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ON EQUALITIES OF BLUES FOR A MULTIPLE RESTRICTED PARTITIONED LINEAR MODEL*

Yunying Huang[1], Bing Zheng[2,†] and Guoliang Chen[1]

Abstract For the multiple restricted partitioned linear model $\mathscr{M} = \{y, X_1\beta_1 + \cdots + X_s\beta_s \mid A_1\beta_1 = b_1, \cdots, A_s\beta_s = b_s, \Sigma\}$, the relationships between the restricted partitioned linear model $\mathscr{M}$ and the corresponding s small restricted linear models $\mathscr{M}_i = \{y, X_i\beta_i \mid A_i\beta_i = b_i, \Sigma\},\ i = 1, \cdots, s$ are studied. The necessary and sufficient conditions for the best linear unbiased estimators (BLUEs) under the full restricted model to be the sums of BLUEs under the s small restricted model are derived. Some statistical properties of the BLUEs are also described.

Keywords Partitioned linear model, restricted models, BLUE, additive decomposition of estimation, Moore-Penrose inverse.

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1. Introduction

Consider a general linear regression model denoted by

$$\mathscr{M} = \{y, X\beta = X_1\beta_1 + \cdots + X_s\beta_s, \Sigma\}, \tag{1.1}$$

where X_i is a known $n \times p_i$ matrix of arbitrary rank with $X = [X_1, \cdots, X_s]$, $i = 1, \cdots, s$, β_i is a $p_i \times 1$ vector of unknown parameters to be estimated and $\beta = [\beta_1', \cdots, \beta_s']'$, $i = 1, \cdots, s$, y is an $n \times 1$ observable random vector with $E(y) = X\beta$ and $Cov(y) = \Sigma$, Σ is a known $n \times n$ nonnegative definite matrix of arbitrary rank. For the full model (1.1), its s small models are given by

$$\mathscr{M}_i = \{y, X_i\beta_i, \Sigma\},\ i = 1, \cdots, s. \tag{1.2}$$

The partitioned linear model in (1.1) is one of the most common representations for modelling data in regression analysis and applications. Such kind of model

[†]the corresponding author.
Email address: yunyinghuang15@163.com (Y. Huang), bzheng@lzu.edu.cn(B. Zheng), glchen@math.ecnu.edu.cn (G. Chen)
[1]School of Mathematical Sciences, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Dongchuan RD 500, Shanghai 200241, China
[2]School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
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frequently occurs in the estimations of partial parameter vectors $\beta_1, \cdots, \beta_s$ and their parametric functions in regression models. Representations of a linear model as some partitioned forms are also used in the study of the relationships between the partitioned model (full model) and its various small or reduced models. This subject was investigated from various aspects, see, e.g., Chu etc [4], Werner and Yapar [26], Groß and Puntanen [8], Tian [21,22], Tian and Takane [24], Huang and Zheng [9], Nurhonen and Puntanen [14], Wang and Liu [25].

Parameters in regression models often satisfy some restrictions, such as the natural restrictions, the stochastic restrictions and some well-known explicit restrictions. The natural restrictions to the unknown parameter vector in a singular linear model were proposed by Rao [17], Baksalary etc [2], Groß [7] and Tian etc [23]. Liu and Wang [12] derived the representations of the BLUEs and the best linear unbiased predictors (BLUPs) of a general mixed linear model through a particular construction from the mixed linear model which uses stochastic restriction. Besides the natural restrictions and the stochastic restrictions, some discussions on well-known explicit restrictions to the unknown parameters can be found in Werner and Yapar [27], Song and Wang [19], Song and Chang [18], Zhang and Tian [29], Jiang and Sun [10]. There are s well-known explicit restrictions on the unknown parameters $\beta_1, \cdots, \beta_s$ which are given by

$$A_1\beta_1 = b_1, \cdots, A_s\beta_s = b_s, \tag{1.3}$$

where $A_i,\ i = 1, \cdots, s$ are known $m_i \times p_i$ matrices with $r(A_i) = m_i$, b_i is a known $m_i \times 1$ vector and these s linear matrix equations are consistent. In such case, model (1.1) together with (1.3) is called a restricted linear model. In the investigation of linear models, parameter constraints are usually handled by transforming an explicitly constrained model into an implicitly constrained model. In regression theory, a constrained linear model is usually handled by transforming into certain implicitly forms by model combination, model reduction by substitution, as well as the Lagrangian multiplier method. Through block matrices, (1.1) and (1.3) can be written as the following implicitly restricted model

$$\mathscr{M}_r = \{y_r, X_r\beta, \Sigma_r\} = \{y_r, X_{r_1}\beta_1 + \cdots + X_{r_s}\beta_s, \Sigma_r\}, \tag{1.4}$$

where $y_r = \begin{bmatrix} y \\ b_1 \\ \vdots \\ b_s \end{bmatrix}$, $X_r = \begin{bmatrix} X_1 & X_2 & \cdots & X_s \\ A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{bmatrix}$ and $\Sigma_r = \begin{bmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$.

(1.2) and (1.3) can be written as the following s small implicitly restricted models

$$\mathscr{M}_{r_i} = \{y_{r_i}, X_{r_i}\beta_i, \Sigma_r\},\ i = 1, \cdots, s, \tag{1.5}$$

where $y_{r_1} = \begin{bmatrix} y \\ b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, y_{r_s} = \begin{bmatrix} y \\ 0 \\ 0 \\ \vdots \\ b_s \end{bmatrix}, X_{r_1} = \begin{bmatrix} X_1 \\ A_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, X_{r_s} = \begin{bmatrix} X_s \\ 0 \\ 0 \\ \vdots \\ A_s \end{bmatrix}$.

In recent years, many works have been devoted to developing additive decompositions of estimators under linear models, such as Tian [21,22], Zhang and Tian [29], Bhimasankaram and Saharay [3], Zhang etc [28], and so on. The necessary and sufficient conditions for the BLUE of $X\beta$ under the full model to be the sum of the BLUEs of $X_1\beta_1$ and $X_2\beta_2$ under the two small models were derived by Tian [21]. Furthermore, Tian [22] gave the necessary and sufficient conditions for the BLUE in a general multiple-partitioned linear model to be the sum of the BLUEs under its k small models. While the necessary and sufficient conditions for the BLUE under the full restricted model to be the sum of BLUEs under the corresponding two small restricted models were derived in [29]. The main purpose of this paper is to investigate the relations among the BLUEs in $\mathscr{M}_r$ and the s small models $\mathscr{M}_{r_1}, \cdots, \mathscr{M}_{r_s}$. In particular, we will derive the necessary and sufficient conditions for

(I) $\mathrm{BLUE}_{\mathscr{M}_r}(X\beta) = \mathrm{BLUE}_{\mathscr{M}_{r_1}}(X_1\beta_1) + \cdots + \mathrm{BLUE}_{\mathscr{M}_{r_s}}(X_s\beta_s)$;

(II) $\mathrm{BLUE}_{\mathscr{M}_r}(X_r\beta) = \mathrm{BLUE}_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1) + \cdots + \mathrm{BLUE}_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s)$

to hold.

Throughout this paper, $\mathbb{R}^{m\times n}$ stands for the collection of all $m\times n$ real matrices. The symbols $A', r(A)$ and $\Re(A)$ stand for the transpose, rank and range (column space) of a matrix $A \in \mathbb{R}^{m\times n}$, respectively. I_m denotes the identity matrix of order m. The symbol $i_+(A)$ denotes the number of positive eigenvalues of a symmetric matrix A counted with multiplicity.

2. Preliminaries

For a given matrix $K \in \mathbb{R}^{q\times p}$, then the product $K\beta$ is a vector of parametric functions, or simply said a parametric function. The mean vectors $X\beta$ and $X_r\beta$ in $\mathscr{M}_r$ are two special cases of $K\beta$. The vector $K\beta$ is said to be estimable under $\mathscr{M}$ if there exists matrix $L_1 \in \mathbb{R}^{q\times n}$ such that $E(L_1y) = K\beta$ holds. It is well known that $K\beta$ is estimable under $\mathscr{M}$ if and only if

$$\Re(K') \subseteq \Re(X'), \tag{2.1}$$

see e.g., Alalouf and Styan [1]. Partition K as $K = [K_1, \cdots, K_s]$, where K_i is a $q \times p_i$ real matrix, $i = 1, \cdots, s$. Then, $K\beta = K_1\beta_1 + \cdots + K_s\beta_s$. Applying (2.1) and [29] to the model in (1.4), we get the following results.

Lemma 2.1. *Let $\mathscr{M}_r$ and X_r be as given in (1.4). Then $K\beta$ is estimable under model $\mathscr{M}_r$ if and only if $\Re(K') \subseteq \Re(X'_r)$, that is* $\Re\begin{bmatrix} K'_1 \\ K'_2 \\ \vdots \\ K'_s \end{bmatrix} \subseteq \Re\begin{bmatrix} X'_{r_1} \\ X'_{r_2} \\ \vdots \\ X'_{r_s} \end{bmatrix} = \Re\begin{bmatrix} X'_1 & A'_1 & 0 & \cdots & 0 \\ X'_2 & 0 & A'_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ X'_s & 0 & 0 & \cdots & A'_s \end{bmatrix}$. *In this case, $\Re(K'_i) \subseteq \Re(X'_{r_i})$, then $K_i\beta_i$ is estimable under model $\mathscr{M}_{r_i}$, $i = 1, \cdots, s$.*

The partial parametric functions $X_i\beta_i$ and $X_{r_i}\beta_i$ in $\mathscr{M}_{r_i}$ are two special cases of $K_i\beta_i$. It can be seen from Lemma 2.1 that $X_i\beta_i$ and $X_{r_i}\beta_i$ are both estimable under model $\mathscr{M}_{r_i}$. The BLUE of $K\beta$ under $\mathscr{M}$, denoted by $\mathrm{BLUE}_{\mathscr{M}}(K\beta)$, is defined to be a linear estimator Gy such that $E(Gy) = K\beta$ and $\mathrm{Cov}(G_1y) - \mathrm{Cov}(Gy)$ is nonnegative definite for any other unbiased estimator G_1y of $K\beta$.

The Moore-Penrose inverse of $A \in \mathbb{R}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution G satisfying the four matrix equations

$$(i) \quad AGA = A, \quad (ii) \quad GAG = G, \quad (iii) \quad (AG)' = AG, \quad (iv) \quad (GA)' = GA.$$

Further, the symbols $P_A = AA^\dagger$, $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$ stand for the three orthogonal projectors onto the range of A, the null spaces of A' and A, respectively. The general solution of the linear matrix equation $AX = B$ can be clearly expressed by the Moore-Penrose inverse of its coefficient matrix, see Penrose [15].

Lemma 2.2 (Theorem 2, [15])**.** *The linear matrix equation $AX = B$ is consistent if and only if $r[A, B] = r(A)$, or equivalently, $AA^\dagger B = B$. In this case, the general solution of the equation can be written as $X = A^\dagger B + (I - A^\dagger A)U$, where U is an arbitrary matrix.*

Some well-known results about the BLUE of an estimable $K\beta$ under model $\mathscr{M}$ were established by Drygas [5] and Rao [17]. The following Lemma will give a new formulation of the BLUE, see Dong etc [6] and Lu etc [11].

Lemma 2.3 (Lemma 1.2, [6], Lemma 2.2, [11])**.** *Let $\mathscr{M}$ be as given in (1.1), and assume that $K\beta$ is estimable under $\mathscr{M}$ and $E(L_0y) = K\beta$. Then*

$$\max_{E(Ly)=K\beta} i_+[Cov(L_0y) - Cov(Ly)] = r\begin{bmatrix} L_0\Sigma \\ X' \end{bmatrix} - r(X) = r(L_0\Sigma E_X). \tag{2.2}$$

Hence,

$$E(L_0y) = K\beta \text{ and } Cov(L_0y) \text{ is minimal} \Leftrightarrow L_0[X, \Sigma E_X] = [K, 0]. \tag{2.3}$$

In this case, the general expression of L_0, denoted by $P_{K;X;\Sigma}$, and $BLUE_{\mathscr{M}}(K\beta)$ can be expressed as

$$BLUE_{\mathscr{M}}(K\beta) = P_{K;X;\Sigma}y = ([K, 0][X, \Sigma E_X]^\dagger + UE_{[X,\Sigma]})y, \tag{2.4}$$

where $U \in \mathbb{R}^{k\times n}$ is arbitrary. Further, the following results hold.

(a) $r[X, \Sigma E_X] = r[X, \Sigma]$ *and* $\Re[X, \Sigma E_X] = \Re[X, \Sigma]$.

(b) $Cov[BLUE_{\mathscr{M}}(K\beta)] = [K, 0][X, \Sigma E_X]^{\dagger}\Sigma([K, 0][X, \Sigma E_X]^{\dagger})'$.

(c) $P_{K;X;\Sigma}$ *is unique if and only if* $r[X, \Sigma] = n$.

(d) $BLUE_{\mathscr{M}}(K\beta)$ *is unique with probability 1 if and only if* $\mathscr{M}$ *is consistent.*

When $s = 2$ in (1.4) and (1.5), the general expressions of the BLUEs under the full model $\mathscr{M}_r$ and the two small models $\mathscr{M}_{r_i}, i = 1, 2$ have been derived by Zhang and Tian [29]. From Lemma 2.3 and [29], the BLUEs under models $\mathscr{M}_r$ and $\mathscr{M}_{r_i}, i = 1, \cdots s$ in (1.4) and (1.5) are given below.

Lemma 2.4. *Let $\mathscr{M}_r$ be as given in (1.4), assume that $K\beta$ is estimable under (1.4), and denote $t = m_1 + \cdots + m_s + n$. Then, the following results hold.*

(a) *The general expression of $BLUE_{\mathscr{M}_r}(K\beta)$ can be written as*

$$BLUE_{\mathscr{M}_r}(K\beta) = P_{K;X_r;\Sigma_r} y_r, \tag{2.5}$$

where $P_{K;X_r;\Sigma_r} = [K, 0][X_r, \Sigma_r E_{X_r}]^{\dagger} + U_0 E_{[X_r,\Sigma_r]}$ and $U_0 \in \mathbb{R}^{k\times t}$ is arbitrary.

(b) *The general expressions of $BLUE_{\mathscr{M}_r}(X\beta)$ and $BLUE_{\mathscr{M}_r}(X_r\beta)$ can be written as*

$$BLUE_{\mathscr{M}_r}(X\beta) = P_{X;X_r;\Sigma_r} y_r \tag{2.6}$$

and

$$BLUE_{\mathscr{M}_r}(X_r\beta) = P_{X_r;\Sigma_r} y_r, \tag{2.7}$$

respectively, where $P_{X;X_r;\Sigma_r} = [X, 0][X_r, \Sigma_r E_{X_r}]^{\dagger} + U E_{[X_r,\Sigma_r]}$, $P_{X_r;\Sigma_r} = [X_r, 0][X_r, \Sigma_r E_{X_r}]^{\dagger} + V E_{[X_r,\Sigma_r]}$, U and V are arbitrary.

Similarly, we can write the BLUEs of $X_i\beta_i$ and $X_{r_i}\beta_i$ under the small model $\mathscr{M}_{r_i}$ in (1.5) as

$$\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_i\beta_i) = P_{X_i;X_{r_i};\Sigma_r} y_{r_i} = P_{X_i;X_{r_i};\Sigma_r} \widehat{I}_{1(i+1)} y_r,\ i = 1, \cdots, s \tag{2.8}$$

and

$$\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i) = P_{X_{r_i};\Sigma_r} y_{r_i} = P_{X_{r_i};\Sigma_r} \widehat{I}_{1(i+1)} y_r,\ i = 1, \cdots, s, \tag{2.9}$$

respectively, where

$$P_{X_i;X_{r_i};\Sigma_r} = [X_i, 0][X_{r_i}, \Sigma_r E_{X_{r_i}}]^{\dagger} + U_i E_{[X_{r_i},\Sigma_r]},\ i = 1, \cdots, s,$$

$$P_{X_{r_i};\Sigma_r} = [X_{r_i}, 0][X_{r_i}, \Sigma_r E_{X_{r_i}}]^{\dagger} + V_i E_{[X_{r_i},\Sigma_r]},\ i = 1, \cdots, s,$$

$$\widehat{I}_{12} = \begin{bmatrix} I_n & 0 & 0 \cdots 0 \\ 0 & I_{m_1} & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & 0 \cdots 0 \end{bmatrix}, \cdots, \widehat{I}_{1(s+1)} = \begin{bmatrix} I_n\ 0\ 0 \cdots & 0 \\ 0\ 0\ 0 \cdots & 0 \\ 0\ 0\ 0 \cdots & 0 \\ \vdots\ \vdots\ \vdots \ddots & \vdots \\ 0\ 0\ 0 \cdots & I_{m_s} \end{bmatrix},$$

U_i and V_i are arbitrary.

The next lemmas give some rank formulas of partitioned matrices which can be used to simplify various matrix expressions involving the Moore-Penrose inverses of matrices.

Lemma 2.5 (Theorem 19, [13]). *Let* $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$, $C \in \mathbb{R}^{q\times n}$, *and* $D \in \mathbb{R}^{q\times k}$. *Then*

$$r[A, B] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{2.10}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C). \tag{2.11}$$

If $\Re(B) \subseteq \Re(A)$ *and* $\Re(C') \subseteq \Re(A')$, *then*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(D - CA^{\dagger}B). \tag{2.12}$$

In particular,

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) \Leftrightarrow \Re(B) \subseteq \Re(A),\ \Re(C') \subseteq \Re(A') \text{ and } D = CA^{\dagger}B. \tag{2.13}$$

Lemma 2.6 ([20]). *Suppose* $\Re(A) \subseteq \Re(B_1)$, $\Re(C_2) \subseteq \Re(C_1)$, $\Re(A') \subseteq \Re(C_1')$ *and* $\Re(B_2') \subseteq \Re(B_1')$. *Then*

$$r(B_2B_1^{\dagger}AC_1^{\dagger}C_2) = r\begin{bmatrix} A & B_1 & 0 \\ C_1 & 0 & C_2 \\ 0 & B_2 & 0 \end{bmatrix} - r(B_1) - r(C_1). \tag{2.14}$$

The following simple and well-known facts will be also useful to simplify various operations on ranges and ranks of matrices:

$$\Re(A) \subseteq \Re(B) \Leftrightarrow r[A, B] = r(B) \Leftrightarrow BB^{\dagger}A = A, \tag{2.15}$$

$$\Re(A_1) = \Re(A_2) \text{ and } \Re(B_1) = \Re(B_2) \Leftrightarrow r[A_1, B_1] = r[A_2, B_2]. \tag{2.16}$$

3. Additive decompositions of BLUEs

Since $P_{K;X;\Sigma}$ and $\mathrm{BLUE}_{\mathscr{M}}(K\beta)$ are not necessarily unique, we use $\{P_{K;X;\Sigma}\}$ and $\{\mathrm{BLUE}_{\mathscr{M}}(K\beta)\}$ to denote the collections of all $P_{K;X;\Sigma}$ and $\mathrm{BLUE}_{\mathscr{M}}(K\beta)$, respectively. The model in (1.1) is said to be consistent if

$$y \in \Re[X, \Sigma]$$

holds with probability 1, see Rao [16, 17]. Then the model (1.4) is said to be consistent if

$$y_r \in \Re[X_r, \Sigma_r]$$

holds with probability 1. In what follows, we assume that the model (1.4) is consistent.

It is noted that the estimators in (2.8) and (2.9) are not really the BLUEs of $X_i\beta_i$ and $X_{r_i}\beta_i,\ i = 1, \cdots, s$ in the s misspecified models in (1.5) under the assumption in (1.4), that is, they are neither unbiased for $X_i\beta_i$ and $X_{r_i}\beta_i,\ i=1,\cdots,s$ in (1.5) nor have the smallest covariance matrices in the Löwner sense. However, their sum can really be the BLUEs of the mean vector $X\beta$ and $X_r\beta$ in (1.4) under some conditions. In this section, we will give the necessary and sufficient conditions for

(I) $\mathrm{BLUE}_{\mathscr{M}_r}(X\beta)=\mathrm{BLUE}_{\mathscr{M}_{r_1}}(X_1\beta_1)+\cdots+\mathrm{BLUE}_{\mathscr{M}_{r_s}}(X_s\beta_s)$;

(II) $\mathrm{BLUE}_{\mathscr{M}_r}(X_r\beta)=\mathrm{BLUE}_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1)+\cdots+\mathrm{BLUE}_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s)$

to hold. Some statistical properties of these BLUEs are also given.

In the following theorem, we will give the expectations of $\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_i\beta_i)$ and $\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i), i=1,\cdots,s$, and discuss the unbiasedness of these BLUEs.

Theorem 3.1. *Let $BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)$ and $BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i), i=1,\cdots,s$ be as given in (2.8) and (2.9). Then the following results hold.*

(a) *The expectations of $BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)$ and $BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)$ under the assumption in (1.4) are given by*

$$E[BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)]=X_i\beta_i+P_{X_i;X_{r_i};\Sigma_r}\widehat{I}_n\widehat{X}_i\widehat{\beta}_i, i=1,\cdots,s \tag{3.1}$$

and

$$E[BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)]=X_{r_i}\beta_i+P_{X_{r_i};\Sigma_r}\widehat{I}_n\widehat{X}_i\widehat{\beta}_i, i=1,\cdots,s, \tag{3.2}$$

where $\widehat{X}_i=[X_1,\cdots,X_{i-1},X_{i+1},\cdots,X_s]$, $\widehat{\beta}_i=[\beta_1',\cdots,\beta_{i-1}',\beta_{i+1}',\cdots,\beta_s']'$ and $\widehat{I}_n=[I_n,0,\cdots,0]', i=1,\cdots,s$.

(b) *The following statements are equivalent:*

(i) *There exists a $BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)$ such that $E[BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)]=X_i\beta_i$ holds.*

(ii) *There exists a $BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)$ such that $E[BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)]=X_{r_i}\beta_i$ holds.*

(iii) *The rank equality*

$$r\begin{bmatrix}\Sigma & X_i\\ 0 & A_i\\ \hat{X}_i' & 0\end{bmatrix}=r\begin{bmatrix}\Sigma & X_i & \hat{X}_i\\ 0 & A_i & 0\end{bmatrix}, i=1,\cdots,s \tag{3.3}$$

holds.

(c) *If Σ is positive definite, then the following statements are equivalent:*

(i) *There exists a $BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)$ such that $E[BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)]=X_i\beta_i$.*

(ii) *There exists a $BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)$ such that $E[BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)]=X_{r_i}\beta_i$.*

(iii) $\Re(X_i'\Sigma^{-1}\hat{X}_i')\subseteq\Re(A_i'),\ i=1,\cdots,s.$

Proof. It can be seen from (2.8) and (2.9) that

$$E[\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_i\beta_i)]=P_{X_i;X_{r_i};\Sigma_r}\widehat{I}_{1(i+1)}X_r\beta$$

$$
\begin{aligned}
&= P_{X_i;X_{r_i};\Sigma_r} \begin{bmatrix} I_n & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & I_{m_i} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 & \cdots & X_i & \cdots & X_s \\ A_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_i & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & A_s \end{bmatrix} \beta \\
&= P_{X_i;X_{r_i};\Sigma_r} \begin{bmatrix} X_1\beta_1 + \cdots + X_s\beta_s \\ 0 \\ \vdots \\ 0 \\ A_i\beta_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= P_{X_i;X_{r_i};\Sigma_r} X_{r_i}\beta_i + P_{X_i;X_{r_i};\Sigma_r} \widehat{I}_n \widehat{X}_i \widehat{\beta}_i \\
&= X_i\beta_i + P_{X_i;X_{r_i};\Sigma_r} \widehat{I}_n \widehat{X}_i \widehat{\beta}_i, \quad i = 1, \cdots, s
\end{aligned}
$$

and

$$
\begin{aligned}
E[\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)] &= P_{X_{r_i};\Sigma_r} \widehat{I}_{1(i+1)} X_r \beta \\
&= P_{X_{r_i};\Sigma_r} \begin{bmatrix} X_1\beta_1 + \cdots + X_s\beta_s \\ 0 \\ \vdots \\ 0 \\ A_i\beta_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= X_{r_i}\beta_i + P_{X_{r_i};\Sigma_r} \widehat{I}_n \widehat{X}_i \widehat{\beta}_i, \quad i = 1, \cdots, s.
\end{aligned}
$$

Hence, the two identities in (a) are established. Observe that

$$P_{X_i;X_{r_i};\Sigma_r} \widehat{I}_n \widehat{X}_i = [X_i, 0][X_{r_i}, \Sigma_r E_{X_{r_i}}]^{\dagger} \widehat{I}_n \widehat{X}_i + U_i E_{[X_{r_i}, \Sigma_r E_{X_{r_i}}]} \widehat{I}_n \widehat{X}_i, \ i = 1, \cdots, s,$$

it can be seen from lemma 2.1 that there exists a U_i such that $P_{X_i;X_{r_i};\Sigma_r} \widehat{I}_n \widehat{X}_i = 0$

if and only if

$$r\begin{bmatrix}[X_i,0][X_{r_i},\Sigma_r E_{X_{r_i}}]^{\dagger}\widehat{I}_n\widehat{X}_i\\ E_{[X_{r_i},\Sigma_r E_{X_{r_i}}]}\widehat{I}_n\widehat{X}_i\end{bmatrix}=r(E_{[X_{r_i},\Sigma_r E_{X_{r_i}}]}\widehat{I}_n\widehat{X}_i),\ i=1,\cdots,s. \tag{3.4}$$

Using (2.10), (2.11) and elementary block matrix operations (EBMOs), we have

$$\begin{aligned}
r\begin{bmatrix}[X_i,0][X_{r_i},\Sigma_r E_{X_{r_i}}]^{\dagger}\widehat{I}_n\widehat{X}_i\\ E_{[X_{r_i},\Sigma_r E_{X_{r_i}}]}\widehat{I}_n\widehat{X}_i\end{bmatrix}&=r\begin{bmatrix}[X_i,0][X_{r_i},\Sigma_r E_{X_{r_i}}]^{\dagger}\widehat{I}_n\widehat{X}_i & 0\\ \widehat{I}_n\widehat{X}_i & [X_{r_i},\Sigma_r E_{X_{r_i}}]\end{bmatrix}-r[X_{r_i},\Sigma_r E_{X_{r_i}}]\\
&=r\begin{bmatrix}0 & -[X_i,0]\\ \widehat{I}_n\widehat{X}_i & [X_{r_i},\Sigma_r E_{X_{r_i}}]\end{bmatrix}-r[X_{r_i},\Sigma_r]\\
&=r\begin{bmatrix}0 & -X_i & 0\\ \widehat{I}_n\widehat{X}_i & X_{r_i} & \Sigma_r\\ 0 & 0 & X'_{r_i}\end{bmatrix}-r(X_{r_i})-r[X_{r_i},\Sigma_r]\\
&=r\begin{bmatrix}\Sigma_r & \widehat{I}_n\widehat{X}_i\\ X'_{r_i} & 0\end{bmatrix}-r[X_{r_i},\Sigma_r]
\end{aligned}$$

and

$$\begin{aligned}
r(E_{[X_{r_i},\Sigma_r E_{X_{r_i}}]}\widehat{I}_n\widehat{X}_i)&=r[X_{r_i},\Sigma_r E_{X_{r_i}},\widehat{I}_n\widehat{X}_i]-r[X_{r_i},\Sigma_r E_{X_{r_i}}]\\
&=r[X_{r_i},\Sigma_r,\widehat{I}_n\widehat{X}_i]-r[X_{r_i},\Sigma_r]
\end{aligned}$$

for $i=1,\cdots,s$. Hence, the statement (i) in (b) holds if and only if

$$r\begin{bmatrix}\Sigma_r & \widehat{I}_n\widehat{X}_i\\ X'_{r_i} & 0\end{bmatrix}=r[X_{r_i},\Sigma_r,\widehat{I}_n\widehat{X}_i], i=1,\cdots,s,$$

that is

$$r\begin{bmatrix}\Sigma & X_i\\ 0 & A_i\\ \hat{X}'_i & 0\end{bmatrix}=r\begin{bmatrix}\Sigma & X_i & \hat{X}_i\\ 0 & A_i & 0\end{bmatrix}, i=1,\cdots,s$$

which implies the equivalence of (i) and (iii) in (b). The equivalence of (ii) and (iii) in (b) can be proved similarly. The results in (c) follow from (b). □

We now consider the sum decomposition of $\mathrm{BLUE}_{\mathscr{M}_r}(X\beta)$ on the equality in (I) and two theorems are given below.

Theorem 3.2. *Let $BLUE_{\mathscr{M}_r}(X\beta)$ and $BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i), i=1,\cdots,s$ be as given in (2.6) and (2.8), and define*

$$D_{\widehat{I}_n}=diag(\widehat{I}_n,\cdots,\widehat{I}_n),\ D_{\Sigma_r}=diag(\Sigma_r,\cdots,\Sigma_r),\ D_{X_r}=diag(X_{r_1},\cdots,X_{r_s}), \tag{3.5}$$

$$\widetilde{X}=\begin{bmatrix}0 & X_2 & \cdots & X_s\\ X_1 & 0 & \cdots & X_s\\ \vdots & \vdots & \ddots & \vdots\\ X_1 & X_2 & \cdots & 0\end{bmatrix}, \textit{where diag represents the diagonal matrix.} \tag{3.6}$$

Then:

(a) *The sum of the BLUEs is given by*

$$BLUE_{\mathscr{M}_{r_1}}(X_1\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_s\beta_s) = (P_{X_1;X_{r_1};\Sigma_r}\widehat{I}_{12} + \cdots + P_{X_s;X_{r_s};\Sigma_r}\widehat{I}_{1(s+1)})y_r \tag{3.7}$$

with the expectation

$$E[BLUE_{\mathscr{M}_{r_1}}(X_1\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_s\beta_s)] = X\beta + [P_{X_1;X_{r_1};\Sigma_r}, \cdots, P_{X_s;X_{r_s};\Sigma_r}]D_{\widehat{I}_n}\widetilde{X}\beta \tag{3.8}$$

under the assumption in $\mathscr{M}_r$.

(b) *The following statements are equivalent:*

(i) *There exist* $BLUE_{\mathscr{M}_{r_1}}(X_1\beta_1), \cdots, BLUE_{\mathscr{M}_{r_s}}(X_s\beta_s)$ *such that*

$$E[BLUE_{\mathscr{M}_{r_1}}(X_1\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_s\beta_s)] = X\beta \tag{3.9}$$

holds under the assumption in $\mathscr{M}_r$.

(ii) *There exist* $BLUE_{\mathscr{M}_{r_1}}(X_1\beta_1), \cdots, BLUE_{\mathscr{M}_{r_s}}(X_s\beta_s)$ *such that*

$$BLUE_{\mathscr{M}_{r_1}}(X_1\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_s\beta_s) \in \{BLUE_{\mathscr{M}_r}(X\beta)\} \tag{3.10}$$

holds under the assumption in $\mathscr{M}_r$.

(iii) $r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \\ 0 & 0 & X \end{bmatrix} = r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \end{bmatrix}$.

(iv) $\Re\begin{bmatrix} 0 \\ 0 \\ X' \end{bmatrix} \subseteq \Re\begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ \widetilde{X}'D'_{\widehat{I}_n} & 0 \\ D'_{X_r} & 0 \end{bmatrix}$ *or* $\Re(X') \subseteq \Re([D'_{X_r}, 0]E_T)$, *where* $T = \begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ \widetilde{X}'D'_{\widehat{I}_n} & 0 \end{bmatrix}$.

Proof. From (2.8) and Theorem 3.1, we can easily get the results in (a). It can be seen from (a) that the statement (i) in (b) holds under the assumption in $\mathscr{M}_r$ if and only if

$$[P_{X_1;X_{r_1};\Sigma_r}, \cdots, P_{X_s;X_{r_s};\Sigma_r}]D_{\widehat{I}_n}\widetilde{X} = 0,$$

that is

$$GD_{\widehat{I}_n}\widetilde{X} + UE_M D_{\widehat{I}_n}\widetilde{X} = 0, \tag{3.11}$$

where

$$G = [[X_1, 0][X_{r_1}, \Sigma_r E_{X_{r_1}}]^\dagger, \cdots, [X_s, 0][X_{r_s}, \Sigma_r E_{X_{r_s}}]^\dagger],$$

$$U = [U_1, \cdots, U_s], \; M = \mathrm{diag}([X_{r_1}, \Sigma_r E_{X_{r_1}}], \cdots, [X_{r_s}, \Sigma_r E_{X_{r_s}}]).$$

However, the equation in (3.11) is solvable for U if and only if

$$r\begin{bmatrix} GD_{\widehat{I}_n}\widetilde{X} \\ E_M D_{\widehat{I}_n}\widetilde{X} \end{bmatrix} = r(E_M D_{\widehat{I}_n}\widetilde{X}). \quad \text{(by Lemma 2.1)} \tag{3.12}$$

Let $N = \mathrm{diag}([X_{r_1}, \Sigma_r], \cdots, [X_{r_s}, \Sigma_r])$ and $L = \mathrm{diag}([0, X'_{r_1}], \cdots, [0, X'_{r_s}])$. Then $M = NF_L$, $r(M) = r(N) = r[D_{\Sigma_r}, D_{X_r}], r(L) = r(D_{X_r})$. Applying (2.10), (2.11) and EBMOs, we have

$$\begin{aligned}
r\begin{bmatrix} GD_{\widehat{I}_n}\widetilde{X} \\ E_M D_{\widehat{I}_n}\widetilde{X} \end{bmatrix} &= r\begin{bmatrix} GD_{\widehat{I}_n}\widetilde{X} & 0 \\ D_{\widehat{I}_n}\widetilde{X} & M \end{bmatrix} - r(M) \\
&= r\begin{bmatrix} 0 & -GM \\ D_{\widehat{I}_n}\widetilde{X} & M \end{bmatrix} - r[D_{\Sigma_r}, D_{X_r}] \\
&= r\begin{bmatrix} 0 & [[X_1, 0], \cdots, [X_s, 0]] \\ D_{\widehat{I}_n}\widetilde{X} & NF_L \end{bmatrix} - r[D_{\Sigma_r}, D_{X_r}] \\
&= r\begin{bmatrix} 0 & [[X_1, 0], \cdots, [X_s, 0]] \\ D_{\widehat{I}_n}\widetilde{X} & N \\ 0 & L \end{bmatrix} - r(L) - r[D_{\Sigma_r}, D_{X_r}] \\
&= r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \\ 0 & 0 & X \end{bmatrix} - r(D_{X_r}) - r[D_{\Sigma_r}, D_{X_r}]
\end{aligned}$$

and

$$\begin{aligned}
r(E_M D_{\widehat{I}_n}\widetilde{X}) &= r[D_{\widehat{I}_n}\widetilde{X}, M] - r(M) \\
&= r[D_{\widehat{I}_n}\widetilde{X}, NF_L] - r[D_{\Sigma_r}, D_{X_r}] \\
&= r\begin{bmatrix} D_{\widehat{I}_n}\widetilde{X} & N \\ 0 & L \end{bmatrix} - r(L) - r[D_{\Sigma_r}, D_{X_r}] \\
&= r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \end{bmatrix} - r(D_{X_r}) - r[D_{\Sigma_r}, D_{X_r}].
\end{aligned}$$

Hence, (3.12) holds if and only if

$$r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \\ 0 & 0 & X \end{bmatrix} = r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \end{bmatrix}.$$

The equivalence of (i) and (iii) in (b) is proved. From (2.15), we know that (iii) and (iv) in (b) are equivalent.

Now we prove (ii) $\Leftrightarrow$ (iii) in (b). It can be seen from (a) that (ii) in (b) holds if and only if $(P_{X_1;X_{r_1};\Sigma_r}\widehat{I}_{12}+\cdots+P_{X_s;X_{r_s};\Sigma_r}\widehat{I}_{1(s+1)})[X_r,\Sigma_r E_{X_r}]=[X,0]$, that is

$$U\begin{bmatrix} E_{[X_{r_1},\Sigma_r E_{X_{r_1}}]}\widehat{I}_{12}[X_r,\Sigma_r E_{X_r}] \\ \vdots \\ E_{[X_{r_s},\Sigma_r E_{X_{r_s}}]}\widehat{I}_{1(s+1)}[X_r,\Sigma_r E_{X_r}] \end{bmatrix}=H, \tag{3.13}$$

where

$$\begin{aligned} H=&[X,0]-[X_1,0][X_{r_1},\Sigma_r E_{X_{r_1}}]^{\dagger}\widehat{I}_{12}[X_r,\Sigma_r E_{X_r}] \\ &-\cdots-[X_s,0][X_{r_s},\Sigma_r E_{X_{r_s}}]^{\dagger}\widehat{I}_{1(s+1)}[X_r,\Sigma_r E_{X_r}] \end{aligned}$$

and $U=[U_1,\cdots,U_s]$. Hence, (3.13) is solvable for U if and only if

$$r\begin{bmatrix} H \\ E_{[X_{r_1},\Sigma_r E_{X_{r_1}}]}\widehat{I}_{12}[X_r,\Sigma_r E_{X_r}] \\ \vdots \\ E_{[X_{r_s},\Sigma_r E_{X_{r_s}}]}\widehat{I}_{1(s+1)}[X_r,\Sigma_r E_{X_r}] \end{bmatrix}=r\begin{bmatrix} E_{[X_{r_1},\Sigma_r E_{X_{r_1}}]}\widehat{I}_{12}[X_r,\Sigma_r E_{X_r}] \\ \vdots \\ E_{[X_{r_s},\Sigma_r E_{X_{r_s}}]}\widehat{I}_{1(s+1)}[X_r,\Sigma_r E_{X_r}] \end{bmatrix}. \tag{3.14}$$

Using (2.10), (2.11) and EBMOs, we obtain

$$\begin{aligned} &r\begin{bmatrix} H \\ E_{[X_{r_1},\Sigma_r E_{X_{r_1}}]}\widehat{I}_{12}[X_r,\Sigma_r E_{X_r}] \\ \vdots \\ E_{[X_{r_s},\Sigma_r E_{X_{r_s}}]}\widehat{I}_{1(s+1)}[X_r,\Sigma_r E_{X_r}] \end{bmatrix} \\ &=r\begin{bmatrix} H & 0 & \cdots & 0 \\ \widehat{I}_{12}[X_r,\Sigma_r E_{X_r}] & [X_{r_1},\Sigma_r E_{X_{r_1}}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{I}_{1(s+1)}[X_r,\Sigma_r E_{X_r}] & 0 & \cdots & [X_{r_s},\Sigma_r E_{X_{r_s}}] \end{bmatrix}-r(M) \\ &=r\begin{bmatrix} [X,0] & [X_1,0] & \cdots & [X_s,0] \\ \widehat{I}_{12}[X_r,\Sigma_r E_{X_r}] & [X_{r_1},\Sigma_r E_{X_{r_1}}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{I}_{1(s+1)}[X_r,\Sigma_r E_{X_r}] & 0 & \cdots & [X_{r_s},\Sigma_r E_{X_{r_s}}] \end{bmatrix}-r[D_{\Sigma_r},D_{X_r}] \\ &=r\begin{bmatrix} [0,0] & [X_1,0] & \cdots & [X_s,0] \\ \left[\widehat{I}_n\widehat{X}_1,0\right] & [X_{r_1},\Sigma_r E_{X_{r_1}}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left[\widehat{I}_n\widehat{X}_s,0\right] & 0 & \cdots & [X_{r_s},\Sigma_r E_{X_{r_s}}] \end{bmatrix}-r[D_{\Sigma_r},D_{X_r}] \end{aligned}$$

$$= r\begin{bmatrix} 0 & X & 0 \\ D_{\widehat{I}_n}\widetilde{X} & D_{X_r} & D_{\Sigma_r}E_{D_{X_r}} \end{bmatrix} - r[D_{\Sigma_r}, D_{X_r}]$$

$$= r\begin{bmatrix} 0 & X & 0 \\ D_{\widehat{I}_n}\widetilde{X} & D_{X_r} & D_{\Sigma_r} \\ 0 & 0 & D'_{X_r} \end{bmatrix} - r(D_{X_r}) - r[D_{\Sigma_r}, D_{X_r}]$$

$$= r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \\ 0 & 0 & X \end{bmatrix} - r(D_{X_r}) - r[D_{\Sigma_r}, D_{X_r}]$$

and

$$r\begin{bmatrix} E_{[X_{r_1},\Sigma_r E_{X_{r_1}}]}\widehat{I}_{12}[X_r, \Sigma_r E_{X_r}] \\ \vdots \\ E_{[X_{r_s},\Sigma_r E_{X_{r_s}}]}\widehat{I}_{1(s+1)}[X_r, \Sigma_r E_{X_r}] \end{bmatrix}$$

$$= r\begin{bmatrix} \widehat{I}_{12}[X_r, \Sigma_r E_{X_r}] & [X_{r_1}, \Sigma_r E_{X_{r_1}}] & 0 & \cdots & 0 \\ \widehat{I}_{13}[X_r, \Sigma_r E_{X_r}] & 0 & [X_{r_2}, \Sigma_r E_{X_{r_2}}] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{I}_{1(s+1)}[X_r, \Sigma_r E_{X_r}] & 0 & 0 & \cdots & [X_{r_s}, \Sigma_r E_{X_{r_s}}] \end{bmatrix} - r(M)$$

$$= r\begin{bmatrix} \widehat{I}_{12}[X_r, 0] & [X_{r_1}, \Sigma_r E_{X_{r_1}}] & 0 & \cdots & 0 \\ \widehat{I}_{13}[X_r, 0] & 0 & [X_{r_2}, \Sigma_r E_{X_{r_2}}] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{I}_{1(s+1)}[X_r, 0] & 0 & 0 & \cdots & [X_{r_s}, \Sigma_r E_{X_{r_s}}] \end{bmatrix} - r[D_{\Sigma_r}, D_{X_r}]$$

$$= r\begin{bmatrix} \left[\widehat{I}_n\widehat{X}_1, 0\right] & [X_{r_1}, \Sigma_r] & 0 & \cdots & 0 \\ \left[\widehat{I}_n\widehat{X}_2, 0\right] & 0 & [X_{r_2}, \Sigma_r] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left[\widehat{I}_n\widehat{X}_s, 0\right] & 0 & 0 & \cdots & [X_{r_s}, \Sigma_r] \\ 0 & [0, X'_{r_1}] & 0 & \cdots & 0 \\ 0 & 0 & [0, X'_{r_2}] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & [0, X'_{r_s}] \end{bmatrix} - r(D_{X_r}) - r[D_{\Sigma_r}, D_{X_r}]$$

$$= r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \end{bmatrix} - r(D_{X_r}) - r[D_{\Sigma_r}, D_{X_r}].$$

Hence, the identity (3.14) is equivalent to

$$r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \\ 0 & 0 & X \end{bmatrix} = r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \end{bmatrix},$$

which implies the equivalence of (ii) and (iii) in (b). □

Theorem 3.3. *Let $BLUE_{\mathscr{M}_r}(X\beta)$ and $BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i), i = 1, \cdots, s$ be as given in (2.6) and (2.8), and let $D_{\widehat{I}_n}$, D_{Σ_r}, D_{X_r} and $\widetilde{X}$ be given as (3.5) and (3.6). Then:*

(a) *The following statements are equivalent:*

(i) *The set inclusion $\{BLUE_{\mathscr{M}_{r_1}}(X_1\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_s\beta_s)\} \subseteq \{BLU E_{\mathscr{M}_r}(X\beta)\}$ holds under the assumption in $\mathscr{M}_r$.*

(ii) $r\begin{bmatrix} D_{\Sigma_r} & D_{X_r} & D_{\widehat{I}_n}\widetilde{X} \\ D'_{X_r} & 0 & 0 \\ 0 & X & 0 \end{bmatrix} = r\begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ D'_{X_r} & 0 \end{bmatrix}$ *and* $r[X_{r_i}, \Sigma_r, \widehat{I}_n\widehat{X}_i] = r[X_{r_i}, \Sigma_r]$, $i = 1, \cdots, s$.

(iii) $[0, X]\begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ D'_{X_r} & 0 \end{bmatrix}^{\dagger}\begin{bmatrix} D_{\widehat{I}_n}\widetilde{X} \\ 0 \end{bmatrix} = 0$ *and* $r[X_{r_i}, \Sigma_r, \widehat{I}_n\widehat{X}_i] = r[X_{r_i}, \Sigma_r]$, $i = 1, \cdots, s$.

(b) *If Σ is positive definite, then $BLUE_{\mathscr{M}_r}(X\beta), BLUE_{\mathscr{M}_{r_1}}(X_1\beta_1), \cdots, BLUE_{\mathscr{M}_{r_s}}(X_s\beta_s)$ are unique and the following statements are equivalent:*

(i) *$BLUE_{\mathscr{M}_r}(X\beta) = BLUE_{\mathscr{M}_{r_1}}(X_1\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_s\beta_s)$ holds under the assumption in $\mathscr{M}_r$.*

(ii) $\Re(X_i'\Sigma^{-1}\hat{X}_i) \subseteq \Re(A_i'), i = 1, \cdots, s$.

Proof. It can be seen that the statement (i) in (a) holds if and only if (3.13) holds for any U, which is equivalent to the following equalities:

$$H = 0 \text{ and } \begin{bmatrix} E_{[X_{r_1}, \Sigma_r E_{X_{r_1}}]}\widehat{I}_{12}[X_r, \Sigma_r E_{X_r}] \\ \vdots \\ E_{[X_{r_s}, \Sigma_r E_{X_{r_s}}]}\widehat{I}_{1(s+1)}[X_r, \Sigma_r E_{X_r}] \end{bmatrix} = 0. \tag{3.15}$$

The second identity in (3.15) is equivalent to

$$r[X_{r_i}, \Sigma_r, \widehat{I}_n\widehat{X}_i] = r[X_{r_i}, \Sigma_r], \; i = 1, \cdots, s.$$

In this case, using (2.11), (2.12) and simplifying by EBMOs, we have

$$r(H) = r\left([X, 0] - [[X_1, 0], \cdots, [X_s, 0]]M^{\dagger}\begin{bmatrix} \widehat{I}_{12}[X_r, \Sigma_r E_{X_r}] \\ \vdots \\ \widehat{I}_{1(s+1)}[X_r, \Sigma_r E_{X_r}] \end{bmatrix}\right)$$

$$
\begin{aligned}
&= r\begin{bmatrix} M & K \\ [[X_1,0],\cdots,[X_s,0]] & [X,0]\end{bmatrix} - r(M)\\
&= r\begin{bmatrix} [X_{r_1},\Sigma_r E_{X_{r_1}}] & \cdots & 0 & \widehat{I}_{12}[X_r,\Sigma_r E_{X_r}] \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & [X_{r_s},\Sigma_r E_{X_{r_s}}] & \widehat{I}_{1(s+1)}[X_r,\Sigma_r E_{X_r}] \\ [X_1,0] & \cdots & [X_s,0] & [X,0]\end{bmatrix} - r[D_{\Sigma_r}, D_{X_r}]\\
&= r\begin{bmatrix} [X_{r_1},\Sigma_r E_{X_{r_1}}] & \cdots & 0 & \left[\widehat{I}_n\widehat{X}_1, 0\right] \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & [X_{r_s},\Sigma_r E_{X_{r_s}}] & \left[\widehat{I}_n\widehat{X}_s, 0\right] \\ [X_1,0] & \cdots & [X_s,0] & [0,0]\end{bmatrix} - r[D_{\Sigma_r}, D_{X_r}]\\
&= r\begin{bmatrix} D_{\Sigma_r}E_{D_{X_r}} & D_{X_r} & D_{\widehat{I}_n}\widetilde{X} \\ 0 & X & 0\end{bmatrix} - r[D_{\Sigma_r}, D_{X_r}]\\
&= r\begin{bmatrix} D_{\Sigma_r} & D_{X_r} & D_{\widehat{I}_n}\widetilde{X} \\ D'_{X_r} & 0 & 0 \\ 0 & X & 0\end{bmatrix} - r\begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ D'_{X_r} & 0\end{bmatrix},
\end{aligned}
$$

where $K = \begin{bmatrix} \widehat{I}_{12}[X_r,\Sigma_r E_{X_r}] \\ \vdots \\ \widehat{I}_{1(s+1)}[X_r,\Sigma_r E_{X_r}]\end{bmatrix}$. Hence, $H = 0$ if and only if

$$
r\begin{bmatrix} D_{\Sigma_r} & D_{X_r} & D_{\widehat{I}_n}\widetilde{X} \\ D'_{X_r} & 0 & 0 \\ 0 & X & 0\end{bmatrix} = r\begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ D'_{X_r} & 0\end{bmatrix},
$$

which is equivalent to

$$
[0, X]\begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ D'_{X_r} & 0\end{bmatrix}^{\dagger}\begin{bmatrix} D_{\widehat{I}_n}\widetilde{X} \\ 0\end{bmatrix} = 0. \quad \text{(by (2.13))}
$$

Summarizing the above discussion, the equivalence of (i), (ii) and (iii) in (a) is proved. The results of (b) can be obtained from (a). □

Similar to the proofs of the Theorem 3.2 and Theorem 3.3, we can get the following two theorems which characterize the equality in (II).

Theorem 3.4. *Let $BLUE_{\mathscr{M}_r}(X_r\beta)$ and $BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i), i = 1,\cdots,s$ be as given in (2.7) and (2.9), and let $D_{\widehat{I}_n}$, D_{Σ_r}, D_{X_r} and $\widetilde{X}$ be as given in (3.5) and (3.6). Then:*

(a) *The sum of the BLUEs is given by*

$$BLUE_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s) = (P_{X_{r_1};\Sigma_r}\widehat{I}_{12} + \cdots + P_{X_{r_s};\Sigma_r}\widehat{I}_{1(s+1)})y_r \tag{3.16}$$

with the expectation

$$E[BLUE_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s)] = X_r\beta + [P_{X_{r_1};\Sigma_r}, \cdots, P_{X_{r_s};\Sigma_r}]D_{\widehat{I}_n}\widetilde{X}\beta \tag{3.17}$$

under the assumption in $\mathscr{M}_r$.

(b) *The following statements are equivalent:*

(i) *There exist* $BLUE_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1), \cdots, BLUE_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s)$ *such that*

$$E[BLUE_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s)] = X_r\beta \tag{3.18}$$

holds under the assumption in $\mathscr{M}_r$.

(ii) *There exist* $BLUE_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1), \cdots, BLUE_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s)$ *such that*

$$BLUE_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s) \in \{BLUE_{\mathscr{M}_r}(X_r\beta)\} \tag{3.19}$$

holds under the assumption in $\mathscr{M}_r$.

(iii) $r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \\ 0 & 0 & X_r \end{bmatrix} = r\begin{bmatrix} D_{\Sigma_r} & D_{\widehat{I}_n}\widetilde{X} & D_{X_r} \\ D'_{X_r} & 0 & 0 \end{bmatrix}$.

(iv) $\Re\begin{bmatrix} 0 \\ 0 \\ X'_r \end{bmatrix} \subseteq \Re\begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ \widetilde{X}'D'_{\widehat{I}_n} & 0 \\ D'_{X_r} & 0 \end{bmatrix}$ *or* $\Re(X'_r) \subseteq \Re([D'_{X_r}, 0]E_T)$, *where* $T = \begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ \widetilde{X}'D'_{\widehat{I}_n} & 0 \end{bmatrix}$.

Theorem 3.5. *Let* $BLUE_{\mathscr{M}_r}(X_r\beta)$ *and* $BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i), i = 1, \cdots, s$ *be as given in (2.7) and (2.9), and let* $D_{\widehat{I}_n}$, D_{Σ_r}, D_{X_r} *and* $\widetilde{X}$ *be given as (3.5) and (3.6). Then:*

(a) *The following statements are equivalent:*

(i) *The set inclusion*

$$\{BLUE_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s)\} \subseteq \{BLUE_{\mathscr{M}_r}(X_r\beta)\}$$

holds under the assumption in $\mathscr{M}_r$.

(ii) $r\begin{bmatrix} D_{\Sigma_r} & D_{X_r} & D_{\widehat{I}_n}\widetilde{X} \\ D'_{X_r} & 0 & 0 \\ 0 & X_r & 0 \end{bmatrix} = r\begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ D'_{X_r} & 0 \end{bmatrix}$ *and* $r[X_{r_i}, \Sigma_r, \widehat{I}_n\widehat{X}_i] = r[X_{r_i}, \Sigma_r]$, $i = 1, \cdots, s$.

(iii) $[0, X_r]\begin{bmatrix} D_{\Sigma_r} & D_{X_r} \\ D'_{X_r} & 0 \end{bmatrix}^{\dagger}\begin{bmatrix} D_{\widehat{I}_n}\widetilde{X} \\ 0 \end{bmatrix} = 0$ *and* $r[X_{r_i}, \Sigma_r, \widehat{I}_n\widehat{X}_i] = r[X_{r_i}, \Sigma_r]$, $i = 1, \cdots, s$.

(b) *If* Σ *is positive definite, then* $BLUE_{\mathscr{M}_r}(X_r\beta)$, $BLUE_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1), \cdots, BLUE_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s)$ *are unique and the following statements are equivalent:*

(i) $BLUE_{\mathscr{M}_r}(X_r\beta) = BLUE_{\mathscr{M}_{r_1}}(X_{r_1}\beta_1) + \cdots + BLUE_{\mathscr{M}_{r_s}}(X_{r_s}\beta_s)$ *holds under the assumption in* $\mathscr{M}_r$.

(ii) $\Re(X_i'\Sigma^{-1}\hat{X}_i) \subseteq \Re(A_i'), i = 1, \cdots, s$.

When setting $s = 2$ in Theorems 3.2-3.5, the corresponding results are given by Zhang and Tian in [29].

In the following, we will give the necessary and sufficient conditions for $\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_i\beta_i)$ and $\mathrm{BLUE}_{\mathscr{M}_{r_j}}(X_j\beta_j)$ to be uncorrelated and $\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)$ and $\mathrm{BLUE}_{\mathscr{M}_{r_j}}(X_{r_j}\beta_j)$ to be uncorrelated, which are stated as follows:

Theorem 3.6. *Let* $BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)$, $BLUE_{\mathscr{M}_{r_j}}(X_j\beta_j)$, $BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)$ *and* $BLUE_{\mathscr{M}_{r_j}}(X_{r_j}\beta_j)$, $i \neq j, i, j = 1, \cdots, s$ *be as given in (2.8) and (2.9). Then*

(a)

$$\begin{aligned} &Cov\{BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i), BLUE_{\mathscr{M}_{r_j}}(X_j\beta_j)\} \\ &= [X_i, 0][X_{r_i}, \Sigma_r E_{X_{r_i}}]^{\dagger}\Sigma_r([X_j, 0][X_{r_j}, \Sigma_r E_{X_{r_j}}]^{\dagger})', i \neq j, i, j = 1, \cdots, s \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &Cov\{BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i), BLUE_{\mathscr{M}_{r_j}}(X_{r_j}\beta_j)\} \\ &= [X_{r_i}, 0][X_{r_i}, \Sigma_r E_{X_{r_i}}]^{\dagger}\Sigma_r([X_{r_j}, 0][X_{r_j}, \Sigma_r E_{X_{r_j}}]^{\dagger})', i \neq j, i, j = 1, \cdots, s. \end{aligned} \tag{3.21}$$

(b) *The following statements are equivalent:*

(i) $Cov\{BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i), BLUE_{\mathscr{M}_{r_j}}(X_j\beta_j)\} = 0,\ i \neq j, i, j = 1, \cdots, s$.
In this case, $BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)$ *and* $BLUE_{\mathscr{M}_{r_j}}(X_j\beta_j)$ *is uncorrelated.*

(ii) $Cov\{BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i), BLUE_{\mathscr{M}_{r_j}}(X_{r_j}\beta_j)\} = 0,\ i \neq j, i, j = 1, \cdots, s$.
In this case, $BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)$ *and* $BLUE_{\mathscr{M}_{r_j}}(X_{r_j}\beta_j)$ *is uncorrelated.*

(iii)

$$r\begin{bmatrix} \Sigma_r & X_{r_j} \\ X'_{r_i} & 0 \end{bmatrix} = r[X_{r_i}, \Sigma_r] + r[X_{r_j}, \Sigma_r] - r(\Sigma), i \neq j, i, j = 1, \cdots, s. \tag{3.22}$$

(c) *If* Σ *is positive definite, then the following statements are equivalent:*

(i) $Cov\{BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i), BLUE_{\mathscr{M}_{r_j}}(X_j\beta_j)\} = 0,\ i \neq j, i, j = 1, \cdots, s$.
In this case, $BLUE_{\mathscr{M}_{r_i}}(X_i\beta_i)$ *and* $BLUE_{\mathscr{M}_{r_j}}(X_j\beta_j)$ *is uncorrelated.*

(ii) $Cov\{BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i), BLUE_{\mathscr{M}_{r_j}}(X_{r_j}\beta_j)\} = 0,\ i \neq j, i, j = 1, \cdots, s$.
In this case, $BLUE_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i)$ *and* $BLUE_{\mathscr{M}_{r_j}}(X_{r_j}\beta_j)$ *is uncorrelated.*

(iii) $r\begin{bmatrix} X_i'\Sigma^{-1}X_j & A_i' \\ A_j & 0 \end{bmatrix} = r(A_i) + r(A_j),\ i \neq j, i, j = 1, \cdots, s$.

Proof. From (2.8) and (2.9), we have

$$\begin{aligned} & Cov\{\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_i\beta_i), \mathrm{BLUE}_{\mathscr{M}_{r_j}}(X_j\beta_j)\} \\ &= P_{X_i;X_{r_i};\Sigma_r}\widehat{I}_{1(i+1)}\Sigma_r\widehat{I}'_{1(j+1)}P'_{X_j;X_{r_j};\Sigma_r} \\ &= [X_i,0][X_{r_i},\Sigma_r E_{X_{r_i}}]^{\dagger}\Sigma_r([X_j,0][X_{r_j},\Sigma_r E_{X_{r_j}}]^{\dagger})', i\neq j, i,j=1,\cdots,s \end{aligned}$$

and

$$\begin{aligned} & Cov\{\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_{r_i}\beta_i), \mathrm{BLUE}_{\mathscr{M}_{r_j}}(X_{r_j}\beta_j)\} \\ &= P_{X_{r_i};\Sigma_r}\widehat{I}_{1(i+1)}\Sigma_r\widehat{I}'_{1(j+1)}P'_{X_{r_j};\Sigma_r} \\ &= [X_{r_i},0][X_{r_i},\Sigma_r E_{X_{r_i}}]^{\dagger}\Sigma_r([X_{r_j},0][X_{r_j},\Sigma_r E_{X_{r_j}}]^{\dagger})', i\neq j, i,j=1,\cdots,s. \end{aligned}$$

Applying (2.11), (2.14) and simplifying by EBMOs, we have

$$\begin{aligned} & r(Cov\{\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_i\beta_i), \mathrm{BLUE}_{\mathscr{M}_{r_j}}(X_j\beta_j)\}) \\ &= r([X_i,0][X_{r_i},\Sigma_r E_{X_{r_i}}]^{\dagger}\Sigma_r([X_j,0][X_{r_j},\Sigma_r E_{X_{r_j}}]^{\dagger})') \\ &= r\begin{bmatrix} \Sigma_r & [X_{r_i},\Sigma_r E_{X_{r_i}}] & 0 \\ \left[X_{r_j},\Sigma_r E_{X_{r_j}}\right]' & 0 & [X_j,0]' \\ 0 & [X_i,0] & 0 \end{bmatrix} - r[X_{r_i},\Sigma_r E_{X_{r_i}}] - r[X_{r_j},\Sigma_r E_{X_{r_j}}] \\ &= r\begin{bmatrix} \Sigma_r & X_{r_i} & \Sigma_r & 0 & 0 \\ X'_{r_j} & 0 & 0 & X'_j & 0 \\ \Sigma_r & 0 & 0 & 0 & X_{r_j} \\ 0 & X_i & 0 & 0 & 0 \\ 0 & 0 & X'_{r_i} & 0 & 0 \end{bmatrix} - r(X_{r_i}) - r(X_{r_j}) - r[X_{r_i},\Sigma_r] - r[X_{r_j},\Sigma_r] \\ &= r\begin{bmatrix} \Sigma_r & X_{r_j} \\ X'_{r_i} & 0 \end{bmatrix} + r(\Sigma) - r[X_{r_i},\Sigma_r] - r[X_{r_j},\Sigma_r]. \end{aligned}$$

Hence, $Cov\{\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_i\beta_i), \mathrm{BLUE}_{\mathscr{M}_{r_j}}(X_j\beta_j)\} = 0$ if and only if

$$r\begin{bmatrix} \Sigma_r & X_{r_j} \\ X'_{r_i} & 0 \end{bmatrix} = r[X_{r_i},\Sigma_r] + r[X_{r_j},\Sigma_r] - r(\Sigma), i\neq j, i,j=1,\cdots,s.$$

Similar to the proof for $Cov\{\mathrm{BLUE}_{\mathscr{M}_{r_i}}(X_i\beta_i), \mathrm{BLUE}_{\mathscr{M}_{r_j}}(X_j\beta_j)\} = 0$, we get the equivalence of (ii) and (iii) in (b). From (b), we can get the results of (c). $\square$

4. Conclusions

In this paper, we have obtained the necessary and sufficient conditions for the BLUEs of $X\beta$ and $X_r\beta$ under the full model $\mathscr{M}_r$ to be the sums of the BLUEs of $X_i\beta_i$ and $X_{r_i}\beta_i$, $i=1,\cdots,s$ under the s small models $\mathscr{M}_{r_i}$, respectively. Moreover, the expectations, the unbiasedness and the covariance matrices of these BLUEs were also derived. The process for the additive decompositions of the BLUEs is mainly based on the theory of generalized inverses of matrices, ranks of matrices and EBMOs.

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