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INFINITELY MANY BOUND STATE SOLUTIONS OF SCHRÖDINGER-POISSON EQUATIONS IN $\mathbb{R}^3$ *

Xu Zhang, Shiwang Ma† and Qilin Xie

Abstract In this paper, we study a system of Schrödinger-Poisson equation

$$\begin{cases} -\Delta u + a(x)u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $p \in (4, 6)$ and $K \geq (\not\equiv)0$. Under some suitable decay assumptions but without any symmetry property on a and K, we obtain infinitely many solutions of this system.

Keywords Schrödinger-Poisson system, infinitely many solutions, without symmetric condition.

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1. Introduction

In this paper, we consider the existence of infinity many solutions for the following Schrödinger-Poisson system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + a(x)u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{P}$$

Systems (P) have been firstly introduced in [4] as a model describing waves interacting with its own electrostatic field in quantum mechanic. For more details on the physical aspects of the problem we refer the readers to [5] and [24].

We deal with the case in which $p \in (4, 6)$ and denote the standard norms of $L^p(\mathbb{R}^3)$ and $L^p(\Omega)$ by $|\cdot|_p$ and $|\cdot|_{p,\Omega}$, respectively. Moreover we make the following assumptions:

(A_1) $a \in C^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $a_0 = \inf_{x\in\mathbb{R}^3} a(x) > 0$.

(A_2) For any $\alpha > 0$, $\lim_{|x|\to\infty} \frac{\partial a}{\partial \mathbf{x}}(x)e^{\alpha|x|} = +\infty$, where $\mathbf{x} = \frac{x}{|x|}$, $x \neq 0$.

(A_3) There exists a constant $\bar{c} > 1$ such that, for $|x| > \bar{c}$

$$|\nabla_{\tau_x} a(x)| \leq \bar{c}\frac{\partial a}{\partial \mathbf{x}}(x),$$

†the corresponding author. Email address:shiwangm@nankai.edu.cn(S. Ma)
School of Mathematical Sciences and LPMC, Nankai University, Weijin road, 300071 Tianjin, China

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where $\nabla_{\tau_x} a(x)$ denoting the component of $\nabla a(x)$ which lies in the hyperplane orthogonal to x and containing x.

(K) $K(x) \geq (\not\equiv) 0$ and there exist constants $\beta, c_\beta > 0$ such that $K(x) \leq c_\beta e^{-\beta|x|}$.

Recently, the nonlinear Schrödinger-Poisson system has been widely studied under variant conditions. The existence and multiplicity results of system (P) have been discussed in many papers. Take for instance, see [1–5, 7]. And the results for (P) with positive non-radial potential and different assumptions on nonlinearities can be found in [2], [27] and [29]. In the case $a = 1$ and $K = 1$, in [22], Ruiz proved (P) does not admit any nontrivial solution for $2 < p \leq 3$ and possesses a positive radial solution for $4 < p < 6$. In [9], Cerami and Vaira considered the system with $a = 1$ and non-autonomous nonlinearities, and they proved the existence of positive ground state solutions via minimization on Nehari manifold and concentration compactness argument. Other existence or multiplicity results can be found in [16, 18, 23, 26, 28] with variant assumptions on the potential and nonlinearities. The semiclassical solutions of the system have also been discussed and we refer the readers to [11, 14, 15, 21] for details.

As to the existence of infinitely many solutions, when $a = K = 1$, Ambrosetti and Ruiz [3] proved that (P) has infinitely many pairs of radial solutions for $3 < p < 6$, and has multiple solutions (but not infinitely) for K mall enough when $2 < p < 3$. Li, Peng and Yan [19] proved the existence of infinitely many non-radial positive solutions of (P) with $a = 1$, radial symmetric K and non-autonomous radial symmetric nonlinearities. Their method is based on a reduction argument and we refer readers to [10] for a similar problem. In [20], Liu, Zhang and Wang consider the system with coercive potential and a more general nonlinearity which covers the case $p \in (3, 6)$ and they obtain infinitely many sign-changing solutions by using minimax arguments in the presence of invariant sets of a descending flow.

To our knowledge, in all the papers mentioned above, there either hold the radial symmetric assumptions or some compact conditions are available on the potential, which makes it naturally possesses the compact embedding. In this paper, we intend to find infinitely many solutions to (P) without any of those symmetric or compactness conditions. Our main result is as follows:

Theorem 1.1. *Assume that $(A_1), (A_2), (A_3)$ and (K) hold. When $p \in (4, 6)$,* (P) *has infinitely many solutions, whose energy can be arbitrarily large.*

Our strategy of proof follows from that of [12] and [8], in which the authors considered the existence of infinitely many solutions for Schrödinger equation with critical growth nonlinearity in bounded domains and with subcritical growth nonlinearity in $\mathbb{R}^N$, respectively. Specifically, we consider a sequence of balls $B_{\rho_n}(0)$ in $\mathbb{R}^3$ with $\rho_n \to +\infty$ as $n \to +\infty$, and consider the related problems on these balls

$$\begin{cases} -\Delta u + a(x)u + K(x)\phi u = |u|^{p-2}u, & \text{in } B_{\rho_n}(0), \\ -\Delta \phi = K(x)u^2, & \text{in } B_{\rho_n}(0), \\ u = 0. & \text{in } \mathbb{R}^3 \setminus B_{\rho_n}(0). \end{cases} \tag{P$_n$}$$

Applying the classical mini-max arguments to the functionals corresponding to (P$_n$), we may obtain the existence of infinitely many solutions. Let $\{u_n\}$ be the sequence consisting of solutions u_n to (P$_n$), corresponding to mini-max classes of the same type, and then we try to pass to the limit. However, since the lack of compactness, we do not know whether such (a *balanced sequence*) $\{u_n\}$ convergence strongly or not. So we arguing indirectly that $\{u_n\}$ (is a *broken balanced sequence*) break up into several parts (one could see [8, 12] for more details for the definitions of balanced sequence and broken balanced sequence). And then we use some uniform decay estimates on the bounded sequence of solutions to (P_n) and a local Pohozaev type inequality to get a contradiction.

The remainder of this paper is organized as follows. In Section 2, we derive a variational setting for the problem (P) and the associated functional. In Section 3, we manage to give a compactness result. In section 4, we prove the existence of the infinitely many solution by using an argument of Morse index.

2. Preliminaries

We consider the Hilbert space $H^1(\mathbb{R}^3)$ with the inner product

$$(u, v) := \int_{\mathbb{R}^3} \nabla u \nabla v + uv dx$$

and corresponding norm $\|u\| = (u, u)^{1/2}$. Let Ω be an open subset of $\mathbb{R}^3$, the space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\mathbb{R}^3)$.

As we know, (P) can be reduced to a single equation with a non-local term. Actually, for each $u \in H^1(\mathbb{R}^3)$, we define an operator T_u on $D^{1,2}(\mathbb{R}^3)$ by

$$T_u(v) = \int_{\mathbb{R}^3} K(x) u^2 v dx.$$

Then the Hölder inequality and (K) yield that there is a constant $C > 0$ such that for every $v \in D^{1,2}(\mathbb{R}^3)$,

$$|T_u(v)| \leq C\|u\|^2 \|v\|_{D^{1,2}(\mathbb{R}^3)}.$$

Hence, by the Lax-Milgram theorem, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla \phi_u \nabla v dx = \int_{\mathbb{R}^3} K(x) u^2 v dx,$$

thus ϕ_u is a weak solution of $-\Delta \phi_u = K(x)u^2$ and can be represented by

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} u^2(y) dy,$$

for x in the interior of supp(u). Moreover, it is obvious that

$$\|\phi\|_{D^{1,2}(\mathbb{R}^3)} = \|T_u\|_{\mathcal{L}(D^{1,2}, \mathbb{R})} \leq C\|u\|^2.$$

Thus, substituting ϕ_u in (P), we can prove that (u, ϕ) is a solution of (P) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + a(x) u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \Phi(u) u^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx,$$

here the operator Φ is defined as

$$\Phi : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3), \qquad \Phi(u) = \phi_u.$$

Setting

$$N(u) = \int_{\mathbb{R}^3} K(x)\Phi(u)u^2 dx.$$

Then we have the following properties, which is useful to our problem.

Lemma 2.1. *When (K) holds, then the following statements hold true:*

(1) Φ is continuous;

(2) Φ maps bounded sets into bounded sets;

(3) $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ implies that $\Phi(u_n) \to \Phi(u)$ in $D^{1,2}(\mathbb{R}^3)$;

(4) $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $N(u_n) \to N(u)$, as $n \to \infty$.

Proof. One can find the proofs in [9] lemma 2.1 for (1), (2); and in [28] Lemma 2.1, Lemma 2.2 for (3), (4). □

We introduce the following inequality related to system (P) as a extreme functional.

$$\begin{cases} -\Delta u + a_\infty u \le u^{p-1} & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u \ge 0 & x \in \mathbb{R}^3, \end{cases} \tag{P_∞}$$

where $a_\infty = \liminf_{|x|\to+\infty} a(x)$.

Let $B_{\rho_n}(0) \subset \mathbb{R}^3$ be an open ball centered at 0 with radius ρ_n. We firstly consider the following system

$$\begin{cases} -\Delta u + a(x)u + K(x)\Phi(u)u = |u|^{p-2}u, & \text{in } B_{\rho_n}(0), \\ u = 0, & \text{in } \mathbb{R}^3 \setminus B_{\rho_n}(0). \end{cases} \tag{$\mathrm{P}_{B_{\rho_n}(0)}$}$$

And then we have

Proposition 2.1. *Under the condition (A_1) and (K), let $(\rho_n)_n$ be a sequence such that $\rho_n \to +\infty$ as $n \to +\infty$ and $(u_n)_n$ be a sequence of nontrivial weak solution of $(\mathrm{P}_{\mathrm{B}_{\rho_n}(0)})$ satisfying $I(u_n) \le C$. Then there is a subsequence (still denoted by $(u_n)_n$) such that there exist $u_0 \in H^1(\mathbb{R}^3)$, an integer $m \ge 0$, nontrivial solution w^i of (P_∞) and sequences (y_n^i) for $1 \le i \le m$ satisfying*

$$\begin{aligned} &u_n \rightharpoonup u_0 \quad \text{in} \quad H^1(\mathbb{R}^3), \quad n \to +\infty; \\ &|u_n| - (|u_0| + \sum_{i=1}^m w^i(\cdot - y_n^i)) \to 0 \quad \text{in} \quad H^1(\mathbb{R}^3), \quad n \to +\infty; \\ &|y_n^i| \to +\infty, \quad |y_n^i - y_n^j| \to +\infty \quad \text{if} \quad i \ne j, \quad n \to +\infty. \end{aligned}$$

Moreover, we agree that in the case $m = 0$ the above holds without w^i and $\{y_n^i\}$ which means $(u_n)_n$ is relatively compact.

Proof. We firstly show that $(u_n)_n$ is bounded.

$$\begin{aligned} C &\ge I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\ &\ge \min(a_0, 1)(\frac{1}{2} - \frac{1}{p})\|u_n\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} K(x)\Phi(u_n)u_n^2 dx \\ &\ge \min(a_0, 1)(\frac{1}{2} - \frac{1}{p})\|u_n\|^2. \end{aligned}$$

Thus, up to a subsequence which we still denote by $(u_n)_n$ we have

$$\begin{aligned} &u_n \rightharpoonup u_0 \quad \text{weakly in} \quad H^1(\mathbb{R}^3);\\ &u_n \to u_0 \quad \text{strongly in} \quad L^q_{loc}(\mathbb{R}^3) \quad \text{for} \quad 1 \le q < 6;\\ &u_n(x) \to u_0(x) \quad \text{a.e. in} \ \ \mathbb{R}^3, \end{aligned}$$

where u_0 is a weak solution of (P).

If $u_n \to u_0$ in $H^1(\mathbb{R}^3)$, we have done. Otherwise $v_n^1 = u_n - u_0$ converges weakly but not strongly to zero in $H^1(\mathbb{R}^3)$. Consequently,

$$\|v_n^1\|^2 = \|u_n\|^2 - \|u_0\|^2 + o(1),$$

as $n \to +\infty$. We claim that $\delta := \limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |v_n^1|^p dx > 0$. Arguing indirectly, if $\delta = 0$, by Lions' Lemma [17], we know for $2 < q < 6$,

$$v_n^1 = u_n - u_0 \to 0 \ \text{ in } \ L^q(\mathbb{R}^3). \tag{2.1}$$

According to (A_1), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla(u_n - u_0)|^2 + a(x)(u_n - u_0)^2 dx\\ =&\int_{\mathbb{R}^3} |\nabla u_n|^2 + a(x)u_n^2 dx - \int_{\mathbb{R}^3} |\nabla u_0|^2 + a(x)u_0^2 dx + o(1)\\ =&\langle I'(u_n), u_n\rangle - \langle I'(u_0), u_0\rangle - N(u_n) + N(u_0) + |u_n|_p^p - |u_0|_p^p + o(1). \end{aligned}$$

By Lemma 2.1 (4), we know $N(u_n) - N(u_0) = o(1)$ as $n \to +\infty$; By Brezis-Lieb Lemma and (2.1), we know $|u_n|_p^p - |u_0|_p^p = o(1)$ as $n \to +\infty$. Combining these with (A_1), we get

$$\min(a_0, 1)\|u_n - u_0\| \le \int_{\mathbb{R}^3} |\nabla(u_n - u_0)|^2 + a(x)(u_n - u_0)^2 dx = o(1), \quad n \to +\infty,$$

which is a contradiction. Thus we can choose $y_n^1 \in \mathbb{R}^3$ such that

$$\int_{B_1(y_n^1)} |v_n^1|^p dx > \frac{\delta}{2}.$$

Since $v_n^1 \rightharpoonup 0$, (y_n^1) must be unbounded. Up to a subsequence, we can assume that $|y_n^1| \to +\infty$ and $v_n^1(\cdot + y_n^1) \rightharpoonup w^1$ in $H^1(\mathbb{R}^3)$. Then it is easy to check that $|w^1|$ is nontrivial and weakly solves (P_∞).

If $v_n^1 - w^1(\cdot - y_n^1) \to 0$, we have done. Otherwise, $v_n^2(x) = v_n^1(x) - w^1(x - y_n^1)$ converges weakly but not strongly to zero in $H^1(\mathbb{R}^3)$. Moreover

$$\|v_n^2\|^2 = \|v_n^1\|^2 - \|w^1\|^2 + o(1) = \|u_n\|^2 - \|u_0\|^2 - \|w^1\|^2 + o(1).$$

Repeating the procedure above, we could obtain sequences $y_n^i \in \mathbb{R}^3$ such that

$$|y_n^i| \to +\infty, \quad |y_n^i - y_n^j| \to +\infty \quad \text{if} \quad i \ne j, \quad n \to +\infty,$$

and also obtain a sequence of functions $v_n^i(x) = v_n^{i-1} - w^{i-1}(x - y_n^{i-1})$ with $i \ge 2$,

$$\begin{aligned} &v_n^i(\cdot + y_n^i) \rightharpoonup w^i \ \text{ in } \ H^1(\mathbb{R}^3), \quad n \to +\infty;\\ &\|v_n^i\|^2 = \|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{i-1} \|w^j\|^2 + o(1). \end{aligned}$$

Here $(|w^j|)_j$ are nontrivial weak solutions of (P_∞). As the result, we know

$$\min(1, a_\infty)\|w^j\|^2 \le |w^j|_p^p,$$

which means that $\|w^j\|$ has a uniformly lower bound $C > 0$. Hence the iteration must stop at some finite steps m, and we obtain as $n \to +\infty$

$$\begin{aligned} &u_n - \sum_{i=1}^m w^i(\cdot - y_n^i) - u_0 \to 0 \ \text{ in } \ H^1(\mathbb{R}^3);\\ &|y_n^i| \to +\infty, \quad |y_n^i - y_n^j| \to, +\infty \quad \text{if} \quad 1 \le i \ne j \le m. \end{aligned}$$

Here $|w^j|$ are nontrivial solutions of (P_∞).

Thus the only we need to show is that

$$|u_n| - (|u_0| + \sum_{i=1}^m |w^i(\cdot - y_n^i)|) \to 0 \quad \text{in} \quad H^1(\mathbb{R}^3), \quad n \to +\infty.$$

For any $\varepsilon > 0$ there exists $R > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^3\setminus B_R(0)} |\nabla u_0|^2 + |u_0|^2 dx < \varepsilon;\\ &\int_{\mathbb{R}^3\setminus B_R(0)} |\nabla w^i|^2 + |w^i|^2 dx < \varepsilon, \quad 1 \le i \le m. \end{aligned}$$

Consequently for sufficiently large n, we have

$$\begin{aligned} \varepsilon &\ge \int_{\mathbb{R}^3} |u_n - (u_0 + \sum_{i=1}^m w^i(\cdot - y_n^i))|^2 dx\\ &= \int_{B_R(0)\cup(\bigcup_{i=1}^m B_R(y_n^i))} |u_n - (u_0 + \sum_{i=1}^m w^i(\cdot - y_n^i))|^2 dx\\ &\quad + \int_{\mathbb{R}^3\setminus B_R(0)\cup(\bigcup_{i=1}^m B_R(y_n^i))} |u_n - (u_0 + \sum_{i=1}^m w^i(\cdot - y_n^i))|^2 dx\\ &\ge \int_{B_R(0)} |u_n - u_0|^2 dx + \sum_{i=1}^m \int_{B_R(y_n^i)} |u_n - w^i(\cdot - y_n^i)|^2 dx - C(m)\varepsilon\\ &\ge \int_{B_R(0)} (|u_n| - |u_0|)^2 dx + \sum_{i=1}^m \int_{B_R(y_n^i)} (|u_n| - |w^i(\cdot - y_n^i)|)^2 dx - (C(m)+1)\varepsilon\\ &\ge \int_{B_R(0)\cup(\bigcup_{i=1}^m B_R(y_n^i))} (|u_n| - |u_0| - \sum_{i=1}^m |w^i(\cdot - y_n^i)|)^2 dx - (2C(m)+1)\varepsilon\\ &\ge \int_{\mathbb{R}^3} (|u_n| - |u_0| - \sum_{i=1}^m |w^i(\cdot - y_n^i)|)^2 dx - (3C(m)+1)\varepsilon. \end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}^3} \left|\nabla(|u_n| - |u_0| - \sum_{i=1}^m |w^i(\cdot - y_n^i)|)\right|^2 dx \le (3C(m)+2)\varepsilon.$$

Noting that $C(m)$ is a positive constant depending only on m, thus the proof is complete. $\square$

Notice that in Proposition 2.2 when $(u_n)_n$ is not relatively compact, we have $m > 0$. In this case we give some definitions which are useful for the estimates in Section 3.

Definition 2.1. Let A be a subset of $\mathbb{R}^3$ and $v \in \mathbb{R}^3$ be a point such that $v \notin A$. Then we define the set

$$\{w \in \mathbb{R}^3 : w = v + \lambda(x - v), x \in A, \lambda \in \mathbb{R}^+\}$$

by cone of vertex v generated by A.

Definition 2.2. Under the assumption of Proposition 2.2, when $(u_n)_n$ is not relatively compact, we have $m > 0$. Moreover for $n \in \mathbb{N}$, there exists $1 \le i_n \le m$ such that $|y_n^{i_n}| = \min_{1\le i\le m} |y_n^i|$. Then we call a sequence $(y_n)_n$ the smallest sequence of $(u_n)_n$ if it satisfies $y_n = y_n^{i_n}$ for all $n \in \mathbb{N}$.

In the following, we consider a noncompact sequence $(u_n)_n$ satisfying the condition of Proposition 2.2. Moreover let $(y_n)_n$ be the associated smallest sequence.

We claim that for $n \in \mathbb{N}$ and $1 \le i \le m$, there exists $R_n > 0$ such that $\mathcal{C}_n$ (the cone of vertex $\frac{y_n}{2}$ generated by $B_{R_n}(y_n)$) satisfies $\partial\mathcal{C}_n \cap B_{\frac{r_n}{2}}(y_n^i) = \emptyset$. Here

$$r_n = \frac{\gamma|y_n|}{2m}, \quad 0 < \gamma < \min(\frac{1}{5}, \frac{1}{4(\bar{c}+1)}),$$

and $\bar{c}$ is the constant in (A_3).

In order to construct such a sequence of $\mathcal{C}_n$, firstly we considering $\mathcal{C}_{1,n}$ which is the cone of vertex $\frac{y_n}{2}$ generated by $B_{r_n}(y_n)$. Obviously, $\partial\mathcal{C}_{1,n} \cap B_{\frac{r_n}{2}}(y_n^{i_n}) = \emptyset$. If additionally there hold

$$\partial\mathcal{C}_{1,n} \cap B_{\frac{r_n}{2}}(y_n^i) = \emptyset, \quad 1 \le i \le m,$$

then let $\mathcal{C}_{1,n}$ be $\mathcal{C}_n$ and we have done. Otherwise, there exists some $1 \le j_1 \le m$ such that $\partial\mathcal{C}_{1,n} \cap B_{\frac{r_n}{2}}(y_n^{j_1}) \ne \emptyset$. Then we define $\mathcal{C}_{2,n}$ be the cone of vertex $\frac{y_n}{2}$ generated by $B_{2r_n}(y_n)$. Using the fact that $|y_n| \le |y_n^{j_1}|$ we obtain $\partial\mathcal{C}_{2,n} \cap B_{\frac{r_n}{2}}(y_n^{j_1}) = \emptyset$. If additionally there hold

$$\partial\mathcal{C}_{2,n} \cap B_{\frac{r_n}{2}}(y_n^i) = \emptyset, \quad 1 \le i \le m,$$

then let $\mathcal{C}_{2,n}$ be $\mathcal{C}_n$. Otherwise, we repeat the procedure and after at most $m-1$ steps, we can obtain $\mathcal{C}_n$, the cone of vertex $\frac{y_n}{2}$ generated by $B_{R_n}(y_n)$ as the claim.

It is obvious that

$$\frac{\gamma|y_n|}{2m} = r_n \le R_n \le mr_n = \frac{\gamma|y_n|}{2}.$$

Consequently, if we denote the width angle of $\mathcal{C}_n$ by θ_n, we have

$$\sin\theta_n = \frac{2R_n}{|y_n|} \in [\frac{\gamma}{m}, \gamma].$$

For $s \in \mathbb{R}$ and $n \in \mathbb{N}$ we define the cone $\mathcal{C}_{s,n}$ as

$$\mathcal{C}_{s,n} = \mathcal{C}_n - s\frac{y_n}{|y_n|},$$

and for $s \in \mathbb{R}^+$, $n \in \mathbb{N}$ a neighbourhood of $\partial\mathcal{C}_n$

$$\mathcal{S}_{2s,n} = \mathcal{C}_{s,n} - \mathcal{C}_{-s,n}.$$

Moreover, we define

$$\mathcal{S}_n = \mathbb{R}^3 \Big\backslash \bigcup_{i=0}^{m} B_{\frac{r_n}{2}-1}(y_n^i)$$

with $y_n^0 = 0$ for $n \in \mathbb{N}$.

3. Compactness Result

In this section, we will show that under our assumptions (A_1)–(A_3) and (K), a sequence $(u_n)_n$ as that in Proposition 2.2 is relatively compact.

Lemma 3.1. *Under conditions (A_1) and (K), let $(\rho_n)_n$ be a sequence such that $\rho_n \to +\infty$ as $n \to +\infty$ and $(u_n)_n$ be a sequence of nontrivial weak solution of $(\mathrm{P}_{B_{\rho_n}(0)})$ satisfying $I(u_n) \leq C$. Then $(u_n)_n$ is bounded in $L^\infty(\mathbb{R}^3)$.*

Proof. From (A_1) and (K), it is easy to see that

$$-\Delta|u_n| + a_0|u_n| \leq |u_n|^{p-1}.$$

Then according to maximum principle, if we set v_n be weak solutions of

$$-\Delta v + a_0 v = |u_n|^{p-1},$$

we have the relation $|u_n| \leq v_n$. Since $(\|u_n\|)_n$ are uniformly bounded, thus by an iterating argument, we conclude $(v_n)_n$ are uniformly bounded in $L^\infty(\mathbb{R}^3)$, which implies that $(u_n)_n$ are uniformly bounded in $L^\infty(\mathbb{R}^3)$. □

Consequently, we have

Corollary 3.1. *Assume (A_1) and (K) hold. Let $(u_n)_n$ be defined as that in Lemma 3.1, then we have*

$$-\Delta|u_n| \leq C_0$$

weakly holds for some positive constant C_0.

Lemma 3.2. *Assume (A_1) and (K) hold. Let $(u_n)_n$ be defined as that in Lemma 3.1, $(x_n)_n$ be a sequence such that $x_n \in \mathcal{S}_n$. If $(u_n)_n$ is not relatively compact, then for any $h \in (0,1)$ there holds*

$$\lim_{n\to+\infty} \sup_{B_{h\sigma_n(x_n)}(x_n)} |u_n(x)| = 0,$$

where

$$\sigma_n(x) = \inf_{1\leq i\leq m} |x - y_n^i|$$

and y_n^i is defined as that in Proposition 2.2.

Proof. Arguing indirectly, we assume that there exist $h \in (0,1)$, $\delta > 0$ and a sequence $(z_n)_n$, such that $z_n \in B_{h\sigma_n(x_n)}(x_n)$ and $|u_n(z_n)| \geq \delta$ for n large enough. Thus by Corollary 3.2, we know

$$\frac{1}{|B_\rho(z_n)|}\int_{B_\rho(z_n)} |u_n| dx > \frac{\delta}{2}$$

for small ρ. Here $|B_\rho(z_n)|$ is the Lebesgue measure of $B_\rho(z_n)$ in $\mathbb{R}^3$. Therefore we have $|u_n(\cdot + z_n)| \rightharpoonup v \neq 0$. On the other hand, according to the choice of σ_n, h and the fact

$$|u_n| - (|u_0| + \sum_{i=1}^{m} w^i(\cdot - y_n^i)) \to 0 \quad \text{in} \quad H^1(\mathbb{R}^3), \quad n \to +\infty,$$

we have $|u_n(\cdot + z_n)| \rightharpoonup 0$. And this is a contradiction. □

Proposition 3.1. *Under the conditions of Lemma 3.3, we have for any $\alpha \in (0, \sqrt{a_\infty})$ there exist a positive constant C_α and $n_\alpha \in \mathbb{N}$ such that*

$$|u_n(x)| \leq C_\alpha e^{-\alpha\sigma_n(x)},$$

when $n > n_\alpha$ and $x \in \mathcal{S}_n$.

Proof. Arguing indirectly, we assume that there exists $\alpha \in (0, \sqrt{a_\infty})$ such that for $k \in \mathbb{N}$ we can find $n_k \geq k$ and $x_k \in \mathcal{S}_{n_k}$ satisfying

$$|u_{n_k}(x_k)| > ke^{-\alpha\sigma_{n_k}(x_k)}.$$

We choose $h \in (\frac{\alpha}{\sqrt{a_\infty}}, 1)$ and $\alpha' \in (\alpha, \sqrt{a_\infty}h)$, according to Lemma 3.3, the following inequality

$$\Delta|u_{n_k}(x)| \geq a(x)|u_{n_k}(x)| - |u_{n_k}(x)|^{p-1} > (\alpha')^2 h^{-2}|u_{n_k}(x)| \geq 0 \tag{3.1}$$

weakly holds for $x \in B_{h\sigma_{n_k}(x_k)}(x_k)$. Noting that $h\sigma_{n_k}(x_k) > 1$ for k large, therefore by mean value inequalities (see [13]) we have

$$|u_{n_k}(x_k)| \leq \int_{B_1(x_k)} |u_{n_k}|dx.$$

Thus if there holds

$$\int_{B_1(x_k)} |u_{n_k}|dx \leq Ce^{-\alpha\sigma_{n_k}(x_k)} \tag{3.2}$$

for a constant $C > 0$, we will get a contradiction. In order to show (3.2), we consider the following functions

$$v_k(\rho) = \int_{B_\rho(x_k)} |u_{n_k}|dx \quad \text{and} \quad w_k(\rho) = \frac{(h\sigma_{n_k}(x_k))^3\omega_3}{e^{\alpha'\sigma_{n_k}(x_k)}} e^{\alpha'\rho/h}$$

with $\omega_3 = \frac{4}{3}\pi$ denoting the Lebesgue measure of 3-dimensional unitary ball. And we claim that for k large enough there holds

$$v_k(\rho) \leq w_k(\rho), \quad \rho \in [0, h\sigma_{n_k}(x_k)]. \tag{3.3}$$

Obviously, for $k \in \mathbb{N}$

$$v_k(0) \leq w_k(0).$$

On the other hand, according to Lemma 3.3, we have

$$\begin{aligned} v_k(h\sigma_{n_k}(x_k)) &\leq |B_{h\sigma_{n_k}(x_k)}(x_k)| \sup_{B_{h\sigma_{n_k}(x_k)}(x_k)} |u_{n_k}(x)| \\ &\leq \omega_3(h\sigma_{n_k}(x_k))^3 = w_k(h\sigma_{n_k}(x_k)). \end{aligned}$$

If our claim (3.3) is false, then $(v_k - w_k)(\rho)$ will have a maximum point $\rho_k \in (0, h\sigma_{n_k}(x_k))$ such that

$$(v_k - w_k)(\rho_k) > 0 \quad \text{and} \quad (v_k - w_k)''(\rho_k) \leq 0.$$

By the definition of v_k,

$$v_k(\rho) = \int_{B_\rho(x_k)} |u_{n_k}(x)| dx = \int_0^\rho \int_{\partial B_r(x_k)} |u_{n_k}| d\sigma dr,$$

we know that

$$v_k'(\rho) = \frac{d}{d\rho} v_k(\rho) = \int_{\partial B_\rho(x_k)} |u_k| d\sigma. \tag{3.4}$$

Then according to (3.4), divergence theorem and the fact

$$\frac{d}{d\rho}(\rho^{-2} \int_{\partial B_\rho(x_k)} |u_{n_k}| d\sigma) = \rho^{-2} \int_{\partial B_\rho(x_k)} \frac{\partial}{\partial \nu} |u_{n_k}| d\sigma,$$

we can deduce

$$\frac{d}{d\rho}(\rho^{-2} v_k'(\rho)) = \rho^{-2} \int_{\partial B_\rho(x_k)} \frac{\partial}{\partial \nu} |u_{n_k}| d\sigma = \rho^{-2} \int_{B_\rho(x_k)} \Delta |u_{n_k}| dx.$$

Combining this and (3.1), we get

$$\begin{aligned}
\frac{v_k''(\rho)}{\rho^2} - 2\frac{v_k'(\rho)}{\rho^3} &= \rho^{-2} \int_{B_\rho(x_k)} \Delta |u_{n_k}| dx \\
&\geq \rho^{-2} \int_{B_\rho(x_k)} \alpha'^2 h^{-2} |u_{n_k}| dx \\
&= \rho^{-2} \alpha'^2 h^{-2} v_k(\rho).
\end{aligned}$$

Since $v_k'(\rho) \geq 0$ (see (3.4)), we have

$$v_k''(\rho) \geq \alpha'^2 h^{-2} v_k(\rho), \quad \rho > 0.$$

Consequently,

$$(v_k - w_k)''(\rho_k) \geq \alpha'^2 h^{-2} (v_k(\rho_k) - w_k(\rho_k)) > 0,$$

which means our claim (3.3) is true. Taking $\rho = 1$ in (3.3) we have

$$\int_{B_1(x_k)} |u_{n_k}| dx = v_k(1) \leq w_k(1) = \frac{(h\sigma_{n_k}(x_k))^3 \omega_3}{e^{\alpha' \sigma_{n_k}(x_k)}} e^{\alpha'/h} \leq c_\alpha e^{-\alpha \sigma_{n_k}(x_k)},$$

which leads to the contradiction and the proof is complete. □

Proposition 3.2. *Under the conditions of Lemma 3.3, let $(y_n)_n$ be a smallest sequence of $(u_n)_n$ defined in Section 2, then there exist positive constants α_1, C_1 such that*

$$\int_{\mathcal{S}_n} |u_n|^q dx \leq C_1 e^{-\alpha_1 |y_n|}, \quad q \in [2, +\infty).$$

Proof. According to Proposition 3.4, for sufficiently large n and $\alpha \in (0, \sqrt{\alpha_\infty})$, we have

$$\begin{aligned}\int_{\mathcal{S}_n} |u_n|^q dx &\le C_\alpha \int_{\mathcal{S}_n} e^{-\alpha q \sigma_n(x)} dx \le C_\alpha \sum_{i=0}^{m} \int_{\mathcal{S}_n} e^{-\alpha q |x - y_n^i|} dx \\ &\le C_\alpha (m+1) \int_{\frac{r_n}{2}-1}^{+\infty} e^{-\alpha q r} r^{n-1} dr \\ &\le C_1 e^{-\alpha_1 |y_n|}.\end{aligned}$$

□

Proposition 3.3. *Under the conditions of Lemma 3.3, let $(y_n)_n$ be the smallest sequence, then there exist positive constants α_0, C_0 and a sequence $(s_n)_n \subset (-\frac{1}{2}, \frac{1}{2})$ such that for all n*

$$\int_{\partial \mathcal{C}_{\mathcal{S}_n, n}} |\nabla u_n|^2 d\sigma \le C_0 e^{-\alpha_0 |y_n|}.$$

Proof. Firstly, we define $\varphi_n \in C^\infty(\mathbb{R}^3, [0,1])$ satisfying

$$\begin{cases} \varphi_n = 1 \quad \text{on} \quad \mathcal{S}_{1,n}, \\ \operatorname{supp}(\varphi_n) \subset \mathcal{S}_{2,n}, \\ \Delta \varphi_n \le C, \quad C \in \mathbb{R}. \end{cases} \tag{3.5}$$

From the definition of u_n, it is easy to see that

$$-\Delta |u_n| + a_0 |u_n| \le |u_n|^{p-1}.$$

Consequently,

$$\int_{\mathcal{S}_{2,n}} -\Delta |u_n| |u_n| \varphi_n dx \le \int_{\mathcal{S}_{2,n}} (|u_n|^p - a_0 |u_n|^2) \varphi_n dx.$$

On the other hand,

$$\begin{aligned}\int_{\mathcal{S}_{2,n}} -\Delta |u_n| |u_n| \varphi_n dx &= \int_{\mathcal{S}_{2,n}} |\nabla u_n|^2 \varphi_n dx + \int_{\mathcal{S}_{2,n}} (\nabla |u_n| \cdot \nabla \varphi_n) |u_n| dx \\ &\ge \int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 dx + \frac{1}{2} \int_{\mathcal{S}_{2,n}} (\nabla (|u_n|^2) \cdot \nabla \varphi_n) dx \\ &= \int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 dx - \frac{1}{2} \int_{\mathcal{S}_{2,n}} (\Delta \varphi_n) |u_n|^2 dx.\end{aligned}$$

Using the fact that $\mathcal{S}_{2,n} \subset \mathcal{S}_n$, (3.5) and Proposition 3.5 we have, for some positive constants C_1 and α_0,

$$\begin{aligned}\int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 dx &\le \frac{1}{2} \int_{\mathcal{S}_{2,n}} (\Delta \varphi_n) |u_n|^2 dx + \int_{\mathcal{S}_{2,n}} (|u_n|^p - a_0 |u_n|^2) \varphi_n dx \\ &\le \frac{C}{2} \int_{\mathcal{S}_n} |u_n|^2 dx + \int_{\mathcal{S}_n} |u_n|^p dx \\ &\le C_1 e^{-\alpha_0 |y_n|},\end{aligned}$$

which leads us to

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin\theta_n \int_{\partial\mathcal{C}_{s,n}} |\nabla u_n|^2 d\sigma ds = \int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 dx \le C_1 e^{-\alpha_0 |y_n|}.$$

Here θ_n denote the width angle of the cone $\mathcal{C}_n$. Thus by using the integral mean value theorem and the fact $\sin\theta_n \ge \frac{\gamma}{m} > 0$ (see the end of Section 2), we get the conclusion. □

In the following, we denote the cone $\mathcal{C}_{s_n,n}$ which we obtained in Proposition 3.6 by $\tilde{\mathcal{C}}_n$ for $n \in \mathbb{N}$. And then we define

$$D_n = \tilde{\mathcal{C}}_n \cap B_{\rho_n}(0).$$

According to Proposition 2.2, we know there exists a positive constant such that $|y_n| - \rho_n \le C$, which implies $\frac{|y_n|}{2} + 1 < \rho_n$ and $D_n \ne \emptyset$ for n sufficiently large.

Obviously, the boundary of D_n:

$$\partial D_n = (\partial D_n)_i \cup (\partial D_n)_e,$$

where $(\partial D_n)_i = \partial\tilde{\mathcal{C}}_n \cap B_{\rho_n}(0)$ and $(\partial D_n)_e = \tilde{\mathcal{C}}_n \cap \partial B_{\rho_n}(0)$. Moreover we define

$$\mathbf{y}_n = \frac{y_n}{|y_n|},$$

with $(y_n)_n$ the smallest sequence of $(u_n)_n$.

Lemma 3.3. *Let $a(x)$ satisfy (A_1), $(u_n)_n$ be nontrivial solutions of $(\mathrm{P}_{\mathrm{B}_{\rho_n}(0)})$ such that $I(u_n) \le C$. If $(u_n)_n$ is not relatively compact, then there holds the identity*

$$\begin{aligned}
&\frac{1}{2}\int_{D_n} u_n^2(\nabla a(x)\cdot \mathbf{y}_n)dx - \int_{D_n} K(x)\Phi(u_n)u_n(\nabla u_n \cdot \mathbf{y}_n)dx\\
&= \frac{1}{2}\int_{\partial D_n} (|\nabla u_n|^2 + a(x)u_n^2)(\nu_n\cdot\mathbf{y}_n)d\sigma - \int_{\partial D_n}(\nabla u_n\cdot\nu_n)(\nabla u_n\cdot\mathbf{y}_n)d\sigma\\
&\quad - \frac{1}{p}\int_{\partial D_n} |u_n|^p(\nu_n\cdot\mathbf{y}_n)d\sigma
\end{aligned}$$

where ν_n denotes the unit outward normal to ∂D_n.

Proof. Since u_n weakly solves $(\mathrm{P}_{\mathrm{B}_{\rho_n}(0)})$, we know

$$\int_{D_n} (-\Delta u_n + a(x)u_n + K(x)\Phi(u_n)u_n - |u_n|^{p-2}u_n)(\nabla u_n\cdot\mathbf{y}_n)dx = 0. \tag{3.6}$$

According to Green's identity we have

$$\begin{aligned}
\int_{D_n} -\Delta u_n(\nabla u_n\cdot\mathbf{y}_n)dx &= \int_{D_n}(\nabla u_n\cdot\nabla(\nabla u_n\cdot\mathbf{y}_n))dx\\
&\quad - \int_{\partial D_n}(\nabla u_n\cdot\nu_n)(\nabla u_n\cdot\mathbf{y}_n)d\sigma.
\end{aligned}\tag{3.7}$$

Using the divergence theorem and the fact that $\mathbf{y}_n$ does not depend on x, we have

$$\begin{aligned}
\int_{D_n}(\nabla u_n\cdot\nabla(\nabla u_n\cdot\mathbf{y}_n))dx &= \frac{1}{2}\int_{D_n}(\nabla|\nabla u_n|^2\cdot\mathbf{y}_n)dx\\
&= \frac{1}{2}\int_{D_n}\mathrm{div}(|\nabla u_n|^2\mathbf{y}_n)dx\\
&= \frac{1}{2}\int_{\partial D_n}|\nabla u_n|^2(\mathbf{y}_n\cdot\nu_n)d\sigma.
\end{aligned}\tag{3.8}$$

Then taking (3.8) into (3.7) we get

$$\int_{D_n} -\Delta u_n(\nabla u_n \cdot \mathbf{y}_n)dx = \frac{1}{2}\int_{\partial D_n} |\nabla u_n|^2(\mathbf{y}_n \cdot \nu_n)d\sigma - \int_{\partial D_n} (\nabla u_n \cdot \nu_n)(\nabla u_n \cdot \mathbf{y}_n)d\sigma. \tag{3.9}$$

From the divergence theorem again, it follows that

$$\begin{aligned}\int_{D_n} a(x)u_n(\nabla u_n \cdot \mathbf{y}_n)dx &= \frac{1}{2}\int_{D_n} a(x)(\nabla(u_n^2) \cdot \mathbf{y}_n)dx \\ &= -\frac{1}{2}\int_{D_n} u_n^2(\nabla a(x) \cdot \mathbf{y}_n)dx + \frac{1}{2}\int_{D_n} \operatorname{div}(a(x)u_n^2\mathbf{y}_n)dx \\ &= -\frac{1}{2}\int_{D_n} u_n^2(\nabla a(x) \cdot \mathbf{y}_n)dx + \frac{1}{2}\int_{\partial D_n} a(x)u_n^2(\mathbf{y}_n \cdot \nu_n)d\sigma.\end{aligned} \tag{3.10}$$

By the same reason, we can also deduce that

$$\int_{D_n} |u_n|^{p-2}u_n(\nabla u_n \cdot \mathbf{y}_n)dx = \frac{1}{p}\int_{D_n} (\nabla|u_n|^p \cdot \mathbf{y}_n)dx = \frac{1}{p}\int_{\partial D_n} |u_n|^p(\mathbf{y}_n \cdot \nu_n)d\sigma. \tag{3.11}$$

Thus the combination of (3.6), (3.9), (3.10) and (3.11) leads us to the conclusion. □

The proofs of the following two lemmas are almost the same as that in [8] Lemma 4.2 and lemma 4.3, for the sake of completeness, we give the details.

Lemma 3.4. *Let conditions $(A_1), (A_3)$ hold and u_n be as in Lemma 3.7, then for large n it follows*

$$\int_{D_n} u_n^2(\nabla a(x) \cdot \mathbf{y}_n)dx \geq \frac{1}{2}\int_{D_n} \frac{\partial a(x)}{\partial \mathbf{x}} u_n^2 dx.$$

Proof. Let $(\mathbf{y}_n)_{\tau_x}$ denote the component of $\mathbf{y}_n$ which lies in the hyperplane orthogonal to x and containing x. Using (A_3), we have

$$\begin{aligned}(\nabla a(x) \cdot \mathbf{y}_n) &= (\nabla a(x) \cdot (\mathbf{y}_n \cdot \mathbf{x})\mathbf{x}) + (\nabla_{\tau_x} a(x) \cdot (\mathbf{y}_n)_{\tau_x}) \\ &\geq \frac{\partial a}{\partial \mathbf{x}}(x)(\mathbf{y}_n \cdot \mathbf{x}) - \bar{c}\frac{\partial a}{\partial \mathbf{x}}(x)|(\mathbf{y}_n)_{\tau_x}| \\ &= \frac{\partial a}{\partial \mathbf{x}}(x)[(\mathbf{y}_n \cdot \mathbf{x}) - \bar{c}|(\mathbf{y}_n)_{\tau_x}|].\end{aligned}$$

In order to show $(\mathbf{y}_n \cdot \mathbf{x}) - \bar{c}|(\mathbf{y}_n)_{\tau_x}| \geq \frac{1}{2}$ in D_n, we firstly consider the case $x \in B_{2R_n}(y_n)$. Then by the definition of R_n in Section 2, we know $|x - y_n| < \gamma|y_n|$ for $x \in B_{2R_n}(y_n)$. Consequently,

$$(\mathbf{y}_n \cdot \mathbf{x}) = \Big(\frac{y_n}{|y_n|} \cdot \frac{y_n + x - y_n}{|x|}\Big) \geq \frac{|y_n| - |x - y_n|}{|x|} \geq \frac{|y_n| - |x - y_n|}{|y_n| + |x - y_n|} \geq \frac{1-\gamma}{1+\gamma}.$$

On the other hand,

$$|(\mathbf{y}_n)_{\tau_x}| = \min_{\lambda \in \mathbb{R}} |\mathbf{y}_n - \lambda x| \leq |\mathbf{y}_n - \frac{x}{|y_n|}| = \frac{|x - y_n|}{|y_n|} < \gamma.$$

Moreover, by homothety the above inequalities also hold for x in $\mathcal{K}$ (the cone of vertex 0 generated by $B_{2R_n}(y_n)$). According to the fact that $D_n \subset \tilde{\mathcal{C}}_n \subset \mathcal{K}$, we deduce the two inequalities are also true for $x \in D_n$. Then from the choice of γ, it follows that $\frac{1}{4}\frac{1-\gamma}{1+\gamma} \geq \bar{c}\gamma$ and hence

$$(\mathbf{y}_n \cdot \mathbf{x}) - \bar{c}|(\mathbf{y}_n)_{\tau_x}| \geq \frac{1-\gamma}{1+\gamma} - \bar{c}\gamma \geq \frac{3}{4}\frac{1-\gamma}{1+\gamma} \geq \frac{1}{2}.$$

□

Lemma 3.5. *Let $(A_1), (A_3)$ hold and u_n be as in Lemma 3.7, then for large n we have*

$$\begin{aligned}&\frac{1}{4}\int_{D_n} \frac{\partial a(x)}{\partial \mathbf{x}} u_n^2 dx - \int_{D_n} K(x)\Phi(u_n)u_n(\nabla u_n \cdot \mathbf{y}_n)dx \\ \leq\ & \frac{1}{2}\int_{(\partial D_n)_i} (|\nabla u_n|^2 + a(x)u_n^2)(\nu_n \cdot \mathbf{y}_n)d\sigma - \int_{(\partial D_n)_i} (\nabla u_n \cdot \nu_n)(\nabla u_n \cdot \mathbf{y}_n)d\sigma \\ &- \frac{1}{p}\int_{(\partial D_n)_i} |u_n|^p(\nu_n \cdot \mathbf{y}_n)d\sigma.\end{aligned}$$

Proof. From Lemma 3.7 and Lemma 3.8, it follows that

$$\begin{aligned}&\frac{1}{4}\int_{D_n} \frac{\partial a(x)}{\partial \mathbf{x}} u_n^2 dx - \int_{D_n} K(x)\Phi(u_n)u_n(\nabla u_n \cdot \mathbf{y}_n)dx \\ \leq\ & \frac{1}{2}\int_{\partial D_n} (|\nabla u_n|^2 + a(x)u_n^2)(\nu_n \cdot \mathbf{y}_n)d\sigma - \int_{\partial D_n} (\nabla u_n \cdot \nu_n)(\nabla u_n \cdot \mathbf{y}_n)d\sigma \\ &- \frac{1}{p}\int_{\partial D_n} |u_n|^p(\nu_n \cdot \mathbf{y}_n)d\sigma.\end{aligned}$$

Thus the only we need to show is that

$$\begin{aligned}&\frac{1}{2}\int_{\partial(D_n)_e} (|\nabla u_n|^2 + a(x)u_n^2)(\nu_n \cdot \mathbf{y}_n)d\sigma - \int_{\partial(D_n)_e} (\nabla u_n \cdot \nu_n)(\nabla u_n \cdot \mathbf{y}_n)d\sigma \\ &- \frac{1}{p}\int_{\partial(D_n)_e} |u_n|^p(\nu_n \cdot \mathbf{y}_n)d\sigma \leq 0.\end{aligned}$$

In fact, since u_n solves $(\mathrm{P}_{\mathrm{B}_{\rho_n}(0)})$, we know that on $\partial(D_n)_e$, $u_n = 0$ and ∇u_n possesses the same direction as ν_n. So that, there hold

$$\int_{\partial(D_n)_e} a(x)u_n^2 d\sigma = \int_{\partial(D_n)_e} |u_n|^p(\nu_n \cdot \mathbf{y}_n)d\sigma = 0$$

and

$$\begin{aligned}&\frac{1}{2}\int_{\partial(D_n)_e} |\nabla u_n|^2(\nu_n \cdot \mathbf{y}_n)d\sigma - \int_{\partial(D_n)_e} (\nabla u_n \cdot \nu_n)(\nabla u_n \cdot \mathbf{y}_n)d\sigma \\ =&-\frac{1}{2}\int_{\partial(D_n)_e} |\nabla u_n|^2(\nu_n \cdot \mathbf{y}_n)d\sigma \leq 0.\end{aligned}$$

Then we get the conclusion. □

And now we can give

Proposition 3.4. *Under conditions* $(A_1)-(A_3)$ *and* (K)*, let* $(\rho_n)_n$ *be a sequence such that* $\rho_n \to +\infty$ *as* $n \to +\infty$ *and* $(u_n)_n$ *be a sequence of nontrivial weak solution of* $(\mathrm{P}_{B_{\rho_n}(0)})$ *satisfying* $I(u_n) \leq C$*. Then* $(u_n)_n$ *is relatively compact in* $H^1(\mathbb{R}^3)$*.*

Proof. Arguing indirectly, we suppose that $(u_n)_n$ is not relatively compact and $(y_n)_n$ is the smallest sequence. So that applying Lemma 3.9 we deduce, for large $n \in \mathbb{N}$

$$\begin{aligned}\frac{1}{4}\int_{D_n} \frac{\partial a(x)}{\partial \mathbf{x}} u_n^2 dx \leq & \frac{1}{2}\int_{(\partial D_n)_i} (|\nabla u_n|^2 + a(x)u_n^2)(\nu_n \cdot \mathbf{y}_n) d\sigma \\ & + \int_{D_n} K(x)\Phi(u_n)u_n(\nabla u_n \cdot \mathbf{y}_n)dx - \int_{(\partial D_n)_i} (\nabla u_n \cdot \nu_n)(\nabla u_n \cdot \mathbf{y}_n) d\sigma \\ & - \frac{1}{p}\int_{(\partial D_n)_i} |u_n|^p (\nu_n \cdot \mathbf{y}_n) d\sigma. \end{aligned} \tag{3.12}$$

Since $(\nu_n \cdot \mathbf{y}_n) \leq 0$ on $(\partial D_n)_i$ and $a(x) \geq a_0 > 0$, it follows that

$$\int_{(\partial D_n)_i} (|\nabla u_n|^2 + a(x)u_n^2)(\nu_n \cdot \mathbf{y}_n) d\sigma \leq 0. \tag{3.13}$$

According to (K) and the fact $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, there holds

$$\int_{D_n} K(x)\Phi(u_n)u_n(\nabla u_n \cdot \mathbf{y}_n)dx \leq |K|_{\infty,D_n}|\Phi(u_n)|_6|u_n|_3|\nabla u_n|_2 \leq C_\beta e^{-\beta|y_n|}. \tag{3.14}$$

By the definition of $\tilde{\mathcal{C}}_n$, $(\partial D_n)_i$ and Proposition 3.6, we know

$$\Big|\int_{(\partial D_n)_i} (\nabla u_n \cdot \nu_n)(\nabla u_n \cdot \mathbf{y}_n) d\sigma\Big| \leq \int_{(\partial D_n)_i} |\nabla u_n|^2 d\sigma \leq \int_{\partial \tilde{\mathcal{C}}_n} |\nabla u_n|^2 d\sigma \leq C_0 e^{-\alpha_0|y_n|}. \tag{3.15}$$

Moreover we claim that there holds

$$\Big|\int_{(\partial D_n)_i} |u_n|^p (\nu_n \cdot \mathbf{y}_n) d\sigma\Big| \leq C_1 e^{-\alpha_1|y_n|} \tag{3.16}$$

for some positive constants C_1 and α_1. If the claim is true, the combination of (3.12)-(3.16) leads us to

$$\int_{D_n} \frac{\partial a(x)}{\partial \mathbf{x}} u_n^2 dx \leq ce^{-\alpha|y_n|}, \tag{3.17}$$

where $c > 0$ is a positive constant and $\alpha = \min(\beta, \alpha_0, \alpha_1)$.

But on the other hand, let $\delta_n = \frac{1}{2}\min_{0\leq i\neq j\leq m}(|y_n^i - y_n^j|, R_n)$, then we have

$$\int_{D_n} \frac{\partial a(x)}{\partial \mathbf{x}} u_n^2 dx \geq \int_{D_n \cap B_{\delta_n}(y_n)} \frac{\partial a(x)}{\partial \mathbf{x}} u_n^2 dx \geq \inf_{D_n \cap B_{\delta_n}(y_n)} \frac{\partial a(x)}{\partial \mathbf{x}} \int_{D_n \cap B_{\delta_n}(y_n)} u_n^2 dx.$$

Since $B_{\delta_n}(y_n) \subset \tilde{\mathcal{C}}_n$, we have

$$D_n \cap B_{\delta_n}(y_n) = (\tilde{\mathcal{C}}_n \cap B_{\rho_n}(0)) \cap B_{\delta_n}(y_n) = B_{\rho_n}(0) \cap B_{\delta_n}(y_n).$$

Taking this into the above inequality and using the fact $u_n \in H_0^1(B_{\rho_n}(0))$, we deduce

$$\begin{aligned}\int_{D_n} \frac{\partial a(x)}{\partial \mathbf{x}} u_n^2 dx &\geq \inf_{D_n \cap B_{\delta_n}(y_n)} \frac{\partial a(x)}{\partial \mathbf{x}} \int_{D_n \cap B_{\delta_n}(y_n)} u_n^2 dx \\ &\geq \inf_{B_{\delta_n}(y_n)} \frac{\partial a(x)}{\partial \mathbf{x}} \int_{B_{\rho_n}(0) \cap B_{\delta_n}(y_n)} u_n^2 dx \\ &= \inf_{B_{\delta_n}(y_n)} \frac{\partial a(x)}{\partial \mathbf{x}} \int_{B_{\delta_n}(y_n)} u_n^2 dx.\end{aligned} \tag{3.18}$$

Then, according to Proposition 2.2 and the definition of y_n and δ_n, we know

$$\liminf_{n\to+\infty} \int_{B_{\delta_n}(y_n)} u_n^2 dx \geq \delta > 0. \tag{3.19}$$

Thus, from (3.17), (3.18) and (3.19), we get for large n

$$\delta \inf_{B_{\delta_n}(y_n)} \frac{\partial a(x)}{\partial \mathbf{x}} \leq c e^{-\alpha |y_n|},$$

which is a contradiction to (A_2) since $|x| \leq 2|y_n|$ for $x \in B_{\delta_n}(y_n)$.

By now, the only we need to show is the claim (3.16). In fact, when n is sufficiently large we have $\partial \tilde{\mathcal{C}}_n \subset \mathcal{S}_n$. Taking account Proposition 3.4, we obtain

$$\begin{aligned}&\Big| \int_{(\partial D_n)_i} |u_n|^p (\nu_n \cdot \mathbf{y}_n) d\sigma \Big| \\ &\leq \int_{\partial \tilde{\mathcal{C}}_n} |u_n|^p d\sigma \leq C_\alpha \int_{\partial \tilde{\mathcal{C}}_n} e^{-p\alpha\sigma_n(x)} d\sigma \leq C_\alpha \sum_{i=1}^m \int_{\partial \tilde{\mathcal{C}}_n} e^{-p\alpha|x-y_n^i|} d\sigma.\end{aligned} \tag{3.20}$$

For $1 \leq k \in \mathbb{N}$ and $1 \leq i \leq m$, we define

$$A_{k,i} = \Big\{ x \in \partial \tilde{\mathcal{C}}_n : 2^{k-1}\frac{r_n}{2} < |x - y_n^i| \leq 2^k \frac{r_n}{2} \Big\}$$

and denote by $|A_{k,i}|$ the 2 dimensional measure of $A_{k,i}$. Then we have

$$|A_{k,i}| \leq C(2^k \frac{r_n}{2})^2, \quad 1 \leq i \leq m.$$

Consequently,

$$\begin{aligned}\int_{\partial \tilde{\mathcal{C}}_n} e^{-p\alpha|x-y_n^i|} d\sigma &\leq \sum_{k=1}^\infty \int_{A_{k,i}} e^{-p\alpha 2^{k-1}\frac{r_n}{2}} \\ &\leq C \sum_{k=1}^\infty e^{-p\alpha 2^{k-1}\frac{r_n}{2}} (2^k \frac{r_n}{2})^2 \\ &\leq c_\alpha r_n^2 e^{-p\alpha \frac{r_n}{2}} \sum_{k=0}^\infty e^{-p\alpha 2^k} 2^{2k} \\ &\leq c'_\alpha e^{-\alpha_1 |y_n|}.\end{aligned}$$

Taking this into (3.20), we can conclude our claim is true. □

4. Proof of The Main Result

In this section, we will show the existence of infinitely many solutions of (P).

Let $(\rho_n)_n$ be an increasing sequence such that $\rho_n \to +\infty$ as $n \to +\infty$ and we consider the problem $(\mathrm{P}_{B_{\rho_n}(0)})$

$$\begin{cases} -\Delta u + a(x)u + K(x)\Phi(u)u = |u|^{p-2}u, & \text{in } B_{\rho_n}(0), \\ u = 0, & \text{in } \mathbb{R}^3 \setminus B_{\rho_n}(0). \end{cases}$$

Firstly, we are going to prove the existence of infinitely many solutions of problem $(\mathrm{P}_{B_{\rho_n}(0)})$. It is clear that the corresponding functional of $(\mathrm{P}_{B_{\rho_n}(0)})$

$$I_n(u) = \frac{1}{2}\int_{B_{\rho_n}(0)} |\nabla u|^2 + a(x)u^2 dx + \frac{1}{4}\int_{B_{\rho_n}(0)} K(x)\Phi(u)u^2 dx - \frac{1}{p}\int_{B_{\rho_n}(0)} |u|^p dx$$

is even and of class C^2. Moreover, by setting

$$J_n(u) = \langle I_n'(u), u\rangle = \int_{B_{\rho_n}(0)} |\nabla u|^2 + a(x)u^2 + K(x)\Phi(u)u^2 dx - \int_{B_{\rho_n}(0)} |u|^p dx,$$

the manifold $\mathcal{N}_n$ defined by

$$\mathcal{N}_n = \{u \in H_0^1(B_{\rho_n}(0)) \setminus \{0\} : J_n(u) = 0\}$$

possesses the following properties.

Lemma 4.1. *Suppose* (A_1) *and* (K) *hold, then we have*

(1) $I_n(u) \geq \tau > 0$ *for* $u \in \mathcal{N}_n$;

(2) $\mathcal{N}_n$ *is a complete* $C^{1,1}$ *manifold;*

(3) $\mathcal{N}_n$ *is a natural constraint, which means any critical point of* $I_n|_{\mathcal{N}_n}$ *(*I_n *constrained on* $\mathcal{N}_n$*) is actually a free critical point of* I_n.

Proof. The proof of this lemma is easy and standard, so that we omit it here. □

Consequently, we only need to find critical point of $I_n|_{\mathcal{N}_n}$. In order to do this, we consider

$$\Gamma_k^n = \{A \subset \mathcal{N}_n : A \text{ is compact}, A = -A, \gamma(A) \geq k\},$$

where $\gamma(A)$ denotes the Krasnoselskii genus of A. Then according to Lemma 4.1 and the well known minimax principle (see [25]), we can conclude that

$$c_k^n = \inf_{A\in\Gamma_k^n} \sup_{u\in A} I_n(u)$$

are critical values of $I_n|_{\mathcal{N}_n}$ and hence are critical values of I_n. Since ρ_n is increasing, we know $\Gamma_k^{n+1} \subset \Gamma_k^n$ which implies

$$c_k^n \geq c_k^{n+1} \geq ... \geq \tau > 0.$$

And then we have

Proposition 4.1. *Under conditions* (A_1) *and* (K), *let* $c_k = \lim_{n\to+\infty} c_k^n$, *then* $c_k \to +\infty$ *as* $k \to +\infty$.

In proving this proposition, we need to use a Morse index argument.

Definition 4.1. We call the augmented Morse index of a critical point u for a functional J, the number (possibly $+\infty$) of eigenvalues of $J''(u)$ less or equal than zero.

Easily to see, under condition (A_1) and (K), the functionals I and I_n are of class C^2. So that the augmented Morse index for I and I_n are well defined, moreover we have:

Lemma 4.2. *Under conditions* (A_1) *and* (K), *for any* $u \in H^1(\mathbb{R}^3)$ *which is a critical point of* I, *the augmented Morse index of* u *for* I *is finite.*

Proof. Arguing indirectly, we assume $I''(u)$ possesses infinitely many eigenfunctions v_n, $n \in \mathbb{N}$ such that $\|v_n\| = 1$, $(v_n, v_m) = 0$ for $n \neq m$ and

$$\langle I''(u)v_n, v_n\rangle \leq 0, \quad n \in \mathbb{N}. \tag{4.1}$$

On the other hand, by direct computation we have

$$\begin{aligned}\langle I''(u)v_n, v_n\rangle = &\int_{\mathbb{R}^3} |\nabla v_n|^2 + a(x)v_n^2 dx - (p-1)\int_{\mathbb{R}^3} |u|^{p-2}v_n^2 dx + \int_{\mathbb{R}^3} K(x)\Phi(u)v_n^2 \\ &+ \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)u(y)v_n(y)}{|x-y|}u(x)v_n(x)dydx. \end{aligned}\tag{4.2}$$

Since there holds (K), we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^3} \frac{K(y)}{|x-y|}u(y)v_n(y)dy\Big| \\ \leq &|K|_\infty\Big(\int_{B_1(x)} \frac{u(y)v_n(y)}{|x-y|}dy + \int_{B_1^c(x)} \frac{u(y)v_n(y)}{|x-y|}dy\Big) \\ \leq &|K|_\infty\Big((\int_{B_1(x)} \frac{1}{|x-y|^2}dy)^{1/2}(\int_{B_1(x)} u^4 dy)^{1/4}(\int_{B_1(x)} v_n^4 dy)^{1/4} \\ &+ (\int_{B_1^c(x)} \frac{1}{|x-y|^4}dy)^{1/4}(\int_{B_1^c(x)} |u|^{8/3}dy)^{3/8}(\int_{B_1^c(x)} |v_n|^{8/3}dy)^{3/8}\Big) \\ \leq &c|K|_\infty\|u\|.\end{aligned}$$

Consequently, noting that $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$,

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\Phi(u)v_n^2 dx + \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)u(y)v_n(y)}{|x-y|}u(x)v_n(x)dydx \\ &-(p-1)\int_{\mathbb{R}^3} |u|^{p-2}v_n^2 dx \to 0\end{aligned}$$

as $n \to +\infty$. Taking this into (4.2) and using (A_1)

$$\langle I''(u)v_n, v_n\rangle = \int_{\mathbb{R}^3} |\nabla v_n|^2 + a(x)v_n^2 dx + o(1) \geq \min(1, a_0) + o(1).$$

Thus, $\langle I''(u)v_n, v_n\rangle > 0$ for n large enough, which contradicts to (4.1). □

And now, we can give the proof of Proposition 4.2.

Proof. [Proof of Proposition 4.2] Arguing by contradiction, we assume that $c_k \to c_0 < +\infty$ as $k \to +\infty$. Then there exists a k_0 such that for any $k \geq k_0$ we can find $n_k > 0$ satisfying

$$c_k^n < c_0 + 1, \quad \text{for} \quad n \geq n_k.$$

Then according to [6] Chapter 2 Theorem 2.3, there exist critical point $w_k \in H_0^1(B_{\rho_{n_k}}(0))$ of I_{n_k} such that

$$I_{n_k}(w_k) = c_k^{n_k}$$

and

$$i_{I_{n_k}}(w_k) \geq k - 1 \tag{4.3}$$

for $k \geq k_0$, where $i_{I_{n_k}}(w_k)$ denote the augmented Morse index of w_k.

Thus according to the construction, we have

$$c_0 + 1 > c_k^{n_k} = I_{n_k}(w_k) - \frac{1}{p}\langle I'_{n_k}(w_k), w_k\rangle \geq \min(1, a_0)\|w_k\|^2,$$

which implies that $(w_k)_k$ is a bounded sequence in $H^1(\mathbb{R}^3)$. Then using Proposition 3.10, we deduce that $w_k \to w$ in $H^1(\mathbb{R}^3)$. Obviously, w is a critical point of I. Due to Lemma 4.4, $i_I(w)$ (the augmented Morse index of w as a critical point of I) is well defined and finite. Consequently, there exist a finite dimensional subspace M of $H^1(\mathbb{R}^3)$ and a positive constant $\delta > 0$ such that

$$\langle I''(w)v, v\rangle \geq \delta\|v\|^2, \quad v \in M^{\perp}. \tag{4.4}$$

On the other hand, when k sufficiently large, we can always find $v_k \in M^{\perp}$ (see (4.3)) such that $\|v_k\| = 1$,

$$\langle I''_{n_k}(w_k)v_k, v_k\rangle \leq 0. \tag{4.5}$$

Since $w_k \to w$ and I is of class C^2, we have for k sufficiently large,

$$\|I''(w_k) - I''(w)\| < \frac{\delta}{2}. \tag{4.6}$$

Thus the combination of (4.4), (4.5) and (4.6) lead us to

$$\begin{aligned}
\delta = \delta\|v_k\|^2 \leq \langle I''(w)v_k, v_k\rangle &= \langle (I''(w) - I''_{n_k}(w_k))v_k, v_k\rangle + \langle I''_{n_k}(w_k)v_k, v_k\rangle \\
&\leq \|I''(w) - I''_{n_k}(w_k)\|\|v_k\|^2 \\
&= \|I''(w) - I''(w_k)\| \\
&< \frac{\delta}{2},
\end{aligned}$$

which is a contradiction. So that we get the conclusion. □

By now, we are ready to finish the proof of the main result:

Proof of Theorem 1.1. According to the argument after Lemma 4.1, we know c_k^n are critical values of I_n for $k \in \mathbb{N}$ and $n \in \mathbb{N}$. So that there exist critical points u_k^n of I_n such that

$$I_n(u_k^n) = c_k^n; \quad \langle I'_n(u_k^n), u_k^n\rangle = 0,$$

which implies for any $k \in \mathbb{N}$, $(u_k^n)_n$ is a bounded sequence in $H^1(\mathbb{R}^3)$. Then according to Proposition 3.10, $(u_k^n)_n$ is relatively compact and (up to a subsequence) converges strongly in $H^1(\mathbb{R}^3)$ to some u_k which satisfies

$$I(u_k) = \lim_{n \to +\infty} c_k^n = c_k \quad \text{and} \quad I'(u_k) = 0.$$

By proposition 4.2, we have $c_k \to +\infty$ which leads us to the conclusion. □

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