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EXISTENCE OF PERIODIC SOLUTIONS FOR HAMILTONIAN SYSTEMS WITH SUPER-LINEAR AND SIGN-CHANGING NONLINEARITIES*

Liqian Jia[1] and Guanwei Chen[1,†]

Abstract In this paper, we consider the existence of periodic solutions for the super quadratic second order Hamiltonian system, and primitive functions of nonlinearities are allowed to be sign-changing. By using some weaker conditions, our result extends and improves some existed results in the literature.

Keywords Second order Hamiltonian system, sign-changing, super quadratic, periodic solutions.

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1. Introduction and Main Result

We consider the following second order Hamiltonian system

$$u''(t) + A(t)u(t) + \nabla H(t, u(t)) = 0, \quad t \in R, \tag{1.1}$$

where $A(\cdot)$ is a continuous T-periodic symmetric matrix, $H : R \times R^N \to R$ is T-periodic ($T > 0$) in its first variable. Moreover, we always assume that $H(t, x)$ is continuous in t for each $x \in R^N$, continuously differentiable in x for each $t \in [0, T]$ and $\nabla H(t, x)$ denotes its gradient with respect to the x variable.

As a special case of dynamical systems, Hamiltonian systems play an important role in the study of gas dynamics, fluid mechanics, relativistic mechanics and nuclear physics. Hamiltonian systems are momentum invariant in classical mechanics of physical systems. They are systems of differential equations which are studied in Hamiltonian mechanics and can be written in the form of Hamilton's equations. And they are usually formulated in terms of Hamiltonian vector fields on a symplectic manifold or Poisson manifold.

Some authors studied autonomous second order Hamiltonian systems, such as [9,11], it is different from the problem (1.1) we focus on. Many authors [1–3,5–8,10, 12,13,15–35] have payed their attentions to the study of periodic solutions for (1.1), which can be divided into the following two cases for $H(t, u)$. The super quadratic

†the corresponding author. Email address: guanweic@163.com(G. Chen)
1School of Mathematical Sciences, University of Jinan, Jinan 250022, China
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case [1–3, 5–7, 10, 12, 13, 15, 17, 18, 20–24, 26, 27, 29–31, 34, 35]:

$$\lim_{|u|\to\infty} \frac{H(t,u)}{|u|^2} = +\infty,$$

and the asymptotically quadratic case [8, 16, 19, 21, 25, 26, 28, 32–34]:

$$\lim_{|u|\to\infty} \frac{H(t,u)}{|u|^2} = L(t), \quad 0 \le L(t) < \infty.$$

Rabinowitz [17] established the existence of periodic solutions of (1.1) with $A(t) = 0$ under the following super quadratic condition (AR-condition): there exist constants $\mu > 2$ and $L > 0$ such that

$$0 < \mu H(t,u) \le (\nabla H(t,u), u), \quad \forall\ |u| \ge L,\ t \in [0,T], \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the inner product in R^N. It is convenient to check the mountain pass geometry and verify the Palais-Smale condition (PS-condition) by the above AR-condition, for the associated Euler functional. For this reason, it has been applied in many literatures, see [1, 13, 18, 29] and references therein.

Now the AR-condition also has been replaced by some more general super quadratic conditions in some papers about second-order Hamiltonian systems recently, such as [2, 3, 5–7, 10, 12, 15, 20–24, 26, 27, 30, 31, 34, 35]. Some of the above authors in [6, 10, 12, 21, 30, 34] obtained infinitely many periodic solutions of (1.1) under the even condition $H(t,-u) = H(t,u)$.

Here, we focus our attention on the existence of period solutions of (1.1) by more general super quadratic conditions than some existed results without the above even condition. And we shall give some comparisons between our result and the results [2, 3, 5, 7, 12, 15, 20, 22–24, 26, 27, 30, 31, 35].

To state our main result, we still need the following assumptions:

($\mathbf{A_1}$) There exist constants $c_1, c_2 > 0$ and $p > 2$ such that

$$|\nabla H(t,u)| \le c_1|u| + c_2|u|^{p-1}, \quad \forall (t,u) \in R \times R^N,$$

where $c_1 < \frac{1}{2\gamma_2^2}$, γ_2 is mentioned in the following Lemma 2.1.

($\mathbf{A_2}$) $\lim_{|u|\to+\infty} \frac{H(t,u)}{|u|^2} = +\infty, \quad \forall (t,u) \in R \times R^N.$

($\mathbf{A_3}$) $(\nabla H(t,u), u) - 2H(t,u) \ge 0$, $\forall (t,u) \in R \times R^N$, and there exist $c_0 > 0$ and $\varrho > 1$ such that

$$|H(t,u)|^{\varrho} \le c_0|u|^{2\varrho}\big[(\nabla H(t,u), u) - 2H(t,u)\big], \quad \forall (t,u) \in R \times R^N, |u| \ge r_0.$$

Now, our main result reads as follows:

Theorem 1.1. *If assumptions* (A_1)–(A_3) *hold, then* (1.1) *has at least a* T*-periodic solution.*

Example 1.1. Let

$$H(t,u)=a(t)[u^4+u^3-\cos(u)],\quad (t,u)\in R\times R^N,$$

where $0<\inf_{t\in[0,T]}a(t)<\sup_{t\in[0,T]}a(t)<+\infty$.

Example 1.2.

$$H(t,u)=\begin{cases} a(t)[a_1|u|^6+a_2|u|^4], & |u|^2\le 2,\\ a(t)[|u|^2\ln(1+|u|^2)+\sin|u|^2-\ln(1+|u|^2)], & |u|^2>2,\end{cases}$$

where $(t,u)\in R\times R^N$, $0<\inf_{t\in[0,T]}a(t)<\sup_{t\in[0,T]}a(t)<+\infty$, and $a_1,a_2>0$ are two suitable constants.

It is not hard to check that the above functions of Example 1.1 and 1.2 satisfy our conditions (A_1)–(A_3), and H of Example 1.1 is sign-changing.

Remark 1.1. Our Theorem 1.1 extends some superlinear results [2, 3, 5, 7, 12, 15, 20, 22–24, 26, 27, 30, 31, 35], the reasons are as follows.

1) Our nonlinearities $H(t,u)$ can be sign-changing, which is more general than the case $(H(t,u)\ge 0,\ \forall(t,u)\in[0,T]\times R^N)$ in papers [2, 5, 7, 22, 26, 27, 30, 35].

2) Papers in [2, 3, 7, 35] all use the condition:
(A_1') There are $d_1>0$ and $\alpha>1$ such that $|\nabla H(t,u)|\le d_1(1+|u|^\alpha)$ and

$$|\nabla H(t,u)|=o(|u|),\quad |u|\to 0,\quad \forall(t,u)\in[0,T]\times R^N, \tag{1.3}$$

clearly, our condition (A_1) is weaker than (A_1'). Besides, the authors in [7, 35] considered two cases for H: sign-changing case and $H(t,u)\ge 0$ case, but the two cases all used the above condition (A_1'), therefore our result is more general.

3) The papers [3, 12, 15, 20, 23, 24, 30, 31] also allowed $H(t,u)$ being sign-changing, but papers [23, 24, 30] used the condition

$$\lim_{|u|\to 0}\frac{H(t,u)}{|u|^2}=0,\quad \forall(t,u)\in[0,T]\times R^N, \tag{1.4}$$

paper [31] used condition (1.3). And several papers of sign-changing case used following conditions which we do not need: papers [15, 24] used the condition

$$H(t,u)\ge 0,\quad \forall|u|\le L,\ t\in[0,T]\text{ for some }L>0; \tag{1.5}$$

author in [3] used the condition

$$H(t,u)\ge\frac{1}{2}a|u|^2,\quad a=\sup(\sigma(B)\cap(-\infty,0))<0,\quad \forall(t,u)\in[0,T]\times R^N; \tag{1.6}$$

and [12, 20] all used the condition

$$2H(t,u)\ge\lambda_{l-1}|u|^2,\quad \forall(t,u)\in[0,T]\times R^N, \tag{1.7}$$

where λ_l is the first positive eigenvalue of B and $\lambda_{l-1}=0$ was allowed.

Moreover, authors in [12, 15, 22, 24, 30] all used the condition: there exist constants $p > 2$ and $q > p - 2$ such that

$$\limsup_{|u|\to+\infty} \frac{H(t,u)}{|u|^p} < \infty, \quad \liminf_{|u|\to+\infty} \frac{(\nabla H(t,u),u) - 2H(t,u)}{|u|^q} > 0. \tag{1.8}$$

It is easy to imply that the function in our Example 1.1 does not satisfy the conditions (1.3)–(1.7), and the function in Example 1.2 does not satisfy (1.8), but they all satisfy our conditions (A_1)–(A_3). Thus our result extends and improves the existing results.

The rest of the present paper is organized as follows. In Section 2, we establish the variational framework associated with (1.1) and give some preliminary lemmas, which are useful in the proof of our main result. Then we give the detailed proof of our result.

2. Variational setting and proof of our result

Throughout this paper we denote by $\|\cdot\|_q$ the usual $L^q(0,T;R^N)$ norm. Let $E := H^1_T$ be the Sobolev space defined by

$$H^1_T := \left\{ u: \ [0,T] \to R^N \,\middle|\, u(0) = u(T), \text{ and } u' \in L^2(0,T;R^N) \right\}$$

with the norm and the corresponding inner product defined by

$$(u,u)_E = \|u\|_E^2 := \int_0^T |u(t)|^2 dt + \int_0^T |u'(t)|^2 dt,$$

where u is absolutely continuous. By Proposition 1.1 in [14], we know there exists a constant $a_0 > 0$ such that

$$\|u\|_\infty := \max_{t\in[0,T]} |u(t)| \le a_0 \|u\|_E, \quad \forall\ u \in H^1_T.$$

Let $B = -\frac{d^2}{dt^2} - A(t)$ be the linearized operator defied by $Bx(t) = -x''(t) - A(t)x(t)$ with T-periodic condition. Then B has a sequence of eigenvalues

$$\lambda_{-m} \le \lambda_{-m+1} \le \cdots \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \cdots \lambda_k \le \cdots$$

with $\lambda_k \to +\infty$ as $k \to \infty$.

Remark 2.1. Obviously, there exists a constant $d_0 > 0$ such that $\lambda_{-m} + d_0 > c > 0$. Let $\nabla \overline{H}(t,u(t)) = \nabla H(t,u(t)) + d_0 u(t)$, then it is easy to check that $\overline{H}$ also satisfies conditions (A_1)–(A_3). Note that the problem (1.1) is equivalent to the following problem

$$u''(t) + \overline{A}(t)u(t) + \nabla \overline{H}(t,u(t)) = 0, \quad t \in R, \quad \overline{A}(t) = A(t) - d_0. \tag{$*$}$$

Thus to prove Theorem 1.1, we only need prove the problem $(*)$ has at least a T-periodic solution under the conditions (A_1)–(A_3).

Lemma 2.1 ([6]). *If E is compactly embedded in $L^q([0,T];\mathrm{R}^N)$ for all $1 \le q \le +\infty$, then by the Sobolev embedding theorem, there exists $\gamma_q > 0$ such that*

$$\|u\|_q \le \gamma_q \|u\|, \quad \forall u \in E.$$

Let $\overline{B}$ be defined by $\overline{B}x(t) = -x''(t) - \overline{A}(t)x(t)$ with T-periodic condition. According to Remark 2.1, we can introduce the following inner product and norm on E: for all $u,\ v \in E$, we can define an equivalent inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ in E by

$$\langle u, v \rangle = (\overline{B}u, v)_{L^2} \quad \text{and} \quad \|u\| = \langle u, u \rangle^{\frac{1}{2}},$$

respectively. Therefore, the corresponding functional of $(*)$ can be written as follows

$$\begin{aligned} I(u) &= \frac{1}{2}\left(\int_0^T |u'(t)|^2 dt - \int_0^T \left(\overline{A}(t)u(t), u(t)\right) dt\right) - \int_0^T \overline{H}(t, u(t))dt \\ &= \frac{1}{2}(\overline{B}u, u)_{L^2} - \int_0^T \overline{H}(t, u(t))dt \\ &= \frac{1}{2}\|u\|^2 - \int_0^T \overline{H}(t, u(t))dt, \quad \forall u \in E. \end{aligned} \tag{2.1}$$

Let $\Psi(u) := \int_0^T \overline{H}(t, u(t))dt$, the hypotheses on H imply that I and Ψ are continuously differentiable, and for all $u,\ v \in E$ we have

$$\langle I'(u), v \rangle = \langle u, v \rangle - \langle \Psi'(u), v \rangle, \qquad \langle \Psi'(u), v \rangle = \int_0^T \left(\nabla \overline{H}(t, u(t)), v(t)\right) dt. \tag{2.2}$$

The hypotheses on H imply that $I,\ \Psi \in C^1(E, R)$ and a standard argument shows that nonzero critical points of I are nontrivial solutions of (1.1).

We shall use the following lemma to prove the Theorem 1.1.

Lemma 2.2 (Mountain Pass Theorem, [4])**.** *Let E be a real Banach space with its dual space E^*, and suppose that $I \in^1 (E, \mathrm{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \mu \le \eta \le \inf_{\|u\|=\rho} I(u)$$

for some $\mu, \eta, \rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le \tau \le 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \ge \eta, \qquad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \quad as \quad n \to \infty. \tag{2.3}$$

Here, we say that $I \in C^1(X, R)$ satisfies $(C)_c$-condition if any sequence $\{u_n\}$ (such that (2.3) holds) has a convergent subsequence. Clearly, by the condition (A_1) and Remark 2.1 ,we have

$$|\overline{H}(t, u)| \le \frac{c_1}{2}|u|^2 + \frac{c_2}{p}|u|^p, \quad \forall (t, u) \in R \times R^N. \tag{2.4}$$

Lemma 2.3. *If assumptions (A_1), (A_2) and (A_3) hold, then I satisfies $(C)_c$-condition.*

Proof. We assume that for any sequence $\{u_n\} \subset E$, $I(u_n) \to c$ and $\|I'(u_n)\|\big(1+\|u_n\|\big) \to 0$. Then $I'(u_n) \to 0$, and

$$\big\langle I'(u_n), u_n \big\rangle \to 0. \tag{2.5}$$

Step 1. We prove the boundedness of $\{u_n\}$ by contradiction, if $\|u_n\| \to \infty$, let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. By the definitions of $I(u)$ and $I'(u)$, for n large, we have

$$\int_0^T \Big[\frac{1}{2}\big(\nabla\overline{H}(t,u_n),u_n\big) - \overline{H}(t,u_n)\Big]dt = I(u_n) - \frac{1}{2}\big\langle I'(u_n),u_n\big\rangle \le c+1. \tag{2.6}$$

By (2.1), $I(u_n) \to c$ and $\|u_n\| \to \infty$, we have

$$\limsup_{n\to\infty} \int_0^T \frac{|\overline{H}(t,u_n)|}{\|u_n\|^2}dt \ge \frac{1}{2}. \tag{2.7}$$

Let

$$\Omega_n(a,b) = \{t \in [0,T] : a \le |u_n(t)| < b\}, \quad 0 \le a < b. \tag{2.8}$$

By $\|v_n\| = 1$, we could assume that $v_n \rightharpoonup v = \{v(t)\}_{t\in[0,T]}$ in E passing to a subsequence, which together with Lemma 2.1 implies $v_n \to v$ in L^q for $1 \le q < \infty$, and $v_n \to v$ on $[0,T]$.

If $v = 0$, then $v_n \to 0$ in L^q, $1 \le q < \infty$, and $v_n \to 0$ on $[0,T]$. It follows from (2.4) that

$$\begin{aligned}\int_{\Omega_n(0,r_0)} \frac{|\overline{H}(t,u_n)|}{|u_n|^2}|v_n|^2dt &\le \Big(\frac{c_1}{2}+\frac{c_2}{p}r_0^{p-2}\Big)\int_{\Omega_n(0,r_0)}|v_n|^2dt\\ &\le \Big(\frac{c_1}{2}+\frac{c_2}{p}r_0^{p-2}\Big)\int_0^T |v_n|^2dt\\ &\to 0.\end{aligned} \tag{2.9}$$

Let $\varrho' = \varrho/(\varrho-1)$. Due to $\varrho > 1$ (see (A_3)), we have that $2\varrho > 2$. So by (A_3), (2.6), the Hölder's inequality and $v_n \to 0$ in L^q for $1 \le q < \infty$, we have

$$\begin{aligned}&\int_{\Omega_n(r_0,\infty)} \frac{|\overline{H}(t,u_n)|}{|u_n|^2}|v_n|^2dt\\ &\le \Big[\int_{\Omega_n(r_0,\infty)}\Big(\frac{|\overline{H}(t,u_n)|}{|u_n|^2}\Big)^{\varrho}dt\Big]^{1/\varrho}\Big[\int_{\Omega_n(r_0,\infty)}|v_n|^{2\varrho'}dt\Big]^{1/\varrho'}\\ &\le (2c_0)^{1/\varrho}\Big[\int_{\Omega_n(r_0,\infty)}\Big(\frac{1}{2}(\nabla\overline{H}(t,u_n),u_n)-\overline{H}(t,u_n)\Big)dt\Big]^{1/\varrho}\Big[\int_{\Omega_n(r_0,\infty)}|v_n|^{2\varrho'}dt\Big]^{1/\varrho'}\\ &\le [2c_0(c+1)]^{1/\varrho}\|v_n\|_{2\varrho'}^2 \to 0.\end{aligned} \tag{2.10}$$

Combining (2.9) with (2.10), we have

$$\int_0^T \frac{|\overline{H}(t,u_n)|}{\|u_n\|^2}dt = \int_{\Omega_n(0,r_0)} \frac{|\overline{H}(t,u_n)|}{|u_n|^2}|v_n|^2dt + \int_{\Omega_n(r_0,\infty)} \frac{|\overline{H}(t,u_n)|}{|u_n|^2}|v_n|^2dt \to 0,$$

which contradicts with (2.7).

If $v \neq 0$, we let $A := \{t \in [0,T] : v(t) \neq 0\}$. For all $t \in A$, by $v_n = \frac{u_n}{\|u_n\|}$ and $\|u_n\| \to \infty$, we have $\lim_{n\to\infty} |u_n| = \infty$. We define

$$\chi_{t,\Omega_n(r_0,\infty)} := \begin{cases} 1, & t \in \Omega_n(r_0,\infty), \\ 0, & t \notin \Omega_n(r_0,\infty), \end{cases} \quad \forall n \in N. \tag{2.11}$$

For large $n \in N$, $A \subset \Omega_n(r_0,\infty)$ and $\lim_{n\to\infty} |u_n| = \infty$ for all $t \in A$, it follows from (2.1), (2.4), (A_2), Remark 2.1, the Fadou's Lemma, $\|v_n\| = 1$, $\|u_n\| \to \infty$, $I(u_n) \to c$ and $\|v_n\|_2 \leq \gamma_2 \|v_n\|$ (see Lemma 2.1) that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|u_n\|^2} \\
&= \lim_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} \\
&= \lim_{n\to\infty} \left[\frac{1}{2} - \int_0^T \frac{\overline{H}(t,u_n)}{(u_n)^2}(v_n)^2 dt\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2} - \int_{\Omega_n(0,r_0)} \frac{\overline{H}(t,u_n)}{(u_n)^2}(v_n)^2 dt - \int_{\Omega_n(r_0,\infty)} \frac{\overline{H}(t,u_n)}{(u_n)^2}(v_n)^2 dt\right] \\
&\leq \limsup_{n\to\infty} \left[\frac{1}{2} + \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right) \int_0^T |v_n|^2 dt - \int_{\Omega_n(r_0,\infty)} \frac{\overline{H}(t,u_n)}{(u_n)^2}(v_n)^2 dt\right] \\
&\leq \frac{1}{2} + \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right)\gamma_2^2 - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{\overline{H}(t,u_n)}{(u_n)^2}(v_n)^2 dt \\
&= \frac{1}{2} + \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right)\gamma_2^2 - \liminf_{n\to\infty} \int_0^T \frac{\overline{H}(t,u_n)}{(u_n)^2}[\chi_{t,\Omega_n(r_0,\infty)}](v_n)^2 dt \\
&\leq \frac{1}{2} + \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right)\gamma_2^2 - \int_0^T \liminf_{n\to\infty} \frac{\overline{H}(t,u_n)}{(u_n)^2}[\chi_{t,\Omega_n(r_0,\infty)}](v_n)^2 dt \\
&= -\infty.
\end{aligned} \tag{2.12}$$

It is a contradiction. So $\{u_n\}$ is bounded in E.

Step 2. The boundedness of $\{u_n\}$ implies that $u_n \rightharpoonup u$ in E passing to a subsequence, where $u = \{u(t)\}_{t\in[0,T]}$. First, we prove

$$\int_0^T \left[\nabla \overline{H}(t,u_n)(u_n - u)\right] dt \to 0, \quad n \to \infty. \tag{2.13}$$

Note that Lemma 2.1 implies that $u_n \to u$ in L^q for all $1 \leq q < \infty$, so we have

$$\|u_n - u\|_2 \to 0, \quad \|u_n - u\|_p \to 0. \tag{2.14}$$

The boundedness of $\{u_n\}$ and Lemma 2.1 imply that $\|u_n\|_q < \infty$ for all $1 \leq q < \infty$, it follows from (A_1), (2.14) and the Hölder's inequality that

$$\begin{aligned}
&\Big|\int_0^T \left[\nabla \overline{H}(t,u_n)(u_n - u)\right] dt\Big| \\
\leq &\int_0^T \left|\nabla \overline{H}(t,u_n)(u_n - u)\right| dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T \big[(c_1|u_n| + c_2|u_n|^{p-1})|u_n - u|\big]dt \\
&= c_1 \int_0^T \big[|u_n||u_n - u|\big]dt + c_2 \int_0^T \big[(|u_n|^{p-1}|u_n - u|\big]dt \\
&\leq c_1\|u_n\|_2\|u_n - u\|_2 + c_2\|u_n\|_p^{p-1}\|u_n - u\|_p \to 0. \qquad (2.15)
\end{aligned}
$$

So (2.13) holds. Therefore, by (2.13), $I'(u_n) \to 0$, $u_n \rightharpoonup u$ in E and the definition of I', we have

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \big\langle I'(u_n), u_n - u\big\rangle \\
&= \lim_{n\to\infty} (u_n, u_n - u) - \lim_{n\to\infty} \int_0^T \big(\nabla \overline{H}(t, u_n)(u_n - u)\big)dt \\
&= \lim_{n\to\infty} \|u_n\|^2 - \|u\|^2 - 0. \qquad (2.16)
\end{aligned}
$$

That is,

$$
\lim_{n\to\infty} \|u_n\| = \|u\|. \qquad (2.17)
$$

It follows from $u_n \rightharpoonup u$ in E that

$$
\|u_n - u\|^2 = (u_n - u, u_n - u) \to 0,
$$

that is, $\{u_n\}$ has a convergent subsequence in E. Thus the proof of I satisfies $(C)_c$-condition is finished. □

Lemma 2.4. *If assumption (A_1) holds, then there exist $\rho, \eta > 0$ such that* $\inf\{I(u)|u \in E, \|u\| = \rho\} > \eta$.

Proof. By Lemma 2.1 and (2.4), for $u \in E$ we have

$$
\begin{aligned}
\Big|\int_0^T \overline{H}(t,u)dt\Big| &\leq \int_0^T \Big|\frac{c_1}{2}|u|^2 + \frac{c_2}{p}|u|^p\Big|dt \\
&= \frac{c_1}{2}\|u\|_2^2 + \frac{c_2}{p}\|u\|_p^p \\
&\leq \frac{\gamma_2^2 c_1}{2}\|u\|^2 + \frac{\gamma_p^p c_2}{p}\|u\|^p. \qquad (2.18)
\end{aligned}
$$

Then from (2.1) and (2.18) we have

$$
\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \int_0^T \overline{H}(t, u(t))dt \\
&\geq \frac{1}{2}\|u\|^2 - \frac{\gamma_2^2 c_1}{2}\|u\|^2 - \frac{\gamma_p^p c_2}{p}\|u\|^p, \quad \forall u \in E. \qquad (2.19)
\end{aligned}
$$

Let $\|u\| = \rho > 0$, it is easy to imply that there exists $\eta > 0$ such that the lemma holds when ρ small enough. The proof is finished. □

Lemma 2.5. *If assumption (A_2) holds, then there exists $v \in E$ with $\|v\| > \rho$ such that $I(v) < 0$, where ρ is given in lemma 2.4.*

Proof. By (2.1) we have

$$\frac{I(su)}{s^2} = \frac{1}{2}\|u\|^2 - \frac{1}{s^2}\int_0^T \overline{H}(t, su)dt.$$

Then it follows from (A_2) and the Fatou's lemma that

$$\begin{aligned}
\lim_{s\to\infty} \frac{I(su)}{s^2} &= \lim_{s\to\infty} \big[\frac{1}{2}\|u\|^2 - \frac{1}{s^2}\int_0^T \overline{H}(t, su)dt\big] \\
&\le \limsup_{s\to\infty} \big[\frac{1}{2}\|u\|^2 - \frac{1}{s^2}\int_0^T \overline{H}(t, su)dt\big] \\
&= \frac{1}{2}\|u\|^2 - \liminf_{s\to\infty} \int_0^T \frac{\overline{H}(t, su)}{s^2u^2} u^2 dt \\
&\le \frac{1}{2}\|u\|^2 - \int_0^T \liminf_{s\to\infty} \frac{\overline{H}(t, su)}{s^2u^2} u^2 dt \\
&= -\infty, \quad \text{as } s \to \infty.
\end{aligned} \tag{2.20}$$

Therefore let $v = s_0 u$, the lemma is proved when $s_0 > 0$ large enough. □

Proof of Theorem 1.1. Lemmas 2.4 and 2.5 imply all the conditions of Lemma 2.2 hold. Therefore, Lemmas 2.2 and 2.3 imply there exists $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) = c > 0$. The proof is finished. □

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