \nu_{2k+1}(2\pi)$, then the origin is called a *k-order fine focus* and $\nu_{2k+1}(2\pi)$ is called the *k-th focus value*; if for all positive integers k, we have $\nu_{2k+1}(2\pi) = 0$, then the origin of system (1.1) is called a *center*. To distinguish between a center and a focus at the origin of (1.1) is the so-called *center-focus problem*. For (1.1) we can always find a Lyapunov function of the form

$$F(x, y) = x^2 + y^2 + \sum_{k=3}^{\infty} F_k(x, y), \tag{1.4}$$

where F_k is a homogeneous polynomial of degree k, so that $\dot{F}(x, y) = \sum_{k=1}^{\infty} V_{2k+1}(x^2 + y^2)^{k+1}$, where V_{2m+1} are real numbers called *Lyapunov constants*. The Lyapunov constants are polynomials whose variables are the coefficients of system (1.1). In the case that $\dot{F} \equiv 0$ we say that the origin of system (1.1) is a center. Then there comes one question, that is the coefficients of formal series F in (1.4) are not unique. In fact, for any integer $m > 1$, the coefficient of one term of F_{2m} can be arbitrarily chosen even though $F_2,\ F_3, \cdots,\ F_{m-1}$ have been determined. Hence, this exact coefficient will effect the latter Lyapunov constants. For real system, it is geometrically obvious that the arbitrary choice of $F_2,\ F_3, \cdots,\ F_{m-1}$ cannot affect the type of the singularity at the origin and, therefore, the local integrability of the system. But for complex system, the fact is not obvious, it was proved in [17, 18]. Furthermore, Liu [13] proved that $\nu_{2m}(2\pi) = \sum_{k=2}^{2m-1} \xi_k \nu_k(2\pi)$ and $\nu_{2m+1}(2\pi) = \sum_{k=1}^{m-1} \eta_k \nu_{2k+1}(2\pi) + \widetilde{\nu}_{2m+1}$, where ξ_k and η_k are polynomials of the coefficients of system (1.1). For the sake of convenience, these two formulas are denoted as $\nu_{2m}(2\pi) \sim 0$ and $\nu_{2m+1}(2\pi) \sim \widetilde{\nu}_{2m+1}$, respectively, where the symbol "$\sim$" represents the mathematically *equivalence relation*. What is more, Liu [13] demonstrated the Lyapunov constant V_{2m+1} and the focal value $\nu_{2m+1}(2\pi)$ are algebraic equivalent as well, that is $V_{2m+1} \sim \nu_{2m+1}(2\pi)/\pi$. The equivalence relationship between the Lyapunov constants and the focal values indicates that, to some extent, these two concepts are just the same. For convenience, the Lyapunov constants and the focal values are collectively referred to as *Lyapunov quantities* in the rest of this paper. Bifurcation theory for finitely smooth planar autonomous differential systems was considered in [5]. Theory of rotated equations is discussed in [6] and was applied to a population model. For a given family of real planar polynomial systems of ordinary differential equations depending on parameters, the authors considered the problem of how to find the systems in the family which become time-reversible after some affine transformation in [7]. Recently, an improvement on the number of limit cycles bifurcating from a nondegenerate center of homogeneous polynomial systems was obtained in [22] by normal form method. Centers for the Kukles homogeneous systems with even degree was studied in [4]. Recently, bifurcation of limit circles in a class of Z_2-equivalent cubic planar differential systems with two nilpotent singular points was studied in [10]. Some new perturbation method was proposed in [11].

The next important step in the investigation of the system is studying limit cycles corresponding to the perturbation in a small neighborhood of the linear center. As it is known, we practically make the order of the Lyapunov quantities as large as possible so as to generate limit cycles as many as possible. Hence, only considering the first nonzero Lyapunov quantity is not nearly enough. It is necessary to investigate the zero roots and their distribution of the Poincaré succession function. Nevertheless, the study involves very laborious computations since the Lyapunov quantities are polynomials in coefficients of the system. These polynomials usually are huge and it is impossible to perform decomposition over the field of characteristic zero.

Compared with the classic methods, i.e., the Poincaré return map and the Lyapunov functions , of computing the Lyapunov quantities, at present, there exist some different methods for determining Lyapunov quantities and the computer realizations of these methods, which permit us to find Lyapunov quantities in the form of symbolic expressions, depending on expansion coefficients of the right-hand of equations of system (1.1). These methods differ in complexity of algorithms and compactness of obtained symbolic expressions. Note that for reduction of symbolic expression and simplification of analysis of system, special transformations of system to complex variables [3,12,16] are often used. Introducing a complex structure on phase plane (x, y) by setting $z = x + iy$, $\mathrm{d}T = i\mathrm{d}t$ and rewriting t instead of T, system (1.1) becomes an analytic complex conjugate system as follows

$$\dot{z} = Z(z, \bar{z}), \qquad \dot{\bar{z}} = -\bar{Z}(z, \bar{z}), \tag{1.5}$$

where $Z(z, \bar{z}) = z + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta} z^{\alpha} \bar{z}^{\beta}$. Amel'kin et al [1] presented in their book that by using a formal change of variables $u = u(z, \bar{z})$ and $v = v(z, \bar{z})$, where $u(z, \bar{z})$ and $v(z, \bar{z})$ are polynomials without zero order terms, system (1.5) can be uniquely reduced to the formal form

$$\dot{u} = u + \sum_{k=1}^{\infty} p_k u^{k+1} v^k, \qquad \dot{v} = -v - \sum_{k=1}^{\infty} q_k v^{k+1} u^k. \tag{1.6}$$

Written as $\mu_k = p_k - q_k$, then μ_k is called the *k-th singular point quantity.* If $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0 \neq \mu_k$, then the origin of (1.5) is called the *k-th weak singular point*. In the case that for all k, $\mu_k = 0$, we say that the origin of system (1.5) is a *complex center*. As mentioned above, Lyapunov quantity is an important detection quantity in the theory of planar dynamical systems. However, it is a difficult task to completely find the Lyapunov quantities for a concrete nonlinear system since we need to perform a great number of integral operations by using the method of succession function or we need to solve large quantities of linear equations by employing the method of formal series. In the past four decades, a lot of algorithms to compute the Lyapunov quantities have been developed [2,9,21] whereas all of them run into troubles from getting the exact expressions of the Lyapunov quantities, these troubles are mainly due to the numerous computations that are involved which break down the capacity of the computers. Liu et al [12, 13] had revealed the algebraic equivalent relationship between the singular point quantities of the origin of system (1.5) and the Lyapunov quantities of the origin of system (1.1), that is $\mu_k \sim \nu_{2k+1}(2\pi)/i\pi$ or $\mu_k \sim V_{2k+1}/i$. Moreover, in the works [16,20] new methods of computation of singular point quantities, which based on constructing linear recursive formulas, are suggested. The advantages of these methods are due to their ideological simplicity and visualization power with the help of computer algebra systems such as Mathematica or Maple. The method appears also in [20].

A very natural extension from the study of limit cycles of system (1.1) is to study systems with the infinity as a Hopf singular point. A similar problem is to investigate the maximal number of limit cycles which may exist in the vicinity of infinity under proper perturbation. But not many results have been obtained on limit cycles for such systems since the analysis for systems with a Hopf infinity is much more complex than system (1.1) with a Hopf origin. For the case of the bifurcation of limit cycles at the infinity, the research is mainly concentrated on the following real planar system

$$\begin{aligned}\dot{x} &= \sum_{k=0}^{2n} X_k(x,y) - y(x^2+y^2)^n,\\ \dot{y} &= \sum_{k=0}^{2n} Y_k(x,y) + x(x^2+y^2)^n.\end{aligned} \tag{1.7}$$

where X_k and Y_k are homogeneous polynomials of degree k. This system has no real singular point on the equator Γ_∞ of the Poincaré compactification on the sphere. Γ_∞ is called infinity on the Gauss sphere or the equator of system (1.7). Concerning the problem of finding the upper bound, called the Hilbert number $I(m)$, on the number of large-amplitude limit cycles which can bifurcate from the infinity of the planar polynomial system (1.7) of degree m, some results have been obtained; see for example [18-25] and references therein. However, the finiteness problem remains unsolved even for $m = 3$. So far, the best result for cubic systems is $I(3) \geq 7$ in [15, 23], while for quintic systems is $I(5) \geq 11$ in [24]. Weak centers and local bifurcations of critical periods at infinity for a class of rational systems was studied in [8]. Furthermore, bifurcation of critical periods of a quintic system was studied in [19]. In this paper, we want to ask: What is an upper bound for the cyclicity of infinity of general septic systems? Although we cannot answer this open problem, we will try to provide a better lower bound in this paper and hope that this will help promote research in this direction.

For general septic systems with a Hopf infinity, the best result obtained so far is 13 large-amplitude limit cycles bifurcating from the infinity of a septic system [25]. In this paper, we shall consider a septic system similar to the one in [25]. In the next section, we formulate the septic system, and then in Section 3 we obtain the first 112 singular point quantities and the sufficient and necessary conditions of center and prove the existence of 16 large-amplitude limit cycles bifurcating from the infinity.

2. The recursive formula and the computation of singular point quantities

In this section, we present a septic system which may yield 16 limit cycles. To achieve this, we start from the following generic septic polynomial system:

$$\begin{cases} \dot{x} = A_{10}x + A_{01}y + A_{20}x^2 + A_{11}xy + (-B_{11} + A_{20})y^2 + A_{30}x^3 \\ \qquad +A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 + A_{32}x^5 - B_{32}x^4y + 2A_{32}x^3y^2 \\ \qquad -2B_{32}x^2y^3 + A_{32}xy^4 - B_{32}y^5 + y(x^2 + y^2)^3, \\ \dot{y} = B_{10}x + B_{01}y + B_{20}x^2 + B_{11}xy + (A_{11} + B_{20})y^2 + B_{30}x^3 \\ \qquad +B_{21}x^2y + B_{12}xy^2 + B_{03}y^3 + B_{32}x^5 + A_{32}x^4y + 2B_{32}x^3y^2 \\ \qquad +2A_{32}x^2y^3 + B_{32}xy^4 + A_{32}y^5 - x(x^2 + y^2)^3, \end{cases} \tag{2.1}$$

where A_{kj} and B_{kj} are real parameters. System (2.1) has a Hopf singularity at the infinity. By means of transformation $u = x + iy,\ v = x - iy,\ \tau = it$ and rewriting t instead of τ, we have its complex conjugate system

$$\begin{cases} \dot{u} = a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{30}u^3 + a_{21}u^2v + a_{03}v^3 + a_{32}u^3v^2 + u^4v^3, \\ \dot{v} = \ b_{10}v + b_{01}u + b_{20}v^2 + b_{11}vu + b_{30}v^3 + b_{21}v^2u + b_{03}u^3 + b_{32}v^3u^2 - v^4u^3 \end{cases} \tag{2.2}$$

with

$$\begin{aligned} a_{10} =& i(A_{10} + B_{01} - iA_{01} + iB_{10})/2, \qquad a_{01} = i(A_{10} - B_{01} + iA_{01} + iB_{10})/2, \\ a_{20} =& i(B_{11} - iA_{11})/2, \qquad\qquad\qquad\quad\ a_{11} = i(2A_{20} - B_{11} + iA_{11} + 2iB_{20})/2, \\ a_{30} =& i(A_{30} - A_{12} + B_{21} - B_{03} - iA_{21} + iA_{03} + iB_{30} - iB_{12})/8, \\ a_{21} =& i(3A_{30} + A_{12} + B_{21} + 3B_{03} - iA_{21} - 3iA_{03} + 3iB_{30} + iB_{12})/8, \\ a_{12} =& i(3A_{30} + A_{12} - B_{21} - 3B_{03} + iA_{21} + 3iA_{03} + 3iB_{30} + iB_{12})/8, \\ a_{03} =& i(A_{30} - A_{12} - B_{21} + B_{03} + iA_{21} - iA_{03} + iB_{30} - iB_{12})/8, \\ a_{32} =& i(A_{32} + iB_{32}). \end{aligned}$$

Obviously, a_{kj} and b_{kj} satisfy the conjugate condition, that is $\bar{a}_{kj} = b_{kj},\ k, j = 0, 1, 2, 3$. We say that (2.2) is the associated system of (2.1) and vice versa. Further, introducing a generalized Bendixson's reciprocal radius transformation given by $u = z/(zw)^4,\ v = w/(zw)^4$ and making a rescaling of the time variable $\mathrm{d}T = (zw)^{21}\mathrm{d}t$ and rewriting t instead of T, system (2.2) becomes

$$\begin{cases} \dot{z} = \ z\big[7+(3a_{32}+4b_{32})w^7z^7+3a_{03}w^{16}z^{12}+(3a_{12}+4b_{30})w^{15}z^{13}+(3a_{21}+4b_{21}) \\ \qquad w^{14}z^{14} + (3a_{30} + 4b_{12})w^{13}z^{15} + 4b_{03}w^{12}z^{16} + (3a_{11} + 4b_{20})w^{18}z^{17} \\ \qquad +(3a_{20}+4b_{11})w^{17}z^{18}+3a_{01}w^{22}z^{20}+(3a_{10}+4b_{10})w^{21}z^{21} + 4b_{01}w^{20}z^{22}\big]/7, \\ \dot{w} = -w\big[7+(4a_{32}+3b_{32})w^7z^7+4a_{03}w^{16}z^{12}+(4a_{12}+3b_{30})w^{15}z^{13}+(4a_{21}+3b_{21}) \\ \qquad w^{14}z^{14} + (4a_{30} + 3b_{12})w^{13}z^{15} + 3b_{03}w^{12}z^{16} + (4a_{11} + 3b_{20})w^{18}z^{17} \\ \qquad +(4a_{20}+3b_{11})w^{17}z^{18}+4a_{01}w^{22}z^{20} + (4a_{10} + 3b_{10})w^{21}z^{21} + 3b_{01}w^{20}z^{22}\big]/7. \end{cases} \tag{2.3}$$

Accordingly, the infinity of system (2.1) reduces to the origin of system (2.3). In other words, the study of bifurcation of limit cycles at the infinity of system (2.1) is equivalent to that of the origin of system (2.3). Before discussing the center-focus problem of the infinity of system (2.1), we introduce some notions and results; for more details, see [13, 16].

Definition 2.1. *(i)* For any positive integer k, $\mu_k = 7\omega_{7k}$ is called the k-th singular point quantity at the infinity of system (2.2), where ω_{7k} is the $7k$-th singular point quantity at the origin of system (2.3).

(ii) If $\mu_1 = \mu_2 = \cdots = \mu_{m-1} = 0 \neq \mu_m$, then the infinity is called a m-th fine singular point of system (2.2).

(iii) If for all k, $\mu_k = 0$, then the infinity of system (2.2) is called a complex center.

Lemma 2.1. *For any positive integer k, $\mu_k \sim \nu_{2k+1}(2\pi)/i\pi$, where μ_k is the k-th singular point quantity at the infinity of system (2.2), while $\nu_{2k+1}(2\pi)$ is the k-th Lyapunov quantity of its associated system (2.1).*

From Theorem 3.2 in [13] or Theorems 5.3.2 and 5.3.3 in [16], we have

Lemma 2.2. *For system (2.3), we can derive uniquely an extended formal power series*

$$F(z, w) = zw \sum_{k=0}^{\infty} f_{7k}(z, w), \tag{2.4}$$

such that

$$\dot{F} = \sum_{m=1}^{\infty} \omega_m (zw)^{7m+1}, \tag{2.5}$$

where $f_{7k}(z, w) = \sum_{\alpha+\beta=7k} c_{\alpha\beta} z^{\alpha} w^{\beta}$, and for any positive integer m, the m-th singular point quantity ω_m at the origin of (2.3) can be determined by the following recursive formulae:

$$
\begin{aligned}
& c_{0,0} = 1, \\
& \textit{when } \alpha = \beta > 0 \textit{ or } \alpha < 0 \textit{ or } \beta < 0,\ c_{\alpha\beta} = 0 \\
& \textit{else} \\
& c_{\alpha\beta} = (b_{01} c_{-22+\alpha,-20+\beta} + 4b_{01}\alpha c_{-22+\alpha,-20+\beta} - 3b_{01}\beta c_{-22+\alpha,-20+\beta} - a_{10} c_{-21+\alpha,-21+\beta} \\
& \quad + b_{10} c_{-21+\alpha,-21+\beta} + 3a_{10}\alpha c_{-21+\alpha,-21+\beta} + 4b_{10}\alpha c_{-21+\alpha,-21+\beta} - 4a_{10}\beta c_{-21+\alpha,-21+\beta} \\
& \quad - 3b_{10}\beta c_{-21+\alpha,-21+\beta} - a_{01} c_{-20+\alpha,-22+\beta} + 3a_{01}\alpha c_{-20+\alpha,-22+\beta} - 4a_{01}\beta c_{-20+\alpha,-22+\beta} \\
& \quad - a_{20} c_{-18+\alpha,-17+\beta} + b_{11} c_{-18+\alpha,-17+\beta} + 3a_{20}\alpha c_{-18+\alpha,-17+\beta} + 4b_{11}\alpha c_{-18+\alpha,-17+\beta} \\
& \quad - 4a_{20}\beta c_{-18+\alpha,-17+\beta} - 3b_{11}\beta c_{-18+\alpha,-17+\beta} - a_{11} c_{-17+\alpha,-18+\beta} + b_{20} c_{-17+\alpha,-18+\beta} \\
& \quad + 3a_{11}\alpha c_{-17+\alpha,-18+\beta} + 4b_{20}\alpha c_{-17+\alpha,-18+\beta} 4a_{11}\beta c_{-17+\alpha,-18+\beta} - 3b_{20}\beta c_{17+\alpha,-18+\beta} \\
& \quad + b_{03} c_{-16+\alpha,-12+\beta} + 4b_{03}\alpha c_{-16+\alpha,-12+\beta} - 3b_{03}\beta c_{-16+\alpha,-12+\beta} - a_{30} c_{-15+\alpha,-13+\beta} \\
& \quad + b_{12} c_{-15+\alpha,-13+\beta} + 3a_{30}\alpha c_{-15+\alpha,-13+\beta} + 4b_{12}\alpha c_{-15+\alpha,-13+\beta} - 4a_{30}\beta c_{-15+\alpha,-13+\beta} \\
& \quad 3b_{12}\beta c_{-15+\alpha,-13+\beta} - a_{21} c_{-14+\alpha,-14+\beta} + b_{21} c_{-14+\alpha,-14+\beta} + 3a_{21}\alpha c_{-14+\alpha,-14+\beta} \\
& \quad + 4b_{21}\alpha c_{-14+\alpha,-14+\beta} - 4a_{21}\beta c_{-14+\alpha,-14+\beta} - 3b_{21}\beta c_{-14+\alpha,-14+\beta} - a_{12} c_{13+\alpha,-15+\beta} \\
& \quad + b_{30} c_{-13+\alpha,-15+\beta} + 3a_{12}\alpha c_{-13+\alpha,-15+\beta} + 4b_{30}\alpha c_{-13+\alpha,-15+\beta} - 4a_{12}\beta c_{-13+\alpha,-15+\beta} \\
& \quad - 3b_{30}\beta c_{-13+\alpha,-15+\beta} - a_{03} c_{-12+\alpha,-16+\beta} + 3a_{03}\alpha c_{-12+\alpha,-16+\beta} - 4a_{03}\beta c_{-12+\alpha,-16+\beta} \\
& \quad - a_{32} c_{-7+\alpha,-7+\beta} + b_{32} c_{-7+\alpha,-7+\beta} + 3a_{32}\alpha c_{-7+\alpha,-7+\beta} + 4b_{32}\alpha c_{-7+\alpha,-7+\beta} \\
& \quad - 4a_{32}\beta c_{-7+\alpha,-7+\beta} - 3b_{32}\beta c_{-7+\alpha,-7+\beta})/7(\alpha - \beta), \\
& \omega_m = (b_{01} c_{-22+m,-20+m} - a_{10} c_{-21+m,-21+m} + b_{10} c_{-21+m,-21+m} - a_{01} c_{-20+m,-22+m} \\
& \quad - a_{20} c_{-18m,-17+m} + b_{11} c_{-18+m,-17+m} - a_{11} c_{-17+m,-18+m} + b_{20} c_{-17+m,-18+m} \\
& \quad + b_{03} c_{-16+m,-12+m} - a_{30} c_{-15+m,-13+m} + b_{12} c_{-15+m,-13+m} - a_{21} c_{-14+m,-14+m}
\end{aligned}
$$

$$+b_{21}c_{-14+m,-14+m}-a_{12}c_{-13+m,-15+m}+b_{30}c_{-13+m,-15+m}-a_{03}c_{-12+m,-16+m}$$
$$-a_{32}c_{-7+m,-7+m}+b_{32}c_{-7+m,-7+m})/7.$$

It can be seen clearly from Definition 2.1 and Lemma 2.1 that the Lyapunov quantities $\{\nu_{2k+1}(2\pi)\}$ of system (2.1) at the infinity can be deduced from the singular point quantities $\{\omega_{7k}\}$ of system (2.3) at the origin. More importantly, the recursive formulae given by Lemma 2.2 are linear with respect to all $c_{\alpha\beta}$, which means that we only need to perform finite many arithmetic operations, i.e., plus, minus, multiply and division, to the coefficients of system (2.3). The calculation is symbolic and can be easily carried out with the help of computer algebra systems such as MATHEMATICA or MAPLE. Unfortunately, another question emerges, the symbolic computation usually results in very large expressions and one can not directly apply these enormous expressions to do further research. In other words, the simplification of singular point quantities is much more difficult to handle with. For example, by using the recursive formulae of Lemma 2.2 and computer algebra system MATHEMATICA to calculate the singular point quantities of the origin of system (2.3), we find that the first thirteen singular point quantities have the terms shown in the following table.

ω_{7k}	ω_{28}	ω_{35}	ω_{42}	ω_{49}	ω_{56}	ω_{63}	ω_{70}	ω_{77}
term	2	16	42	116	244	524	1018	1936

ω_{7k}	ω_{84}	ω_{91}	ω_{98}	ω_{105}	ω_{112}
term	3480	6180	10572	17822	29268

This table tells us that for the computation of the singular point quantities, to find a method for the simplification of ω_{7k} under the conditions $\omega_7 = \omega_{14} = \cdots = \omega_{7(k-1)} = 0$, is a key step. For the sake of simplicity we suppose system (2.3) with $a_{01} = rb_{30}, b_{01} = ra_{30}, a_{11} \neq 3b_{20}, b_{11} \neq 3a_{20}, a_{12} \neq 3b_{30}, b_{12} \neq 3a_{30}$ and $a_{30}b_{30} \neq 0$. Meanwhile, we apply the recursive formulae presented in Lemma 2.1 and utilize computer algebra system MATHEMATICA to do symbolic computation, the singular point quantities of system (2.3) are obtained as follows:

Theorem 2.1. *The first 112 singular point quantities of the origin of system (2.3) are given by*

$$\begin{aligned}
\omega_7 &= (-a_{32}+b_{32})/7,\\
\omega_{14} &= (-a_{21}+b_{21})/7,\\
\omega_{21} &= (-a_{10}+b_{10})/7,\\
\omega_{28} &= (a_{12}a_{30}-b_{12}b_{30})/7,\\
\omega_{35} &= (a_{11}a_{20}-b_{11}b_{20})/7,\\
\omega_{42} &= (-3a_{03}a_{30}^2 3b_{03}b_{30}^2+a_{03}a_{30}b_{12}-a_{12}b_{03}b_{30})/14,\\
\omega_{49} &= (-3a_{12}a_{20}^2+3b_{12}b_{20}^2+2a_{30}a_{11}^2-2b_{30}b_{11}^2+3a_{30}b_{20}^2-3b_{30}a_{20}^2\\
&\quad +7a_{20}b_{11}b_{30}-7a_{11}b_{20}a_{30}+b_{11}a_{12}a_{20}-a_{11}b_{12}b_{20})/14,\\
\omega_{56} &= (a_{11}^2a_{30}-b_{11}^2b_{30}+4a_{30}b_{20}^2-4b_{30}a_{20}^2-4a_{11}b_{20}a_{30}+4b_{11}a_{20}b_{30})r/7,\\
\omega_{63} &= 3\sqrt{h_{03}}(3a_{30}a_{11}^2-3b_{30}b_{11}^2+10a_{30}b_{20}^2-10b_{30}a_{20}^2-11a_{11}a_{30}b_{20}+11b_{11}b_{30}a_{20})/14,\\
\omega_{70} &= 8(a_{32}+b_{32})H/63,
\end{aligned}$$

$$
\begin{aligned}
\omega_{77} &= 2(21a_{21} + 21b_{21} - \sqrt{h_{03}})H/189,\\
\omega_{84} &= (a_{30}b_{20}^2 + a_{20}^2 b_{30} + 6a_{10}h_{30} + 6b_{10}h_{30})H/(21h_{30}),\\
\omega_{91} &= (83247h_{03} - 744800h_{30})H/1666980,\\
\omega_{98} &= -10(3a_{10}\sqrt{h_{03}} + 3b_{10}\sqrt{h_{03}} - 68h_{20})H/1701,\\
\omega_{105} &= (-78668415a_{10}^2 - 157336830h_{10} - 78668415b_{10}^2 + 367721042h_{30}\sqrt{h_{03}})\\
&\quad \times H/220271562,\\
\omega_{112} &= 904063960h_{30}\sqrt{h_{03}}(a_{10} + b_{10})H/114631119,
\end{aligned}
$$

where $H = \sqrt{h_{03}}(-a_{11}a_{30}b_{20} + 2a_{30}b_{20}^2 - 2a_{20}^2 b_{30} + b_{11}b_{30}a_{20})$, $h_{30} = a_{30}b_{30}$, $h_{03} = a_{03}b_{03}$, $h_{20} = a_{20}b_{20}$, $h_{10} = a_{10}b_{10}$. $\omega_k = 0$, $k \neq 7j$, $j \in N$, $j < 16$. *In the above expression of* ω_{7k}, *it is assumed that:* $\omega_7 = \omega_{14} = \cdots = \omega_{7(k-1)} = 0$ $(k = 2, 3, \cdots, 16)$.

3. Center conditions and bifurcation of limit cycles

Theorem 3.1. *For system (2.3), the first 112 singular point quantities are zero if and only if one of the following three conditions holds:*

$$
\begin{aligned}
&(i) && a_{10} = b_{10}, a_{21} = b_{21}, a_{32} = b_{32}, a_{12}a_{30} = b_{12}b_{30}, a_{03}a_{30}^2 = b_{03}b_{30}^2,\\
& && a_{30}a_{11}^2 = b_{30}b_{11}^2, a_{20}b_{20} = 0,\\
&(ii) && a_{10} = b_{10}, a_{21} = b_{21}, a_{32} = b_{32}, a_{11}a_{20} = b_{11}b_{20}, a_{12}a_{30} = b_{12}b_{30},\\
& && a_{03}a_{30}^2 = b_{03}b_{30}^2, a_{30}b_{20}^2 = b_{30}a_{20}^2,\\
& && a_{20}b_{20} \neq 0,\\
&(iii) && a_{10} = b_{10}, a_{21} = b_{21}, a_{32} = b_{32}, a_{11} = qb_{20}, b_{11} = qa_{20},\\
& && a_{12} = (2q-1)b_{30}, b_{12} = (2q-1)a_{30},\\
& && a_{01} = b_{01} = 0, a_{03}b_{03} = 0, a_{20}b_{20} \neq 0 \quad (q \in R \text{ and } q \neq 2, 3).
\end{aligned}
$$

Proof. The sufficiency is evident. Let us prove the necessity. By $\omega_{28} = 0$, we have $a_{12}a_{30} = b_{12}b_{30}$. Since $a_{30}b_{30} \neq 0$, there exists a real constant p, such that $a_{12} = pb_{30}$ and $b_{12} = pa_{30}$. then

$$\omega_{42} = (a_{03}a_{30}^2 - b_{03}b_{30}^2)(-3 + p)/14, \tag{3.1}$$

from the above $\omega_{42} = 0$ and $p \neq 3$, there exists a real constant c, such that $a_{03} = cb_{30}^2$ and $b_{03} = ca_{30}^2$. If $a_{20}b_{20} = 0$, then

$$\omega_{49} = (a_{11}^2 a_{30} - b_{11}^2 b_{30})/7, \tag{3.2}$$

$$\omega_{7k} = 0, \ k > 7, \tag{3.3}$$

that is the condition (i) holds.

If $a_{20}b_{20} \neq 0$, then by $\omega_{35} = 0$, there must exist a real constant q, such that $a_{11} = qb_{20}$ and $b_{11} = qa_{20}$, and let $a_{32} = b_{32} = e$, $a_{21} = b_{21} = s$ and $a_{10} = b_{10} = l$, then

$$
\begin{aligned}
\omega_{49} &= -\, I_0(1 + p - 2q)(-3 + q)/14,\\
\omega_{56} &= I_0(-2 + q)^2 r/7,
\end{aligned}
$$

$$\begin{aligned}
\omega_{63} &= -\,3H(-5+3q)/14,\\
\omega_{70} &= 16He/63,\\
\omega_{77} &= 2H(h_{30}c+42s)/189,\\
\omega_{84} &= H(J_0+12h_{30}l)/(21h_{30}),\\
\omega_{91} &= h_{30}(-744800+83247c^2h_{30})H/1666980,\\
\omega_{98} &= 20H(34h_{20}+3ch_{30}l)/1701,\\
\omega_{105} &= -\,h_{30}H(183860521ch_{30}^2+157336830l^2)/110135781,\\
\omega_{112} &= 180812792lh_{30}H/114631119,
\end{aligned} \tag{3.4}$$

where $I_0 = a_{30}b_{20}^2 - b_{30}a_{20}^2$, $J_0 = a_{30}b_{20}^2 + b_{30}a_{20}^2$, $H = I_0h_{30}(-2+q)c$, $h_{20} = a_{20}b_{20}$, $h_{30} = a_{30}b_{30}$.

From Eq. (3.4), we conclude the condition (ii) or (iii) holds.

Theorem 3.2. *All the singular point quantities of system (2.3) at the origin are zero if and only if one of the conditions of Theorem 3.1 holds.*

Proof. Necessity is evident, we prove the sufficiency. When the condition (i) or (ii) in Theorem 3.1 holds, we can prove that system (2.3) satisfies the extended symmetric principle in [12,16]. If the condition (iii) holds, then, system (2.3) has an integrating factor $F(z,w) = (zw)^{\frac{2-q}{q-1}}$ if $q \neq 1$; system (2.3) admits a first integral $G(z,w) = zw = C$ if $q = 1$ (C is a constant). This completes the proof of the sufficiency.

Corollary 3.1. *The infinity of system (2.1) or the origin of system (2.3) is a center if and only if one of the three conditions in Theorem 3.1 holds.*

Theorem 3.3. *The infinity of system (2.1) is a 33 order fine focus if and only if the origin of system (2.3) is a 112 order fine singular point, that is the coefficients of system (2.3) satisfy*

$$\begin{aligned}
&a_{01} = b_{01} = 0,\ a_{32} = b_{32} = 0,\ a_{10} = b_{10} = l > 0,\ a_{21} = b_{21} = -\sqrt{h_{03}}/42,\\
&a_{03} = cb_{30}^2,\ b_{03} = ca_{30}^2,\ a_{11} = 5/3b_{20},\ b_{11} = 5/3a_{20},\ a_{12} = 7/3b_{30},\ b_{12} = 7/3a_{30},\\
&a_{20}^2b_{30} + b_{20}^2a_{30} = -12h_{30}l,\ h_{20} = 3l\sqrt{h_{03}}/34,\ h_{03} = 744800h_{30}/83247,\\
&1457034322674112000 + 15576487793376021c^3l^2 = 0,\ c < 0,\ I_0 \neq 0.
\end{aligned} \tag{3.5}$$

In order to prove the existence of 16 large-amplitude limit cycles bifurcated from the infinity of perturbed system of (2.1), we need to check the Jacobian determinant of Lyapunov quantities. We remind that due to the algebraic equivalent relationship between $\{\nu_{2k+1}\}$ and $\{\omega_{7k}\}$ given in Lemma 2.1 and Definition 2.1, the Jacobian determinant evaluated in the infinity of system (2.1), based on the $\{\nu_{2k+1}\}$, is equivalent to that of $\{\omega_{7k}\}$ presented in Theorem 2.1. That is, the Jacobian determinant of $\{\nu_{2k+1}\}$ is non-zero if and only if that of $\{\omega_{7k}\}$ is non-zero. We choose $a_{30} = b_{30} = 1$. Then from Theorem 3.3, we have

$$h_{20} = -\frac{3}{34}cl, \qquad a_{20}^2 + b_{20}^2 = -12l. \tag{3.6}$$

And from ω_{91} and ω_{105} of Eq. (3.4), we have

$$c \approx -2.991131718303069, \qquad l \approx 1.869591885272554. \tag{3.7}$$

The Jacobian determinant of the function group

$$(\omega_7,\ \omega_{14},\ \omega_{21},\ \omega_{28},\ \omega_{35},\ \omega_{42},\ \omega_{49},\ \omega_{56},\ \omega_{63},\ \omega_{70},\ \omega_{77},\ \omega_{84},\ \omega_{91},\ \omega_{98},\ \omega_{105}),$$

with respect to the variables

$$(r,\ a_{32},\ b_{32},\ a_{21},\ b_{21},\ a_{10},\ b_{10},\ a_{12},\ b_{12},\ a_{20},\ b_{20},\ a_{11},\ b_{11},\ a_{03},\ b_{03})$$

is

$$\begin{aligned}J =& -959787471455911936000000000000000(a_{20}-b_{20})^8(a_{20}+b_{20})^8\\ &(-38908415499142539880967324160000a_{20}^2\\ &-38908415499142539880967324160000b_{20}^2\\ &+21628622938250861117625771931200a_{20}^2c\\ &+21628622938250861117625771931200b_{20}^2c\\ &-4781853926415648644341270400 0l\\ &+10097339662407233533123008000cl\\ &+27564388993403760633400907333 1a_{20}^4cl\\ &+27564388993403760633400907333 1b_{20}^4cl)\\ &/939141165583243214977468514744301725015172325735473605241332989482957.\end{aligned} \tag{3.8}$$

By Eqs. (3.6), (3.7) and (3.8), we finally get

$$\begin{aligned}J =& 123429635664594091997293381479045391180574396884647268371742\\ &17566875339028619639995489396738765514342400000000000000000\\ &\times(23510776476564465806+16577575797293413 47c)l\\ &/121170010027225078527809129001534914944096001952695923\\ &565936474924083792638387363622022574437526173045367056 2\\ &513569467464182485979\\ \approx& 3.53319\times 10^7.\end{aligned} \tag{3.9}$$

Hence, from (3.9) and Theorem 4.7 in [16], we have the following conclusion.

Theorem 3.4. *If the origin of system (2.3) is a 112-order singular point, then, by a small perturbation of system (2.3), there exist 16 small-amplitude limit cycles in a small neighborhood of the origin of system (2.3). Correspondingly, there exist 16 large-amplitude limit cycles in a small enough neighborhood of the infinity of system (2.1).*

Acknowledgements

We would like to thank the referee for his/her helpful comments.

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