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MULTIVALUED FIXED POINT IN BANACH ALGEBRA USING CONTINUOUS SELECTION AND ITS APPLICATION TO DIFFERENTIAL INCLUSION*

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Abstract In this paper, we provide some fixed point results using continuous selection given by Poonguzali *et al.* [15]. Also, using the selection theorem we discuss the existence of fixed point for the product of two multivalued mappings, that is, of the form $Ax \cdot Bx$. Using those fixed point results, we give the existence of solution for a newly developed differential inclusion.

Keywords Perfectly normal, Hausdorff metric, set-valued nonexpansive map, fixed point, differential inclusion.

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1. Introduction and preliminaries

Continuous selection plays an important role in differential inclusion. Existence of continuous selection for multivalued mapping was first studied by Michael [13]. After Michael, many others started working in this area because of its wide application in differential equations, etc. For its applications, one can refer to [1,2,6–8,12,17–19]. Rybinski [16] and Dhage [3,4] used this continuous selection to give the existence of solution to Krasnoselki type operators. In this approach, Poonguzali *et al.* [15] provided the existence of continuous selection for some special type of multivalued mapping, which is a more general class of functions which contains set-valued contractions as a subclass. Using the existence result, we derive some important fixed point results. Again, using those fixed point results, we show the existence of solution of a differential inclusion. Until now in literature, existence of fixed point for product of two operators were proved for contraction type mappings.

In this paper, we are going to prove the existence of fixed point for a different type of mapping which does not comes under contraction type.

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Let X be any normed linear space. Then $B = \{x \in X : \|y\| \leq 1\}$ represents the closed unit ball and $B^0 = \{x \in X : \|x\| < 1\}$ represents the open unit ball in X. Here the following notations are used frequently in this paper

$$\begin{aligned}
\mathcal{P}(X) &= \{A \subset X : A \neq \emptyset\},\\
\mathcal{P}_{cl}(X) &= \{A \in \mathcal{P}(X) : A \textit{ is closed}\},\\
\mathcal{P}_{cv}(X) &= \{A \in \mathcal{P}(X) : A \textit{ is convex}\},\\
\mathcal{P}_{cp}(X) &= \{A \in \mathcal{P}(X) : A \textit{ is compact}\}\\
\mathcal{P}_{bd}(X) &= \{A \in \mathcal{P}(X) : A \textit{ is bounded}\}.
\end{aligned}$$

In [3], Dhage proved the following selection theorem.

Theorem 1.1. *Let S be a closed convex and bounded subset of the Banach space X, and let $A : X \to \mathcal{P}_{cl,cv,bd}(X)$, $B : S \to \mathcal{P}_{cp,cv}(X)$ be two multivalued operators such that*

(a) A is a multivalued contraction;

(b) B is l.s.c and compact;

(c) $AxBy$ is a convex subset of X and $x \in AxBy \implies x \in S$ for all $y \in S$;

(d) $Mk \leq 1$, where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.

Then the operator inclusion $x \in AxBx$ has a solution in S.

For $x \in X$, $A, B \in \mathcal{P}(X)$,

$$\begin{aligned}
d(x, A) &= \inf\{\|x - u\| : u \in A\},\\
\delta(A, B) &= \sup\{d(x, B) : x \in A\}.
\end{aligned}$$

Definition 1.1. Let A and B are any two subsets of a metric space X. Then the Hausdorff distance between A and B is defined as

$$\mathcal{D}(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

Let us consider the mapping $H : X \times Y \to \mathcal{P}_{cl,cv}(Y)$. Then the fixed point set is defined as $P_H(x) := \{y \in Y : y \in H(x, y)\}$.

Definition 1.2. A multivalued mapping $F : X \to \mathcal{P}(Y)$ is called lower semicontinuous (l.s.c) at $x_0 \in X$ if and only if for every $\epsilon > 0$ and $z \in F(x_0)$ there exists a neighborhood U_z containing x_0 with the property that

$$z \in \cap\{F(x) + \epsilon B^0 : x \in U_z\}$$

or equivalently, F is said to be l.s.c if $x_n \to x$ and for any $y \in Fx$, then there exists $y_n \in Fx_n$ such that $y_n \to y$.

Definition 1.3. A multivalued mapping F is said to be weakly lower semicontinuous (w.l.s.c) at $x_0 \in X$ if and only if for every $\epsilon > 0$ and for every neighborhood V containing x_0, there exists a point $x_1 \in V$ so that for every $z \in F(x_1)$ there is a neighborhood U_z containing x_0 satifying the condition that

$$z \in \cap\{F(x) + \epsilon B^0 : x \in U_z\}.$$

Naturally, F is l.s.c (w.l.s.c) if and only if F is l.s.c (w.l.s.c) at every $x \in X$. Also, it is easy to see that the fact that F is l.s.c implies that F is w.l.s.c, but the converse is not true.

A topological space X is said to be paracompact if every open cover of X has a locally finite refinement. A cover $\{U_\beta\}_{\beta \in J}$ is called a refinement of $\{W_\alpha\}_{\alpha \in I}$ if for all $\beta \in J$, there exists $\alpha \in I$ such that $U_\beta \subset W_\alpha$. Also, a collection $\{A_i : i \in I\}$ of subsets of X is locally finite if and only if for each $x \in X$ there is an open $U \ni x$ with $|\{i \in I : A_i \cap U \neq \emptyset\}|$ is finite. A topological space X is said to be perfectly normal if it is normal and every closed subset is a G_δ subset. A multivalued mapping $H : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ is said to be contraction in second variable if it satisfies

$$\mathcal{D}(H(x, y_1), H(x, y_2)) \leq K\|y_1 - y_2\| \; for \; x \in X, \; y_1, y_2 \in Y, \tag{1.1}$$

where $K < 1$. In a similar way, a multivalued mapping $H : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ is said to be nonexpansive in second variable if it satisfies

$$\mathcal{D}(H(x, y_1), H(x, y_2)) \leq \|y_1 - y_2\| \; for \; x \in X, \; y_1, y_2 \in Y.. \tag{1.2}$$

Observe that if a multivalued mapping satisfies (1.1) then it satisfies (1.2) but the converse does not hold.

In 1989, Rybinski [16] proved the following theorem.

Theorem 1.2. *Let X be a paracompact and perfectly normal topological space and Y be a closed subset of a Banach space $(Z, \|\cdot\|)$. Assume that $H : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ satisfies a multivalued contraction condition in second variable and also satisfies the condition that for every $y \in Y$ the multivalued mapping $H(\cdot, y)$ is w.l.s.c. Then there exists a continuous mapping $h : X \times Y \to Y$ such that $h(x, y) \in X \times Y$.*

In [15], Poonguzali *et al.* generalized the above result, that is, they proved the existence of continuous selection by assuming weaker condition than contraction condition. The statement of their result is as follows.

Theorem 1.3 ([15]). *Let X_1 be a paracompact and perfectly normal topological space and X_2 be a Banach space. Assume that*

(a) $F : X_1 \times X_2 \to \mathcal{P}_{cl,cv}(X_2)$ satisfies the property (1.2);

(b) for any given $x \in X$, the mapping F satifies

$$\mathcal{D}(F(x, v_1), F(x, v_2)) \leq \lambda[d(v_1, F(x, v_1)) + d(v_2, F(x, v_2))]$$

where $\lambda < \frac{1}{2}$;

(c) for each $x_2 \in X_2$, the mapping $F(\cdot, x_2)$ is w.l.s.c.

Then there exists a continuous mapping $f : X_1 \times X_2 \to X_2$ such that $f(x_1, x_2) \in P_H(x_1)$ for every $(x_1, x_2) \in X_1 \times X_2$.

In this paper, we have extended Theorem 1.1 to w.l.s.c mapping and relaxed the condition (a) of Theorem 1.1.

2. Fixed point results using selection theorem

In this section, we show how the selection theorem is more useful in the theory of fixed points.

Lemma 2.1. *Let X be a metric space, $\{F_n\} \subset \mathcal{C}l(X)$ and $F \in \mathcal{C}l(X)$ such that $\lim_{n\to\infty} \mathcal{D}(F_n, F) = 0$. If $x_n \in F_n$ and $x_n \to x$, then $x \in F$.*

Proof. Let x_n be in F_n for each $n \in \mathbb{N}$ such that x_n converges to some $x \in X$. Now consider

$$\begin{aligned} d(x,F) &\leq d(x,x_n) + d(x_n,F_n) + d(F_n,F) \\ &\leq d(x,x_n) + d(x_n,F_n) + \mathcal{D}(F_n,F) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This shows that $x \in \bar{F}$. Since F is closed, $x \in F$. □

Theorem 2.1. *Let $\Lambda : Y \to X$ be an operator, $H : \Lambda(Y) \times Y \to \mathcal{P}_{cl,cv}(Y)$ be as in Theorem 1.3. Then there exists an element $y \in Y$ such that $y \in H(\Lambda(y), y)$.*

Proof. By Theorem 1.3, there exists $h : X \times Y \to Y$. Let $y \in Y$. Now define $g : Y \to Y$ by

$$g(w) = h(\Lambda(w), y).$$

Since g maps a nonempty compact convex subset to a compact subset, by application of Schauder theorem, g has a fixed point. Hence there exists a point $y \in Y$ such that

$$y = g(y) = h(\Lambda(y), y) \in H(\Lambda(y), h(\Lambda(y), y)) = H(\Lambda(y), y),$$

as desired. □

Corollary 2.1. *Let all the assumptions of Theorem 1.2 be satisfied. Assume additionally that Y is convex and compact and $\Gamma : Y \to X$ is a continuous mapping such that $\Gamma(Y)$ is a relatively compact subset of X. Then there exists a point $w \in Y$ such that $w \in H(\Gamma(w), w)$.*

Krasnoselki was the first person who started studying hybrid fixed point theorems in Banach spaces. There are cases where we are unable to apply either Banach contraction theorem or Schauder theorem. For example, many problems in analysis may split in the form $H = T + S$, where T is a contraction in some sense and S is a compact mapping, but H has neither of this properties. So, in such cases, we have to develop the theory to overcome such kind of problems. In that approach, Krasnoselki started the study of hybrid fixed point theory. Followed by him, many researchers gained interest in this area because of its wide application in perturbation theory [9, 20]. The fixed point theory for set-valued mapping is an equally important area in analysis. Several researchers have worked actively in this area. For the past few years, the set-valued analogue of Krasnoselki fixed point results was attracted by many researchers. Now, we are going to see such kind of fixed point result. The following lemma is useful in proving our results:

Lemma 2.2 ([3])**.** *If $A, B \in \mathcal{P}_{bd,cl}(X)$, then $\mathcal{D}(AC, BC) \leq \mathcal{D}(0, C)\mathcal{D}(A, B)$.*

Theorem 2.2 ([11])**.** *Let Z be a uniformly convex Banach space and S be a closed convex bounded nonempty subset of a Banach space Z. Let $T : S \to \mathcal{P}_{cp}(S)$ be a nonexpansive set-valued mapping. Then T has a fixed point.*

Now, we are going to prove our main results.

Theorem 2.3. *Let K be a compact convex subset of a uniformly convex Banach space $(Z, \|\cdot\|)$ and let $A : X \to \mathcal{P}_{cl,cv,bd}(X)$, $B : K \to \mathcal{P}_{cv}(X)$ be two multivalued operators such that* *(a) A is a multivalued Lipschitz operator with Lipschitz constant k;*

(b) $\mathcal{D}(A(x_1), A(x_2)) \leq \lambda[d(x_1, Ax_1) + d(x_2, Ax_2)]$ where $\lambda < \frac{1}{2}$;

(c) B is w.l.s.c;

(d) $AxBy$ is a convex subset of K for all $x, y \in K$;

(e) $Mk \leq 1$, where $M = \|B(K)\| = \sup\{\|Bx\| : x \in K\}$.

Then the operator inclusion $x \in AxBx$ has a solution in K.

Proof. Define a multivalued operator $H : K \times K \to \mathcal{P}_{cl,cv}(K)$ by $H(x, y) = AxBy$ *for all* $x, y \in K$. Now we claim that for all $y \in X$, H satisfies the condition (1.2). Let $x_1, x_2 \in K$. Consider

$$\begin{aligned}\mathcal{D}(H(x_1, y), H(x_2, y)) &= \mathcal{D}(A(x_1)B(y), A(x_2)B(y)) \\ &\leq \mathcal{D}(A(x_1), A(x_2))\mathcal{D}(0, By) \\ &\leq k\|x_1 - x_2\|B(K) \\ &\leq kM\|x_1 - x_2\| \\ &\leq \|x_1 - x_2\|.\end{aligned}$$

Hence $H_y(\cdot) := H(\cdot, y)$ satisfies the condition (1.2) on K. Again, consider

$$\begin{aligned}\mathcal{D}(H(x_1, y), H(x_2, y)) &\leq \mathcal{D}(A(x_1)B(y), A(x_2)B(y)) \\ &\leq \mathcal{D}(A(x_1), A(x_2))d(0, By) \\ &\leq \lambda[d(x_1, Ax_1) + d(x_2, Ax_2)]d(0, By) \\ &\leq \lambda[d(x_1, Ax_1)d(0, By) + d(x_2, Ax_2)d(0, By)] \\ &\leq \lambda[d(x_1, Ax_1By) + d(x_2, Ax_2By)].\end{aligned}$$

By Theorem 2.2, $Fix(H_y) = \{x \in K : x \in A(x)B(y)\}$ is nonempty. Let $(x_n) \in Fix(H_y)$ with $x_n \to x$. Then $x_n \in A(x_n)B(y)$. Therefore,

$$\mathcal{D}(A(x_n)B(y), A(x)B(y)) \leq \|x_n - x\| \to 0 \text{ as } n \to \infty.$$

By Lemma 2.1, $x \in A(x)B(y)$, which shows that $Fix(H_y)$ is closed. Also, note that $H(x, .)$ is w.l.s.c and so H satisfies all the conditions of Theorem 1.3. Thus there exists $h : K \times K \to K$ such that $h(x, y) \in A(h(x, y))B(y)$. Now define $L : K \to \mathcal{P}_{cl}(K)$ by

$$L(y) = Fix(H_y).$$

Consider the single-valued mapping $l : K \to K$ defined by $l(x) = f(x, x)$ for all $x \in K$. Then l is continuous and satisfies the property that $l(x) = f(x, x) \in A(f(x, x))B(x) = A(l(x))B(x)$ for all $x \in K$. Since l is continuous on a compact set, l is a compact mapping. Also, l satisfies all the condition of Schauder theorem and so there exists $p \in K$ such that $l(p) = p$. Hence

$$l(p) \in A(l(p))B(p) = A(p)B(p),$$

as desired. □

The following example is to show condition (b) of Theorem 2.3 is not a contraction condition.

Example 2.1. Consider $X := (\mathbb{R}^2, \|.\|_2)$. Then, define $A : X \to \mathcal{P}_{cl,cv,bd}(X)$ by

$$A((x,y)) := \begin{cases} \{0\} \times [0, \frac{y}{4}] & (x,y) \in \{0\} \times [0,1] \\ (0,0) & otherwise. \end{cases}.$$

Then A satisfies condition (b) of Theorem 2.3, but not a contraction mapping.

For the better understanding of Theorem 2.3, we provide the following example.

Example 2.2. Consider $X := (\mathbb{R}^2, \|.\|_2)$ and choose $K := \{0\} \times [0,1]$. Then, define $A : X \to \mathcal{P}_{cl,cv,bd}(X)$ and $B : K \to \mathcal{P}_{cl}(X)$ by $A((x,y)) := \{0\} \times [\frac{y}{8}, \frac{y}{4}]$ $(x,y) \in X$ and $B((x,y)) := (x,y)$ respectively.

To see that A satisfies condition (b) in Theorem 2.3, choose $x_1, x_2 \in X$. Then $x_1 = (a_1, b_1)$ and $x_2 = (a_2, b_2)$ and also either $b_1 \le b_2$ or $b_2 \le b_1$. Without loss of generality, assume $b_1 \le b_2$. Then $\mathcal{D}(Ax_1, Ax_2) = \|\frac{b_2}{4} - \frac{b_1}{8}\|$, $d(x_1, Ax_1) = \sqrt{a_1^2 + (b_1 - u)^2}$, and $d(x_2, Ax_2) = \sqrt{a_2^2 + (b_2 - v)^2}$, where $u \in [\frac{b_1}{8}, \frac{b_1}{4}]$, $v \in [\frac{b_1}{8}, \frac{b_1}{4}]$. Then it is not hard to prove that $d(x_1, Ax_1) \ge \frac{7}{8} b_1$ and $d(x_2, Ax_2) \ge \frac{7}{8} b_2$. Now, take $\lambda = \frac{2}{7}$, then we can easily see that $\mathcal{D}(Ax_1, Ax_2) \le \lambda[d(x_1, Ax_1) + d(x_2, Ax_2)]$. Since $\mathcal{D}(Ax_1, Ax_2) = \|\frac{b_2}{4} - \frac{b_1}{8}\|$, it is easy to see that A is a Lipschitz operator with Lipschitz constant $k = \frac{3}{8}$. Also, A and B satisfy all the conditions of Theorem 2.3. Hence $(0,0)$ is a solution for the inclusion equation $x \in Ax\ Bx$.

Corollary 2.2. *Let K be a compact convex subset of a Banach space $(Z, \|\cdot\|)$ and let $A : X \to \mathcal{P}_{cl,cv,bd}(X)$, $B : K \to \mathcal{P}_{cv}(X)$ be two multivalued operators such that*

- *(a) A is a multivalued Lipschitz operator with Lipschitz constant k;*
- *(b) B is w.l.s.c;*
- *(c) $AxBy$ is a convex subset of K for all $x, y \in K$;*
- *(d) $Mk < 1$, where $M = \|B(K)\| = \sup\{\|Bx\| : x \in K\}$.*

Then the operator inclusion $x \in AxBx$ has a solution in K.

3. Application to differential inclusion

Let $T = [0,a]$ be a closed interval with $a > 0$. Now consider the differential inclusion (shortly DI)

$$\begin{cases} \left(\dfrac{x(t)}{f(t,x(t))}\right)' \in H(t, x(t)) \ a.e.\ t \in T, \\ x(0) = p \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $f : T \times \mathbb{R} \to \mathbb{R}^*$, $\mathbb{R}^*$ denotes the set $\mathbb{R} \setminus \{0\}$ and $H : T \times \mathbb{R} \to \mathcal{P}_{cl,cv}(\mathbb{R})$.

A solution to DI (3.1) means that there should exist a function $x \in C(J, \mathbb{R})$ and should satisfy the following properties:

- (i) the function x should be absolutely continuous;
- (ii) the function $t \to \dfrac{x(t)}{f(t,x(t))}$ should be a differentiable function;
- (iii) $\left(\dfrac{x(t)}{f(t,x(t))}\right)' = v(t)$, $t \in T$ for some $v \in L^1(T, \mathbb{R})$ and also $v(t) \in H(t, x(t))$ a.e. $t \in T$.

Special cases of the above DI are the following:

(1) If $f(t,x)=1$, then the DI becomes $\begin{cases} x' \in H(t,x) \ a.e. \ t \in T, \\ x(0)=p \in \mathbb{R}. \end{cases}$

(2) If $H(t,x)=\{g(t,x)\}$, then the DI reduces to the following case

$$\begin{cases} \left(\dfrac{x(t)}{f(t,x(t))}\right)' = g(t,x(t)) \ a.e. \ t \ \in T, \\ x(0)=p \in \mathbb{R}. \end{cases}$$

The above DI's have been studied by many researchers (see [5, 14]).

In this section, we are going to prove the existence of solution to the DI (3.1) (which is newly developed DI in literature) under mixed Lipscitz and Caratheodory conditions. Let $X := C(T,\mathbb{R})$ with sup norm, that is, if $f \in C(T,\mathbb{R})$, then $\|f\| = \sup_{t\in T}|f(t)|$. Define the multiplication operation "$\cdot$" on X by $(f\cdot g)(t) = f(t)g(t)$ for all $t \in T$. Then X forms a Banach algebra with respect to the sup norm and multiplication operation.

Definition 3.1. Let $F : T \to \mathcal{P}(\mathbb{R})$ be any multivalued mapping. Then F is said to be measurable if for every $x \in \mathbb{R}$, the function $t \to d(x, F(t))$ is a measurable function.

Definition 3.2. Let $F : T \to \mathcal{P}_{cp}(\mathbb{R})$ be any measurable multivalued mapping. Then F is said to be integrably bounded if there exists a function $l \in L^1(T,\mathbb{R})$ such that $\|v\| \le l(t)$ $a.e.$ $t \in T$ for all $v \in F(t)$.

It is to be noted that if F is an integrably bounded multivalued function, then the set S^1_F which contains all Lebesgue integrable selections of F is closed and nonempty (see [7]).

Definition 3.3. A multivalued mapping $\beta : T\times\mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is called Caratheodory if

(i) $t \to \beta(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) $x \to \beta(t,x)$ is upper semi-continuous almost everywhere for $t \in T$.

Definition 3.4. A Caratheodory multivalued function $\beta(t,x)$ is called L^1- Caratheodory if there exists a function $l \in L^1(T,\mathbb{R})$ such that

$$\beta(t,x)\| \le l(t) \ a.e. \ t \in T$$

for all $x \in \mathbb{R}$, and the function l is called a growth function of β on $T\times\mathbb{R}$.

Define $S^1_\beta(x) = \{v \in L^1(T,\mathbb{R}) : v(t) \in \beta(t,x(t)) \ a.e. \ t \in T\}$. Here, the following lemma due to Lasota and Opial [10] is very important for our result.

Lemma 3.1. *Let E be a Banach space. If $dim(E) < \infty$ and $\beta : T\times E \to \mathcal{P}_{cp}(E)$ is L^1- Caratheodory, then $S^1_\beta(x) \neq \emptyset$ for each $x \in E$.*

We consider the following hypotheses in the sequel.

(P1) The function f is bounded on $T\times\mathbb{R}\to\mathbb{R}$ with bound L.

(P2) Let $f : T \times \mathbb{R} \to \mathbb{R}^*$ be continuous. If there exists a bounded function $h : T \to \mathbb{R}$ with bound $\|h\|$, then

$$|f(t,x) - f(t,y)| \leq h(t)|x - y| \ a.e. t \in T$$

for all $x, y \in \mathbb{R}$.

(P3) Also, f satisfies

$$|f(t,x) - f(t,y)| \leq k[|x - f(t,x)| + |y - f(t,y)] \ a.e. \ t \in T$$

for all $x, y \in \mathbb{R}$ and $k < \frac{1}{2}$.

(P4) $H : T \times \mathbb{R} \to \mathcal{P}_{cp,cv}(R)$ is an L^1-Caratheodory multivalued mapping with growth function l.

(P5) $x \to S^1_H(x)$ is l.s.c.

Theorem 3.1. *Assume that the hypotheses* $(P1)$–$(P5)$ *hold. If*

$$\|h\| \left(|\frac{p}{f(0,p)}| + \|l\|_{L^1} \right) \leq 1,$$

then the DI has a solution on T.

Proof. Take $X := C(T, \mathbb{R})$. Now define $K := \{x \in X : \|x\| \leq M\}$, which is equicontinuous, where $M = L \left(|\frac{p}{f(0,p)}| + \|l\|_{L^1} \right)$. It is clear that K is a compact convex subset of X. Define multivalued mappings A and B on K as follows:

$$A(x(t)) = f(t, x(t))$$

and

$$B(x(t)) = \{z \in X : z(t) = \frac{p}{f(0,p)} + \int_0^t w(s)ds, w \in S^1_H(x)\}$$

for all $t \in T$. Then the DI (3.1) is equivalent to the following problem

$$x(t) \in Ax(t)Bx(t), \ t \in T.$$

Now, our aim is to show that the multivalued operators A and B satisfy all the conditions of Theorem 2.3. Firstly, A and B are well-defined, since $S^1_H(x)$ is nonempty for all $x \in X$. It is clear that $A : K \to \mathcal{P}_{cl,cv,bd}(X)$. Let $z_1, z_2 \in Bx$. Then there exist $w_1, w_2 \in S^1_H(x)$ such that

$$z_1(t) = \frac{p}{f(0,p)} + \int_0^t w_1(s)ds, \ t \in T$$

and

$$z_2(t) = \frac{p}{f(0,p)} + \int_0^t w_2(s)ds.$$

Take any $\lambda \in [0, 1]$. Then

$$\lambda z_1(t) + (1 - \lambda)z_2(t) = \lambda \left(\frac{p}{f(0,p)} + \int_0^t w_1(s)ds \right) + (1 - \lambda) \left(\frac{p}{f(0,p)} + \int_0^t w_2(s)ds \right)$$

$$= \frac{p}{f(0,p)} + \int_0^t [\lambda w_1(s) + (1-\lambda)w_2(s)]ds$$
$$= \frac{p}{f(0,p)} + \int_0^t w(s)ds,$$

where $w(s) = \lambda w_1(s) + (1-\lambda)w_2(s) \in H(s,x)$ for all $s \in T$. This shows that Bx is convex valued for all $x \in K$. Hence $B : K \to \mathcal{P}_{cv}(X)$.

Next we claim that A is a multivalued Lipschitz mapping. Let $x_1, x_2 \in K$. Then

$$\begin{aligned}\|Ax_1 - Ax_2\| &= \sup_{t\in T} |Ax_1(t) - Ax_2(t)| \\ &= \sup_{t\in T} |f(t,x_1(t)) - f(t,x_2(t))| \\ &\le \sup_{t\in T} h(t)|x_1(t) - x_2(t)| \\ &\le \|h\|\|x_1 - x_2\|,\end{aligned}$$

which shows that A is a multivalued Lipschitz operator on K. Using $(P3)$, one can easily see that $\|Ax_1 - Ax_2\| \le k[\|x_1 - Ax_1\| + \|x_2 - Ax_2\|]$, where $k < \frac{1}{2}$. Next, we are going to show that B is l.s.c. Let $(x_n) \in K$ with $x_n \to x$ and $y \in Bx$. Then $y(t) = \dfrac{p}{f(0,p)} + \int_0^t w(s)ds,\ w \in S^1_H(x)$. By (P4), there exist $w_n \in S^1_H(x_n)$ for all $n \in \mathbb{N}$ such that $w_n \to w$.

Now, our aim is to extract an uniformly convergent subsequence. For that, define $\mathcal{L} : L^1(T,\mathbb{R}) \to C(T,\mathbb{R})$ by $\mathcal{L}(v) := \int_0^t v(s)ds$. Since $\mathcal{L}$ is continuous, $\mathcal{L}w_n(t) \to \mathcal{L}w(t)$ pointwise on T as $n \to \infty$. Let $t_1, t_2 \in T$. Then

$$\|\mathcal{L}w_n(t_1) - \mathcal{L}w_n(t_2)\| \le |\int_0^{t_1} w_n(s)ds - \int_0^{t_2} w_n(s)ds| \le |\int_{t_1}^{t_2} w_n(s)ds|.$$

Observe that the right hand side converges to 0 as $t_1 \to t_2$. This shows that $(\mathcal{L}w_n)$ is equicontinuous and so by Ascoli theorem there exists an uniformly convergent subsequence (w_{n_k}) such that $\mathcal{L}w_{n_k} \to \mathcal{L}w$ uniformly. Now, define $y_{n_k}(t) = \dfrac{p}{f(0,p)} + \int_0^t w_{n_k}(s)ds,\ w_{n_k} \in S^1_H(x_{n_k})$. Then $y_{n_k} \to y$. Hence B is l.s.c.

Finally, we have to show that $AxBx$ is a convex subset of K for all $x \in K$. Let $x \in K$ be taken arbitrarily and let $v_1, v_2 \in K$. Then there exists $S^1_H(x)$ such that

$$v_1 = [f(t,x(t))]\left(\frac{p}{f(0,p)} + \int_0^t w_1(s)ds\right),$$

$$v_2 = [f(t,x(t))]\left(\frac{p}{f(0,p)} + \int_0^t w_2(s)ds\right).$$

Let $\lambda \in [0,1]$. Then

$$\begin{aligned}&\lambda v_1 + (1-\lambda)v_2 \\ =&\lambda[f(t,x(t))]\left(\frac{p}{f(0,p)} + \int_0^t w_1(s)ds\right) + (1-\lambda)[f(t,x(t))]\left(\frac{p}{f(0,p)} + \int_0^t w_2(s)ds\right) \\ =&[f(t,x(t))]\left(\lambda\frac{p}{f(0,p)} + \int_0^t \lambda w_1(s)ds\right)\end{aligned}$$

$$+ [f(t,x(t))]\left((1-\lambda)\frac{p}{f(0,p)}+\int_0^t(1-\lambda)w_1(s)ds\right)$$
$$=[f(t,x(t))]\left(\frac{p}{f(0,p)}+\int_0^t\lambda w_1(s)+(1-\lambda)w_2(s)ds\right).$$

Since $H(t,x(t))$ is convex, $z=\lambda w_1+(1-\lambda)w_2\in H(t,x(t))$ for all $t\in T$. This gives $\lambda v_1+(1-\lambda)v_2\in AxBx$. Hence $AxBx$ is a convex subset of X. By the condition,

$$Mk=\|h\|\left(|\frac{p}{f(0,p)}|+\|l\|_{L^1}\right)\leq 1$$

and so A and B satisfy all the conditions of Theorem 2.3. Therefore, the inclusion equation $x\in AxBx$ has a solution. □

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