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MINIMIZATION PRINCIPLE AND GENERALIZED FOURIER SERIES FOR DISCONTINUOUS STURM-LIOUVILLE SYSTEMS IN DIRECT SUM SPACES*

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Abstract By modifing the Green's function method we study certain spectral aspects of discontinuous Sturm-Liouville problems with interior singularities. Firstly, we define four eigen-solutions and construct the Green's function in terms of them. Based on the Green's function we establish the uniform convergeness of generalized Fourier series as eigenfunction expansion in the direct sum of Lebesgue spaces L_2 where the usual inner product replaced by new inner product. Finally, we extend and generalize such important spectral properties as Parseval equation, Rayleigh quotient and Rayleigh-Ritz formula (minimization principle) for the considered problem.

Keywords Sturm-Liouville problems, boundary-transmission conditions, Rayleigh quotient, minimization principle.

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1. Introduction

Many physical processes, such as the vibration of strings, the interaction of atomic particles, electrodynamics of complex medium, aerodynamics, polymer rheology or the earth's free oscillations yields Sturm-Liouville eigenvalue problems(see, for example, [8,11,21–23] and references cited therein). For instance, the one-dimensional form of the advection-dispersion equation for a nonreactive dissolved solute in a saturated, homogeneous, isotropic porous medium under steady, uniform flow is

$$f_s + \nu f_x = C f_{xx}, \quad x \in (0, K), \; s > 0$$

where $f(x, s)$ is the concentration of the solute, ν is the average linear groundwater velocity, C is the coefficient of hydrodynamic dispersion,and K is the length of the

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aquifer. Using the method of separation of variables the problem can be written in the form as Sturm-Liouville eigenvalue problem

$$[p(x)u']' + \lambda r(x)u = 0, \quad x \in (0, K), \quad u(0) = 0, \quad u'(K) = 0$$

where $p(x) = r(x) = \exp(-\nu x/C)$. Usually, many interesting applications of Sturm-Liouville theory arise in quantum mechanics. For instance, for a single quantum-mechanical particle of mass m moving in one space dimension in a potential $V(x)$, the time-dependent Shrödinger equation is

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + V(x)\psi.$$

Looking for separable solutions $\psi(x,t) = \varphi(x)e^{-iEt/\hbar}$, we find that $\varphi(x)$ satisfies the differential equation

$$-\frac{\hbar^2}{2m}\varphi'' + V(x)\varphi = E\varphi.$$

That is a Sturm-Liouville equation of the form $-u'' + qu = \lambda u$ The coefficient q is proportional to the potential V and the eigenvalue parameter λ in proportional to the energy E. The Rayleigh quotient is the basis of an important approximation method that is used in solid mechanics as well as in quantum mechanics. Recall that the Rayleigh quotient $R(u)$ is defined as

$$R(u) = \frac{\langle Lu, u\rangle_H}{\langle u, u\rangle_H}$$

for the eigenvalue problem $Lu = \lambda u$, where L is a differential operator in a inner-product space H. From this expression it follows that $\lambda_i = R(u_i)$ for any eigen-pair (λ_i, u_i) of the differential operator L. Although any eigenvalue can be related to its eigenfunction by the Rayleigh quotient, this quotient cannot be used to explicitly determine the eigenvalue since eigenfunction is unknown. However, interesting and significant results can be obtained from the Rayleigh quotient without solving the differential equation(i.e. even in the case when the eigenfunction is not known). For example, it can be quite useful in estimating the eigenvalue. It is the purpose of this paper to extend and generalize such important spectral properties as Rayleigh quotient, eigenfunction expansion,Parseval equality and Rayleigh-Ritz formula(minimization principle) for Sturm-Liouville problems with interior singularities. The development of classical, rather than the operatoric, Sturm-Liouville theory in the years after 1950 can be found in various sources; in particular in the texts of Atkinson [1], Coddington and Levinson [9], Levitan and Sargsjan [16] and Zettl [28]. However in different areas of applied mathematics and physics many problems arise in the form of boundary value problems involving interior singularities(see, for example [6, 11, 27]. In this paper we shall investigate certain spectral problems arising in the theory of the convergence of the eigenfunction expansion for one nonclassical eigenvalue problem which consists of the Sturm-Liouville equation

$$L(u) := -(p(x)u')'(x) + q(x)u(x) = \lambda\rho u(x), \quad x \in [\alpha, \gamma) \cup (\gamma, \beta] \tag{1.1}$$

together with boundary conditions (BCs) at the endpoints $x = \alpha, \beta$

$$u(\alpha) = u(\beta) = 0, \tag{1.2}$$

and the transmission conditions at the interior point $\gamma \in (\alpha, \beta)$

$$\cos\mu u(\gamma+0) + \sin\mu u(\gamma-0) = 0, \tag{1.3}$$
$$\cos\sigma u'(\gamma+0) + \sin\sigma u'(\gamma-0) = 0, \tag{1.4}$$

where $p(x) = p_1^2$ for $x \in [\alpha, \gamma)$, $p(x) = p_2^2$ for $x \in (\gamma, \beta], p_1, p_2 \in \mathbb{R}$ the potential $q(x)$ and weight function $\rho(x)$ is real-valued functions, both are continuous in $[\alpha, \gamma)$ and $(\gamma, \beta]$ with finite limits $q(\gamma \mp 0)$, and $\rho(\gamma \mp 0)$, respectively. λ is a complex spectral parameter, μ and σ are real numbers. Throughout in this paper we assume that $\cot\mu\cot\sigma > 0$. Since the values of the solutions and their derivatives at the interior point γ is not defined, an important question is how to introduce a new Hilbert space such a way that the considered problem can be interpreted as self-adjoint problem in this space. Note that boundary value problems together with supplementary transmission conditions appear frequently in various fields of physics and technics. For example, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see, [21] and the references listed therein). Another completely different field is that of "hydraulic fracturing" (see, [10]) used in order to increase the flow of oil from a reservoir into a producing oil well. Some problems with transmission conditions arise in thermal conduction problems for a thin laminated plate (i.e., a plate composed by materials with different characteristics piled in the thickness, see, [24]). Similar problems with point interactions are also studied in [4, 18], etc. Some aspects of spectral problems for differential equations having singularities with classical boundary conditions at the endpoints were studied among others in [2, 3, 5, 7, 12–15, 17, 19, 20, 25] where further references can be found.

2. The Green's function and eigenvalues of the problem

With a view to constructing the Green's function we shall define two basic solutions $\upsilon_1(x, \lambda)$ and $\nu_1(x, \lambda)$ on the left interval $\Omega_1 := [\alpha, \gamma)$(so-called left-hand solutions) and two "basic" solutions $\upsilon_2(x, \lambda)$ and $\nu_2(x, \lambda)$ on the right interval $\Omega_2 := (\gamma, \beta]$ (so-called right-hand solutions) by the own technique as following. Let $\upsilon_1(x, \lambda)$ and $\nu_2(x, \lambda)$ be the solutions of the equation (1.1) on Ω_1 and Ω_2 satisfying the initial conditions

$$\upsilon_1(\alpha, \lambda) = 0, \ \frac{\partial \upsilon_1(\alpha, \lambda)}{\partial x} = 1, \tag{2.1}$$

and

$$\nu_2(\beta, \lambda) = 0, \ \frac{\partial \nu_2(\beta, \lambda)}{\partial x} = 1, \tag{2.2}$$

respectively. In terms of these solutions we shall define the other solutions $\upsilon_2(x, \lambda)$ and $\nu_1(x, \lambda)$ by initial conditions

$$\upsilon_2(\gamma+0, \lambda) = -\tan\mu\upsilon_1(\gamma-0, \lambda), \frac{\partial \upsilon_2(\gamma+0, \lambda)}{\partial x} = -\tan\sigma\frac{\partial \upsilon_1(\gamma-0, \lambda)}{\partial x} \tag{2.3}$$

and

$$\nu_1(\gamma-0, \lambda) = -\cot\mu\nu_2(\gamma+0, \lambda), \ \frac{\partial \nu_1(\gamma-0, \lambda)}{\partial x} = -\cot\sigma\frac{\partial \nu_2(\gamma+0, \lambda)}{\partial x}, \tag{2.4}$$

respectively. The existence and uniqueness of these solutions follow from the well-known existence and uniqueness theorem of ordinary differential equation theory. Moreover, applying the same procedure as in [20] it is easy to show that each of the one-hand solutions $\upsilon_i(x,\lambda)$ and $\nu_i(x,\lambda)$(i=1,2) is entire function of complex spectral parameter λ for each fixed x. Taking in view the initial conditions (2.1)-(2.4) and well-known fact that each of the Wronskians $W[\upsilon_i(x,\lambda),\nu_i(x,\lambda)]$(i=1,2) is independent of variable x and denoting $\Delta_i(\lambda) := W[\upsilon_i(.,\lambda),\nu_i(.,\lambda)]$ we have

$$\begin{aligned}\Delta_2(\lambda) &= W[\upsilon_2(\gamma+0,\lambda),\nu_2(\gamma+0,\lambda)]\\ &= \tan\mu\tan\sigma W[\upsilon_1(\gamma-0,\lambda),\nu_1(\gamma-0,\lambda)]\\ &= \tan\mu\tan\sigma\Delta_1(\lambda).\end{aligned}$$

Now we shall define the characteristic function $\Delta(\lambda)$ as

$$\Delta(\lambda) := \Delta_2(\lambda) = \tan\mu\tan\sigma\Delta_1(\lambda). \tag{2.5}$$

The following Theorems 2.1 and 2.2 are follows from the results of our previous works [17] and [4] respectively. Therefore we shall formulate them without proofs.

Theorem 2.1. *The eigenvalues of the BVTP (1.1)-(1.4) coincide with the zeros of* $\Delta(\lambda)$. *All zeros of* $\Delta(\lambda)$ *are real and simple(i.e.all eigenvalues are real and simple). Moreover there are indefinitely many eigenvalues* $\{\lambda_n\}$ *such that* $\lambda_1 < \lambda_2 < ...$ *and* $\lambda_n \to \infty$ *as* $n \to \infty$.

Remark 2.1. Since all eigenvalues of the BVTP (1.1)-(1.4) are real,simple and all coefficients of this problem are real valued the corresponding eigenfunctions can be chosen to be real-valued. Taking in view this fact, from now on we can assume that all eigenfunctions are real-valued.

Theorem 2.2. *For* λ *not an eigenvalue, the nonhomogeneous equation* $L(u) = \lambda\rho u + f$ *together with the boundary-value-transmission conditions (1.2)-(1.4) has a unique solution* $u = u_f(x,\lambda)$ *for which the formula*

$$u_f(x,\lambda) = \frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0} G(x,\xi;\lambda)f(\xi)d\xi + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} G(x,\xi;\lambda)f(\xi)d\xi \tag{2.6}$$

holds, where the function $G(x,\xi;\lambda)$ *is defined for* $x,\xi \in \Omega_1\cup\Omega_2$ *by the formula*

$$G(x,\xi;\lambda) = \begin{cases} \dfrac{\cot\mu}{p_1^2\Delta_1(\lambda)}\upsilon_1(x,\lambda)\nu_1(\xi,\lambda), & for\ \alpha\le x\le\xi<\gamma\\ \dfrac{\cot\mu}{p_1^2\Delta_1(\lambda)}\upsilon_1(\xi,\lambda)\nu_1(x,\lambda), & for\ \alpha\le\xi\le x<\gamma\\ \dfrac{\cot\mu}{p_1^2\Delta_1(\lambda)}\upsilon_1(x,\lambda)\nu_2(\xi,\lambda), & for\ \alpha\le x<\gamma<\xi\le\beta\\ \dfrac{\tan\sigma}{p_2^2\Delta_2(\lambda)}\upsilon_1(\xi,\lambda)\nu_2(x,\lambda), & for\ \alpha\le\xi<\gamma<x\le\beta\\ \dfrac{\tan\sigma}{p_2^2\Delta_2(\lambda)}\upsilon_2(x,\lambda)\nu_2(\xi,\lambda), & for\ \gamma<x\le\xi\le\beta\\ \dfrac{\tan\sigma}{p_2^2\Delta_2(\lambda)}\upsilon_2(\xi,\lambda)\nu_2(x,\lambda), & for\ \gamma<\xi\le x\le\beta \end{cases} \tag{2.7}$$

which is called the Green's function for the (1.1)-(1.4).

Theorem 2.3. *Let* $p_1^2 = p_2^2 \cot\mu \cot\sigma$ *and* $q(x) \geq 0$ *on* $\Omega_1 \cup \Omega_2$. *Then, all the eigenvalues of the problem (1.1)-(1.4) are positive.*

Proof. Let λ be an eigenvalue and $u(x)$ be the corresponding eigenfunction. Multiplying (1.1) by $u(x)$ and integrating by parts on Ω_1 and Ω_2 we have

$$\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0} u[(pu')' - qu + \lambda\rho u]dx + p_1^2\cot\sigma\int_{\gamma+0}^\beta u[(pu')' - qu + \lambda\rho u]dx$$
$$= \frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0} [-pu'^2 - qu^2 + \lambda\rho u^2]dx + p_1^2\cot\sigma\int_{\gamma+0}^\beta [-pu'^2 - qu^2 + \lambda\rho u^2]dx$$
$$+puu'|_\alpha^{\gamma-0} + puu'|_{\gamma+0}^\beta = 0. \tag{2.8}$$

Since $p_1^2 = p_2^2 \cot\mu\cot\sigma$, from (1.2)-(1.4) we get easily that

$$puu'|_\alpha^{\gamma-0} + puu'|_{\gamma+0}^\beta = 0.$$

Consequently

$$\lambda = \frac{\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}[pu'^2 + qu^2]dx + p_1^2\cot\sigma\int_{\gamma+0}^\beta[pu'^2 + qu^2]dx}{\frac{p_2^2}{\cot\mu}\int_\alpha^\gamma \rho u^2 dx + p_1^2\cot\sigma\int_\gamma^\beta \rho u^2 dx} > 0 \tag{2.9}$$

since $p > 0, q \geq 0$, and $\rho > 0$ on $\Omega_1 \cup \Omega_2$. Show that $\lambda \neq 0$. Otherwise, from (2.9) $u' \equiv 0$, i.e. $u = const$. Then from (1.2) $u \equiv 0$, a contradiction. Hence $\lambda > 0$. $\square$

3. Uniform convergeness of eigenfunction expansion

From now on we shall suppose that $p_1^2 = p_2^2 \cot\mu\cot\sigma$. Then by Theorem 4.1 all eigenvalues are positive. Since $\lambda = 0$ is not an eigenvalue the homogeneous equation $L(u) = 0$ with the boundary-transmission conditions (1.2)-(1.4) has only the trivial solution. Then by Theorem 2.2, the Green's function $G(x,\xi) := G(x,\xi;0)$ exist and the associated nonhomogeneous problem

$$L[u] = -f, u(\alpha) = u(\beta) = 0, \tag{3.1}$$
$$u(\gamma - 0) = -\cot\mu u(\gamma - 0), u'(\gamma + 0) = -\cot\sigma u'(\gamma - 0) \tag{3.2}$$

has a unique solution given by

$$u(x) = \frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0} G(x,\xi)f(\xi)d\xi + p_1^2\cot\sigma\int_{\gamma+0}^\beta G(x,\xi)f(\xi)d\xi. \tag{3.3}$$

If we set $f(x) = \lambda\rho(x)u(x)$, in (3.1) and (3.3), then we have that the eigenfunction $u(x)$ of (1.1)-(1.4) with corresponding eigenvalue λ satisfies

$$u(x) = \lambda\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0} G(x,\xi)\rho(\xi)u(\xi)d\xi + \lambda p_1^2\cot\sigma\int_{\gamma+0}^\beta G(x,\xi)\rho(\xi)u(\xi)d\xi. \tag{3.4}$$

Denoting $\psi(x) = \sqrt{\rho(x)}u(x)$, $\mu = \frac{1}{\lambda}$ and

$$T\psi(x) = \frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0} \sqrt{\rho(x)}G(x,\xi)\sqrt{\rho(\xi)}\psi(\xi)d\xi$$

$$+p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\sqrt{\rho(x)}G(x,\xi)\sqrt{\rho(\xi)}\psi(\xi)d\xi \tag{3.5}$$

the equation (3.4) is changed to the integral equation $T\psi(x) = \mu\psi(x)$ by with continuous and symmetric kernel $\sqrt{\rho(x)}G(x,\xi)\sqrt{\rho(\xi)}$.

Remark 3.1. The equation (3.4) is changed to the integral equation $T\psi(x) = \mu\psi(x)$ by Let $\lambda_0 \neq 0$ is an eigenvalue of the BVTP (1.1)-(1.3) and $\phi_0(x)$ is the corresponding eigenfunction. Then $\psi_0(x) = \sqrt{\rho(x)}\phi_0(x)$ is the eigenfunction of the integral operator T corresponding to the eigenvalue $\mu_0 = \frac{1}{\lambda_0}$. Conversely if (μ_0, ψ_0) is the eigen-pair of the integral operator T, then $(\frac{1}{\mu_0}, \frac{\psi_0}{\sqrt{\rho(x)}})$ is the eigen-pair of the BVTP (1.1)-(1.4).

In the direct sum space $\mathcal{H} = L_2(\Omega_1) \oplus L_2(\Omega_2)$ we define a new inner-product, which associated with the considered BVTP (1.1)-(1.4), by

$$\langle f, g\rangle_{\mathcal{H}} := \frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}\rho(x)f(x)\overline{g(x)}dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho(x)f(x)\overline{g(x)}dx \tag{3.6}$$

for $f(x), g(x) \in \mathcal{H}$. By virtue of the well-konown Theorem about integral operators with continuous, symmetric kernel(see,for example [26]) it follows that there exists a sequence of eigenvalues $\{\mu_k\}$ of (3.5) with corresponding eigenfunctions $\{\psi_k\}$ such that

$$\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}\psi_k^2(x)dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\psi_k^2(x)dx = 1.$$

Since $\lambda_k = \frac{1}{\mu_k}$ are eigenvalues of (1.1)-(1.3) and λ_k is positive and simple we have $0 < \lambda_1 < \lambda_2 < ... < \lambda_k < ...$ and $\lambda_k \to \infty$ and

$$\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}\rho(x)u_k^2(x)dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho(x)u_k^2(x)dx = 1$$

where $\{\phi_k(x)\}$ is a sequence of normalized eigenfunctions of (1.1)-(1.4) with respect to the weight function $\rho(x)$. Now we shall introduce to the consideration the Banach space $\oplus C^k(\Omega)$ defined by

$$\oplus C^k(\Omega) := \{f = \begin{cases} f_{(1)}(x) \text{ for } x \in \Omega_1 \\ f_{(2)}(x) \text{ for } x \in \Omega_2 \end{cases} : f_{(1)} \in C^k[\alpha,\gamma], f_{(2)} \in C^k[\gamma,\beta]\}, k = 0, 1, \ldots,$$

$$\|f\|_{\oplus C^k} = \max\{\|f_{(1)}\|_{C^k[\alpha,\gamma]}, \|f_{(2)}\|_{C^k[\gamma,\beta]}\}$$

and the Hilbert space $\oplus L_2(\Omega)$ defined by

$$\oplus L_2(\Omega) := \{f = \begin{cases} f_{(1)}(x) \text{ for } x \in \Omega_1 \\ f_{(2)}(x) \text{ for } x \in \Omega_2 \end{cases} : f_{(i)} \in L_2(\Omega_i)(i = 1, 2),$$

$$\langle f, g\rangle_{\oplus L_2(\Omega)} = \frac{p_2^2}{\cot\mu}\langle f_{(1)}, g_{(1)}\rangle_{L_2(\Omega_1)} + p_1^2\cot\sigma\langle f_{(2)}, g_{(2)}\rangle_{L_2(\Omega_2)}\}.$$

Let $R(T)$ be the range of integral operator T acting in $\oplus C(\Omega)$.

Theorem 3.1. *Let $f \in R(T)$. Then the Fourier series of f with respect to $\{\psi_i(x)\}$ in the Hilbert Space $\oplus L_2(\Omega)$ i.e. the series*

$$\sum_{i=1}^{\infty}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} f\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} f\psi_i ds\}\psi_i(x)$$

converges to f in the Banach space $\oplus C(\Omega)$ i.e. uniformly on $[\alpha,\gamma)\cup(\gamma,\beta]$.

Proof. Since $f(x) \in R(T)$, there exist $u \in \oplus C(\Omega)$ such that $f = Tu$. Consider the sequence u_n given by

$$u_n(x) = u(x) - \sum_{i=1}^{n}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} u\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} u\psi_i ds\}\phi_i.$$

It is obvious that for each $i = 1,...n$.

$$\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} u_n\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} u_n\psi_i ds = 0.$$

From the extremal principles it follows that $\|Tu_n\|_{\oplus L_2(\Omega)} \leq |\mu_{n+1}|\|u_n\|_{\oplus L_2(\Omega)}$. Since $\mu_n \to 0 \; as \; n \to \infty$ we have $\{Tu_n\} \to \infty \; as \; n \to \infty$. From this and the equality $T\psi_i = \mu_i\psi_i$ it follows that

$$\begin{aligned} f = Tu &= \sum_{i=1}^{\infty}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} u\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} u\psi_i ds\}T\phi_i \\ &= \sum_{i=1}^{\infty}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} uT\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} uT\psi_i ds\}\psi_i \\ &= \sum_{i=1}^{\infty}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} Tu\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} Tu\psi_i ds\}\psi_i \end{aligned} \tag{3.7}$$

where the above convergence hold in the Hilbert space $\oplus L_2(\Omega)$. If $m > n$, then

$$\begin{aligned} &\sum_{i=n}^{m}\mu_i\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} u\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} u\psi_i ds\}\psi_i \\ &= T[\sum_{i=n}^{m}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} u\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} u\psi_i ds\}\psi_i]. \end{aligned} \tag{3.8}$$

Taking in view (2.1)-(2.5), (2.7) and (3.7) it easy to see that there is a constant $M > 0$ such that $|Tu| \leq M\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}|u|^2 ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}|u|^2 ds\}^{\frac{1}{2}}$ for all $u \in \oplus C(\Omega)$. Consequently

$$\begin{aligned} &|\sum_{i=n}^{m}\mu_i\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} u\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} u\psi_i ds\}\psi_i(x)| \\ &\leq M[\sum_{i=n}^{m}[\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} u\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} u\psi_i ds]^2]^{\frac{1}{2}}. \end{aligned} \tag{3.9}$$

The right-hand side of this inequality tends to zero as $n, m \to \infty$ by the Bessel's inequality. Hence the series

$$\sum_{i=1}^{\infty}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} Tu\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} Tu\psi_i)ds\}\psi_i(x) \tag{3.10}$$

is convergent to a function $\psi(x)$ in the Banach space $\oplus C(\Omega)$(i.e. uniformly on $\Omega_1 \cup \Omega_2$). By (3.7) this series converges to Tu in the Hilbert space $\oplus L_2(\Omega)$. Since $Tu \in \oplus C(\Omega)$ and the embedding $\oplus C(\Omega) \subset \oplus L_2(\Omega)$ is continuous, it follows immediately that $\psi(x) = (Tu)(x)$ for all $x \in \Omega_1 \cup \Omega_2$. Thus

$$(Tu)(x) = \sum_{i=1}^{\infty}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} Tu\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} Tu\psi_i dx\}\psi_i(x) \tag{3.11}$$

where the above series converges in the Banach space $\oplus C(\Omega)$, i.e. uniformly on $[\alpha, \gamma) \cup (\gamma, \beta]$. □

Theorem 3.2. *Let $f \in \oplus C^2(\Omega)$ and satisfy the boundary-transmission conditions (1.2)-(1.3). Then*

$$f(x) = \sum_{i=1}^{\infty}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} f\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} f\psi_i ds\}\psi_i(x)$$

where the series being convergent in the Banach space $\oplus C(\Omega)$.

Proof. Let $f \in \oplus C^2(\Omega)$ and satisfy the boundary-transmission conditions (1.2)-(1.4). Then $L[f] \in \oplus C(\Omega)$. Denote $u = L[f]$. From Theorem 3.1 it follows that

$$\begin{aligned} f(x) = -Tu(x) &= -\sum_{i=1}^{\infty}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} Tu\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} Tu\psi_i ds\}\psi_i(x) \\ &= \sum_{i=1}^{\infty}\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0} f\psi_i ds + p_1^2\cot\sigma\int_{\gamma+0}^{\beta} f\psi_i ds\}\psi_i(x) \end{aligned} \tag{3.12}$$

where the above series converges in the Banach space $\oplus C(\Omega)$, i.e. uniformly on $[\alpha, \gamma) \cup (\gamma, \beta]$. □

4. Generalized Fourier series and generalized Parseval's equation

Now we are ready to prove the next important result.

Theorem 4.1. *The orthonormal set $\{\psi_i(.)\}$ is a complete in the Hilbert space $L_2(\Omega)$.*

Proof. Let $f \in \oplus L_2(\Omega)$ and $\epsilon > 0$ be given arbitrary number. Since $C_0^\infty[a, b]$(the set of infinitely differentiable functions on $[a, b]$ each of which vanishes of some neighborhoods of the end-points $x = a$ and $x = b$) is dense in the Lebesgue space $L_2[a, b]$ $(-\infty < a < b < +\infty)$ it follows that there exists a function $f_\epsilon \in \oplus C^2(\Omega)$ satisfying (1.2)-(1.4) such that

$$\|f - f_\epsilon\|_{\oplus L_2(\Omega)} < \frac{\epsilon}{3}. \tag{4.1}$$

By the previous theorem there exists an integer $n_0 = n_0(\epsilon)$ such that for all $m \geq n_0(\epsilon)$

$$\|f_\epsilon - \sum_{i=1}^{m} \{ \frac{p_2^2}{\cot \mu} \int_{\alpha}^{\gamma-0} f_\epsilon \psi_i ds + p_1^2 \cot \sigma \int_{\gamma}^{\beta+0} f_\epsilon \psi_i ds \} \psi_i \|_{\oplus L_2(\Omega)} \leq \frac{\epsilon}{3}. \quad (4.2)$$

Denote by $C_{\epsilon,i}$ the Fourier coefficients of the function $f - f_\epsilon$ with respect to the orthonormal set $\{\psi_i(x)\}$ in the Hilbert space $\oplus L_2(\Omega)$ i.e.

$$C_{\epsilon,i} := \langle f - f_\epsilon, \psi_i \rangle_{\oplus L_2(\Omega)}.$$

Then by Bessel's inequality

$$\sum_{i=1}^{m} |\langle f - f_\epsilon, \psi_i \rangle_{\oplus L_2(\Omega)}|^2 = \sum_{i=1}^{m} |C_{\epsilon,i}|^2 \leq \|f - f_\epsilon\|^2_{\oplus L_2(\Omega)} \quad (4.3)$$

for any integer m. From (4.1) and (4.3) it follows that

$$\| \sum_{i=1}^{m} \langle f - f_\epsilon, \psi_i \rangle \psi_i \|^2_{\oplus L_2(\Omega)} = \sum_{i=1}^{m} |C_{\epsilon,i}|^2 \leq (\frac{\epsilon}{3})^2 \quad (4.4)$$

for $m \geq n_0(\epsilon)$. Now by applying the triangle inequality and using (4.1),(4.2) and (4.4) we have that for any $m \geq n_0(\epsilon)$

$$\begin{aligned} \|f - \sum_{i=1}^{m} \langle f, \psi_i \rangle_{\oplus L_2(\Omega)} \psi_i \|_{\oplus L_2(\Omega)} &\leq \|f - f_\epsilon\|_{L_2(\Omega)} + \|f_\epsilon - \sum_{i=1}^{m} \langle f_\epsilon, \psi_i \rangle_{\oplus L_2(\Omega)} \psi_i \|_{L_2(\Omega)} \\ &+ \| \sum_{i=1}^{m} \langle f_\epsilon - f, \psi_i \rangle_{\oplus L_2(\Omega)} \psi_i \|_{\oplus L_2(\Omega)} < \epsilon. \end{aligned}$$

Consequently the orthonormal set $\{\psi_i(.)\}$ is a complete in the Hilbert space $\oplus L_2(\Omega)$. □

Since any complete orthonormal set of a Hilbert space form a orthonormal basis of this space, we have the next Corollaries.

Corollary 4.1. *For any $f \in \oplus L_2(\Omega)$*

$$f = \sum_{i=1}^{\infty} \{ \frac{p_2^2}{\cot \mu} \int_{\alpha}^{\gamma-0} f \psi_i ds + p_1^2 \cot \sigma \int_{\gamma+0}^{\beta} f \psi_i ds \} \psi_i$$

where, the series converges in the Hilbert space $\oplus L_2(\Omega)$.

Corollary 4.2 (Generalized Parseval's equality)**.** *For any $f \in \oplus L_2(\Omega)$, we have the following generalized Parseval's equality*

$$\|f\|^2_{\oplus L_2(\Omega)} = \sum_{n=1}^{\infty} | \frac{p_2^2}{\cot \mu} \int_{\alpha}^{\gamma-0} f \psi_n ds + p_1^2 \cot \sigma \int_{\gamma+0}^{\beta} f \psi_n ds |^2.$$

5. Minimization principle of eigenvalues

Recall that the minimum eigenvalue of Sturm-Liouville BVP is called the principal eigenvalue and the corresponding eigenfunction is called the principal eigenfunction.

Theorem 5.1. *The principal eigenvalue for the boundary value -transmission problem (1.2)-(1.4) is the minimum value of the functional*

$$R(u)=\frac{\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}[p_1^2u'^2+qu^2]dx+p_1^2\cot\sigma\int_{\gamma+0}^{\beta}[p_2^2u'^2+qu^2]dx}{\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}\rho u^2dx+p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho u^2dx} \tag{5.1}$$

for all functions $u\in\oplus C^2(\Omega)$ satisfying (1.2)-(1.4). Moreover, the minimum is achieved for the principal eigenfunction.

Proof. Let $\varphi(x)\in\oplus C^2(\Omega)$ with $\varphi(\alpha)=\varphi(\beta)=0, \varphi(\gamma-0)=-\cot\mu\varphi(\gamma+0)$, $\varphi'(\gamma-0)=-\cot\sigma\varphi'(\gamma+0)$. Then by Corollary 4.1

$$\varphi(x)=\sum_{n=1}^{\infty}c_n(\varphi)\phi_n(x)$$

with

$$c_n(\varphi)=\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}\rho\varphi\phi_nds+p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho\varphi\phi_nds$$

where the convergence is uniform on $[\alpha,\gamma)\cup(\gamma,\beta]$. Now, by integration by parts we get

$$\begin{aligned}&\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}L[\varphi]\phi_ndx+p_1^2\cot\sigma\int_{\gamma+0}^{\beta}L[\varphi]\phi_ndx\\ &=\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}\varphi L[\phi_n]dx+p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\varphi L[\phi_n]dx\\ &=-\lambda_n\{\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}\rho\phi_n\varphi dx+p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho\phi_n\varphi dx\}\\ &=-c_n(\varphi)\lambda_n.\end{aligned} \tag{5.2}$$

Then by using Corollary 4.1 and Corollary 4.2 we have

$$\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}\rho\varphi^2dx+p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho\varphi^2dx=\sum_{n=1}^{\infty}c_n^2(\varphi) \tag{5.3}$$

and

$$\begin{aligned}&\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}[p_1^2\varphi'^2+q\varphi^2]dx+p_1^2\cot\sigma\int_{\gamma+0}^{\beta}[p_2^2\varphi'^2+q\varphi^2]dx\\ &=-\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}\varphi L[\varphi]dx-p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\varphi L[\varphi]dx\\ &=-\frac{p_2^2}{\cot\mu}\int_\alpha^{\gamma-0}\{\sum_{n=1}^{\infty}c_n(\varphi)\phi_n\}L[\varphi]dx-p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\{\sum_{n=1}^{\infty}c_n(\varphi)\phi_n\}L[\varphi]dx\end{aligned}$$

$$= -\sum_{n=1}^{\infty} c_n(\varphi)\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}\phi_n L[\varphi]dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\phi_n L[\varphi]dx\}$$
$$= -\sum_{n=1}^{\infty} c_n(\varphi)\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}L[\phi_n]\varphi dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}L[\phi_n]\varphi dx\}$$
$$= \sum_{n=1}^{\infty} c_n(\varphi)\lambda_n\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}\rho\varphi\phi_n dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho\varphi\phi_n dx\} = \sum_{n=1}^{\infty}\lambda_n c_n^2(\varphi)$$
$$\geq \lambda_1\sum_{n=1}^{\infty} c_n^2(\varphi) = \lambda_1\{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}\rho\varphi^2 dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho\varphi^2 dx\}. \tag{5.4}$$

Hence,

$$\lambda_1 \leq \frac{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}[p_1^2\varphi'^2 + q\varphi^2]dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}[p_2^2\varphi'^2 + q\varphi^2]dx}{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}\rho\varphi^2 dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho\varphi^2 dx}.$$

Let $\varphi = \phi_1$. Then we have $c_n(\varphi) = 0$ for $n = 2, 3, ...$ and

$$\lambda_1 = \frac{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}[p_1^2\phi_1'^2 + q\phi_1^2]dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}[p_2^2\phi_1'^2 + q\phi_1^2]dx}{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}\rho\phi_1^2 dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho\phi_1^2 dx}.$$

Consequently $\lambda_1 = \min R(\varphi)$ for the class $\varphi(x) \in \oplus C^2(\Omega)$ with $\varphi(\alpha) = \varphi(\beta) = 0, \varphi(\gamma-0) = -\cot\mu\varphi(\gamma+0), \quad \varphi'(\gamma-0) = -\cot\sigma\varphi'(\gamma-0)$ and its minimizing function is $\varphi = \phi_1(x)$. □

Theorem 5.2. *The n-th eigenvalue λ_n of the BVTP (1.1)-(1.4) is the minimum value of the functional $R(\varphi)$(Rayleigh quotient defined by (5.1)) for all functions $\varphi \in \oplus C^2(\Omega)$ satisfying boundary-transmission conditions (1.2)-(1.4) and $\langle\varphi, \phi_i\rangle_{\mathcal{H}} = 0, i = 1, 2, ..., n-1$. where $\phi_1, \phi_2, ..., \phi_{n-1}$ are the first $n-1$ normalized eigenfunctions. The minimum is achived for the n-th normalized eigenfunction $\phi_n(x)$.*

Proof. Let $\varphi(x) \in \oplus C^2(\Omega)$ with $\varphi(\alpha) = \varphi(\beta) = 0, \varphi(\gamma-0) = -\cot\mu\varphi(\gamma+0), \quad \varphi'(\gamma-0) = -\cot\sigma\varphi'(\gamma+0)$ and $\langle\varphi, \phi_i\rangle_{\mathcal{H}} = 0, i = 1, ...n-1$. Then by (5.4) we have

$$\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}[p_1^2\varphi'^2 + q\varphi^2]dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}[p_2^2\varphi'^2 + q\varphi^2]dx = \sum_{k=n}^{\infty}\lambda_k c_n^2(\varphi)$$
$$\geq \lambda_n\sum_{k=n}^{\infty} c_k^2(\varphi).$$

Further, by generalized Parseval's equation we have

$$\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}p_1^2\varphi'^2 dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}p_2^2\varphi'^2 dx = \sum_{k=n}^{\infty} c_k^2(\varphi). \tag{5.5}$$

Hence, as in the proof of the previous Theorem 5.1, we have

$$\lambda_n = \min\frac{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}[p_1^2\varphi'^2 + q\varphi^2]dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}[p_2^2\varphi'^2 + q\varphi^2]dx}{\frac{p_2^2}{\cot\mu}\int_{\alpha}^{\gamma-0}\rho\varphi^2 dx + p_1^2\cot\sigma\int_{\gamma+0}^{\beta}\rho\varphi^2 dx}$$

for all functions $\varphi(x) \in \oplus C^2(\Omega)$ satisfying boundary-transmission conditions (1.2)-(1.4) and $\langle \varphi, \phi_i \rangle_{\mathcal{H}} = 0, i = 1, ...n-1$, where the minimum is achived only for the n-th eigenfunction $\phi_n(x)$. $\square$

Remark 5.1. We have shown that among the class of $\oplus C^2(\Omega)$ functions $\varphi(x)$ with $\varphi(\alpha) = \varphi(\beta) = 0, \varphi(\gamma-0) = -\cot\mu\varphi(\gamma+0), \ \ \varphi'(\gamma+0) = -\cot\sigma\varphi'(\gamma+0)$. λ_1 is the minimum value for $R(\varphi)$ and $\phi_1(x)$ is the minimizing function. But the functional $R(\varphi)$ is still well-defined even if $\varphi \in \oplus C^1(\Omega)$ and no assumption on φ''. Hence it is natural to consider the class of admissible functions $D := \{\varphi | \phi \in \oplus C^1(\Omega), \varphi(\alpha) = \varphi(\beta) = 0, \varphi(\gamma-0) = -\cot\mu\varphi(\gamma+0), \ \ \varphi'(\gamma+0) = -\cot\sigma\varphi'(\gamma+0)\}$; It is possible that for the new class D we may obtain a new minimizing function which no longer satisfies the BVTP (1.1)-(1.4). However, in this particular functional, we can apply the theory of calculus of variations to show that any function $\widetilde{u} \in D$ that gives a minimum value to the functional $R(\varphi)$ must be in $\oplus C^2(\Omega)$ and satisfy the BVTP (1.1)-(1.4), (see [8]),i.e. $\min_{\phi \in D} R(\phi) = R(\phi_1) = \lambda_1$ where λ_1 is the principal eigenvalue and ϕ_1 is the corresponding principal eigenfunction.

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