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EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATION WITH *P*-LAPLACIAN THROUGH VARIATIONAL METHOD*

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Abstract In this paper, a class of fractional differential equation with p-Laplacian operator is studied based on the variational approach. Combining the mountain pass theorem with iterative technique, the existence of at least one nontrivial solution for our equation is obtained. Additionally, we demonstrate the application of our main result through an example.

Keywords Fractional differential equation, p-Laplacian operator, variational method, mountain pass theorem, iterative technique.

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1. Introduction

In this paper, we consider the following p-Laplacian fractional differential boundary value problem (BVP for short) with Dirichlet's boundary value condition:

$$\begin{cases} {}_tD_T^\alpha\left(\frac{1}{\omega(t)^{p-2}}\varphi_p(\omega(t){}_0D_t^\alpha u(t))\right)+\lambda u(t)=f(t,u,{}_0^cD_t^\alpha u(t))+h(u(t)), \\ u(0)=u(T)=0,\ \text{a.e. } t\in[0,T], \end{cases} \tag{1.1}$$

where $\frac{1}{p}<\alpha\leq 1$, λ is a non-negative real parameter, ${}_0^cD_t^\alpha$ is the left Caputo derivative, ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$ denote the left and right standard Riemann-Liouville fractional derivatives, respectively. $\omega(t)\in L^\infty[0,T]$ with $\omega_0=\operatorname{ess\,inf}_{[0,T]}\omega(t)>0$ and $\omega^0=\operatorname{ess\,sup}_{[0,T]}\omega(t)$. The functions $\varphi_p(s)=|s|^{p-2}s$, $p\geq 2$, $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, $h:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L>0$, i.e.,

$$\mid h(x_1)-h(x_2)\mid\leq L\mid x_1-x_2\mid, \tag{1.2}$$

for every $x_1,x_2\in\mathbb{R}$, and $h(0)=0$.

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Fractional calculus is a broader concept, since it is a generalization of arbitrary order derivatives and integrals. With the development of fractional differential equation (FDE for short), a growing number of researchers have been aroused to discuss the existence of solutions for nonlinear FDE owing to the vast application space in different areas of science and engineering, such as physics, mechanics, chemistry, economics, engineering and biological sciences, etc. For details, see [7, 8, 14, 17, 18]. In recent years, the existence of solutions for nonlinear FDE has been established with all kinds of classical tools, such as fixed-point theorems, the method of upper and lower solutions, the topological degree theory and the critical point theory, etc. (see [1–3, 9, 12] and references therein). In [1], by using the Schauder fixed point theorem, the existence results were obtained for the fractional differential equation with three-point boundary conditions. By means of the Leray-Schauder degree theory and upper and lower solutions method, the existence of multiple solutions was proved for the fractional BVP in [12]. Especially, because of the practicability and effectiveness of variational methods and critical point theory, more and more scholars have paid attention to tackling the existence of solutions for fractional BVP by applying those tools, such as [4, 10, 11, 13, 23, 24], although it is often difficult to develop appropriate function spaces and variational frameworks for FDE containing both left and right fractional derivatives. For example, in [13], under suitable assumptions, the existence of at least one solution for the following FDE was obtained by applying the mountain pass theorem

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)),\ a.e.\ t\in[0,T], \\ u(0)=u(T)=0, \end{cases} \tag{1.3}$$

where ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville derivatives with order $0<\alpha\le 1$, respectively. $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$, $\nabla F(t,u(t))$ is the gradient of F at u. Recently, in [10], Heidarkhani et al. investigated the existence results for FDE with the following form

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t){}_0D_t^{\alpha_i}u_i(t)) = F_{u_i}(t,u_1,...,u_n)+h_i(u_i(t)),\ t\in(0,T), \\ u_i(0)=u_i(T)=0, \end{cases} \tag{1.4}$$

for $1\le i\le n$, where $a_i(t)\in L^{\infty}[0,T]$ with $\overline{a}_i = \operatorname{ess\,inf}_{[0,T]} a_i(t)>0$, for $1\le i\le n$. $F:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ is measurable with respect to t, for all $u\in\mathbb{R}^n$, continuously differentiable in u, for any $t\in[0,T]$ such that $F(t,0,...,0)=0$ for any $t\in[0,T]$, $h_i:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function, $1\le i\le n$. Based on variational methods, the existence of one weak solution for BVP (1.4) was established.

In addition, the existence of solutions for fractional BVP with generalized p-Laplacian operator has been discussed via using variational methods in recent years. Chen in [5] considered the existence of at least one weak solution for a class of p-Laplacian type FDE by using variational method as below

$$\begin{cases} {}_tD_T^{\alpha}\varphi_p({}_0D_t^{\alpha}u(t)) = f(t,u(t)),\ t\in[0,T], \\ u(0)=u(T)=0, \end{cases} \tag{1.5}$$

where $0<\alpha\le 1$, ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville derivatives, respectively. $\varphi_p(s)=\mid s\mid^{p-2}s$, $p>1$.

However, with the advent of the fractional derivative contained in the nonlinearity f, we are not able to deal with the existence of solutions of BVP just relying on variational method and critical point theory directly. Therefore, in this paper, combining the variational method with iterative technique, the existence results are obtained for a class of generalized p-Laplacian type fractional boundary value problem with nonlinear function f including the fractional derivative ${}_0^cD_t^\alpha$.

The main contributions of our work include three points. Firstly, the suitable function space and the variational framework are developed reasonably for BVP (1.1). Then, a new criteria on the existence of solutions is obtained for BVP (1.1). Secondly, the nonlocal and nonlinear differential operator ${}_tD_T^\alpha \varphi_p({}_0D_t^\alpha)$ can be reduced to the linear differential operator ${}_tD_T^\alpha {}_0D_t^\alpha$ under $p = 2$. Thus, the content of this article is discussed based on the space of $L^p([0, T], \mathbb{R})$ $(2 \leq p < \infty)$, which is a generalization for the existing results based on the inner product space of $L^2([0, T], \mathbb{R})$. Finally, comparing with the published relevant results, some looser assumptions are given to guarantee the existence of solutions for BVP (1.1) in this paper. For instance, the literature [6] discussed a class of fractional equation whose nonlinear function f includes the fractional derivative, and the complex parameter conditions $P_0 < 1$ and $\frac{Q_0}{1-P_0} < 1$ were required to ensure the existence of solutions of the equation. In our assumptions, the analogous restricted conditions do not appear. Hence, the conclusion obtained in the paper is more convenience for application and differ from the results mentioned above.

The organization of this paper is as follows. Section 2 shows a brief review of fractional calculus and the construct of theoretical framework. In section 3, the main result is proposed to guarantee the existence of solutions of BVP (1.1). Then, we demonstrate the application of our result through an example in Section 4. Finally, a conclusion is given in Section 5.

2. Preliminaries and lemmas

In this section, some associated definitions and basic lemmas are introduced, which will be used throughout this paper.

Let $L^p([0, T], \mathbb{R})$ $(1 \leq p < \infty)$ be the space of functions for which the p-th power of the absolute value is Lebesgue integrable with the norm

$$\|x\|_{L^p} = \left(\int_0^T \mid x(t) \mid^p dt\right)^{\frac{1}{p}}, \quad \forall\, x \in L^p([0, T], \mathbb{R}), \text{ a.e. } t \in [0, T], \tag{2.1}$$

$C([0, T], \mathbb{R})$ be the space of continuous functions with the norm $\| x \|_\infty = \max_{t\in[0,T]} |x(t)|$.

Definition 2.1 ([14, 19]). Let x be a function on $[0, T]$. Define the left and right Riemann-Liouville fractional integrals with order $0 < \alpha \leq 1$ by

$${}_0D_t^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \eta)^{\alpha-1} x(\eta) d\eta,$$

and

$${}_tD_T^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (\eta - t)^{\alpha-1} x(\eta) d\eta,$$

respectively.

Definition 2.2 ([14, 19]). The left and right Riemann-Liouville fractional derivatives with order α are represented as

$$ {}_0D_t^{\alpha}x(t) = \frac{d}{dt}{}_0D_t^{\alpha-1}x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\eta)^{-\alpha}x(\eta)d\eta, $$

and

$$ {}_tD_T^{\alpha}x(t) = (-1)\frac{d}{dt}{}_tD_T^{\alpha-1}x(t) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T (t-\eta)^{-\alpha}x(\eta)d\eta, $$

where $0 < \alpha \leq 1$ and x is a function defined on $[0, T]$.

Literatures [14] and [21] show that the Riemann-Liouville fractional integrals satisfy the following property.

Property 2.1. If $f \in L^p([0,T],\mathbb{R})$, $g \in L^q([0,T],\mathbb{R})$ and $p \geq 1, q \geq 1, \frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ or $p \neq 1, q \neq 1, \frac{1}{p}+\frac{1}{q} = 1+\alpha$. Then

$$ \int_0^T ({}_0D_t^{-\alpha}f(t))g(t)dt = \int_0^T ({}_tD_T^{-\alpha}g(t))f(t)dt,\ \alpha > 0. $$

Nextly, the suitable function space and the variational framework are developed to apply variational method.

Definition 2.3. Let $0 < \alpha \leq 1$, and $2 \leq p < \infty$. The fractional derivative space E_p^{α} is defined by the closure $C_0^{\infty}([0,T],\mathbb{R})$, i.e., $E_p^{\alpha} = \overline{C_0^{\infty}([0,T],\mathbb{R})}$ with the norm

$$ \|u\|_{\alpha,p} = \left(\int_0^T \mid u(t)\mid^p dt + \int_0^T \omega(t)\mid {}_0D_t^{\alpha}u(t)\mid^p dt\right)^{\frac{1}{p}},\quad \forall\ u \in E_p^{\alpha}. \tag{2.2} $$

Remark 2.1. Obviously, E_p^{α} is the space of functions $u(t) \in L^p([0,T],\mathbb{R})$ with an α-order Riemann-Liouville fractional derivative ${}_0D_t^{\alpha}u(t) \in L^p([0,T],\mathbb{R})$ and $u(0) = u(T) = 0$.

Property 2.2. From [14], the following properties hold

$$ {}_0D_t^{\alpha}u(t) = {}_0^cD_t^{\alpha}u(t),\quad {}_tD_T^{\alpha}u(t) = {}_t^cD_T^{\alpha}u(t),\quad \forall\ u(t) \in E_p^{\alpha},\quad \text{a.e. } t \in [0,T], $$

where ${}_a^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives with order α, respectively. (See [14] for a detailed introduction of Caputo fractional derivatives and integrals).

Lemma 2.1 ([13]). *Let $0 < \alpha \leq 1$, and $1 < p < \infty$. For any $u \in E_p^{\alpha}$, we have*

$$ \| u \|_{L^p} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \| {}_0D_t^{\alpha}u \|_{L^p}, \tag{2.3} $$

furthermore, when $\alpha > \frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q} = 1$, we have

$$ \| u \|_{\infty} \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \| {}_0D_t^{\alpha}u \|_{L^p}\ . \tag{2.4} $$

By Lemma 2.1, we obtain

$$\| u \|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)(\omega_0)^{\frac{1}{p}}} \left(\int_0^T \omega(t) \mid {}_0D_t^\alpha u \mid^p dt \right)^{\frac{1}{p}}, \quad 0 < \alpha \leq 1, \tag{2.5}$$

$$\| u \|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\omega_0)^{\frac{1}{p}}((\alpha-1)q+1)^{\frac{1}{q}}} \left(\int_0^T \omega(t) \mid {}_0D_t^\alpha u \mid^p dt \right)^{\frac{1}{p}}, \ \frac{1}{p} < \alpha \leq 1. \tag{2.6}$$

Denote $\Lambda = \frac{T^\alpha}{\Gamma(\alpha+1)(\omega_0)^{\frac{1}{p}}}$ and $\overline{\Lambda} = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\omega_0)^{\frac{1}{p}}((\alpha-1)q+1)^{\frac{1}{q}}}$.

Based on (2.5), the norm of (2.2) is equivalent to

$$\|u\|_{\alpha,p} = \left(\int_0^T \omega(t) \mid {}_0D_t^\alpha u(t) \mid^p dt \right)^{\frac{1}{p}}, \quad \forall u \in E_p^\alpha. \tag{2.7}$$

Lemma 2.2 (Lemma 9, [15])**.** *The fractional derivative space E_p^α is a reflexive and separable Banach space.*

Lemma 2.3 ([13])**.** *Let $\frac{1}{p} < \alpha \leq 1$, and $1 < p < \infty$. Assume that the sequence $\{u_k\}_{k\in\mathbb{N}}$ converges weakly to u in E_p^α, that is $u_k \rightharpoonup u$, as $k \to \infty$. Then $u_k \to u$ in $C([0,T],\mathbb{R})$ as $k \to \infty$, which means that $\| u_k - u \|_\infty \to 0$, as $k \to \infty$.*

Lemma 2.4. *Let $u(t) \in E_p^\alpha$. According to [15], the following relationship*

$$\int_0^T {}_tD_T^\alpha \left(\frac{1}{\omega(t)^{p-2}}\varphi_p(\omega(t)_0D_t^\alpha u(t))\right)v(t)dt = \int_0^T \frac{1}{\omega(t)^{p-2}}\varphi_p(\omega(t)_0D_t^\alpha u(t))_0D_t^\alpha v(t)dt$$

holds, for any $v(t) \in E_p^\alpha$.

Hence, the definition of weak solution for the BVP (1.1) can be given as below.

Definition 2.4. We say $u(t) \in E_p^\alpha$ is a weak solution of the BVP (1.1). If the following identity

$$\begin{aligned} &\int_0^T \frac{1}{\omega(t)^{p-2}}\varphi_p(\omega(t)_0D_t^\alpha u(t))_0D_t^\alpha v(t) + \lambda u(t)\cdot v(t) - h(u(t))v(t)dt \\ =&\int_0^T f(t, u(t), {}_0D_t^\alpha u(t))v(t)dt \end{aligned}$$

holds, for any $v(t) \in E_p^\alpha$.

In order to obtain our result, let us first consider the functional $I_\xi : E_p^\alpha \to \mathbb{R}$ for any fixed $\xi(t) \in E_p^\alpha$ as follows

$$\begin{aligned} I_\xi(u(t)) =& \frac{1}{p}\|u\|_{\alpha,p}^p + \frac{1}{2}\int_0^T \lambda \mid u(t) \mid^2 dt - \int_0^T H(u(t))dt \\ &- \int_0^T F(t, u(t), {}_0D_t^\alpha \xi(t))dt, \end{aligned} \tag{2.8}$$

where $u(t) \in E_p^\alpha$, $F(t,x,y) = \int_0^x f(t,s,y)ds$ and $H(x) = \int_0^x h(s)ds$, for $x, y \in \mathbb{R}$.

Since E_p^α is compactly embedded in $C([0,T],\mathbb{R})$ and f is continuous, we can know that I_ξ is a continuous and Fréchet differentiable functional on E_p^α. The Fréchet derivative of I_ξ at the point $u \in E_p^\alpha$ is given as

$$\begin{aligned}\langle I'_\xi(u(t)), v(t)\rangle = &\int_0^T \frac{1}{\omega(t)^{p-2}}\varphi_p(\omega(t)_0D_t^\alpha u(t))_0D_t^\alpha v(t) + \lambda u(t)\cdot v(t)dt\\ &- \int_0^T h(u(t))\cdot v(t)dt - \int_0^T f(t,u(t),{}_0D_t^\alpha \xi(t))\cdot v(t)dt, \qquad (2.9)\end{aligned}$$

for any $v(t) \in E_p^\alpha$, a.e. $t \in [0,T]$.

Lemma 2.5. *Let $0 < \alpha \leq 1$, and $2 \leq p < \infty$. We say $u(t)$ is a classical solution of BVP (1.1). If the function $u(t) \in E_p^\alpha$ is a nontrivial weak solution of BVP (1.1).*

Proof. In fact, if $u(t) \in E_p^\alpha$ is a nontrivial weak solution of BVP (1.1), then, Definition 2.4 is satisfied for any $v(t) \in E_p^\alpha$. According to Lemma 2.4, we have

$$\begin{aligned}&\int_0^T \frac{1}{\omega(t)^{p-2}}\varphi_p(\omega(t)_0D_t^\alpha u(t))_0D_t^\alpha v(t)dt\\ =&\int_0^T {}_tD_T^\alpha\left(\frac{1}{\omega(t)^{p-2}}\varphi_p(\omega(t)_0D_t^\alpha u(t))\right)v(t)dt. \qquad (2.10)\end{aligned}$$

Combining Definition 2.4 with (2.10), yields

$${}_tD_T^\alpha\left(\frac{1}{\omega(t)^{p-2}}\varphi_p(\omega(t)_0D_t^\alpha u(t))\right) + \lambda u(t) = f(t,u,{}_0^cD_t^\alpha u(t)) + h(u(t)),$$

for a.e. $t \in [0,T]$. Namely, u satisfies the equation of (1.1).

Moreover, $u(t) \in E_p^\alpha = \overline{C_0^\infty([0,T],\mathbb{R})}$ means that $u(0) = u(T) = 0$, i.e., the boundary value condition of (1.1) holds. Hence, $u(t)$ is a classical solution of BVP (1.1).

Conversely, if $u(t) \in E_p^\alpha$ is a nontrivial classical solution of BVP (1.1), $u(t)$ is also a weak solution of BVP (1.1) obviously. The proof is completed. □

Lemma 2.6 (Multiple Hölder inequality, [16])**.** *If $f_i \in L^{q_i}(E)$, where E is a measurable space, $i = 1,...,n$, and $\sum_{i=1}^n \frac{1}{q_i} = 1$, where $q_i \geq 1$, then*

$$\| \Pi_{i=1}^n f_i \|_{L^1} \leq \Pi_{i=1}^n \| f_i \|_{L^{q_i}} .$$

Definition 2.5 (P.S. condition)**.** Let E be a Banach space. We say functional $I \in C^1(E,\mathbb{R})$ satisfy the Palais-Smale (P.S. for short) condition, if for any sequence $\{u_k\}_{k=1}^\infty \subset E$, for which $\{I(u_k)\}_{k=1}^\infty$ is bounded and $\lim_{k\to\infty} I'(u_k) = 0$, possesses a convergent subsequence in E.

Theorem 2.1 (Mountain pass theorem, [20])**.** *Let E be a real Banach space and functional $I \in C^1(E,\mathbb{R})$ satisfying the P.S. condition. Suppose that*

(i) $I(0) = 0$;

(ii) *There exist $\rho > 0$ and $\sigma > 0$ such that $I(z) \geq \sigma$ for every $z \in E$ with $\| z \| = \rho$;*

(iii) *There exists $z_1 \in E$ with $\| z_1 \| \geq \rho$ such that $I(z_1) < \sigma$.*

Then, functional I possesses a critical value $z^* \geq \sigma$. Moreover, z^* can be characterized as

$$z^* = \inf_{g \in \Omega} \max_{z \in g([0,1])} I(z),$$

where $\Omega = \{g \in C([0,1], E) \mid g(0) = 0, g(1) = z_1\}$.

3. Main results

In this section, the existence of solutions for BVP (1.1) is established by using Theorem 2.1 and iterative technique.

Firstly, some necessary assumptions are stated, which will be used in the further discussion of the main result.

(H_1) There exist constants $\tau > p$, $a \geq 0, b \geq 0, d \geq 0$ and $0 < \beta, \overline{\beta} < p$, such that

$$\tau F(t,x,y) - f(t,x,y)x \leq a \mid x \mid^{\beta} + b \mid y \mid^{\overline{\beta}} + d,$$

for $x, y \in \mathbb{R}$, a.e. $t \in [0,T]$.

(H_2) There exist non-negative constants s_1, s_2, s_1', s_2', δ and $\zeta > p, \gamma > p$, $0 < \eta_1 < p$, $0 < \eta_2 \leq p-1$ and functions $c(t)$, $\vartheta(t) \in L^1([0,T], \mathbb{R}^+)$, such that

$$F(t,x,y) \leq s_1 \mid x \mid^{\zeta} + s_2 x^{\gamma} \mid y \mid^{\eta_1} - c(t), \ \mid x \mid \leq \delta, \ y \in \mathbb{R}, \tag{3.1}$$

$$F(t,x,y) \geq s_1' \mid x \mid^{\zeta} - s_2' x \mid y \mid^{\eta_2} - \vartheta(t), \quad \text{for } x, y \in \mathbb{R}. \tag{3.2}$$

(H_3) There exist nonnegative constants M_1, M_2 such that

$$\mid f(t,x,y) - f(t,x',y') \mid \leq M_1 \mid x - x' \mid + M_2 \mid y - y' \mid,$$

for $x, x' \in [-G^*, G^*]$, $y, y' \in \mathbb{R}$, a.e. $t \in [0,T]$, where G^* is introduced in the sequel.

In order to describe easily for the further analysis, some notations are given as below. Denote

$$u_0(t) = \begin{cases} \frac{\Gamma(2-\alpha)}{T} t, & t \in [0, \frac{T}{4}[, \\ \Gamma(2-\alpha), & t \in [\frac{T}{4}, \frac{3T}{4}], \\ \frac{\Gamma(2-\alpha)}{T}(T-t), & t \in]\frac{3T}{4}, T], \end{cases} \ \widetilde{u_0} = \frac{u_0}{\|u_0\|_{\alpha,p}}, \ A = (\frac{TL}{2} + \frac{T\lambda}{2})\overline{\Lambda}^2,$$

$$B = \frac{s_2'}{(\omega_0)^{\frac{\eta_2}{p}}} \parallel \widetilde{u_0} \parallel_{L^{\frac{p}{p-\eta_2}}}, D = s_1' \parallel \widetilde{u_0} \parallel_{L^{\zeta}}^{\zeta}, \ W = \frac{p-\eta_2}{p}(B)^{\frac{p}{p-\eta_2}} \cdot (\frac{6\eta_2\tau}{\tau - p})^{\frac{\eta_2}{p-\eta_2}},$$

$$d_1 = (\frac{T\tau L}{2} + LT)\overline{\Lambda}^2, \ d_2 = aT\overline{\Lambda}^{\beta}, \ d_3 = b\left(\frac{T^{p-\overline{\beta}}}{(\omega_0)^{\overline{\beta}}}\right)^{\frac{1}{p}},$$

$$d_1^* = \frac{p-2}{p}\left(\frac{12}{\tau-p}\right)^{\frac{2}{p-2}} d_1^{\frac{p}{p-2}}, \ d_2^* = \frac{p-\beta}{p}\left(\frac{6\beta}{\tau-p}\right)^{\frac{\beta}{p-\beta}} d_2^{\frac{p}{p-\beta}},$$

$$d_3^* = \frac{p-\overline{\beta}}{p}\left(\frac{6\overline{\beta}}{\tau-p}\right)^{\frac{\overline{\beta}}{p-\overline{\beta}}} d_3^{\frac{p}{p-\overline{\beta}}},$$

$$G = \left(\frac{6p}{2(\tau-p)}(\tau C_3 + \tau \parallel \vartheta \parallel_{L^1} + d_1^* + d_2^* + d_3^* + dT)\right)^{\frac{1}{p}}, \ G^* = \overline{\Lambda} G.$$

Theorem 3.1. *Let $\frac{1}{p} < \alpha \leq 1$, $2 \leq p < \infty$. Suppose that the conditions $(H_1)-(H_3)$ hold, and $\lambda - L \geq 0$. Then, BVP (1.1) has at least one nontrivial solution on E_p^α.*

Proof. The proof will be shown as four steps.

Step 1. We claim that functional I_ξ satisfies the *P.S.* condition.

Suppose that $\{u_k\}_{k=1}^\infty \subset E_p^\alpha$ is a sequence such that $\{I_\xi(u_k)\}_{k=1}^\infty$ is bounded and $I'_\xi(u_k) \to 0$ as $k \to \infty$.

For any fixed $\xi(t) \in E_p^\alpha$ with $\| \xi \|_{\alpha,p} \leq G$. Combining (1.2) with $h(0) = 0$, one has $| h(u) | \leq L | u |$ for any $u \in \mathbb{R}$. Then, based on (1.2), (2.6), (2.8), (2.9) and (H_1), we have

$$\begin{aligned}
&\tau I_\xi(u_k(t)) - \langle I'_\xi(u_k(t)), u_k(t)\rangle \qquad (3.3)\\
=&(\frac{\tau}{p} - 1) \| u_k \|_{\alpha,p}^p + (\frac{\lambda\tau}{2} - \lambda)\int_0^T | u_k(t) |^2 \, dt + \int_0^T h(u_k(t))u_k(t) - \tau H(u_k(t))dt\\
&+ \int_0^T f(t, u_k(t), {}_0D_t^\alpha \xi(t))u_k(t) - \tau F(t, u_k(t), {}_0D_t^\alpha \xi(t))dt\\
\geq&(\frac{\tau}{p} - 1) \| u_k \|_{\alpha,p}^p - \int_0^T L | u_k(t) |^2 \, dt - \frac{T\tau L}{2}\overline{\Lambda}^2 \| u_k \|_{\alpha,p}^2\\
&- \int_0^T a | u_k(t) |^\beta + b | {}_0D_t^\alpha \xi(t)) |^{\overline{\beta}} \, dt - dT\\
\geq&(\frac{\tau}{p} - 1) \| u_k \|_{\alpha,p}^p - (\frac{T\tau L}{2} + LT)\overline{\Lambda}^2 \| u_k \|_{\alpha,p}^2 - aT\overline{\Lambda}^\beta \| u_k \|_{\alpha,p}^\beta\\
&- b\bigg(\frac{T^{p-\overline{\beta}}}{(\omega_0)^{\overline{\beta}}}\bigg)^{\frac{1}{p}} \| \xi \|_{\alpha,p}^{\overline{\beta}} - dT,
\end{aligned}$$

where

$$\int_0^T | {}_0D_t^\alpha \xi(t) |^{\overline{\beta}} \, dt \leq T^{\frac{p-\overline{\beta}}{p}} \cdot (\int_0^T | {}_0D_t^\alpha \xi(t)) |^p \, dt)^{\frac{\overline{\beta}}{p}} \leq \bigg(\frac{T^{p-\overline{\beta}}}{(\omega_0)^{\overline{\beta}}}\bigg)^{\frac{1}{p}} \| \xi \|_{\alpha,p}^{\overline{\beta}} .$$

Recalling $I_\xi(u_k(t))$ is bounded and $I'_\xi(u_k(t)) \to 0$ as $k \to \infty$ on E_p^α, we have $\{u_k\}_{k=1}^\infty \subset E_p^\alpha$ is bounded. Since E_p^α is a reflexive space, there exists a weakly convergent subsequence such that $u_{k_i} \rightharpoonup u_0$ in E_p^α. For convenience, we still take $\{u_{k_i}\}$ as $\{u_k\}$. In view of the fact that $u_k \rightharpoonup u_0$ and $I'_\xi(u_k(t)) \to 0$ as $k \to \infty$ on E_p^α, we derive

$$\begin{aligned}
&\langle I'_\xi(u_k(t)) - I'_\xi(u_0(t)), u_k(t) - u_0(t)\rangle\\
=&\langle I'_\xi(u_k(t)), u_k(t) - u_0(t)\rangle - \langle I'_\xi(u_0(t)), u_k(t) - u_0(t)\rangle\\
\leq& \| I'_\xi(u_k) \|_{-\alpha,q} \cdot \| u_k - u_0 \|_{\alpha,p} - \langle I'_\xi(u_0(t)), u_k(t) - u_0(t)\rangle\\
\to& 0, \quad \text{as } k \to \infty,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\langle I'_\xi(u_k(t)) - I'_\xi(u_0(t)), u_k(t) - u_0(t)\rangle \qquad (3.4)\\
=&\int_0^T \frac{1}{\omega^{p-2}(t)}(\varphi_p(\omega(t){}_0D_t^\alpha u_k(t)) - \varphi_p(\omega(t){}_0D_t^\alpha u_0(t))){}_0D_t^\alpha(u_k(t) - u_0(t))\\
&+ \lambda(u_k(t) - u_0(t))^2 dt - \int_0^T (h(u_k(t)) - h(u_0(t)))(u_k(t) - u_0(t))dt
\end{aligned}$$

$$-\int_0^T (f(t,u_k(t),{}_0D_t^\alpha\xi(t)) - f(t,u_0(t),{}_0D_t^\alpha\xi(t)))(u_k(t)-u_0(t))dt$$
$$\to 0, \quad \text{as} \quad k\to\infty.$$

Since $u_k(t)\to u_0(t)$ in $C([0,T],\mathbb{R})$ as $k\to\infty$ and f is continuous, h is Lipschitz continuous, one has

$$\begin{cases} u_k(t)-u_0(t)\to 0,\ t\in[0,T],\\ (f(t,u_k(t),{}_0D_t^\alpha\xi(t)) - f(t,u_0(t),{}_0D_t^\alpha\xi(t)))(u_k(t)-u_0(t))\to 0,\\ (h(u_k(t))-h(u_0(t)))(u_k(t)-u_0(t))\to 0,\end{cases}$$

as $k\to\infty$. Hence, according to (3.4), we obtain

$$\int_0^T \frac{1}{\omega^{p-2}(t)}(\varphi_p(\omega(t){}_0D_t^\alpha u_k(t))-\varphi_p(\omega(t){}_0D_t^\alpha u_0(t))){}_0D_t^\alpha(u_k(t)-u_0(t))dt\to 0, k\to\infty. \tag{3.5}$$

It is well known that there exist nonnegative constants a_1 and a_2, for each υ_1, $\upsilon_2\in\mathbb{R}^n$, the following inequalities hold (see [22])

$$\langle|\ \upsilon_1\ |^{p-2}\ \upsilon_1-|\ \upsilon_2\ |^{p-2}\ \upsilon_2,\upsilon_1-\upsilon_2\rangle\geq\begin{cases} a_1\mid\upsilon_1-\upsilon_2\mid^p, & p\geq 2,\\ a_1\frac{|\upsilon_1-\upsilon_2|^2}{(|\upsilon_1|+|\upsilon_2|)^{2-p}}, & 1<p\leq 2,\end{cases} \tag{3.6}$$

and

$$||\ \upsilon_1\ |^{p-2}\ \upsilon_1-|\ \upsilon_2\ |^{p-2}\ \upsilon_2\ |\leq\begin{cases} a_2\mid\upsilon_1-\upsilon_2\mid(|\ \upsilon_1\ |+|\ \upsilon_2\ |)^{p-2}, & p\geq 2,\\ a_2\mid\upsilon_1-\upsilon_2\mid^{p-1}, & 1<p\leq 2.\end{cases} \tag{3.7}$$

Recalling $p\geq 2$, from (3.6), there exists $l_1\in\mathbb{R}^+$ such that

$$\begin{aligned}&\int_0^T \frac{1}{\omega^{p-2}(t)}(\varphi_p(\omega(t){}_0D_t^\alpha u_k(t))-\varphi_p(\omega(t){}_0D_t^\alpha u_0(t))){}_0D_t^\alpha(u_k(t)-u_0(t))dt\\ \geq& l_1\int_0^T\frac{1}{\omega^{p-1}(t)}\mid\omega(t){}_0D_t^\alpha u_k(t)-\omega(t){}_0D_t^\alpha u_0(t)\mid^p dt\\ =&l_1\parallel u_k-u_0\parallel_{\alpha,p}^p .\end{aligned} \tag{3.8}$$

Then, from (3.5) and (3.8), we assert $\parallel u_k-u_0\parallel_{\alpha,p}^p\to 0$ as $k\to\infty$, which means that $u_k\to u_0$ in E_p^α. Hence, functional I_ξ satisfies the P.S. condition.

Step 2. We will verify that functional I_ξ satisfies the geometry conditions of mountain pass theorem.

Let $\rho\leq\frac{\delta}{\overline{\Lambda}}$, where δ is defined in (3.1). From (2.6), we have

$$\parallel u\parallel_\infty\leq\overline{\Lambda}\parallel u\parallel_{\alpha,p}=\overline{\Lambda}\rho\leq\delta,\ \forall\ u\in E_p^\alpha,\ \parallel u\parallel_{\alpha,p}=\rho, \tag{3.9}$$

then, combining (2.8), (2.5) and (3.1), and noting $\lambda-L\geq 0$, we obtain

$$I_\xi(u(t))\geq\frac{1}{p}\|u\|_{\alpha,p}^p+\frac{\lambda}{2}\int_0^T\mid u(t)\mid^2 dt-\frac{L}{2}\int_0^T\mid u(t)\mid^2 dt \tag{3.10}$$

$$
\begin{aligned}
&-\int_0^T s_1 \mid u(t) \mid^{\zeta} + s_2 u^{\gamma}(t) \mid {}_0D_t^{\alpha}\xi(t) \mid^{\eta_1} - c(t)dt \\
\geq &\frac{1}{p}\|u\|_{\alpha,p}^p - s_1 T^{\frac{p-\zeta}{p}} \parallel u \parallel_{L^p}^{\zeta} - s_2 \parallel u \parallel_{\infty}^{\gamma} T^{\frac{p-\eta_1}{p}} \parallel {}_0D_t^{\alpha}\xi \parallel_{L^p}^{\eta_1} + \parallel c \parallel_{L^1} \\
\geq &\frac{1}{p}\rho^p - T^{\frac{p-\zeta}{p}} s_1 \Lambda^{\zeta}\rho^{\zeta} - s_2 \overline{\Lambda}^{\gamma} T^{\frac{p-\eta_1}{p}} \frac{G^{\eta_1}}{(\omega_0)^{\frac{\eta_1}{p}}}\rho^{\gamma} + \parallel c \parallel_{L^1}
\end{aligned}
$$

for any $u(t) \in E_p^{\alpha}$ with $\parallel u \parallel_{\alpha,p} = \rho$. Noting $\zeta, \gamma > p$. Choose ρ small enough, then, we can obtain a constant $\sigma > 0$ such that $I_{\xi}(u(t)) \geq \sigma$ with $\parallel u \parallel_{\alpha,p} = \rho$. Hence, the condition (ii) of Theorem 2.1 holds.

On the other hand, choose $\widetilde{u_0}(t) = \frac{u_0(t)}{\|u_0\|_{\alpha,p}} \in E_p^{\alpha}$ with $\parallel \widetilde{u_0} \parallel_{\alpha,p} = 1$, and

$$
u_0(t) = \begin{cases} \frac{\Gamma(2-\alpha)}{T} t, & t \in [0, \frac{T}{4}], \\ \Gamma(2-\alpha), & t \in [\frac{T}{4}, \frac{3T}{4}], \\ \frac{\Gamma(2-\alpha)}{T}(T-t), & t \in]\frac{3T}{4}, T]. \end{cases} \tag{3.11}
$$

From Definition 2.2 and (3.11), we have

$$
{}_0D_t^{\alpha}u_0(t) = \frac{1}{T}\begin{cases} t^{1-\alpha}, & t \in [0, \frac{T}{4}], \\ t^{1-\alpha} - (t - \frac{T}{4})^{1-\alpha}, & t \in [\frac{T}{4}, \frac{3T}{4}], \\ t^{1-\alpha} - (t - \frac{T}{4})^{1-\alpha} - (t - \frac{3T}{4})^{1-\alpha}, & t \in]\frac{3T}{4}, T]. \end{cases}
$$

Then, for any $\mu \in \mathbb{R}^+$, due to (2.8), (2.6), (3.2) and Holder inequality, we deduce

$$
\begin{aligned}
I_{\xi}(\mu\widetilde{u_0}(t)) \leq &\frac{\mu^p}{p} \parallel \widetilde{u_0} \parallel_{\alpha,p}^p + \left(\frac{TL}{2} + \frac{T\lambda}{2}\right) \parallel \mu\widetilde{u_0} \parallel_{\infty}^2 \\
&- \int_0^T s_1' \mid \mu\widetilde{u_0}(t) \mid^{\zeta} - s_2'(\mu\widetilde{u_0}(t)) \mid {}_0D_t^{\alpha}\xi(t) \mid^{\eta_2} - \vartheta(t)dt \\
\leq &\frac{\mu^p}{p} + \left(\frac{TL}{2} + \frac{T\lambda}{2}\right)\mu^2\overline{\Lambda}^2 + s_2'\mu \int_0^T \widetilde{u_0}(t) \mid {}_0D_t^{\alpha}\xi(t) \mid^{\eta_2} dt \\
&- s_1'\mu^{\zeta} \parallel \widetilde{u_0} \parallel_{L^{\zeta}}^{\zeta} + \parallel \vartheta \parallel_{L^1} . \\
\leq &\frac{\mu^p}{p} + \left(\frac{TL}{2} + \frac{T\lambda}{2}\right)\mu^2\overline{\Lambda}^2 - s_1'\mu^{\zeta} \parallel \widetilde{u_0} \parallel_{L^{\zeta}}^{\zeta} + \parallel \vartheta \parallel_{L^1} \\
&+ s_2'\mu\left(\int_0^T \mid \widetilde{u_0}(t) \mid^{\frac{p}{p-\eta_2}} dt\right)^{\frac{p-\eta_2}{p}} \cdot \left(\int_0^T \mid {}_0D_t^{\alpha}\xi(t) \mid^p dt\right)^{\frac{\eta_2}{p}} \\
\leq &\frac{\mu^p}{p} + \mu^2\left(\frac{TL}{2} + \frac{T\lambda}{2}\right)\overline{\Lambda}^2 + \mu\frac{s_2'G^{\eta_2}}{(\omega_0)^{\frac{\eta_2}{p}}} \parallel \widetilde{u_0} \parallel_{L^{\frac{p}{p-\eta_2}}} - \mu^{\zeta}s_1' \|\widetilde{u_0}\|_{L^{\zeta}}^{\zeta} + \|\vartheta\|_{L^1} .
\end{aligned} \tag{3.12}
$$

Note that $\zeta > p \geq 2$. We can obtain that $I_{\xi}(\mu\widetilde{u_0}(t)) \to -\infty$ as $\mu \to \infty$. Choose μ_0 large enough and take $e(t) = \mu_0\widetilde{u_0}(t)$ such that $\parallel e \parallel_{\alpha,p} > \rho$ and $I_{\xi}(e(t)) \leq 0$. Hence, the condition (iii) of Theorem 2.1 holds.

Obviously, $I_{\xi}(0) = 0$. Thus, from Theorem 2.1, there exists a critical point $\overline{u}(t) \in E_p^{\alpha}$ such that $I_{\xi}(\overline{u}(t)) \geq \sigma > 0$. Since I_{ξ} is also Fréchet differentiable on E_p^{α}, we have $I_{\xi}'(\overline{u}(t)) = 0$.

Step 3. We can establish a sequence $\{u_k\}_{k=1}^{\infty} \subset E_p^{\alpha}$ to satisfy $I'_{u_{k-1}}(u_k(t)) = 0$ and $I_{u_{k-1}}(u_k(t)) \geq \sigma$ with $\| u_k \|_{\alpha,p} \leq G$, for all $k \in \mathbb{N}$.

For a fixed point $x_0(t) \in E_p^{\alpha}$ with $\| x_0 \|_{\alpha,p} \leq G$, there exists $\overline{x}(t) \in E_p^{\alpha}$ to ensure $I'_{x_0}(\overline{x}(t)) = 0$ and $I_{x_0}(\overline{x}(t)) \geq \sigma$ under the conclusion obtained in *Step 2*. Now, we prove that $\| u_k \|_{\alpha,p} \leq G$, for all $k \in \mathbb{N}$.

In fact, according to (3.12), one has

$$\begin{aligned} I_{x_0}(\overline{x}(t)) &\leq \max_{0\leq\mu<\infty} I_{x_0}(\mu\widetilde{u_0}(t)) \qquad (3.13)\\ &\leq \max_{0\leq\mu<\infty} \frac{\mu^p}{p} + \mu^2\left(\frac{TL}{2} + \frac{T\lambda}{2}\right)\overline{\Lambda}^2 + \| \vartheta \|_{L^1} + \mu s_2' \frac{G^{\eta_2}}{(\omega_0)^{\frac{\eta_2}{p}}} \| \widetilde{u_0} \|_{L^{\frac{p}{p-\eta_2}}} \\ &\quad - \mu^{\zeta} s_1' \| \widetilde{u_0} \|_{L^{\zeta}}^{\zeta} \\ &= \max_{0\leq\mu<\infty} \frac{\mu^p}{p} + \mu^2 A + \mu G^{\eta_2} B - \mu^{\zeta} D + \| \vartheta \|_{L^1}, \end{aligned}$$

where $A = (\frac{TL}{2} + \frac{T\lambda}{2})\overline{\Lambda}^2$, $B = \frac{s_2'}{(\omega_0)^{\frac{\eta_2}{p}}} \| \widetilde{u_0} \|_{L^{\frac{p}{p-\eta_2}}}$, $D = s_1' \| \widetilde{u_0} \|_{L^{\zeta}}^{\zeta}$.

Based on Young inequality, taking $q = \frac{p}{p-\eta_2}$, $q' = \frac{p}{\eta_2}$ and $\varepsilon_0 = (\frac{\tau-p}{6\eta_2\tau})^{\frac{\eta_2}{p}}$, we have

$$\mu G^{\eta_2} B \leq \frac{1}{q}(\frac{1}{\varepsilon_0}\mu B)^q + \frac{1}{q'}(\varepsilon_0 G^{\eta_2})^{q'} = \frac{p-\eta_2}{p}(\mu B)^{\frac{p}{p-\eta_2}} \cdot (\frac{6\eta_2\tau}{\tau-p})^{\frac{\eta_2}{p-\eta_2}} + \frac{\eta_2}{p}(\frac{\tau-p}{6\eta_2\tau})G^p. \tag{3.14}$$

Define $W = \frac{p-\eta_2}{p}(B)^{\frac{p}{p-\eta_2}} \cdot (\frac{6\eta_2\tau}{\tau-p})^{\frac{\eta_2}{p-\eta_2}}$. Combining (3.13) with (3.14), we obtain

$$I_{x_0}(\overline{x}) \leq \max_{0\leq\mu<\infty} \frac{\mu^p}{p} + \mu^2 A + \mu^{\frac{p}{p-\eta_2}} W - \mu^{\zeta} D + \frac{\tau-p}{6p\tau} G^p + \| \vartheta \|_{L^1} .$$

Denote

$$\psi(\mu) = \max_{0\leq\mu<\infty} \frac{\mu^p}{p} + \mu^2 A + \mu^{\frac{p}{p-\eta_2}} W - \mu^{\zeta} D.$$

When $0 \leq \mu < 1$, one has

$$\psi(\mu) \leq \frac{1}{p} + A + W := C_1.$$

In addition, when $1 \leq \mu < \infty$, noting $p \geq 2$, $\zeta > p$ and $0 < \eta_2 \leq p-1$, we derive

$$\psi(\mu) \leq (\frac{1}{p} + A + W)\mu^p - D\mu^{\zeta} := \overline{\psi}(\mu).$$

Then, $\overline{\psi}'(\mu) = p(\frac{1}{p} + A + W)\mu^{p-1} - \zeta D\mu^{\zeta-1}$, i.e., there exists $\overline{\mu} = \left(\frac{p(\frac{1}{p}+A+W)}{\zeta D}\right)^{\frac{1}{\zeta-p}}$ such that $\overline{\psi}'(\overline{\mu}) = 0$ and $\overline{\psi}(\overline{\mu}) = \max_{1\leq\mu<\infty} \overline{\psi}(\mu) := C_2$. Take $C_3 = \max\{C_1, C_2\}$, we have

$$I_{x_0}(\overline{x}(t)) \leq C_3 + \frac{\tau-p}{6p\tau} G^p + \| \vartheta \|_{L^1} . \tag{3.15}$$

On the other hand, based on (3.3), yields

$$\tau I_{x_0}(\overline{x}(t)) - \langle I'_{x_0}(\overline{x}(t)), \overline{x}(t) \rangle \tag{3.16}$$

$$\geq (\frac{\tau}{p}-1) \parallel \overline{x} \parallel_{\alpha,p}^{p} - (\frac{T\tau L}{2}+LT)\overline{\Lambda}^{2} \parallel \overline{x} \parallel_{\alpha,p}^{2}$$
$$- aT\overline{\Lambda}^{\beta} \parallel \overline{x} \parallel_{\alpha,p}^{\beta} - b\left(\frac{T^{p-\overline{\beta}}}{(\omega_0)^{\overline{\beta}}}\right)^{\frac{1}{p}} \parallel x_0 \parallel_{\alpha,p}^{\overline{\beta}} - dT$$
$$\geq (\frac{\tau}{p}-1) \parallel \overline{x} \parallel_{\alpha,p}^{p} - d_1 \parallel \overline{x} \parallel_{\alpha,p}^{2} - d_2 \parallel \overline{x} \parallel_{\alpha,p}^{\beta} - d_3 G^{\overline{\beta}} - dT,$$

where $d_1 = (\frac{T\tau L}{2}+LT)\overline{\Lambda}^{2}$, $d_2 = aT\overline{\Lambda}^{\beta}$, $d_3 = b\left(\frac{T^{p-\overline{\beta}}}{(\omega_0)^{\overline{\beta}}}\right)^{\frac{1}{p}}$.

At this point, taking account of (3.15), (3.16) and $I'_{x_0}(\overline{x}) = 0$, we have

$$(\frac{\tau}{p}-1) \parallel \overline{x} \parallel_{\alpha,p}^{p} \leq \tau(C_3 + \frac{\tau-p}{6p\tau}G^p + \parallel \vartheta \parallel_{L^1}) + d_1 \parallel \overline{x} \parallel_{\alpha,p}^{2} + d_2 \parallel \overline{x} \parallel_{\alpha,p}^{\beta} + d_3 G^{\overline{\beta}} + dT. \quad (3.17)$$

Applying the Young inequality, we deduce

$$d_1 \parallel \overline{x} \parallel_{\alpha,p}^{2} \leq \frac{p-2}{p}\left(\frac{12}{\tau-p}\right)^{\frac{2}{p-2}} d_1^{\frac{p}{p-2}} + \frac{\tau-p}{6p} \parallel \overline{x} \parallel_{\alpha,p}^{p} := d_1^* + \frac{\tau-p}{6p} \parallel \overline{x} \parallel_{\alpha,p}^{p},$$
$$d_2 \parallel \overline{x} \parallel_{\alpha,p}^{\beta} \leq \frac{p-\beta}{p}\left(\frac{6\beta}{\tau-p}\right)^{\frac{\beta}{p-\beta}} d_2^{\frac{p}{p-\beta}} + \frac{\tau-p}{6p} \parallel \overline{x} \parallel_{\alpha,p}^{p} := d_2^* + \frac{\tau-p}{6p} \parallel \overline{x} \parallel_{\alpha,p}^{p},$$
$$d_3 G^{\overline{\beta}} \leq \frac{p-\overline{\beta}}{p}\left(\frac{6\overline{\beta}}{\tau-p}\right)^{\frac{\overline{\beta}}{p-\overline{\beta}}} d_3^{\frac{p}{p-\overline{\beta}}} + \frac{\tau-p}{6p}G^p := d_3^* + \frac{\tau-p}{6p}G^p,$$

which means that

$$\frac{\tau-p}{p} \parallel \overline{x} \parallel_{\alpha,p}^{p} \leq \tau C_3 + \frac{\tau-p}{3p}G^p + \tau \parallel \vartheta \parallel_{L^1} + d_1^* + d_2^* + d_3^* + dT + \frac{\tau-p}{3p} \parallel \overline{x} \parallel_{\alpha,p}^{p},$$

that is

$$\parallel \overline{x} \parallel_{\alpha,p}^{p} \leq \frac{3p}{2(\tau-p)}(\tau C_3 + \tau \parallel \vartheta \parallel_{L^1} + d_1^* + d_2^* + d_3^* + dT) + \frac{1}{2}G^p.$$

Since $G = \left(\frac{6p}{2(\tau-p)}(\tau C_3 + \tau \parallel \vartheta \parallel_{L^1} + d_1^* + d_2^* + d_3^* + dT)\right)^{\frac{1}{p}}$, one has $\parallel \overline{x} \parallel_{\alpha,p}^{p} \leq G^p$, i.e., $\parallel \overline{x} \parallel_{\alpha,p} \leq G$.

Suppose that $\parallel u_{k-1} \parallel_{\alpha,p} \leq G$, similar to the proof procedure above, we obtain that $\parallel u_k \parallel_{\alpha,p} \leq G$. Hence, $\parallel u_k \parallel_{\alpha,p} \leq G$, for all $k \in \mathbb{N}$. From (2.6), we confirm that $\parallel u_k \parallel_{\infty} \leq \overline{\Lambda} G := G^*$.

Step 4. We will point out that $\{u_k\}_{k=1}^{\infty}$ converges to $u^* \in E_p^{\alpha}$, and u^* is a solution of BVP(1.1) on E_p^{α}.

According to the conclusion obtained in Step 3, we have $\{u_k\}_{k=1}^{\infty} \subset E_p^{\alpha}$ is bounded. Since E_p^{α} is a reflexive space, there exists a weakly convergent subsequence such that $u_{k_i} \rightharpoonup u^*$ on E_p^{α} as $k_i \to \infty$. Without loss of generality, take $\{u_{k_i}\}$ as $\{u_k\}$. Then, from Lemma 2.3, one has $u_k \to u^*$ in $C([0,T],\mathbb{R})$, as $k \to \infty$.

Suppose that the sequence $\{u_k\}_{k=1}^{\infty}$ is divergent on E_p^{α}. Then, there exists a number $\varepsilon_0 > 0$, for any positive number N such that for each $k, k' > N$, we have $\parallel u_{k'} - u_k \parallel_{\alpha,p} \geq \varepsilon_0$.

Moreover, from (3.8), there exists $l_2 \in \mathbb{R}^+$ such that

$$\int_0^T \frac{1}{\omega^{p-2}(t)}(\varphi_p(\omega(t)_0D_t^\alpha u_{k'}(t)) - \varphi_p(\omega(t)_0D_t^\alpha u_k(t)))_0D_t^\alpha(u_{k'}(t) - u_k(t))dt$$
$$\geq l_2 \parallel u_{k'} - u_k \parallel_{\alpha,p}^p .$$

Therefore, we have

$$\begin{aligned}
&\langle I'_{u_{k'-1}}(u_{k'}(t)) - I'_{u_{k-1}}(u_k(t)), u_{k'}(t) - u_k(t)\rangle \qquad (3.18)\\
=&\int_0^T \frac{1}{\omega^{p-2}(t)}(\varphi_p(\omega(t)_0D_t^\alpha u_{k'}(t)) - \varphi_p(\omega(t)_0D_t^\alpha u_k(t)))_0D_t^\alpha(u_{k'}(t) - u_k(t))\\
&+ \lambda(u_{k'}(t) - u_k(t))^2dt - \int_0^T (h(u_{k'}(t)) - h(u_k(t)))(u_{k'}(t) - u_k(t))dt\\
&- \int_0^T (f(t, u_{k'}(t), {}_0D_t^\alpha u_{k'-1}(t)) - f(t, u_k(t), {}_0D_t^\alpha u_{k-1}(t)))(u_{k'}(t) - u_k(t))dt\\
\geq& l_2 \parallel u_{k'} - u_k \parallel_{\alpha,p}^p + \int_0^T \lambda(u_{k'}(t) - u_k(t))^2 - (h(u_{k'}(t)) - h(u_k(t)))(u_{k'}(t) - u_k(t))dt\\
&- \int_0^T (f(t, u_{k'}(t), {}_0D_t^\alpha u_{k'-1}(t)) - f(t, u_k(t), {}_0D_t^\alpha u_{k-1}(t)))(u_{k'}(t) - u_k(t))dt.
\end{aligned}$$

Recalling $\langle I'_{u_{k-1}}(u_k(t)), u_{k'}(t) - u_k(t)\rangle = 0$, $\langle I'_{u_{k'-1}}(u_{k'}(t)), u_{k'}(t) - u_k(t)\rangle = 0$ and $\lambda \geq L$, then, combining (3.18), (1.2), (2.6) and (H_3), yields

$$\begin{aligned}
&l_2 \parallel u_{k'} - u_k \parallel_{\alpha,p}^p\\
\leq&\int_0^T (h(u_{k'}(t)) - h(u_k(t)))(u_{k'}(t) - u_k(t))dt - \int_0^T \lambda(u_{k'}(t) - u_k(t))^2dt\\
&+ \int_0^T (f(t, u_{k'}(t), {}_0D_t^\alpha u_{k'-1}(t)) - f(t, u_k(t), {}_0D_t^\alpha u_{k-1}(t)))(u_{k'}(t) - u_k(t))dt\\
\leq&\int_0^T (L - \lambda) \mid u_{k'}(t) - u_k(t) \mid^2 dt + \int_0^T \Big(M_1 \mid u_{k'}(t) - u_k(t) \mid\\
&+ M_2 \mid {}_0D_t^\alpha u_{k'-1}(t) - {}_0D_t^\alpha u_{k-1}(t) \mid \Big) \mid u_{k'}(t) - u_k(t) \mid dt\\
\leq& M_1 \int_0^T \mid u_{k'}(t) - u_k(t) \mid\mid u_{k'}(t) - u_k(t) \mid dt + M_2 \int_0^T \mid {}_0D_t^\alpha u_{k'-1}(t)\\
&- {}_0D_t^\alpha u_{k-1}(t) \mid\mid u_{k'}(t) - u_k(t) \mid dt\\
\leq& M_1T \parallel u_{k'} - u_k \parallel_\infty \parallel u_{k'} - u_k \parallel_\infty + M_2T^{\frac{p-1}{p}} \parallel u_{k'} - u_k \parallel_\infty \parallel {}_0D_t^\alpha(u_{k'-1} - u_{k-1}) \parallel_{L^p}\\
\leq& 2M_1TG^* \parallel u_{k'} - u_k \parallel_\infty + \frac{2M_2T^{\frac{p-1}{p}}G}{\omega_0^{\frac{1}{p}}} \parallel u_{k'} - u_k \parallel_\infty\\
=&\left(2M_1TG^* + \frac{2M_2T^{\frac{p-1}{p}}G}{\omega_0^{\frac{1}{p}}}\right) \parallel u_{k'} - u_k \parallel_\infty\\
:=& C^* \parallel u_{k'} - u_k \parallel_\infty,
\end{aligned}$$

which means that $\parallel u_{k'} - u_k \parallel_\infty \geq \frac{l_2}{C^*} \parallel u_{k'} - u_k \parallel_{\alpha,p}^p \geq \frac{l_2}{C^*}\varepsilon_0^p := \varepsilon'_0$, i.e., there exists a number $\varepsilon'_0 > 0$, for any positive number N such that for each $k, k' > N$, we have

$\| u_{k'} - u_k \|_\infty \geq \varepsilon_0'$. It is contradict with the fact that $\{u_k(t)\}$ strongly converges to $u^*(t)$ in $C([0,T],\mathbb{R})$ as $k \to \infty$. Hence, we obtain that the sequence $\{u_k(t)\}$ converges to $u^*(t)$ on E_p^α as $k \to \infty$.

In the following, we claim that $I'_{u^*}(u^*(t)) = 0$. In fact, in view of (3.7), there exists a nonnegative constant l_3 such that

$$\begin{aligned}&|| {}_0D_t^\alpha u_k(t) |^{p-2} {}_0D_t^\alpha u_k(t) - | {}_0D_t^\alpha u^*(t) |^{p-2} {}_0D_t^\alpha u^*(t) | \\ \leq & l_3 | {}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u^*(t) | (| {}_0D_t^\alpha u_k(t) | + | {}_0D_t^\alpha u^*(t) |)^{p-2},\ p \geq 2.\end{aligned} \tag{3.19}$$

Then, for any $v(t) \in E_p^\alpha$, from (3.19), we derive

$$\begin{aligned}&| \int_0^T \frac{1}{\omega^{p-2}(t)} (\varphi_p(\omega(t)_0D_t^\alpha u_k(t)) - \varphi_p(\omega(t)_0D_t^\alpha u^*(t)))_0D_t^\alpha v(t)dt | \qquad (3.20)\\ \leq & \int_0^T \omega(t) || {}_0D_t^\alpha u_k(t) |^{p-2} {}_0D_t^\alpha u_k(t) - | {}_0D_t^\alpha u^*(t) |^{p-2} {}_0D_t^\alpha u^*(t) | \cdot | {}_0D_t^\alpha v(t) | dt \\ \leq & \int_0^T \omega(t) l_3 | {}_0D_t^\alpha (u_k(t) - u^*(t)) | (| {}_0D_t^\alpha u_k(t) | + | {}_0D_t^\alpha u^*(t) |)^{p-2} | {}_0D_t^\alpha v(t) | dt.\end{aligned}$$

According to multiple Hölder inequality presented in Lemma 2.6, we obtain

$$\begin{aligned}&\int_0^T \omega(t) | {}_0D_t^\alpha (u_k(t) - u^*(t)) | (| {}_0D_t^\alpha u_k(t) | + | {}_0D_t^\alpha u^*(t) |)^{p-2} | {}_0D_t^\alpha v(t) | dt \\ \leq & \| \omega(t)_0D_t^\alpha (u_k - u^*) \|_{L^p} \cdot \| {}_0D_t^\alpha (| u_k | + | u^* |) \|_{L^p}^{p-2} \cdot \| {}_0D_t^\alpha v \|_{L^p} \\ \leq & (\omega^0)^{1-\frac{1}{p}} \| u_k - u^* \|_{\alpha,p} \cdot \| {}_0D_t^\alpha (| u_k | + | u^* |) \|_{L^p}^{p-2} \cdot \| {}_0D_t^\alpha v \|_{L^p} .\end{aligned} \tag{3.21}$$

Then, (3.20) is written as

$$\begin{aligned}&| \int_0^T \frac{1}{\omega^{p-2}(t)} (\varphi_p(\omega(t)_0D_t^\alpha u_k(t)) - \varphi_p(\omega(t)_0D_t^\alpha u^*(t)))_0D_t^\alpha v(t)dt | \\ \leq & l_3(\omega^0)^{1-\frac{1}{p}} \| u_k - u^* \|_{\alpha,p} \cdot \| {}_0D_t^\alpha (| u_k | + | u^* |) \|_{L^p}^{p-2} \cdot \| {}_0D_t^\alpha v \|_{L^p} .\end{aligned}$$

Since $u_k(t) \to u^*(t)$ in E_p^α as $k \to \infty$, we can easily observe that

$$\begin{aligned}&\lim_{k\to\infty} \int_0^T \frac{1}{\omega^{p-2}(t)} \varphi_p(\omega(t)_0D_t^\alpha u_k(t))_0D_t^\alpha v(t)dt \\ = & \int_0^T \frac{1}{\omega^{p-2}(t)} \varphi_p(\omega(t)_0D_t^\alpha u^*(t))_0D_t^\alpha v(t)dt.\end{aligned} \tag{3.22}$$

Additionally, based on (1.2) and (2.6), we have

$$\begin{aligned}&| \int_0^T \lambda u_k(t)v(t) - h(u_k(t))v(t)dt - \int_0^T \lambda u^*(t)v(t) - h(u^*(t))v(t)dt | \\ \leq & \int_0^T \lambda | u_k(t) - u^*(t) || v(t) | + L | u_k(t) - u^*(t) || v(t) | dt \\ \leq & T(\lambda + L)\overline{\Lambda}^2 \| u_k - u^* \|_{\alpha,p} \| v \|_{\alpha,p} .\end{aligned}$$

By using $u_k(t) \to u^*(t)$ in E_p^α as $k \to \infty$, for any $v(t) \in E_p^\alpha$, a.e. $t \in [0,T]$, we have

$$\lim_{k\to\infty} \int_0^T \lambda u^k(t)v(t) - h(u_k(t))v(t)dt = \int_0^T \lambda u^*(t)v(t) - h(u^*(t))v(t)dt. \tag{3.23}$$

On the other hand, based on (H_3), (2.5), (2.6) and the Hölder inequality, for every $v(t) \in E_p^\alpha$, a.e. $t \in [0,T]$, one has

$$\begin{aligned}
&\mid \int_0^T (f(t,u^*(t),{}_0D_t^\alpha u^*(t)) - f(t,u_k(t),{}_0D_t^\alpha u_{k-1}(t)))v(t)dt \mid \\
\le &\int_0^T \bigg(M_1 \mid u_k(t) - u^*(t) \mid + M_2 \mid {}_0D_t^\alpha u_{k-1}(t) - {}_0D_t^\alpha u^*(t) \mid \bigg) \mid v(t) \mid dt \\
\le &M_1 \parallel u_k - u^* \parallel_{L^p} \parallel v \parallel_{L^{\frac{p}{p-1}}} + M_2 \parallel v \parallel_{L^{\frac{p}{p-1}}} \parallel {}_0D_t^\alpha (u_{k-1} - u^*) \parallel_{L^p} \\
\le &M_1 \Lambda \parallel u_k - u^* \parallel_{\alpha,p} \parallel v \parallel_{L^{\frac{p}{p-1}}} + M_2 \parallel v \parallel_{L^{\frac{p}{p-1}}} \frac{\parallel u_{k-1} - u^* \parallel_{\alpha,p}}{\omega_0^{\frac{1}{p}}}.
\end{aligned}$$

Clearly, sequence $u_k(t) \to u^*(t)$ in E_p^α as $k \to \infty$, which means that

$$\lim_{k\to\infty} \int_0^T f(t,u_k(t),{}_0D_t^\alpha u_{k-1}(t))v(t)dt = \int_0^T f(t,u^*(t),{}_0D_t^\alpha u^*(t))v(t)dt. \quad (3.24)$$

Combining (3.22)–(3.24) with $I'_{u_{k-1}}(u_k(t))v(t) = 0$, for any $v(t) \in E_p^\alpha$, yields

$$\begin{aligned}
&\int_0^T \frac{1}{\omega^{p-2}(t)} \varphi_p(\omega(t)_0 D_t^\alpha u^*(t))_0 D_t^\alpha v(t) + \lambda u^*(t)v(t)dt \\
= &\int_0^T h(u^*(t))v(t)dt + \int_0^T f(t,u^*(t),{}_0D_t^\alpha u^*(t))v(t)dt.
\end{aligned}$$

Namely, $I'_{u^*}(u^*(t))v(t) = 0$, for any $v(t) \in E_p^\alpha$, and we can also guarantee that $\lim\limits_{k\to\infty} I'_{u_{k-1}}(u_k(t)) = I'_{u^*}(u^*(t))$. Hence, $u^*(t)$ is a solution of BVP(1.1) on E_p^α. The proof is completed. □

Remark 3.1. It is well known that the nonlocal and nonlinear differential operator ${}_tD_T^\alpha \varphi_p({}_0D_t^\alpha)$ can be reduced to the linear differential operator ${}_tD_T^\alpha {}_0D_t^\alpha$ under $p = 2$. Thus, the contents of our paper based on the space of $L^p([0,T],\mathbb{R})$ $(2 \le p < \infty)$ are more general comparing with the existing relevant results based on the inner product space of $L^2([0,T],\mathbb{R})$. Moreover, we present some looser assumptions to establish the existence of solutions for BVP (1.1), which guarantee the conclusion obtained in the paper more convenience for application. For example, in reference [6], the complex parameter conditions $P_0 < 1$ and $\frac{Q_0}{1-P_0} < 1$ are required to ensure the existence of solutions for the equation. The analogous restricted conditions do not appear in our assumptions. So far, little work has been done for the existence of solutions of p-Laplacian fractional boundary value problem with nonlinear function f including the fractional derivative. Therefore, it is worth studying further.

4. Example

Let $p = 3$, $\lambda = 1$, $h(u(t)) = \frac{1}{2}\sin u(t)$. Then, BVP (1.1) becomes the following form

$$\begin{cases} {}_tD_T^\alpha \left(\frac{1}{\omega(t)} \varphi_3(\omega(t)_0 D_t^\alpha u(t)) \right) + u(t) = f(t,u,{}_0^cD_t^\alpha u(t)) + \frac{1}{2}\sin u(t), \\ u(0) = u(T) = 0, \text{ a.e. } t \in [0,T]. \end{cases} \quad (4.1)$$

It is easy to observe that $\frac{1}{2}\sin u(t) \leq \frac{1}{2} \mid u(t) \mid$, i.e., $L = \frac{1}{2}$, which means that $\lambda - L > 0$. Define $F(t, x(t), y(t)) = e^{-t}x^4 + tx^4(\sin y)^2$. Then, $f(t, x(t), y(t)) = 4e^{-t}x^3 + 4tx^3(\sin y)^2$.

We claim that the conditions of (H_1)–(H_3) in Theorem 3.1 hold.

(i)

$$\tau F(t, x, y) - f(t, x, y)x = 0, \quad x, y \in \mathbb{R}, \quad \text{a.e. } t \in [0, T],$$

where $\tau = 4, a = b = d = 0$;

(ii)

$$\begin{cases} F(t, x, y) \leq x^4 + Tx^4y^2, & \text{for } x, y \in \mathbb{R}, \text{ a.e. } t \in [0, T], \\ F(t, x, y) \geq \frac{1}{e^T}x^4 - Txy^2, & \text{for } x, y \in \mathbb{R}, \text{ a.e. } t \in [0, T], \end{cases} \tag{4.2}$$

where $\zeta = 4$, $\gamma = 4$, $\eta_1 = 2$, $\eta_2 = 2$, $s_1 = 1$, $s_2 = T$, $s_1' = \frac{1}{e^T}$, $s_2' = T$, $c(t) = \vartheta(t) = 0$.

(iii)

$$\begin{aligned} \mid f(t, x, y) - f(t, x', y') \mid &\leq \mid f(t, x, y) - f(t, x', y) \mid + \mid f(t, x', y) - f(t, x', y') \mid \\ &\leq (12 + 12T)(G^*)^2 \mid x - x' \mid + (4 + 8T)(G^*)^3 \mid y - y' \mid, \end{aligned}$$

for $x, x' \in [-G^*, G^*]$, $y, y' \in \mathbb{R}$, a.e. $t \in [0, T]$, where $M_1 = (12 + 12T)(G^*)^2$ and $M_2 = (4 + 8T)(G^*)^3$.

Hence, all the conditions of Theorem 3.1 are satisfied. Namely, BVP (4.1) exists one nontrivial solution on E_p^α, for $p = 3$.

5. Conclusion

In this paper, a class of fractional differential equation with p-Laplacian has been investigated. Combining the mountain pass theorem with iterative technique, the existence of at least one nontrivial solution for BVP (1.1) has been obtained. The reasonably function space and variational framework for BVP (1.1) have been developed to apply the variational approach. And iterative method has been used to obtain the solution of our equation. Finally, we have illustrated the application of our main result through an example.

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