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OSCILLATION OF SECOND ORDER NONLINEAR DYNAMIC EQUATIONS WITH A NONLINEAR NEUTRAL TERM ON TIME SCALES*

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Abstract In this article, we consider the oscillation of second order nonlinear dynamic equations with a nonlinear neutral term on time scales. Some new sufficient conditions which insure that any solution of the equation oscillates are established by means of an inequality technique and Riccati transformation. This paper improves and generalizes some known results. Several illustrative examples are given throughout.

Keywords Oscillation, neutral, time scales, dynamic equation.

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1. Introduction

The theory of the calculus on time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis in 1988 [5] in order to unify continuous and discrete analysis. In recent years, there has been an increasing interest in obtaining sufficient conditions for oscillatory behavior of different classes of dynamic equations on time scales, we refer the reader to [2,7,8] and the references cited therein.

The neutral functional differential equation arises in the design of high-speed computer lossless transmission lines. It also finds wide applications in certain high-tech fields, such as control, communication, mechanical engineering, biomedicine, physics, mechanics, economics and so on. Also the neutral dynamic equation having a nonlinearity in the neutral term arises in many applications.

B. Baculikova et al. [3] studied the oscillation of the second-order nonlinear neutral differential equation

$$(a(t)[z'(t)]^{\gamma})'(t) + q(t)x^{\beta}(\delta(t))) = 0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$, and two oscillation criteria were presented.

R. P. Agarwal et al. [1] studied the oscillation of the second-order differential equation with a sublinear neutral term

$$(r(t)(x(t) + p(t)x^{\alpha}(\tau(t)))')' + q(t)x(\sigma(t)) = 0, \ t \geq t_0,$$

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and derived some oscillation results.

S. H. Saker [9] studied the oscillation of the second-order nonlinear neutral delay dynamic equation

$$(r(t)([y(t)+p(t)y(t-\tau)]^{\Delta})^{\gamma})^{\Delta}+f(t,y(t-\delta))=0$$

on time scales and presented some necessary and sufficient conditions for oscillation.

Motivated by the above articles, in this article, we consider the following nonlinear dynamic equation

$$(a(t)(x(t)+p(t)x^{\alpha}(m(t)))^{\Delta})^{\Delta}+q(t)f(x(r(t)))=0,\ \ t\in I, \tag{1.1}$$

where $I=[t_0,\infty)_{\mathbb{T}}$ and $0<\alpha\leq 1$ is the ratio of two odd positive integers, and $\sigma(t)$ is a jump operator on time scales, $\sigma(t)\geq t$. Assume that the following conditions are satisfied:

(H1) $a\in C^1_{rd}(I,(0,\infty)_{\mathbb{T}})$, $p\in C_{rd}(I,[0,\infty)_{\mathbb{T}})$, $q\in C_{rd}(I,(0,\infty)_{\mathbb{T}})$, for all $t\in I$;

(H2) $m,r\in C^1_{rd}(I,\mathbb{R})$, $m(t)\leq t$, $r(t)\leq t$, $r^{\Delta}(t)>0$ and $\lim\limits_{t\to\infty}m(t)=\lim\limits_{t\to\infty}r(t)=\infty$;

(H3) $f\in C(\mathbb{R},\mathbb{R})$ such that $xf(x)>0$ and $f(x)/x^{\beta}\geq k>0$, for $x\neq 0$, $k>0$ is a constant, β is the ratio of two odd positive integers.

A function x is called a solution of (1.1), and $x\in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, $T_x\geq t_0$, which has the property $a(t)(x(t)+p(t)x^{\alpha}(m(t)))\in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)_{\mathbb{T}}$. Let $\mathbb{T}$ be a time scale with $\sup\mathbb{T}=\infty$. We restrict our attention to those solutions of (1.1) which exist on $I=[t_0,\infty)_{\mathbb{T}}$ and satisfy the condition

$$\sup\{|x(t)|:T\leq t\}>0\ \ \text{for}\ \ T\geq T_x.$$

As usual, a solution x of (1.1) is said to be oscillatory, if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. The equation (1.1) is called oscillatory, if all its solutions are oscillatory. Otherwise, it is called nonoscillatory.

Throughout this paper we assume that

$$\int^{\infty}\frac{1}{a(t)}\Delta t<\infty. \tag{1.2}$$

And we investigate the oscillatory behavior of (1.1) under the condition (1.2). By using a Riccati transformation and an inequality technique, we present some new sufficient conditions which ensure that any solution of (1.1) oscillates. Specifically, we study neutral dynamic equation and the constants α and β are independent with each other.

2. Preliminaries

In this section, we will present some necessary background. Without loss of generality, we can only deal with the positive solutions of equation (1.1) since the proof of the other case is similar. For the sake of simplicity, we define

$$z(t)=x(t)+p(t)x^{\alpha}(m(t)),\quad A(t)=\int_t^{\infty}\frac{1}{a(s)}\Delta s\ \ \text{and}\ \ R(t)=\int_{t_0}^{t}\frac{1}{a(s)}\Delta s.$$

Next, we state and prove the following lemmas.

Lemma 2.1. *If $x(t)$ is an eventually positive solution of (1.1), then one of the following two cases holds for all sufficiently large t:*

$$(1)\ z(t) > 0,\ z^{\Delta}(t) > 0,\ (a(t)z^{\Delta}(t))^{\Delta} < 0, \tag{2.1}$$

$$(2)\ z(t) > 0,\ z^{\Delta}(t) < 0,\ (a(t)z^{\Delta}(t))^{\Delta} < 0. \tag{2.2}$$

Proof. From the definition of $z(t)$, (H1) and (H3), we have $z(t) > 0$. By (1.1), then

$$(a(t)z^{\Delta}(t))^{\Delta} = -q(t)f(x(r(t))) \leq -kq(t)x^{\beta}(r(t)) < 0 \tag{2.3}$$

for all sufficiently large t. Thus $a(t)z^{\Delta}(t)$ is decreasing, which implies $a(t)z^{\Delta}(t)$ does not change sign eventually, so as $z^{\Delta}(t)$, then there exists a $t_1 \geq t_0$ such that either $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$ for any $t \geq t_1$. This completes the proof. □

Lemma 2.2. *Suppose that (2.1) of Lemma 2.1 holds. If $x(t)$ is an eventually positive solution of equation (1.1), then there exists a $T \geq t_1$ such that*

$$x(t) \geq (1 - p_1(t))z(t), \tag{2.4}$$

where $p_1(t) = p(t)A^{\alpha-1}(t)$, $t \geq T$.

Proof. By the definition of $z(t)$ and (H1), we get $z(t) \geq x(t)$ for all $t \geq t_1 \geq t_0$. Since $m(t) \leq t$, then

$$x(t) = z(t) - p(t)x^{\alpha}(m(t)) \geq z(t) - p(t)z^{\alpha}(m(t)) \geq z(t) - p(t)z^{\alpha}(t),\ t \geq t_1. \tag{2.5}$$

From (2.1), we see that $z(t)$ is positive and increasing, since $A(t)$ is positive, decreasing and tends to zero as $t \to \infty$, there exists a $T \geq t_1$ such that

$$z(t) \geq A(t),\ t \geq T \geq t_1 \geq t_0. \tag{2.6}$$

By (2.5) and (2.6), we have

$$x(t) \geq z(t)\left(1 - p(t)A^{\alpha-1}(t)\right),\ t \geq T.$$

This completes the proof. □

Lemma 2.3. *Suppose that (2.2) of Lemma 2.1 holds. If $x(t)$ is an eventually positive solution of equation (1.1), then there exists a $T \geq t_1$ such that*

$$x(t) \geq (1 - p_2(t))z(t), \tag{2.7}$$

where $p_2(t) = p(t)A^{\alpha}(m(t))A^{\alpha-2}(t)$, $t \geq T$.

Proof. Since (2.3), we have $(a(t)z^{\Delta}(t))^{\Delta} < 0$, then

$$z^{\Delta}(s) \leq \frac{a(t)}{a(s)}z^{\Delta}(t), \quad s \geq t \geq t_1. \tag{2.8}$$

Integrating (2.8) from t to l, then

$$z(l) - z(t) \leq a(t)z^{\Delta}(t)\int_t^l \frac{\Delta s}{a(s)}.$$

Letting $l \to \infty$, we obtain

$$\frac{a(t)z^{\Delta}(t)}{z(t)}A(t) \geq -1, \ t \geq t_1, \tag{2.9}$$

thus

$$\frac{z(t)}{A(t)} \geq -a(t)z^{\Delta}(t) > 0, \ t \geq t_1, \tag{2.10}$$

i.e.,

$$z^{\Delta}(t)A(t) \geq -a^{-1}(t)z(t), \ t \geq t_1. \tag{2.11}$$

By (2.11), we have

$$\left(\frac{z(t)}{A(t)}\right)^{\Delta} = \frac{z^{\Delta}(t)A(t) - z(t)A^{\Delta}(t)}{A(\sigma(t))A(t)} = \frac{z^{\Delta}(t)A(t) + z(t)a^{-1}(t)}{A(\sigma(t))A(t)} \geq 0,$$

which means that $\frac{z(t)}{A(t)}$ is nondecreasing. Therefore, $\frac{z(t)}{A(t)} \geq \frac{z(m(t))}{A(m(t))}$, i.e.,

$$\frac{A(m(t))}{A(t)}z(t) \geq z(m(t)).$$

What's more, we get

$$\begin{aligned} x(t) &\geq z(t) - p(t)z^{\alpha}(m(t)) \geq z(t) - p(t)z^{\alpha}(t)\frac{A^{\alpha}(m(t))}{A^{\alpha}(t)} \\ &= z(t)\left(1 - p(t)z^{\alpha-1}(t)\frac{A^{\alpha}(m(t))}{A^{\alpha}(t)}\right), \ t \geq t_1. \end{aligned} \tag{2.12}$$

By (2.10) and $-a(t)z^{\Delta}(t)$ is positive and increasing, $A(t)$ is positive, decreasing and tends to zero as $t \to \infty$, there exists a $T \geq t_1$ such that

$$\frac{z(t)}{A(t)} \geq -a(t)z^{\Delta}(t) \geq A(t), \ t \geq T \geq t_1. \tag{2.13}$$

Combining (2.13) with (2.12), we have

$$x(t) \geq z(t)\left(1 - p(t)z^{\alpha-1}(t)\frac{A^{\alpha}(m(t))}{A^{\alpha}(t)}\right) \geq z(t)\left(1 - p(t)A^{\alpha}(m(t))A^{\alpha-2}(t)\right), \ t \geq T. \tag{2.14}$$

This completes the proof. □

3. Oscillation Results

We are now in a position to state and prove our main results in this paper.

Theorem 3.1. *Assume that (1.2) holds, $\beta \geq 1$ and $\max\{p_1(r(t)), p_2(r(t))\} < 1$, $t \in I$, where $p_1(t)$ and $p_2(t)$ are defined as in Lemma 2.1 and Lemma 2.2. Suppose that the condition $r(\sigma(t)) = \sigma(r(t))$ holds. If there exists a positive, nondecreasing and Δ-differentiable function $b(t)$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[k(1 - p_1(r(\sigma(s))))^{\beta} b(s)q(\sigma(s)) - \frac{a(r(s))(b^{\Delta}(s))^2}{4\beta r^{\Delta}(s)b(s)M^{\beta-1}}\right] \Delta s = \infty, \tag{3.1}$$

or

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[k(1-p_2(r(s)))^{\beta}A^{\beta}(s)q(s)+\frac{\beta K^{\beta-1}}{L^{2\beta-2}A(s)a(s)}\right]\Delta s=\infty, \tag{3.2}$$

where $K>0$ and $M>0$ are any positive constants, then every solution of (1.1) is oscillatory.

Proof. By contradiction, suppose that (1.1) is nonoscillatory and $x(t)$ is a nonoscillatory solution for (1.1). Without loss of generalization, assume that $x(t)$ is eventually positive, and there exists a $t_1\ge t_0$ such that $x(t)>0$, $x(m(t))>0$ and $x(r(t))>0$ for any $t\ge t_1$. By the definition of $z(t)$, we have $z(t)>0$, $t\ge T\ge t_1$, where T is chosen so that (2.1) and (2.2) of Lemma 2.1 hold for all $t\ge T$. We shall show that in each case we are led to a contradiction.

Case (1): Suppose that (2.1) of Lemma 2.1 holds, by (2.3) and (2.4), we obtain

$$(a(t)z^{\Delta}(t))^{\Delta}+kq(t)(1-p_1(r(t)))^{\beta}z^{\beta}(r(t))<0,\ \text{for all } t\ge T. \tag{3.3}$$

Define the following Riccati transformation:

$$w(t)=b(t)\frac{a(\sigma(t))z^{\Delta}(\sigma(t))}{z^{\beta}(r(t))},\ t\ge T. \tag{3.4}$$

Then $w(t)\ge 0$ for $t\ge T$, and

$$\begin{aligned} w^{\Delta}(t)&=\left[b(t)\frac{a(\sigma(t))z^{\Delta}(\sigma(t))}{z^{\beta}(r(t))}\right]^{\Delta}\\ &=b^{\Delta}(t)\frac{a(\sigma(\sigma(t)))z^{\Delta}(\sigma(\sigma(t)))}{z^{\beta}(r(\sigma(t)))}+b(t)\left[\frac{a(\sigma(t))z^{\Delta}(\sigma(t))}{z^{\beta}(r(t))}\right]^{\Delta}\\ &=\frac{b^{\Delta}(t)}{b(\sigma(t))}w(\sigma(t))+b(t)\frac{[a(\sigma(t))z^{\Delta}(\sigma(t))]^{\Delta}}{z^{\beta}(r(\sigma(t)))}-b(t)\frac{a(\sigma(t))z^{\Delta}(\sigma(t))[z^{\beta}(r(t))]^{\Delta}}{z^{\beta}(r(t))z^{\beta}(r(\sigma(t)))}. \end{aligned} \tag{3.5}$$

By the corollary of the Keller chain rule [4] and the condition $r(\sigma(t))=\sigma(r(t))$ ([6] Lemma 2.2), for $\beta\ge 1$, we have

$$\begin{aligned} (z^{\beta}(r(t)))^{\Delta}&=\beta\int_0^1\left[hz(r(\sigma(t)))+(1-h)z(r(t))\right]^{\beta-1}(z(r(t)))^{\Delta}dh\\ &\ge\beta\int_0^1\left[hz(r(t))+(1-h)z(r(t))\right]^{\beta-1}(z(r(t)))^{\Delta}dh\\ &=\beta z^{\beta-1}(r(t))z^{\Delta}(r(t))r^{\Delta}(t), \end{aligned}$$

then

$$\begin{aligned} &-b(t)\frac{a(\sigma(t))z^{\Delta}(\sigma(t))[z^{\beta}(r(t))]^{\Delta}}{z^{\beta}(r(t))z^{\beta}(r(\sigma(t)))}\\ \le&-b(t)\frac{a(\sigma(t))z^{\Delta}(\sigma(t))\beta z^{\beta-1}(r(t))z^{\Delta}(r(t))r^{\Delta}(t)}{z^{\beta}(r(t))z^{\beta}(r(\sigma(t)))}\\ =&-b(t)\frac{a(\sigma(t))z^{\Delta}(\sigma(t))\beta z^{\Delta}(r(t))r^{\Delta}(t)}{z(r(t))z^{\beta}(r(\sigma(t)))}\\ \le&-\frac{b(t)}{b(\sigma(t))}\frac{b(\sigma(t))a(\sigma(\sigma(t)))z^{\Delta}(\sigma(\sigma(t)))\beta z^{\Delta}(r(t))r^{\Delta}(t)}{z^{\beta}(r(\sigma(t)))z(r(t))}\\ =&-\frac{b(t)}{b(\sigma(t))}\frac{\beta w(\sigma(t))z^{\Delta}(r(t))r^{\Delta}(t)}{z(r(t))}, \end{aligned} \tag{3.6}$$

thus, using (3.6) and (3.3) in (3.5), we get

$$\begin{aligned}
&w^{\Delta}(t)\\
\leq&\frac{b^{\Delta}(t)}{b(\sigma(t))}w(\sigma(t))+b(t)\frac{[a(\sigma(t))z^{\Delta}(\sigma(t))]^{\Delta}}{z^{\beta}(r(\sigma(t)))}-\frac{b(t)}{b(\sigma(t))}\frac{\beta w(\sigma(t))z^{\Delta}(r(t))r^{\Delta}(t)}{z(r(t))}\\
<&\frac{b^{\Delta}(t)}{b(\sigma(t))}w(\sigma(t))+b(t)\frac{-kq(\sigma(t))(1-p_1(r(\sigma(t))))^{\beta}z^{\beta}(r(\sigma(t)))}{z^{\beta}(r(\sigma(t)))}\\
&-\frac{b(t)}{b(\sigma(t))}\frac{\beta w(\sigma(t))z^{\Delta}(r(t))r^{\Delta}(t)}{z(r(t))}\\
=&\frac{b^{\Delta}(t)}{b(\sigma(t))}w(\sigma(t))-kq(\sigma(t))b(t)(1-p_1(r(\sigma(t))))^{\beta}-\frac{b(t)}{b(\sigma(t))}\frac{\beta w(\sigma(t))z^{\Delta}(r(t))r^{\Delta}(t)}{z(r(t))}.
\end{aligned} \tag{3.7}$$

Because $z(t)$ is increasing, and there exists a constant $M>0$ such that $z(t)\geq M$, $t\geq T$. Thus, by (3.7), we have

$$\begin{aligned}
&w^{\Delta}(t)\\
<&\frac{b^{\Delta}(t)}{b(\sigma(t))}w(\sigma(t))-kq(\sigma(t))b(t)(1-p_1(r(\sigma(t))))^{\beta}-\frac{\beta w^2(\sigma(t))b(t)r^{\Delta}(t)M^{\beta-1}}{b^2(\sigma(t))a(r(t))}.
\end{aligned} \tag{3.8}$$

Then by completing the square in (3.8), we get

$$\begin{aligned}
w^{\Delta}(t)<&-kq(\sigma(t))b(t)(1-p_1(r(\sigma(t))))^{\beta}\\
&-\left(\frac{\beta b(t)r^{\Delta}(t)M^{\beta-1}}{b^2(\sigma(t))a(r(t))}w^2(\sigma(t))-\frac{b^{\Delta}(t)}{b(\sigma(t))}w(\sigma(t))\right)\\
=&-kq(\sigma(t))b(t)(1-p_1(r(\sigma(t))))^{\beta}+\frac{a(r(t))(b^{\Delta}(t))^2}{4\beta b(t)r^{\Delta}(t)M^{\beta-1}}\\
&-\left(\sqrt{\frac{\beta b(t)r^{\Delta}(t)M^{\beta-1}}{b^2(\sigma(t))a(r(t))}}w(\sigma(t))-\sqrt{\frac{a(r(t))(b^{\Delta}(t))^2}{4\beta b(t)r^{\Delta}(t)M^{\beta-1}}}\right)^2\\
\leq&-kq(\sigma(t))b(t)(1-p_1(r(\sigma(t))))^{\beta}+\frac{a(r(t))(b^{\Delta}(t))^2}{4\beta b(t)r^{\Delta}(t)M^{\beta-1}},\ t\geq T.
\end{aligned} \tag{3.9}$$

Integrating (3.9) from T to t, we obtain

$$\int_T^t\left[kq(\sigma(s))b(s)(1-p_1(r(\sigma(s))))^{\beta}-\frac{a(r(s))(b^{\Delta}(s))^2}{4\beta b(s)r^{\Delta}(s)M^{\beta-1}}\right]\Delta s<w(T),$$

which contradicts with (3.1) as $t\to\infty$.

Case (2): Suppose that (2.2) of Lemma 2.1 holds, by (2.3) and (2.7), we obtain

$$(a(t)z^{\Delta}(t))^{\Delta}+kq(t)(1-p_2(r(t)))^{\beta}z^{\beta}(r(t))<0,\ \text{for all } t\geq T. \tag{3.10}$$

Define the following Riccati transformation:

$$u(t)=\frac{a(t)z^{\Delta}(t)}{z^{\beta}(t)},\ t\geq T. \tag{3.11}$$

Then $u(t)\leq 0$, $t\geq T$. By (2.9), we have

$$\frac{-a(t)z^{\Delta}(t)(-a(t)z^{\Delta}(t))^{\beta-1}A^{\beta}(t)}{z^{\beta}(t)}\leq 1,\ t\geq T. \tag{3.12}$$

So by $-a(t)z^{\Delta}(t) > 0$ and (3.11), we obtain

$$-\frac{1}{L^{\beta-1}} \leq u(t)A^{\beta}(t) \leq 0, \tag{3.13}$$

where $L = -a(T)z^{\Delta}(T)$. Differentiating (3.11), we get

$$\begin{aligned} u^{\Delta}(t) &= \left[\frac{a(t)z^{\Delta}(t)}{z^{\beta}(t)}\right]^{\Delta} = \frac{[a(t)z^{\Delta}(t)]^{\Delta}z^{\beta}(t) - a(t)z^{\Delta}(t)[z^{\beta}(t)]^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))} \\ &= \frac{[a(t)z^{\Delta}(t)]^{\Delta}}{z^{\beta}(\sigma(t))} - \frac{a(t)z^{\Delta}(t)[z^{\beta}(t)]^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}. \end{aligned} \tag{3.14}$$

By the corollary of the Keller chain rule [4] and $\beta \geq 1$ we have

$$\begin{aligned} (z^{\beta}(t))^{\Delta} &= \beta \int_0^1 [hz(\sigma(t)) + (1-h)z(t)]^{\beta-1} z^{\Delta}(t)dh \\ &\leq \beta \int_0^1 [hz(\sigma(t)) + (1-h)z(\sigma(t))]^{\beta-1} z^{\Delta}(t)dh = \beta z^{\beta-1}(\sigma(t))z^{\Delta}(t), \end{aligned} \tag{3.15}$$

using (3.15) and (3.10) in (3.14),

$$\begin{aligned} u^{\Delta}(t) &\leq \frac{[a(t)z^{\Delta}(t)]^{\Delta}}{z^{\beta}(\sigma(t))} - \frac{\beta a(t)(z^{\Delta}(t))^2}{z^{\beta}(t)z(\sigma(t))} \\ &< \frac{-kq(t)(1-p_2(r(t)))^{\beta}z^{\beta}(r(t))}{z^{\beta}(\sigma(t))} - \frac{\beta a(t)(z^{\Delta}(t))^2}{z^{\beta+1}(t)} \\ &\leq \frac{-kq(t)(1-p_2(r(t)))^{\beta}z^{\beta}(\sigma(t))}{z^{\beta}(\sigma(t))} - \frac{\beta u^2(t)}{a(t)}z^{\beta-1}(t) \\ &= -kq(t)(1-p_2(r(t)))^{\beta} - \frac{\beta u^2(t)}{a(t)}z^{\beta-1}(t). \end{aligned} \tag{3.16}$$

By (2.10) and $-a(t)z^{\Delta}(t)$ is positive and increasing, there exists a constant $K > 0$ such that

$$\frac{z(t)}{A(t)} \geq -a(t)z^{\Delta}(t) \geq K, \ t \geq T. \tag{3.17}$$

Using (3.17) in (3.16), we obtain

$$u^{\Delta}(t) < -kq(t)(1-p_2(r(t)))^{\beta} - \frac{\beta K^{\beta-1}(t)A^{\beta-1}(t)}{a(t)}u^2(t), \ t \geq T. \tag{3.18}$$

Multiplying (3.18) by $A^{\beta}(t)$ and integrating the resulting inequality from T to t, we have

$$\begin{aligned} &A^{\beta}(t)u(t) - A^{\beta}(T)u(T) + \int_T^t kq(s)(1-p_2(r(s)))^{\beta}A^{\beta}(s)\Delta s \\ &+ \int_T^t \frac{\beta K^{\beta-1}A^{2\beta-1}(s)}{a(s)}u^2(s)\Delta s < 0. \end{aligned}$$

By using (3.13), we get

$$\int_T^t \left[kq(s)(1-p_2(r(s)))^{\beta}A^{\beta}(s) + \frac{\beta K^{\beta-1}}{L^{2\beta-2}A(s)a(s)}\right]\Delta s < A^{\beta}(T)u(T) + \frac{1}{L^{\beta-1}}.$$

This contradicts (3.2) and the proof is complete. $\square$

Example 3.1. As an illustrative example, we consider the following equation

$$\left[t^2\left(x(t)+\frac{1}{4}x(\frac{t}{2})\right)'\right]'+4x(\frac{t}{3})=0,\ t\geq 1. \tag{3.19}$$

Here $\mathbb{T}=\mathbb{R}^+$, and $a(t)=t^2$, $p(t)=\frac{1}{4}$, $q(t)=4$, $m(t)=\frac{t}{2}$, $f(t)=t$, $r(t)=\frac{t}{3}$, $\alpha=1$. Let $\beta=1$, $k=1$. Then $A(t)=\frac{1}{t}$, $p_1(r(t))=\frac{1}{4}$, $p_2(r(t))=\frac{1}{2}$, and $\int_1^\infty \frac{1}{t^2}dt=1<\infty$. By taking $b(t)=1$, we get

$$\begin{aligned}&\limsup_{t\to\infty}\int_1^t\left[k(1-p_1(r(\sigma(s))))^\beta b(s)q(\sigma(s))-\frac{a(r(s))(b'(s))^2}{4\beta r'(s)b(s)M^{\beta-1}}\right]ds\\ =&\limsup_{t\to\infty}\int_1^t 3ds=\infty,\end{aligned}$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\int_1^t\left[k(1-p_2(r(s)))^\beta A^\beta(s)q(s)+\frac{\beta K^{\beta-1}}{L^{2\beta-2}A(s)a(s)}\right]ds\\ =&\limsup_{t\to\infty}\int_1^t\frac{1}{s}ds=\infty.\end{aligned}$$

It is easy to verify that all conditions of Theorem 3.1 are satisfied, thus we conclude that every solution of the equation (3.19) is oscillatory.

Theorem 3.2. *Assume that (1.2) holds,* $0<\beta<1$ *and* $\max\{p_1(r(t)),p_2(r(t))\}<1$, $t\in I$, *where* $p_1(t)$ *and* $p_2(t)$ *are defined as in Lemma 2.1 and Lemma 2.2. If*

$$\int_{t_0}^\infty kq(s)(1-p_1(r(s)))^\beta R^\beta(r(s))\Delta s=\infty, \tag{3.20}$$

or

$$\limsup_{t\to\infty}\int_{t_0}^t\left[kK^{\beta-1}(1-p_2(r(s)))^\beta A(s)q(s)+\frac{1}{A(s)a(s)}\right]\Delta s=\infty, \tag{3.21}$$

where $K>0$ *is any positive constant, then every solution of (1.1) is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1, we see that two cases of Lemma 2.1 hold.

Case (1): Suppose that (2.1) holds. Proceeding as in the proof of Case(1) in Theorem 3.1, we arrive at (3.3). From (2.3), we have $(a(t)z^\Delta(t))^\Delta<0$, then there exists a $T\in[t_0,\infty)_{\mathbb{T}}$, and $s,t\in[T,\infty)_{\mathbb{T}}$ with $t\geq s$, such that $z^\Delta(s)\geq\frac{a(t)}{a(s)}z^\Delta(t)$. Integrating the inequality from t_0 to t, we get

$$z(t)\geq R(t)a(t)z^\Delta(t),\ t\geq T\geq t_1. \tag{3.22}$$

Using (3.22) in (3.3) we get

$$(a(t)z^\Delta(t))^\Delta+kq(t)(1-p_1(r(t)))^\beta R^\beta(r(t))\left(a(r(t))z^\Delta(r(t))\right)^\beta<0,\ t\geq T. \tag{3.23}$$

Set $w(t)=a(t)z^\Delta(t)>0$. Then by (3.23) we see that $w(t)$ is a positive solution of the following inequality

$$w^\Delta(t)+kq(t)(1-p_1(r(t)))^\beta R^\beta(r(t))w^\beta(r(t))<0,\ t\geq T. \tag{3.24}$$

Because $r(t) \leq t$ and $0 < \beta < 1$, then integrating (3.24) from t to ∞, and by condition (3.20), we get

$$\lim_{t\to\infty} w^{1-\beta}(t) = -\infty,$$

which contradicts with positivity of $w(t)$. Then the inequality (3.24) has no positive solution, which is a contradiction.

Case (2): Suppose that (2.2) of Lemma 2.1 holds. Proceeding as in the proof of Case(2) of Theorem 3.1, we get (3.10). Define the following Riccati transformation:

$$u(t) = \frac{a(t)z^{\Delta}(t)}{z(t)}, \ t \geq T. \tag{3.25}$$

Then $u(t) \leq 0$, $t \geq T$. Differentiating (3.25), and by (3.10) and (3.25), we get

$$\begin{aligned}
u^{\Delta}(t) &= \left[\frac{a(t)z^{\Delta}(t)}{z(t)}\right]^{\Delta} = \frac{(a(t)z^{\Delta}(t))^{\Delta}z(t) - a(t)z^{\Delta}(t)z^{\Delta}(t)}{z(t)z(\sigma(t))}\\
&= \frac{(a(t)z^{\Delta}(t))^{\Delta}}{z(\sigma(t))} - \frac{a(t)(z^{\Delta}(t))^2}{z(t)z(\sigma(t))}\\
&< \frac{-kq(t)(1-p_2(r(t)))^{\beta}z^{\beta}(r(t))}{z(\sigma(t))} - \frac{a(t)(z^{\Delta}(t))^2}{z^2(t)}\\
&\leq -kq(t)(1-p_2(r(t)))^{\beta}\frac{z^{\beta}(\sigma(t))}{z(\sigma(t))} - \frac{u^2(t)}{a(t)}\\
&= -kq(t)(1-p_2(r(t)))^{\beta}z^{\beta-1}(\sigma(t)) - \frac{u^2(t)}{a(t)}.
\end{aligned} \tag{3.26}$$

Because $z(t)$ is positive and decreasing, there exists a constant $K > 0$ such that

$$z(t) \leq K, \quad t \geq T. \tag{3.27}$$

Since $0 < \beta < 1$, using (3.27) in (3.26), we get

$$u^{\Delta}(t) < -kq(t)(1-p_2(r(t)))^{\beta}K^{\beta-1} - \frac{u^2(t)}{a(t)}, \ t \geq T. \tag{3.28}$$

Multiplying (3.28) by $A(t)$ and then integrating the resulting inequality from T to t, we obtain

$$\begin{aligned}
&A(t)u(t) - A(T)u(T) + \int_T^t kK^{\beta-1}q(s)(1-p_2(r(s)))^{\beta}A(s)\Delta s\\
&+ \int_T^t \frac{u^2(s)}{a(s)}A(s)\Delta s < 0.
\end{aligned} \tag{3.29}$$

By using (2.9), we get

$$\int_T^t \left[kK^{\beta-1}q(s)(1-p_2(r(s)))^{\beta}A(s) + \frac{1}{a(s)A(s)}\right]\Delta s < 1 + A(T)u(T).$$

This contradicts with (3.21) as $t \to \infty$. This completes the proof. ☐

Example 3.2. As an illustrative example, we consider the following equation

$$\left[e^t\left(x(t)+e^{-4t}x\big(\frac{t}{3}\big)\right)'\right]'+e^{3t}x^{\frac{1}{5}}\big(\frac{t}{2}\big)=0,\ t\geq 1. \tag{3.30}$$

Here $\mathbb{T}=\mathbb{R}^+$, and $a(t)=e^t$, $p(t)=e^{-4t}$, $q(t)=e^{3t}$, $m(t)=\frac{t}{3}$, $f(t)=t^{\frac{1}{5}}$, $r(t)=\frac{t}{2}$, $\alpha=1$. Let $\beta=\frac{1}{5}$, $k=1$. Then $A(t)=e^{-t}$, $p_1(r(t))=e^{-2t}$, $p_2(r(t))=e^{-\frac{5t}{3}}$, and $R(r(t))=e^{-1}-e^{-\frac{t}{2}}$. It is easy to verify that all conditions of Theorem 3.2 are satisfied, thus we conclude that every solution of equation (3.30) is oscillatory.

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