

STOCHASTIC VIRUS DYNAMICS WITH BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE

Junyi Tu² and Yuncheng You^{1,†}

Abstract Stochastic virus dynamics modeled by a system of stochastic differential equations with Beddington-DeAngelis functional response and driven by white noise is investigated. The global existence of positive solutions and the existence of stationary distribution are proved. Upper and lower bounds of the pathwise and asymptotic moments for the positive solutions are sharply estimated. The absorbing property in time average is shown and the moment Lyapunov exponents are proved to be nonpositive.

Keywords Stochastic virus dynamics, Beddington-DeAngelis functional response, stationary distribution, dynamic estimation, moment Lyapunov exponent.

MSC(2010) 34K15, 34K20, 92A15.

1. Introduction

In this paper we shall consider the stochastic viral dynamics modeled by the following stochastic differential equations

$$\begin{aligned} dx &= \left(\lambda - \delta x - \frac{\beta x z}{1 + ax + bz} \right) dt + \sigma_1 x dB_1(t), \\ dy &= \left(\frac{\beta x z}{1 + ax + bz} - qy \right) dt + \sigma_2 y dB_2(t), \\ dz &= (ky - \gamma z) dt + \sigma_3 z dB_3(t), \end{aligned} \quad (1.1)$$

where the susceptible cells whose density is denoted by $x(t)$ are generated at a constant rate λ , die at a density-proportional rate δx , and become infected with a rate $\beta x z / (1 + ax + bz)$ in terms of the Beddington-DeAngelis functional response, the infected cells of the density $y(t)$ are produced at the same rate from the susceptible cells and die at a density-dependent rate qy , while the virus-free cells $z(t)$ are released from the infected cells at a rate ky and die at a rate γz , cf. [4, 8, 16]. All these involved parameters and the intensity coefficients σ_i 's of the stochastic driving terms are positive constants.

[†]the corresponding author. Email address: you@mail.usf.edu (Y. You)

¹Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

²Department of Computer Science and Engineering, University of South Florida, Tampa, FL 33620, USA

In the classical infectious disease model, a bilinear incidence as βxz was often used. However, the actual incidence rate is most likely not linear over the entire range of the densities x and z . The nonlinear incidence rate with this Holling type II response $\beta xz/(1 + ax + bz)$ was introduced by Beddington [1] and DeAngelis et al. [3] in the study of HIV-1 virus and uninfected CD4⁺T cells. This incidence rate is more reasonable than a bilinear rate because it incorporates the behavioral change and crowding effect of the infective cells or particles.

As indicated in recent researches on epidemic dynamics, infection dynamics, as well as population dynamics, cf. [2, 4, 5, 14, 16, 17] and references therein, it is found that the deterministic models are subject to some limitations in modeling the virus transmission of infectious or epidemic diseases. Virus or disease transmission dynamics are influenced by the random effect of environmental noise, immunodeficiency, antibiotic resistance, even weather conditions, or uncertain fluctuations. These researches also showed that stochastic differential equation models with the white noise generated by Brownian motion can provide some additional degree of realism in dealing with continuous fluctuations of randomness affecting the birth and death rates, transmission coefficients, and other parameters in the system. The stochastic models with this kind of Wiener noise likely serve better in predicting the future dynamics than the deterministic models. It is commented in [5, page 879] that adding the stochastic Wiener noise to the parameters of relevant rates is a well-established way of introducing stochastic environmental noise into biologically realistic population dynamic models. Thus it is reasonable to include the additive or multiplicative white noises to the death rates as $-\delta x + \sigma_1 x dB_1(t)$, $-qy + \sigma_2 y dB_2(t)$ and $-\gamma z + \sigma_3 z dB_3(t)$ in the model equations (1.1) for tackling the effect of environmental fluctuations.

The global stability of virus dynamics for the corresponding deterministic system

$$\begin{aligned}\frac{dx}{dt} &= \lambda - \delta x - \frac{\beta xz}{1 + ax + bz}, \\ \frac{dy}{dt} &= \frac{\beta xz}{1 + ax + bz} - qy, \\ \frac{dz}{dt} &= ky - \gamma z,\end{aligned}\tag{1.2}$$

was analyzed in [8] and the following results are proved: When the comprehensive reproductive ratio of the virus

$$R_0 = \frac{k\beta\lambda}{\delta q\gamma + aq\gamma\lambda} \leq 1,$$

the disease-free equilibrium $E_0 = (\lambda/\delta, 0, 0)$ is globally asymptotically stable; When $R_0 > 1$, then the endemic equilibrium $E_1 = (x_0, y_0, z_0)$ is globally asymptotically stable, where

$$x_0 = \frac{q\gamma + kb\lambda}{k\beta - aq\gamma + bk\delta}, \quad y_0 = \frac{k\beta\lambda - a\lambda q\gamma - \delta q\gamma}{q(k\beta - aq\gamma + bk\delta)}, \quad z_0 = \frac{k(k\beta\lambda - a\lambda q\gamma - \delta q\gamma)}{q\gamma(k\beta - aq\gamma + bk\delta)}.$$

Dynamics of a stochastic predator-prey system with Beddington-DeAngelis functional response and multiplicative or additive noise was studied in [9, 18]. It was shown that if the noise is large, both species in the system may go to extinction, which does not occur for the deterministic system. Besides, in [21] and [27], stationary distribution of stochastic population systems and stochastic Hopf bifurcation

of the predator-prey system with Beddington-DeAngelis response have been investigated. Stochastic dynamics of predator-prey system with Leslie-Gower response and lévy jumps was studied in [19].

The main objective of this article is to study the pathwise, time-averaging, and asymptotic dynamics for the almost surely positive solutions in terms of construction of Lyapunov functions, proving the existence of a stationary distribution, conducting sharp estimates of stochastic upper and lower bounds of trajectories, application of the ergodic property, and evaluation of the moment Lyapunov exponents in regard to stochastic stability.

The basic concepts and some underlying results for stochastic ordinary differential equations can be found in [10] and [22]. In this paper, let $(\Omega, F, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, which is right continuous and \mathcal{F}_0 contains all P -null sets. Assume that $B_i(t), i = 1, 2, 3$, are independent standard Brownian motion defined on this probability space. We use the notation

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}.$$

For an n -dimensional stochastic differential equation driven by the white noise, which will be called nonautonomous if functions f and g are explicitly dependent of time t ,

$$dX(t) = f(X(t), t) dt + g(X(t), t) dB, \quad \text{for } t \geq t_0, \tag{1.3}$$

we shall denote by $C^{2,1}(\mathbb{R}^n \times [t_0, \infty), \mathbb{R}_+ \cup \{0\})$ the family of all nonnegative functions $V(x, t)$ defined on $\mathbb{R}^n \times [t_0, \infty)$ and continuously differentiable in x to the second order and in t to the first order. Define the differential operator L associated with (1.3) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^n f_i(x, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n [g(x, t)g^T(x, t)]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then the action of the operator L on $V \in C^{2,1}(\mathbb{R}^n \times [t_0, \infty); \mathbb{R}_+ \cup \{0\})$ is

$$LV(X(t), t) = V_t(X, t) + V_x(X, t)f(X, t) + \frac{1}{2} \text{Tr} [g^T(X, t)V_{xx}(X, t)g(X, t)], \tag{1.4}$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{n \times n}$. By the Itô formula, we have the following differential formula, which corresponds to the the total derivative in deterministic case,

$$dV(X(t), t) = LV(X(t), t) dt + V_x(X(t), t)g(X(t), t) dB(t).$$

Definition 1.1. [10, 21] Let $P_{X_0,t}(\cdot)$ be the probability measure induced by a stochastic process $\{X(t)\}_{t \geq 0}$ in \mathbb{R}_+^n (or \mathbb{R}^n) over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with initial state $X(0) = X_0$, namely,

$$P_{X_0,t}(S) = \mathbb{P}\{X(t, \omega) \in S\}, \quad S \in \mathcal{B}(\mathbb{R}_+^n),$$

where $\mathcal{B}(\cdot)$ stands for the σ -algebra of all Borel sets. If there is a probability measure $\mu(\cdot)$ on the measurable space $(\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}_+^n))$ such that

$$P_{X_0,t}(\cdot) \longrightarrow \mu(\cdot) \quad \text{as } t \rightarrow \infty,$$

in distribution for any $X_0 \in \mathbb{R}_+^n$, then $X(t)$ has a stationary distribution $\mu(\cdot)$.

Definition 1.2. A set S of solutions of a SDE (1.3) is said to have the p -th moment absorbing property in time average, if there is a positive constant $M = M(p) > 0$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t |X(s, X_0)|^p ds \leq M$$

for all solutions in S . This property means the trajectories of a random dynamical system will be confined in a fixed bounded region in time average.

Definition 1.3. The p -th moment Lyapunov exponent of a solution $X(t, X_0)$ of a SDE (1.3) is defined by

$$\Lambda(p) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log E|X(t, X_0)|^p, \quad X_0 \neq 0, \quad p \geq 1.$$

In Section 2, we shall prove the global existence of positive solutions to the system of the model equations (1.1) for the stochastic virus dynamics. In Section 3, it will be shown that a stationary distribution exists. In Section 4, we present the pathwise and asymptotic moment estimation of the upper and lower bounds of the positive solutions. In Section 5, we show the absorbing property of in time average. In Section 6, we show that the moment Lyapunov exponents are nonpositive for this dynamical system.

2. Global Existence of Positive Solutions

First we prove the existence and uniqueness of a global positive solution for the initial value problem of the SDE (1.1).

Theorem 2.1. *Under the condition $q \geq k$, for any initial data $(x_0, y_0, z_0) \in \mathbb{R}_+^3$, there exists a unique positive solution $(x(t), y(t), z(t))$ to the system (1.1) such that the solution will remain in \mathbb{R}_+^3 for all $t \geq 0$ with probability one.*

Proof. Since the coefficients of the equations in (1.1) are locally Lipschitz continuous, for any $\omega \in \Omega$ and any given initial data $(x_0, y_0, z_0) \in \mathbb{R}_+^3$, there is a unique local solution $(x(t), y(t), z(t))$ on $t \in [0, \tau_e)$, where $[0, \tau_e)$ is the maximal existence interval pathwise depending on the initial data.

We now show that the solution exists globally, namely, $\tau_e = \infty$ a.s. Let an integer $m_0 > 0$ be sufficiently large such that $(x_0, y_0, z_0) \in [1/m_0, m_0]^3$. For each integer $m > m_0$ we define the stopping time

$$\tau_m = \inf \left\{ t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{m}, m \right), y(t) \notin \left(\frac{1}{m}, m \right), \text{ or } z(t) \notin \left(\frac{1}{m}, m \right) \right\}. \tag{2.1}$$

We set the infimum of empty set = ∞ . Obviously, τ_m is increasing as $m \rightarrow \infty$. Set $\tau_\infty = \lim_{m \rightarrow \infty} \tau_m$.

It suffices to prove that $\tau_\infty = \infty$ for any initial data a.s. If this statement is false, then there would be constants $\varepsilon \in (0, 1)$ and $T > 0$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon. \tag{2.2}$$

Since $\tau_\infty \geq \tau_m$, there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon, \quad \text{for any } m > m_1.$$

Consider the function $V(x, y, z)$ defined by

$$V(x, y, z) = (x - 1 - \log x) + (y - 1 - \log y) + (z - 1 - \log z). \quad (2.3)$$

Using the Itô formula, we have

$$\begin{aligned} dV &= \left(1 - \frac{1}{x}\right) \left(\lambda - \delta x - \frac{\beta x z}{1 + ax + bz} + \sigma_1 x dB_1(t)\right) \\ &\quad + \left(1 - \frac{1}{y}\right) \left(\frac{\beta x z}{1 + ax + bz} - qy + \sigma_2 y dB_2(t)\right) \\ &\quad + \left(1 - \frac{1}{z}\right) (ky - \gamma z + \sigma_3 z dB_3(t)) + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) dt \\ &= LV dt + \sigma_1(x - 1) dB_1(t) + \sigma_2(y - 1) dB_2(t) + \sigma_3(z - 1) dB_3(t), \end{aligned} \quad (2.4)$$

where the condition $q \geq k$ implies that

$$\begin{aligned} LV &= \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \lambda + \delta + q + \gamma - \delta x - \frac{\lambda}{x} - qy + ky - \gamma z \\ &\quad + \frac{\beta z}{1 + ax + bz} - \frac{\beta x z}{y(1 + ax + bz)} \\ &= \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \lambda + \delta + q + \gamma + (k - q)y + \frac{\beta z}{1 + ax + bz} \\ &\leq \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \lambda + \delta + q + \gamma + \frac{\beta}{b}. \end{aligned} \quad (2.5)$$

Therefore, we obtain

$$dV \leq K dt + \sigma_1 x dB_1(t) + \sigma_2 y dB_2(t) + \sigma_3 z dB_3(t), \quad (2.6)$$

where

$$K = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \lambda + \delta + q + \gamma + \frac{\beta}{b}.$$

By integrating both sides of (2.6) from 0 to $\tau_m \wedge T$ and then taking the expectation on both sides, it yields

$$E[V(x(\tau_m \wedge T), y(\tau_m \wedge T), z(\tau_m \wedge T))] \leq V(x_0, y_0, z_0) + KT.$$

Let $\Omega_m = \{\tau_m \leq T\}$, then $P(\Omega_m) \geq \varepsilon$. Note that for every $\omega \in \Omega_m$ at least one of $x(\tau_m, \omega)$, $y(\tau_m, \omega)$, $z(\tau_m, \omega)$ equals to m or $1/m$, then $V(x(\tau_m, \omega), y(\tau_m, \omega), z(\tau_m, \omega))$ is no less than

$$m - 1 - \log m \quad \text{or} \quad \frac{1}{m} - 1 + \log m.$$

Thus we get

$$V(x(\tau_m, \omega), y(\tau_m, \omega), z(\tau_m, \omega)) \geq (m - 1 - \log m) \wedge \left(\frac{1}{m} - 1 + \log m\right).$$

It follows that

$$\begin{aligned} V(x_0, y_0, z_0) + KT &\geq E[I_{\Omega_m}(\omega)V(x(\tau_m \wedge T), y(\tau_m \wedge T), z(\tau_m \wedge T))] \\ &= E[I_{\Omega_m}(\omega)V(x(\tau_m, \omega), y(\tau_m, \omega), z(\tau_m, \omega))] \\ &\geq \varepsilon \left[(m - 1 - \log m) \wedge \left(\frac{1}{m} - 1 + \log m\right) \right], \end{aligned} \quad (2.7)$$

where $I_{\Omega_m}(\omega)$ is the indicator function of $\Omega_m(\omega)$. Let $m \rightarrow \infty$. Then it leads to a contradiction

$$\infty > V(x(0), y(0), z(0)) + KT = \infty.$$

Therefore we have $\tau_e = \infty$ for any given initial data almost surely, which completes the proof. \square

According to [22, Theorem 2.9.1], this positive solution $X(t) = (x(t), y(t), z(t))^T$ is a Markov process with time-homogeneous transition function.

3. Stationary Distribution

The system of stochastic autonomous differential equations (1.1) can be written in the vector-matrix form

$$dX = f(X) dt + g(X) dB, \tag{3.1}$$

where $B(t) = (B_1(t), B_2(t), B_3(t))^T$ is the independent 3-dimensional Wiener process and

$$f(X) = \begin{pmatrix} \lambda - \delta x - \frac{\beta xz}{1+ax+bz} \\ \frac{\beta xz}{1+ax+bz} - qy \\ ky - \gamma z \end{pmatrix}, \quad g(X) = \begin{pmatrix} \sigma_1 x & 0 & 0 \\ 0 & \sigma_2 y & 0 \\ 0 & 0 & \sigma_3 z \end{pmatrix}.$$

Its diffusion matrix is given by

$$A(x, y, z) = \begin{pmatrix} \sigma_1^2 x^2 & 0 & 0 \\ 0 & \sigma_2^2 y^2 & 0 \\ 0 & 0 & \sigma_3^2 z^2 \end{pmatrix}.$$

Here we shall prove the existence of a stationary distribution for a positive solution of the system (1.1). We mention the following assumption in cf. [10, p.107].

Assumption U. There is a bounded open domain $U \subset \mathbb{R}_+^3$ with regular boundary and the following properties:

- I. In a neighborhood of U , the smallest eigenvalue of the diffusion matrix $A(x, y, z)$ is uniformly bounded away from zero.
- II. For any $X = (x, y, z) \in \mathbb{R}_+^3 \setminus U$, the mean time τ at which a path from X reaches the set U is finite and $\sup_K E_X[\tau] < \infty$ for every compact set $K \subset \mathbb{R}_+^3$.

Lemma 3.1. *Suppose the Assumption U is satisfied. Then the Markov process $X(t, X_0)$ given by the pathwise solution of (3.1) has a stationary distribution $\mu(\cdot)$ such that $\mu(\mathbb{R}^3 \setminus \mathbb{R}_+^3) = 0$ and for any nonnegative continuous function $Q(X)$ it holds that*

$$\int_{\mathbb{R}_+^3} E[Q(X(t, \xi))] d\mu(\xi) = \int_{\mathbb{R}_+^3} Q(\xi) d\mu(\xi), \quad t \geq 0. \tag{3.2}$$

Moreover, for any integrable function $F(X)$ with respect to μ , it holds that

$$\lim_{T \rightarrow \infty} P \left\{ \frac{1}{T} \int_0^T F(X(t, \xi)) dt = \int_{\mathbb{R}_+^3} F(\xi) d\mu(\xi) \right\} = 1, \quad \text{for all } \xi \in \mathbb{R}_+^3. \tag{3.3}$$

The proof of this lemma is referred to Theorems 4.4.1 and 4.4.2 in [10].

Theorem 3.1. *If there exists a positive equilibrium point $(x_0, y_0, z_0) \in \mathbb{R}_+^3$ for the deterministic system (1.2), then there exists a stationary distribution with respect to the positive solutions of the system (1.1).*

Proof. Consider a new Lyapunov function

$$V(x, y, z) = \frac{1 + bz_0}{1 + ax_0 + bz_0} \left(x - x_0 - x_0 \log \frac{x}{x_0} \right) + y - y_0 - y_0 \log \frac{y}{y_0} + \frac{q}{k} \left(z - z_0 - z_0 \ln \frac{z}{z_0} \right). \quad (3.4)$$

By the differential formula (1.4), we obtain

$$\begin{aligned} LV(x(t), y(t), z(t)) &= \frac{1 + bz_0}{1 + ax_0 + bz_0} \left(1 - \frac{x_0}{x} \right) \left(\lambda - \delta x - \frac{\beta x z}{1 + ax + bz} \right) \\ &\quad + \frac{(1 + bz_0)x_0\sigma_1^2}{2(1 + ax_0 + bz_0)} + \left(1 - \frac{y_0}{y} \right) \left(\frac{\beta x z}{1 + ax + bz} - qy \right) \\ &\quad + \frac{q}{k} \left(1 - \frac{z_0}{z} \right) (ky - \gamma z) + \frac{y_0\sigma_2^2}{2} + \frac{qz_0\sigma_3^2}{2k} \\ &= \lambda - \delta x - \frac{q\gamma}{k} z - \frac{(1 + ax + bz_0)x_0}{(1 + ax_0 + bz_0)x} \left(\lambda - \delta x - \frac{\beta x z}{1 + ax + bz} \right) \\ &\quad - \frac{y_0}{y} \left(\frac{\beta x z}{1 + ax + bz} - qy \right) - \frac{qz_0}{kz} (ky - \gamma z) + \sigma. \end{aligned} \quad (3.5)$$

Since (x_0, y_0, z_0) is a positive equilibrium point, we have

$$\begin{aligned} \lambda &= \delta x_0 + qy_0 \\ \gamma/k &= y_0/z_0 \\ qy_0 &= \beta x_0 z_0 / (1 + ax_0 + bz_0). \end{aligned} \quad (3.6)$$

It follows that

$$\begin{aligned} &LV(x(t), y(t), z(t)) \\ &= \delta x_0 + qy_0 - \delta x - \frac{qy_0 z}{z_0} - \frac{(1 + ax + bz_0)x_0}{(1 + ax_0 + bz_0)x} \delta x_0 \\ &\quad - \frac{(1 + ax + bz_0)x_0}{(1 + ax_0 + bz_0)x} qy_0 + \frac{(1 + ax + bz_0)}{(1 + ax_0 + bz_0)} \delta x_0 + \frac{(1 + ax + bz_0)z_0}{(1 + ax_0 + bz_0)z} qy_0 \\ &\quad - \frac{y_0}{y} \left(\frac{\beta x z}{1 + ax + bz} - qy \right) - \frac{qz_0}{kz} (ky - \gamma z) + \sigma \\ &= \delta x_0 \left(1 - \frac{x}{x_0} - \frac{(1 + ax + bz_0)x_0}{(1 + ax_0 + bz_0)x} + \frac{(1 + ax + bz_0)}{(1 + ax_0 + bz_0)} \right) \\ &\quad + qy_0 \left(1 - \frac{(1 + ax + bz_0)x_0}{(1 + ax_0 + bz_0)x} + \frac{(1 + ax + bz_0)z_0}{(1 + ax + bz)z} \right) \\ &\quad + qy_0 \left(1 - \frac{(1 + ax_0 + bz_0)x_0 y_0 z}{(1 + ax + bz)x_0 y_0 z_0} \right) + qy_0 \left(1 - \frac{z}{z_0} - \frac{y z_0}{y_0 z} \right) + \sigma \\ &= -Q(x(t), y(t), z(t)) + \sigma, \end{aligned} \quad (3.7)$$

where

$$\sigma := \frac{(1 + bz_0)x_0\sigma_1^2}{2(1 + ax_0 + bz_0)} + \frac{y_0\sigma_2^2}{2} + \frac{qz_0\sigma_3^2}{2k} \tag{3.8}$$

and

$$-Q(x, y, z) = -\frac{\delta(1 + bz_0)}{x(1 + ax_0 + bz_0)}(x - x_0)^2 - \frac{qy_0b(1 + ax)(z - z_0)^2}{z_0(1 + ax + bz)(1 + ax + bz_0)} + qy_0 \left[4 - \frac{(1 + ax + bz_0)x_0}{(1 + ax_0 + bz_0)x} - \frac{(1 + ax_0 + bz_0)xy_0z}{(1 + ax + bz)x_0yz_0} - \frac{yz_0}{y_0z} - \frac{(1 + ax + bz)}{(1 + ax + bz_0)} \right]. \tag{3.9}$$

It can be verified by reciprocal calculation that

$$4 - \frac{(1 + ax + bz_0)x_0}{(1 + ax_0 + bz_0)x} - \frac{(1 + ax_0 + bz_0)xy_0z}{(1 + ax + bz)x_0yz_0} - \frac{yz_0}{y_0z} - \frac{(1 + ax + bz)}{(1 + ax + bz_0)} \leq 0.$$

Hence we have the observation that $Q(x, y, z)$ is continuous and $Q(x, y, z) \geq 0$. Moreover,

- $Q(x, y, z) = 0$, if $(x, y, z) = (x_0, y_0, z_0)$;
- $Q(x, y, z) > 0$, if otherwise;
- $Q(x, y, z) \rightarrow \infty$ as $(x, y, z) \rightarrow \infty$.

Take $U = \{(x, y, z) : Q(x, y, z) < \sigma\} \cap \mathbb{R}_+^3$, which is a bounded open set. Then $LV(x, y, z)$ is nonpositive for any $(x, y, z) \in \mathbb{R}_+^3/U$, which implies that the second condition in the aforementioned Assumption U is satisfied. Moreover, let $M = \min_{(x,y,z) \in \bar{U}} \{\sigma_1^2 x^2, \sigma_2^2 y^2, \sigma_3^2 z^2\} (> 0)$. Then we have

$$\sigma_1^2 x^2 \xi_1^2 + \sigma_2^2 y^2 \xi_2^2 + \sigma_3^2 z^2 \xi_3^2 \geq M|\xi|^2 \quad \text{for all } (x, y, z) \in U \text{ and } \xi \in \mathbb{R}^3.$$

Thus the first condition in the Assumption U is also satisfied. Therefore, by Lemma 3.1, the system of SDE (1.1) has a unique stationary distribution $\mu(\cdot)$ on \mathbb{R}_+^3 . \square

The ergodic property in terms of this stationary distribution will be shown in Section 5.

4. Asymptotic Moment Estimation

First we prove an auxiliary lemma useful for dynamic estimation of the moments of the positive solutions to the stochastic viral system (1.1).

Lemma 4.1. *The solution of an initial value problem of the Bernoulli equation*

$$\begin{aligned} \frac{dv}{dt} &= pv(t) \left[-\left(\delta - \frac{\sigma^2}{2}(p-1) \right) + \lambda v^{-\frac{1}{p}}(t) \right], \\ v(0) &= x_0 \end{aligned} \tag{4.1}$$

is given by

$$v(t) = \left\{ x_0^{1/p} e^{-(\delta - \frac{\sigma^2}{2}(p-1))t} + \frac{\lambda}{\delta - \frac{\sigma^2}{2}(p-1)} \left(1 - e^{-(\delta - \frac{\sigma^2}{2}(p-1))t} \right) \right\}^p. \tag{4.2}$$

Proof. Divided the equation (4.1) by $v^{1-\frac{1}{p}}(t)$, we get

$$\frac{1}{v^{1-\frac{1}{p}}(t)} \frac{dv(t)}{dt} \leq -p \left(\delta - \frac{\sigma^2}{2}(p-1) \right) v^{\frac{1}{p}}(t) + p\lambda.$$

Let $w = v^{\frac{1}{p}}(t)$, then we come up with the linear ordinary differential equation

$$\frac{dw}{dt} = - \left(\delta - \frac{\sigma^2}{2}(p-1) \right) w + \lambda$$

and the solution (4.2) is obtained. \square

We now derive an asymptotic upper bound for the p -th moment of each component solution of the system of SDE (1.1).

Theorem 4.1. *Suppose the following conditions are satisfied,*

$$\delta + \frac{\sigma_1^2}{2} - \frac{\sigma_1^2}{2}p > 0, \quad q + \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2}p > 0, \quad \text{and} \quad \gamma + \frac{\sigma_3^2}{2} - \frac{\sigma_3^2}{2}p > 0. \quad (4.3)$$

Let $(x(t), y(t), z(t))$ be a solution of the system of stochastic viral equations (1.1) with any initial data $(x_0, y_0, z_0) \in R_+^3$. Then for any $p > 1$ it holds that

$$\limsup_{t \rightarrow \infty} E[x^p(t)] \leq L_1(p), \quad \limsup_{t \rightarrow \infty} E[y^p(t)] \leq L_2(p), \quad \limsup_{t \rightarrow \infty} E[z^p(t)] \leq L_3(p), \quad (4.4)$$

where

$$\begin{aligned} L_1(p) &= \left(\frac{\lambda}{\delta - \frac{\sigma_1^2}{2}(p-1)} \right)^p, \\ L_2(p) &= \left(\frac{4\beta b\lambda}{b(2\delta - \sigma_1^2(p-1))(2q - \sigma_2^2(p-1))} \right)^p, \\ L_3(p) &= \frac{8k\beta\lambda}{b(2\delta - \sigma_1^2(p-1))(2q - \sigma_2^2(p-1))(2\gamma - \sigma_3^2(p-1))}. \end{aligned}$$

Proof. By the Itô formula, for the x -component solution of (1.1) we have

$$\begin{aligned} d(x^p) &= px^{p-1}dx + \frac{1}{2}p(p-1)x^{p-2}(dx)^2 \\ &= px^{p-1} \left[\left(\lambda - \delta x - \frac{\beta xz}{1+ax+bz} \right) dt + \sigma_1 x dB_1(t) \right] + \frac{1}{2}p(p-1)x^p \sigma_1^2 dt \\ &= px^{p-1} \left(\lambda - \delta x - \frac{\beta xz}{1+ax+bz} + \frac{\sigma_1^2}{2}(p-1)x \right) dt + px^p \sigma_1 dB_1(t) \\ &\leq px^{p-1} \left(\lambda - \delta x + \frac{\sigma_1^2}{2}(p-1)x \right) dt + px^p \sigma_1 dB_1(t), \quad t \geq 0. \end{aligned}$$

Consequently,

$$x^p \leq x_0^p + \int_0^t px^{p-1}(s) \left(\lambda - \delta x(s) + \frac{\sigma_1^2}{2}(p-1)x(s) \right) ds + \int_0^t px^p(s) \sigma_1 dB_1(s). \quad (4.5)$$

Taking the expectation on both sides of (4.5), we have

$$E[x^p(t)] \leq x_0^p + \int_0^t \left[p\lambda E[x^{p-1}(s)] - p \left(\delta - \frac{\sigma_1^2}{2}(p-1) \right) E[x^p(s)] \right] ds. \quad (4.6)$$

Therefore, the p -moment of the x -component solution satisfies the differential equation

$$\begin{aligned} \frac{dE[x^p(t)]}{dt} &\leq p\lambda E[x^{p-1}] - p \left(\delta - \frac{\sigma_1^2}{2}(p-1) \right) E[x^p] \\ &\leq -p \left(\delta - \frac{\sigma_1^2}{2}(p-1) \right) E[x^p] + p\lambda E[x^p]^{1-\frac{1}{p}}. \end{aligned}$$

Define $u(t) = E[x^p]$ and we have

$$\frac{du(t)}{dt} \leq -p \left(\delta - \frac{\sigma_1^2}{2}(p-1) \right) u(t) + p\lambda u^{1-\frac{1}{p}}(t). \quad (4.7)$$

According to the condition (4.3), we have $\delta + \frac{\sigma_1^2}{2} - \frac{\sigma_1^2}{2}p \geq 0$. By the exponential decay in (4.2) of Lemma 4.1 and by the comparison theorem of ODE, we have

$$\limsup_{t \rightarrow \infty} E[x^p(t)] = \limsup_{t \rightarrow \infty} u(t) \leq L_1(p) = \left(\frac{\lambda}{\delta - \frac{\sigma_1^2}{2}(p-1)} \right)^p. \quad (4.8)$$

Similarly we can treat the y -component of the positive solutions of (1.1) as follows,

$$\begin{aligned} d(y^p) &= py^{p-1}dy + \frac{1}{2}p(p-1)y^{p-2}(dy)^2 \\ &= py^{p-1} \left(\frac{\beta xz}{1+ax+bz} - qy + \frac{\sigma_2^2}{2}(p-1) \right) dt + py^{p-1}\sigma_2y dB_2(t) \\ &\leq py^{p-1} \left(\frac{\beta x}{b} - qy + \frac{\sigma_2^2}{2}(p-1)y \right) dt + px^p\sigma_2 dB_2(t) \end{aligned}$$

and consequently,

$$E[y^p(t)] \leq y_0^p + \int_0^t \left[\frac{p\beta}{b} E[x(s)y^{p-1}(s)] - p \left(q - \frac{\sigma_2^2}{2}(p-1) \right) E[y^p(s)] \right] ds. \quad (4.9)$$

Hence,

$$\begin{aligned} \frac{dE[y^p(t)]}{dt} &\leq -p \left(q - \frac{\sigma_2^2}{2}(p-1) \right) E[y^p(t)] + \frac{p\beta}{b} E[x(t)y^{p-1}(t)] \\ &\leq -p \left(q - \frac{\sigma_2^2}{2}(p-1) \right) E[y^p(t)] + \frac{p\beta}{b} E[x(t)]E[y^{p-1}(t)] \\ &\leq -p \left(q - \frac{\sigma_2^2}{2}(p-1) \right) E[y^p(t)] + \frac{2p\beta\lambda}{b(2d - \sigma_1^2(p-1))} E[y(t)]^{1-\frac{1}{p}}. \end{aligned} \quad (4.10)$$

By Lemma 4.1 and its proof, the solution of the following Bernoulli equation

$$\frac{dv}{dt} = pv(t) \left[- \left(q - \frac{\sigma_2^2}{2}(p-1) \right) + \frac{2p\beta\lambda}{b(2d - \sigma_1^2(p-1))} v^{-\frac{1}{p}}(t) \right] \quad (4.11)$$

with $v(0) = y_0$ is given by $v(t) = [w(t)]^p$, where

$$\frac{dw}{dt} = - \left(q - \frac{\sigma_2^2}{2}(p-1) \right) w + \frac{2\beta\lambda}{b(2\delta - \sigma_1^2(p-1))}.$$

Thus the solution of (4.11) is given by

$$v^p(t) = y_0^{1/p} e^{-\left(q - \frac{\sigma_2^2}{2}(p-1)\right)t} + \frac{4\beta\lambda}{b(2\delta - \sigma_1^2(p-1))(2q - \sigma_2^2(p-1))} \left(1 - e^{-\left(q - \frac{\sigma_2^2}{2}(p-1)\right)t} \right).$$

According to the condition (4.3), we have $q - \frac{\sigma_2^2}{2}(p-1) > 0$. Due to the exponential decay and by the comparison theorem, we get

$$\limsup_{t \rightarrow \infty} E[y^p(t)] \leq L_2(p) = \left(\frac{4\beta\lambda}{b(2\delta - \sigma_1^2(p-1))(2q - \sigma_2^2(p-1))} \right)^p. \quad (4.12)$$

Finally, for the z -component solution we have

$$\begin{aligned} d(z^p) &= pz^{p-1}dx + \frac{1}{2}p(p-1)z^{p-2}(dz)^2 \\ &= pz^{p-1}[(ky - \gamma z)dt + \sigma_3 z dB_3(t)] + \frac{1}{2}p(p-1)z^p\sigma_3^2 dt \\ &= pz^{p-1} \left(ky - \gamma z + \frac{\sigma_3^2}{2}(p-1)z \right) dt + pz^p\sigma_3 dB_3(t) \end{aligned}$$

and

$$E[z^p(t)] \leq z_0^p + \int_0^t \left[pk E[y(s)z^{p-1}(s)] - p \left(\gamma - \frac{\sigma_3^2}{2}(p-1) \right) E[z^p(s)] \right] ds. \quad (4.13)$$

Hence,

$$\begin{aligned} \frac{dE[z^p(t)]}{dt} &\leq pk E[y(t)]E[z^{p-1}(t)] - p \left(\gamma - \frac{\sigma_3^2}{2}(p-1) \right) E[z^p(t)] \\ &\leq -p \left(\gamma - \frac{\sigma_3^2}{2}(p-1) \right) E[z^p(t)] + \frac{4pk\beta\lambda}{b(2\delta - \sigma_1^2(p-1))(2q - \sigma_2^2(p-1))} E[z^p]^{1-\frac{1}{p}}. \end{aligned} \quad (4.14)$$

Similarly, since $\gamma + \frac{\sigma_3^2}{2} - \frac{\sigma_3^2}{2}p > 0$, through solution of the corresponding Bernoulli equation similar to Lemma 4.1 and the comparison argument, we obtain

$$\limsup_{t \rightarrow \infty} E[z^p(t)] \leq L_3(p) = \left(\frac{8k\beta\lambda}{b(2\delta - \sigma_1^2(p-1))(2q - \sigma_2^2(p-1))(2\gamma - \sigma_3^2(p-1))} \right)^p. \quad (4.15)$$

The proof is completed. \square

Next we derive specific pathwise upper bound and lower bound for each of the three component solutions of the system (1.1) for the study of stochastic virus dynamics and stability.

Theorem 4.2. *Every positive solution $(x(t), y(t), z(t))$ of the system (1.1) with any initial data $(x_0, y_0, z_0) \in \mathbb{R}_+^3$ satisfies the estimates*

$$\Phi_u(t) \leq x(t) \leq \Phi_u(t), \quad \Psi_l(t) \leq y(t) \leq \Psi_u(t), \quad \Gamma_l(t) \leq z(t) \leq \Gamma_u(t), \quad t \geq 0, \quad a.s. \tag{4.16}$$

where the upper and lower bounds $\Phi_u(t), \Phi_l(t), \Psi_u(t), \Psi_l(t), \Gamma_u(t)$ and $\Gamma_l(t)$ are the stochastic processes shown by (4.17) through (4.22) below.

Proof. Since the solutions are positive in probability one, from (1.1) we have

$$dx \leq (\lambda - \delta x) dt + \sigma_1 x dB_1, \quad t \geq 0.$$

We can assert and verify that the stochastic process given by

$$\begin{aligned} \Phi_u(t) = & \lambda \int_0^t \exp \left\{ - \left(\delta + \frac{\sigma_1^2}{2} \right) (t-s) + \sigma_1 (B_1(t) - B_1(s)) \right\} ds \\ & + x_0 \exp \left\{ - \left(\delta + \frac{\sigma_1^2}{2} \right) t + \sigma_1 B_1(t) \right\} \end{aligned} \tag{4.17}$$

is the unique solution of stochastic differential equation

$$\frac{d\Phi}{dt} = (\lambda - \delta\Phi(t))dt + \sigma_1\Phi(t) dB_1$$

with the initial condition $\Phi(0) = x_0$. By the comparison theorem for the pathwise solutions of stochastic differential equations, it holds that

$$x(t) \leq \Phi_u(t), \quad t \geq 0, \quad a.s.$$

For the y -component solution, we have

$$dy = \left(\frac{\beta x z}{1 + ax + bz} - qy \right) dt + \sigma_2 y dB_2(t) \geq (-\beta - qy) dt + \sigma_2 y dB_2(t).$$

Here we can claim and check that the stochastic process given by

$$\begin{aligned} \Psi_l(t) = & -\beta \int_0^t \exp \left\{ - \left(q + \frac{\sigma_2^2}{2} \right) (t-s) + \sigma_2 (B_2(t) - B_2(s)) \right\} ds \\ & + y_0 \exp \left\{ - \left(q + \frac{\sigma_2^2}{2} \right) t + \sigma_2 B_2(t) \right\} \end{aligned} \tag{4.18}$$

is the unique solution of the initial value problem

$$\frac{d\Psi}{dt} = (-\beta - q\Psi(t)) dt + \sigma_2\Psi(t) dB_2,$$

with $\Psi(0) = y_0$. By the comparison argument, we obtain

$$y(t) \geq \Psi_l(t), \quad t \geq 0, \quad a.s.$$

On the other hand,

$$\begin{aligned} dy = & \left(\frac{\beta x(t)}{\frac{1+ax(t)}{z(t)} + b} - qy(t) \right) dt + \sigma_2 y(t) dB_2(t) \\ \leq & \left(\frac{\beta\Phi_u(t)}{b} - qy(t) \right) dt + \sigma_2 y(t) dB_2(t). \end{aligned}$$

Similarly, by the comparison argument, we can get the stochastic process given by

$$\begin{aligned} \Psi_u(t) &= \frac{\beta}{b} \int_0^t \Phi_u(s) \exp \left\{ - \left(q + \frac{\sigma_2^2}{2} \right) (t-s) + \sigma_2 (B_2(t) - B_2(s)) \right\} ds \\ &\quad + y_0 \exp \left\{ - \left(q + \frac{\sigma_2^2}{2} \right) t + \sigma_2 B_2(t) \right\}, \end{aligned} \quad (4.19)$$

as the upper bound for the y -component solution,

$$y(t) \leq \Psi_u(t), \quad t \geq 0, \quad a.s.$$

For the component solution $z(t)$, we have

$$dz = (ky - \gamma z) dt + \sigma_3 z dB_3 \geq -\gamma z dt + \sigma_3 z dB_3.$$

Accordingly, the stochastic process given by

$$\Gamma_l(t) = z_0 \exp \left\{ - \left(\gamma + \frac{\sigma_3^2}{2} \right) t + \sigma_3 B_3(t) \right\} \quad (4.20)$$

is the unique solution of the initial value problem

$$\frac{d\Gamma}{dt} = -\gamma \Gamma(t) dt + \sigma_3 \Gamma(t) dB_3$$

with $\Gamma(0) = z_0$ and we have

$$z(t) \geq \Gamma_l(t), \quad t \geq 0, \quad a.s.$$

On the other hand,

$$dz = (ky - \gamma z) dt + \sigma_3 z dB_3(t) \leq (k\Psi_2(t) - \gamma z) dt + \sigma_3 z dB_3.$$

Thus we have

$$z(t) \leq \Gamma_u(t), \quad t \geq 0, \quad a.s.$$

where

$$\begin{aligned} \Gamma_u(t) &= k \int_0^t \Psi_u(s) \exp \left\{ - \left(\gamma + \frac{\sigma_3^2}{2} \right) (t-s) + \sigma_3 (B_3(t) - B_3(s)) \right\} ds \\ &\quad + z_0 \exp \left\{ - \left(\gamma + \frac{\sigma_3^2}{2} \right) t + \sigma_3 B_3(t) \right\}. \end{aligned} \quad (4.21)$$

Finally, by the first stochastic differential equation in (1.1) and $z(t) \leq \Gamma_u(t)$, we notice that

$$dx \geq \left(\lambda - \delta x - \frac{\beta \Gamma_u(t)}{a} \right) dt + \sigma_1 x dB_1.$$

Hence we can deduce by the similar argument that

$$x(t) \geq \Phi_l(t), \quad t \geq 0, \quad a.s.$$

where

$$\begin{aligned} \Phi_l(t) &= x_0 \exp \left\{ - \left(\delta + \frac{\sigma_1^2}{2} \right) t + \sigma_1 B_1(t) \right\} \\ &\quad + \int_0^t \left(\lambda - \frac{\beta \Gamma_2(t)}{a} \right) \exp \left\{ - \left(\delta + \frac{\sigma_1^2}{2} \right) (t-s) + \sigma_1 (B_1(t) - B_1(s)) \right\} ds. \end{aligned} \quad (4.22)$$

The proof is completed. \square

5. Absorbing Property in Time Average

Here we prove the absorbing property of the positive solutions in time average for the stochastic viral system (1.1) based on the asymptotic moment estimation and the stationary distribution shown in previous sections.

Theorem 5.1. *Under the same condition as in Theorem 3.1, if (4.3) is satisfied and $p > 1$, then*

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x^p(s) ds &= \int_{R_+^3} \xi^p d\mu(\xi, \eta, \zeta) \leq L_1(p), \quad a.s. \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y^p(s) ds &= \int_{R_+^3} \eta^p d\mu(\xi, \eta, \zeta) \leq L_2(p), \quad a.s. \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t z^p(s) ds &= \int_{R_+^3} \zeta^p d\mu(\xi, \eta, \zeta) \leq L_3(p), \quad a.s. \end{aligned} \tag{5.1}$$

where $(x(t), y(t), z(t))$ is any solution of the system (1.1) with initial data in R_+^3 and $L_i(p), 1 \leq i \leq 3$, are given in (4.4). Consequently, the p -th moments of the pathwise positive solutions exist for $p > 1$ with respect to the stationary distribution μ .

Proof. By the Birkhoff ergodic property, for any $N > 0$, it holds that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t [x^p(s) \wedge N] ds &= \int_{R_+^3} (\xi^p \wedge N) d\mu(\xi, \eta, \zeta) \quad a.s. \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t [y^p(s) \wedge N] ds &= \int_{R_+^3} (\eta^p \wedge N) d\mu(\xi, \eta, \zeta) \quad a.s. \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t [z^p(s) \wedge N] ds &= \int_{R_+^3} (\zeta^p \wedge N) d\mu(\xi, \eta, \zeta) \quad a.s. \end{aligned}$$

Using the dominated convergence theorem and by Theorem 4.1, we have

$$\begin{aligned} E \left[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t [x^p(s) \wedge N] ds \right] &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t E[x^p(s) \wedge N] ds \leq L_1(p), \\ E \left[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t [y^p(s) \wedge N] ds \right] &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t E[y^p(s) \wedge N] ds \leq L_2(p), \\ E \left[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t [z^p(s) \wedge N] ds \right] &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t E[z^p(s) \wedge N] ds \leq L_3(p), \end{aligned}$$

where $L_i(p), 1 \leq i \leq 3$, are given in (4.4). Therefore,

$$\begin{aligned} E \left[\int_{R_+^3} (\xi^p \wedge N) d\mu(\xi, \eta, \zeta) \right] &\leq L_1(p), \\ E \left[\int_{R_+^3} (\eta^p \wedge N) d\mu(\xi, \eta, \zeta) \right] &\leq L_2(p), \end{aligned}$$

$$E \left[\int_{\mathbb{R}_+^3} (\zeta_3^p \wedge N) d\mu(\xi, \eta, \zeta) \right] \leq L_3(p).$$

Let $N \rightarrow \infty$, then we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x^p(s) ds &= \int_{\mathbb{R}_+^3} \xi^p d\mu(\xi, \eta, \zeta) = \int_{\mathbb{R}_+^3} E[x(t, \xi)]^p d\mu(\xi, \eta, \zeta) \leq L_1(p), \quad a.s. \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y^p(s) ds &= \int_{\mathbb{R}_+^3} \eta^p d\mu(\xi, \eta, \zeta) = \int_{\mathbb{R}_+^3} E[y(t, \eta)]^p d\mu(\xi, \eta, \zeta) \leq L_2(p), \quad a.s. \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t z^p(s) ds &= \int_{\mathbb{R}_+^3} \zeta^p d\mu(\xi, \eta, \zeta) = \int_{\mathbb{R}_+^3} E[z(t, \zeta)]^p d\mu(\xi, \eta, \zeta) \leq L_3(p), \quad a.s. \end{aligned}$$

where we used (3.2) and (3.3) in Lemma 3.1. Thus (5.1) is proved. □

6. Moment Lyapunov Exponent

Lyapunov exponent and moment Lyapunov exponent are important characteristics for study of the almost sure stability and the moment stability of a stochastic dynamical system generated by stochastic differential equations. However, the actual evaluation of the moment Lyapunov exponents will be very difficult except by few approximation approaches of the asymptotic expansions, cf. [10, Appendix B].

Here for this stochastic virus model, we can make a calculation in the following result based on the upper and lower bound estimates.

Theorem 6.1. *Under the condition $\delta > \sigma_1^2/2, q > \sigma_2^2/2$ and $\gamma > \sigma_3^2/2$, the mean-square moment Lyapunov exponent of the positive solution trajectories of the system (1.1) is nonpositive,*

$$\Lambda(2) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log E \|X(t, X_0)\|_{\mathbb{R}_+^3}^2 \leq 0, \tag{6.1}$$

where $X(t, X_0) = (x(t, x_0), y(t, y_0), z(t, z_0))$ is the solution with any initial data $X_0 \in \mathbb{R}_+^3$.

Proof. Step 1. (4.17) shows that

$$\Phi_u(t) = \left[\lambda \int_0^t \exp \left\{ \left[\delta + \frac{\sigma_1^2}{2} \right] s - \sigma_1 B_1(s) \right\} ds + x_0 \right] \exp \left\{ - \left[\delta + \frac{\sigma_1^2}{2} \right] t + \sigma_1 B_1(t) \right\}.$$

Note that

$$S(t) = S(0) \exp \left\{ - \left[\delta + \frac{\sigma_1^2}{2} \right] t + \sigma_1 B_1(t) \right\}, \quad t \geq 0, \tag{6.2}$$

is a geometric Brownian motion, which is a solution of the linear SDE

$$dS = -\delta S(t) dt + \sigma_1 S(t) dB_1(t).$$

Hence we can calculate

$$E[S(t)] = S(0) e^{-\delta t} \quad \text{and} \quad E|S(t)|^2 = |S(0)|^2 \exp\{(\sigma_1^2 - 2\delta)t\}, \quad t \geq 0. \tag{6.3}$$

Accordingly we have

$$E \left| x_0^2 \exp \left\{ -2 \left[\delta + \frac{\sigma_1^2}{2} \right] t + 2\sigma_1 B_1(t) \right\} \right| = x_0^2 \exp\{(\sigma_1^2 - 2\delta)t\} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Then by the Fubini theorem and the Jensen inequality of integral as well as the time-shifting property of Brownian motion, we can derive

$$\begin{aligned} E|\Phi_u(t)|^2 &\leq 2\lambda^2 \int_0^t E \left[\exp \left\{ -2 \left[\delta + \frac{\sigma_1^2}{2} \right] (t-s) + 2\sigma_1(B_1(t) - B_1(s)) \right\} \right] ds \\ &\quad + 2x_0^2 \exp\{(\sigma_1^2 - 2\delta)t\} \\ &= 2\lambda^2 \int_0^t \exp\{(\sigma_1^2 - 2\delta)(t-s)\} ds + 2x_0^2 \exp\{(\sigma_1^2 - 2\delta)t\} \\ &= \frac{2\lambda^2}{2\delta - \sigma_1^2} (1 - \exp\{(\sigma_1^2 - 2\delta)t\}) + 2x_0^2 \exp\{(\sigma_1^2 - 2\delta)t\} \\ &\leq \frac{2\lambda^2}{2\delta - \sigma_1^2} + 2x_0^2 \exp\{(\sigma_1^2 - 2\delta)t\}, \quad t \geq 0. \end{aligned} \tag{6.4}$$

In view of the simple property $\log(1+x) \leq x$ for all $x > -1$ and $\delta > \sigma_1^2/2$, it follows that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \log (E|x(t, x_0)|^2) &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log (E|\Phi_u(t)|^2) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\frac{2\lambda^2}{2\delta - \sigma_1^2} + 2x_0^2 \exp\{(\sigma_1^2 - 2\delta)t\} \right] \\ &= \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\frac{2\lambda^2}{2\delta - \sigma_1^2} \left(1 + \frac{x_0^2(2\delta - \sigma_1^2)}{\lambda^2} \exp\{(\sigma_1^2 - 2\delta)t\} \right) \right] \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \frac{2\lambda^2}{2\delta - \sigma_1^2} + \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left(1 + \frac{x_0^2(2\delta - \sigma_1^2)}{\lambda^2} \exp\{(\sigma_1^2 - 2\delta)t\} \right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \left(\frac{x_0^2(2\delta - \sigma_1^2)}{\lambda^2} \exp\{(\sigma_1^2 - 2\delta)t\} \right) = 0. \end{aligned} \tag{6.5}$$

Step 2. For the y -component of the solution, from (4.19) we get

$$\begin{aligned} E|\Psi_u(t)|^2 &\leq \frac{2\beta^2}{b^2} \int_0^t E \left[\Phi_u^2(s) \exp \left\{ -2 \left[q + \frac{\sigma_2^2}{2} \right] (t-s) + 2\sigma_2(B_2(t) - B_2(s)) \right\} \right] ds \\ &\quad + 2y_0^2 \exp\{(\sigma_2^2 - 2q)t\}. \end{aligned} \tag{6.6}$$

Since $B_1(t), B_2(t), B_3(t)$ are independent Brownian motions, from (6.3) and (6.4) we can deduce

$$\begin{aligned} &E \left[\Phi_u^2(s) \exp \left\{ -2 \left[q + \frac{\sigma_2^2}{2} \right] (t-s) + 2\sigma_2(B_2(t) - B_2(s)) \right\} \right] \\ &= E[\Phi_u^2(s)] E \left[\exp \left\{ -2 \left[q + \frac{\sigma_2^2}{2} \right] (t-s) + 2\sigma_2(B_2(t) - B_2(s)) \right\} \right] \\ &\leq \left[\frac{2\lambda^2}{2\delta - \sigma_1^2} (1 - \exp\{(\sigma_1^2 - 2\delta)s\}) + 2x_0^2 \exp\{(\sigma_1^2 - 2\delta)s\} \right] \exp\{(\sigma_2^2 - 2q)(t-s)\}. \end{aligned} \tag{6.7}$$

Substitute (6.7) into (6.6). In view of the condition $\delta > \sigma_1^2/2$ and $q > \sigma_2^2/2$, we obtain

$$\begin{aligned}
 & E|\Psi_u(t)|^2 \\
 & \leq \frac{2\beta^2}{b^2} \int_0^t \left(\frac{2\lambda^2}{2\delta - \sigma_1^2} + 2x_0^2 \exp\{\sigma_1^2 - 2\delta\}s \right) \exp\{(\sigma_2^2 - 2q)(t-s)\} ds + 2y_0^2 \exp\{(\sigma_2^2 - 2q)t\} \\
 & \leq \frac{2\beta^2}{b^2} \left(\frac{2\lambda^2}{(2\delta - \sigma_1^2)(2q - \sigma_2^2)} + 2x_0^2 \int_0^t \exp\{\sigma_1^2 - 2\delta\}s \exp\{(\sigma_2^2 - 2q)(t-s)\} ds \right) \\
 & \quad + 2y_0^2 \exp\{(\sigma_2^2 - 2q)t\} \\
 & \leq \frac{4\beta^2\lambda^2}{b^2(2\delta - \sigma_1^2)(2q - \sigma_2^2)} + \frac{4\beta^2x_0^2}{b^2(2q - \sigma_2^2)} + 2y_0^2 \exp\{(\sigma_2^2 - 2q)t\}, \quad t \geq 0.
 \end{aligned} \tag{6.8}$$

Therefore, similar to (6.5), we have

$$\begin{aligned}
 & \limsup_{t \rightarrow \infty} \frac{1}{t} \log (E|y(t, y_0)|^2) \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log (E|\Psi_u(t)|^2) \\
 & \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log [C_1 + 2y_0^2 \exp\{(\sigma_2^2 - 2q)t\}] \\
 & \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log C_1 + \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left(1 + \frac{2y_0^2}{C_1} \exp\{(\sigma_2^2 - 2q)t\} \right) \\
 & \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \left(\frac{2y_0^2}{C_1} \exp\{(\sigma_2^2 - 2q)t\} \right) = 0,
 \end{aligned} \tag{6.9}$$

where

$$C_1 = \frac{4\beta^2\lambda^2}{b^2(2\delta - \sigma_1^2)(2q - \sigma_2^2)} + \frac{4\beta^2x_0^2}{b^2(2q - \sigma_2^2)}.$$

Step 3. For the z -component of the solution, from (4.21) we have

$$\begin{aligned}
 E|\Gamma_u(t)|^2 & \leq 2k^2 \int_0^t E \left[\Psi_u^2(s) \exp \left\{ -2 \left[\gamma + \frac{\sigma_3^2}{2} \right] (t-s) + 2\sigma_3(B_3(t) - B_3(s)) \right\} \right] ds \\
 & \quad + 2z_0^2 \exp\{(\sigma_3^2 - 2\gamma)t\}.
 \end{aligned} \tag{6.10}$$

Then in turn we substitute (6.8) into (6.10). By the independence of $B_2(t)$ and $B_3(t)$ together with all the condition $\delta > \sigma_1^2/2$, $q > \sigma_2^2/2$ and $\gamma > \sigma_3^2/2$, we get

$$\begin{aligned}
 & E|\Gamma_u(t)|^2 \\
 & \leq 2k^2 \int_0^t \left(\frac{4\beta^2\lambda^2}{b^2(2\delta - \sigma_1^2)(2q - \sigma_2^2)} + \frac{4\beta^2x_0^2}{b^2(2q - \sigma_2^2)} + 2y_0^2 \exp\{(\sigma_2^2 - 2q)s\} \right) \cdot \\
 & \quad \cdot E \left[\exp \left\{ -2 \left[\gamma + \frac{\sigma_3^2}{2} \right] (t-s) + 2\sigma_3(B_3(t) - B_3(s)) \right\} \right] ds + 2z_0^2 \exp\{(\sigma_3^2 - 2\gamma)t\} \\
 & \leq 2k^2 \int_0^t (C_1 + 2y_0^2) \exp\{(\sigma_3^2 - 2\gamma)(t-s)\} ds + 2z_0^2 \exp\{(\sigma_3^2 - 2\gamma)t\} \\
 & \leq \frac{2k^2}{(2\gamma - \sigma_3^2)} (C_1 + 2y_0^2) + 2z_0^2 \exp\{(\sigma_3^2 - 2\gamma)t\}, \quad t \geq 0.
 \end{aligned} \tag{6.11}$$

Consequently and similarly, we obtain

$$\begin{aligned}
 & \limsup_{t \rightarrow \infty} \frac{1}{t} \log (E|z(t, z_0)|^2) \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log (E|\Gamma_u(t)|^2) \\
 & \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\frac{2k^2}{(2\gamma - \sigma_3^2)} (C_1 + 2y_0^2) + 2z_0^2 \exp\{(\sigma_3^2 - 2\gamma)t\} \right] \\
 & \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\frac{2k^2}{(2\gamma - \sigma_3^2)} (C_1 + 2y_0^2) \right] \\
 & \quad + \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[1 + \frac{2z_0^2}{C_2} \exp\{(\sigma_3^2 - 2\gamma)t\} \right] \\
 & \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \left(\frac{2z_0^2}{C_2} \exp\{(\sigma_3^2 - 2\gamma)t\} \right) = 0
 \end{aligned} \tag{6.12}$$

where $C_2 = 2k^2(C_1 + 2y_0^2)/(2\gamma - \sigma_3^2)$. Combining (6.5), (6.9) and (6.12), we see that (6.1) is proved. \square

Theorem 6.2. *Under the same assumption as in Theorem 6.1, for any $p > 1$, the p -th moment Lyapunov exponent of the positive solution trajectories of the system (1.1) is nonpositive,*

$$\Lambda(p) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log E\|X(t, X_0)\|_{\mathbb{R}_+^3}^p \leq 0, \tag{6.13}$$

where $X(t, X_0) = (x(t, x_0), y(t, y_0), z(t, z_0))$ is the solution with any initial data $X_0 \in \mathbb{R}_+^3$.

Proof. One can prove (6.13) by the same approach as in the proof of Theorem 6.1 with the fact that the p -th moment of the geometric Brownian motion (6.2) is given by

$$E|S(t)|^p = |S(0)|^p \exp \left[p \left(\frac{(p-1)}{2} \sigma_1^2 - \delta \right) t \right], \quad t \geq 0, \tag{6.14}$$

cf. [22, p. 106]. The detailed proof is omitted here for simplicity. \square

7. Conclusion

The aim of this paper is to investigate the virus dynamics modeled by the susceptible-infected-removed (SIR) equations with the Beddington-DeAngelis functional response and the stochastic multiplicative and independent white noises. Here we studied the pathwise, time-averaging, and asymptotic dynamics for the almost surely positive solutions of this system.

Starting with the proof of global existence of the positive solutions by construction of a linear-log Lyapunov function and using the stochastic Itô formula, the existence of a stationary distribution with respect to the positive solutions is established by construction of another sophisticated Lyapunov function. Through conducting sharp estimates, we obtained the asymptotic upper and lower bounds of the moments of stochastic trajectories.

Based on the asymptotic moment estimation and the stationary distribution as well as the ergodicity, we further proved the absorbing property in time average for

the solution trajectories of this stochastic virus model. Finally we made the non-positive evaluation of the mean-square and any order moment Lyapunov exponents. These results will be useful for a potential study in regard to prediction of virus persistence and extinction asymptotically in a long run.

Recently there have been stochastic epidemic models driven with noise of Lévy processes (sometimes called telephone noise or telegraph noise [2, 11, 23, 25]) proposed and studied. Many good results on epidemic dynamics and prediction of persistence and extinction of diseases or virus for SIR and SIRS type equations with Markov switching have been reported, cf. [6, 7, 12, 13, 25, 26] and references therein. Besides there are researches of longtime dynamics on the budworm growth and predator-prey models with harvesting and distributed delays also involving stochastic noises of regime-switching, cf. [2, 15, 20, 28]. It is expected that the Beddington-DeAngelis dynamics with Lévy type noise will also be investigated.

References

- [1] J. R. Beddington, *Mutual interference between parasites or predators and its effect on searching efficiency*, J. Anim. Ecology, 1975, 44(1) 331–340.
- [2] S. Bonaccorsi and S. Ottaviano, *Epidemics on networks with heterogeneous population and stochastic infection rates*, Mathematical Biosciences, 2016, 279, 43–52.
- [3] D. L. DeAngelis, R. A. Goldstein, R. V. O'Neill, *A model for trophic interaction*, Ecology, 1975, 56(2), 881–892.
- [4] N. T. Dieu, *Asymptotic properties of a stochastic SIR epidemic model with Beddington-DeAngelis incidence rate*, J. Dynamics and Differential equations, 2018, 30(1), 93–106.
- [5] A. Gray, D. Greenhalgh, L. Hu, X. Mao and J. Pan, *A stochastic differential equation on SIS epidemic model*, SIAM J. Appl. Math., 2011, 71(3), 876–902.
- [6] D. Greenhalgh, Y. Liang and X. Mao, *Modeling the effect of telegraph noise in the SIRS epidemic model using Markovian switching*, Physica A, 2016, 462(15), 684–704.
- [7] A. Gray, D. Greenhalgh, X. Mao and J. Pan, *The SIS epidemic model with Markovian switching*, J. Math. Anal. Appl., 2012, 394(2), 496–516.
- [8] G. Huang, W. MA and Y. Takeuch, *Global properties for virus dynamics model with Beddington-DeAngelis functional response*, Applied Mathematics Letters, 2009, 22(11), 1690–1693.
- [9] C. Ji and D. Jiang, *Dynamics of a stochastic density dependent predator-prey system with beddington-DeAngelis functional response*, J. Math. Anal. Appl., 2011, 381(1), 441–453.
- [10] R. Z. Khasminskii, *Stochastic Stability of Differential Equations, 2nd edition*, Stochastic Modeling and Applied Probability, Vol. 66, Springer-Verlag, Berlin, 2012.
- [11] E. Kussell and S. Leibler, *Phenotype diversity, population growth, and information in fluctuating environments*, Science, 2005, 309(5743), 2075–2078.
- [12] A. Lahrouz and A. Settati, *Asymptotic properties of switching diffusion epidemic model with varying population size*, Applied Mathematics and Computations, 2013, 219(24), 11134–11148.

-
- [13] D. Li, S. Liu and J. Cui, *Threshold dynamics and ergodicity of an SIRS epidemic model with Markovian switching*, J. Differential Equations, 2017, 263(12), 8873–8915.
- [14] J. Li, M. Shan, M. Banerjee and A. wang, *Stochastic dynamics of feline immunodeficiency virus within cat populations*, Journal of the Franklin Institute, 2016, 353(16), 4191–4212.
- [15] M. Liu, X. He and J. Yu, *Dynamics of a stochastic regime-switching predator-prey model with harvesting and distributed delays*, Nonlinear Analysis: Hybrid Systems, 2018, 28, 87–104.
- [16] Q. Liu and D. Jiang, *Stationary distribution of a stochastic SIS epidemic model with double diseases and the Beddington-DeAngelis incidence*, Chaos, 2017, 27(8), 083126.
- [17] Q. Liu, D. Jiang, T. Hayat and A. Alsaedi, *Dynamics of a stochastic tuberculosis model with antibiotic resistance*, Chaos, Solitons and fractals, 2018, 109, 223–230.
- [18] M. Liu and K. Wang, *Global stability of a nonlinear stochastic predator-prey system with Beddington-DeAngelis functional response*, Communications on Nonlinear Science and Numerical Simulation, 2011, 16, 1114–1121.
- [19] M. Liu and K. Wang, *Dynamics of a Leslie-Gower Holling-type II predator-prey system with Lévy jumps*, Nonlinear Analysis, 2013, 85, 204–213.
- [20] M. Liu and Y. Zhu, *Stability of a budworm growth model with random perturbation*, Applied Mathematics Letters, 2018, 79, 13–19.
- [21] X. Mao, *Stationary distribution of stochastic population systems*, Systems and Control Letters, 2011, 60(6), 398–405.
- [22] X. Mao, *Stochastic Differential Equations and Applications, 2nd Edition*, Horwood, Chichester, 2007.
- [23] B.W. Stuck and B. Kleiner, *A statistical analysis of telephone noise*, The Bell System Technical Journal, 1974, 53(7) , 1263–1320.
- [24] T.V. Ton and A. Yagi, *Dynamics of a stochastic predator-prey model with the Beddington-DeAngelis functional response*, Communications on Stochastic Analysis, 2011, 5(2), 371–386.
- [25] X. Zhang, D. Jiang, T. Hayat and B. Ahmad, *Dynamics of a stochastic SIS model with double epidemic disease driven by Lévy jumps*, Physica A, 2017, 471(1), 767–777.
- [26] Y. Zhou and W. Zhang, *Threshold of a stochastic SIR epidemic model with Lévy jumps*, Physica A, 2016, 446(15), 204–216.
- [27] X. Zou, D. Fan, K. Wang, *Stationary distribution and stochastic Hopf bifurcation for a predator-prey system with noises*, Discrete and Continuous Dynamical Systems, Series B, 2013, 18(5), 1507–1519.
- [28] X. Zou and K. Wang, *Optimal harvesting for a stochastic regime-switching logistic diffusion system with jumps*, Nonlinear Analysis: Hybrid Systems, 2014, 13, 32–44.