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OPERATOR SPLITTING FOR NUMERICAL SOLUTIONS OF THE RLW EQUATION

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Abstract In this study, the numerical behavior of the one-dimensional Regularized Long Wave (RLW) equation has been sought by the Strang splitting technique with respect to time. For this purpose, cubic B-spline functions are used with the finite element collocation method. Then, single solitary wave motion, the interaction of two solitary waves and undular bore problems have been studied and the effectiveness of the method has been investigated. The new results have been compared with those of some of the previous studies available in the literature. The stability analysis has also been taken into account by the von Neumann method.

Keywords Collocation method, RLW equation, strang splitting, solitary waves.

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1. Introduction

In science, nonlinear partial differential equations often represent wave events that are motivated by certain physical initial/boundary conditions [12]. In this work, we will consider a one-dimensional RLW equation given with the physical boundary conditions as follows

$$U_t + U_x + \varepsilon U U_x - \mu U_{xxt} = 0 \tag{1.1}$$

$U \to 0$ when $x \to \pm\infty$. Where t is the time, x is the position coordinate, $U(x,t)$ is the wave height (amplitude), and ε and μ are the positive parameters.

The RLW equation was first appeared when calculating the development of the "undular bore" problem by Peregrine. The RLW equation is a nonlinear dispersive wave equation which is a more conventional than the KdV equation in observing the wave phenomena. This equation is most commonly used in order to model physical phenomena such as shallow water waves and plasma waves [17].

Several researhers have solved the RLW equation using various methods and techniques. Among others, Kutluay [13] and Esen [6] have solved the equation using both finite difference and finite elements method. Mei and Chen [14] have used explicit multistep method, Oruç and *et al.* [16] have utilized Haar wavelet method and Islam and *et al.* [11] have presented a meshfree technique using the radial basis functions (RBFs) in order to obtain the numerical solutions of the equation. Moreover; Dağ and *et al.* in the papers [2] and [3] have developed cubic B-spline collocation and quintic B-Spline Galerkin finite element methods for obtaining numerical

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solutions of the present equation. Besides, while Zaki [27] has applied a combination of the splitting method and cubic B-spline finite elements, and Raslan [19] has used cubic B-spline collocation method for approximate solutions of the equation. Dağ and Özer [5] have solved the regularized long wave equation numerically giving a new algorithm based on a kind of space-time least square finite element method, in which a combination of cubic B-splines is used as an approximate function. Saka et.al. [20] have used both sextic and septic B-spline collocation algorithms for the numerical solutions of the RLW equation. Doğan [4] has solved the equation by Galerkin's method using linear space finite elements. Saka and Dağ [21] have used Galerkin finite element method based on quartic B-splines to find a numerical solution of the regularized long wave equation. Saka et. al. [22] have obtained the numerical solutions of the equation using space-splitting technique and quadratic B-spline Galerkin finite element method. Mokhtari and Mohammadi [15] have presented a mashfree technique based on a global collocation method using Sinc basis functions to obtain numerical solutions of the problem. Saka and Dağ [23] have studied the collocation method based on quartic B-spline interpolation and also solved time-split RLW equation with quartic B-spline collocation method.

In recent times, solitary waves, especially soliton waves, have become both experimental and theoretically very interesting and outstanding. A soliton is a very special type of solitary wave, which has a continuous form, can be placed in a region and interaction with another soliton, and can be separated unchanged without a change of phase [1]. After splitting the RLW equation with respect to time and applying the discretization process, we use the Strang splitting technique [24], using cubic B-spline collocation method to solve single solitary wave motion, two solitary waves interactions, undular bore problems. We are going to examine the error norms L_2 and L_∞, together with the three conservation constants.

2. Formulation of Splitting Methods

A scientist doing numerical computation often faces new and complex equations that require the development of an effective solution method. If everthing goes well and the equation is of a well-known type, finding an easy-to-implement method is fairly straightforward. But otherwise, it is difficult to find a good method in most cases and it can be difficult to apply this method. One way of dealing with complex problems is "divide and conquer". In the context of evolution type equations, the operator splitting idea has been a very successful approach. The underlying idea behind such an approach is that all model evolution operators are formally written as the sum of the evolution operators for each term that is being modeled. In other words, when one splits the model into a series of sub-equations, simpler and more practical algorithms for each sub-equation occur. Then the applied numerical method is applied to each sub-problem and numerical schemes are obtained and these schemes are combined by operator splitting [8].Traditional splitting methods, which have a wide use in solving real life problems, are very common.

We are going to dwell on the situation in which we have the following the Cauchy problem

$$\frac{dU(t)}{dt} = AU(t) + BU(t), \quad t \in [0, T], \quad U(0) = U_0.$$

Here, an initial function $U_0 \in X$ is given, A and B are assumed to be bounded linear operators in the Banach space X together with $A, B : X \to X$. There is also a norm associated with the space X denoted by $\|.\|_X$, and if both A and B are matrixes, then it is called Euclidean norm [7].

2.1. First Order Schemes (Lie-Trotter Splitting)

Let us first present the general forms of the first order schemes with respect to the splitting time step Δt and known as Lie-Trotter splitting as presented in [25].

- $(A-B)$ Splitting

$$\begin{aligned} \frac{dU^*}{dt} &= AU^*, \quad U^*(0) = U_0 \quad \text{on } [0,\Delta t]\,, \\ \frac{dU^{**}}{dt} &= BU^{**}, \quad U^{**}(0) = U^*(\Delta t) \quad \text{on } [0,\Delta t]\,, \end{aligned}$$

where the final values are obtained by using $U^{**}(\Delta t)$.

- $(B-A)$ Splitting

This method is characterized by reversing the sequence of successive integration for the operators A and B.

Replacing the original problem with sub-problems naturally gives rise to an error called local splitting error. Using Taylor series expansion, local splitting error for Lie-Trotter splitting method is

$$\begin{aligned} E_n &= \frac{1}{\Delta t}\left(e^{(A+B)\Delta t} - e^{B\Delta t}e^{A\Delta t}\right)U(t_n) \\ &= \frac{1}{\Delta t}\left[\frac{\Delta t^2}{2}(AB-BA)\,U(t_n) + O(\Delta t^3)\right] \\ &= \frac{\Delta t}{2}[A,B]\,U(t_n) + O\left(\Delta t^2\right). \end{aligned}$$

Here $[A,B] = AB - BA$. As a result, the Lie-Trotter method is a first-order method if A and B are not commute [25].

2.2. Second order Strang Splitting scheme

In order to improve the accuracy, Strang [24] has proposed a symettrizing splitting scheme

$$\begin{aligned} \frac{dU^*}{dt} &= AU^*, \quad U^*(0) = U_0 \quad \text{on } [0,\Delta t/2]\,, \\ \frac{dU^{**}}{dt} &= BU^{**}, \quad U^{**}(0) = U^*(\Delta t/2) \quad \text{on } [0,\Delta t]\,, \\ \frac{dU^{***}}{dt} &= AU^{***}, \quad U^{***}(0) = U^{**}(\Delta t) \quad \text{on } [0,\Delta t/2]\,, \end{aligned} \tag{2.1}$$

where the final values are obtained by $U^{***}(\Delta t/2)$. This scheme is called $(A-B-A)$ and the scheme $(B-A-B)$ can be derived in a similar manner. Again, using Taylor series expansion, this scheme has a local splitting error

$$\begin{aligned} E_n &= \left[e^{A\frac{\Delta t}{2}}e^{B\Delta t}e^{A\frac{\Delta t}{2}} - e^{(A+B)\Delta t}\right]U(t_n) \\ &= \frac{\Delta t^2}{24}\left([A,[B,A]] - 2\,[B,[A,B]]\right)U(t_n) + O(\Delta t^3) \end{aligned}$$

is a second-order scheme and is used in practice for many applications.

3. Method of Solution

To examine the numerical behavior of the RLW equation (1), the solution domain is constrained on a closed interval $[a, b]$. The initial

$$U(x, 0) = f(x), \qquad a \le x \le b$$

and the homogenous boundary conditions

$$\begin{aligned} U(a,t) = 0, \quad U(b,t) = 0, \qquad t \ge 0 \\ U_x(a,t) = 0, \quad U_x(b,t) = 0 \end{aligned}$$

are taken as stated above, and $f(x)$ is a predefined function. The partition of the interval $[a, b]$ in terms of nodal points x_m are defined as follows

$$a = x_0 \le x_1 \le ... \le x_N = b,$$

where $h = x_m - x_{m-1} = \frac{b-a}{N}$, $m = 0, 1, ..., N$. Cubic B-spline functions are defined as $\phi_m(x)$, $m = -1, ..., N+1$. On this fragmentation, we will need cubic B-spline functions at x_m node points $\phi_m(x)$ given as follows.

$$\phi_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3 & [x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3 & [\ x_{m-1}, x_m\ ] \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3 & [\ x_m, x_{m+1}\ ] \\ (x_{m+2} - x)^3 & [x_{m+1}, x_{m+2}] \\ 0 & \text{otherwise.} \end{cases}$$

Then, they form a base on the range $[a, b]$ consisting of the set of B-splines $\{\phi_{-1}, \phi_0, ..., \phi_N, \phi_{N+1}\}$ [18]. A general approach to the function $U_N(x, t)$ in terms of cubic B-spline base functions and element parameters δ_j as follows

$$U_N(x,t) = \sum_{j=-1}^{N+1} \delta_j \phi_j. \tag{3.1}$$

Therefore, over the element $[x_m, x_{m+1}]$ the approximate solution $U_N(x, t)$ can be expressed as follows:

$$U_N(x,t) = \sum_{j=m-1}^{m+2} \delta_j \phi_j, \tag{3.2}$$

where time dependent parameters δ_j are obtained using collocation finite element method together with the boundary and initial conditions [2]. The values of $\phi(x)$ and its first and second derivatives $\phi'(x)$, $\phi''(x)$ at the nodal points can be found using cubic B-spline base functions and Eqs.(3.1)-(3.2). Using the expression in Eq. (3.1), the first order derivative $U_m^{'}$, the second order derivative $U_m^{''}$ and the values of U_m are obtained as follows at the nodal points

$$\begin{aligned} U_m &= \delta_{m-1} + 4\delta_m + \delta_{m+1}, \\ U_m^{'} &= \frac{3}{h}(\delta_{m+1} - \delta_{m-1}), \\ U_m^{''} &= \frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}). \end{aligned}$$

The time splitted RLW equation is taken as follows

$$U_t - \mu U_{xxt} + U_x = 0, \tag{3.3}$$

$$U_t - \mu U_{xxt} + UU_x = 0. \tag{3.4}$$

If the values of $U_m, U_m^{'}$ and $U_m^{''}$ at nodal points x_m are used in (3.3) and (3.4) and basic necessary operations are performed, we obtain the following first order ordinary differential equation systems

$$\overset{\circ}{\delta}_{m-1} + 4\overset{\circ}{\delta}_m + \overset{\circ}{\delta}_{m+1} - \frac{6\mu}{h^2}(\overset{\circ}{\delta}_{m-1} - 2\overset{\circ}{\delta}_m + \overset{\circ}{\delta}_{m+1}) + \frac{3}{h}(\delta_{m+1} - \delta_{m-1}) = 0, \tag{3.5}$$

$$\overset{\circ}{\delta}_{m-1} + 4\overset{\circ}{\delta}_m + \overset{\circ}{\delta}_{m+1} - \frac{6\mu}{h^2}(\overset{\circ}{\delta}_{m-1} - 2\overset{\circ}{\delta}_m + \overset{\circ}{\delta}_{m+1}) + \frac{3z_m\varepsilon}{h}(\delta_{m+1} - \delta_{m-1}) = 0, \tag{3.6}$$

here $^\circ$ denotes derivation with respect to t and the value of z_m is taken as follows for linearization process

$$z_m = \delta_{m-1} + 4\delta_m + \delta_{m+1}.$$

Instead of the parameter δ_m, $(\delta_m^{n+1} + \delta_m^n)/2$ is written and instead of time-varying parameters $\overset{\circ}{\delta}_m$, $\left(\delta_m^{n+1} - \delta_m^n\right)/\Delta t$ is written in Eqs. (3.5) and (3.6), the following equations

$$a_1\delta_{m-1}^{n+1} + b_1\delta_m^{n+1} + c_1\delta_{m+1}^{n+1} = c_1\delta_{m-1}^n + b_1\delta_m^n + a_1\delta_{m+1}^n, \tag{3.7}$$

$$a_2\delta_{m-1}^{n+1} + b_2\delta_m^{n+1} + c_2\delta_{m+1}^{n+1} = c_2\delta_{m-1}^n + b_2\delta_m^n + a_2\delta_{m+1}^n, \tag{3.8}$$

$$a_1 = 1 - \frac{6\mu}{h^2} - \frac{3\Delta t}{2h},\ b_1 = 4 + \frac{12\mu}{h^2},\ c_1 = 1 - \frac{6\mu}{h^2} + \frac{3\Delta t}{2h},$$

$$a_2 = 1 - \frac{6\mu}{h^2} - \frac{3z_m\Delta t\epsilon}{2h},\ b_2 = 4 + \frac{12\mu}{h^2},\ c_2 = 1 - \frac{6\mu}{h^2} + \frac{3z_m\Delta t\epsilon}{2h}$$

are obtained. The equations given in (3.7) and (3.8) consist of $(N+1)$ equations and $(N+3)$ unknown δ_j parameters. Using the boundary conditions $U(a,t) = 0$ and $U(b,t) = 0$, we obtain the following equalities for parameters δ_{-1} and δ_{N+1}

$$\delta_{-1} = -4\delta_0 - \delta_1,\ \delta_{N+1} = -4\delta_N - \delta_{N-1}. \tag{3.9}$$

If the parameters δ_{-1} and δ_{N+1} are eliminated from systems (3.7) and (3.8) using identities (3.9) , $(N+1)\times(N+1)$ dimensional tridiagonal band matrix systems are obtained. A unique solution of these systems can be obtained using the Thomas algorithm. In order to solve these systems, it is necessary to use δ_m^0 initial parameters in (3.7) and (3.8) after the initial parameters $U(x,0) = f(x)$ are obtained. If we call (3.7) and (3.8) systems A and B respectively, then the results will be obtained using the splitting scheme $(A - B - A)$ as stated in (2.1).

3.1. Initial Condition

The initial vector δ_m^0 will be formed using the initial condition $U(x,0) = f(x)$ as follows

$$U(x_m,0) = U_N(x_m,0),\ \ m = 0(1)N, \tag{3.10}$$

$$U_m = \delta_{m-1}^0 + 4\delta_m^0 + \delta_{m+1}^0,$$

$$\begin{aligned}
U_0 &= \delta_{-1}^0 + 4\delta_0^0 + \delta_1^0, \\
U_1 &= \delta_0^0 + 4\delta_1^0 + \delta_2^0, \\
&\vdots \\
U_N &= \delta_{N-1}^0 + 4\delta_N^0 + \delta_{N+1}^0.
\end{aligned}$$

This system consists of $(N+1)$ equations and $(N+3)$ unknown δ_m^0 parameters. The parameters δ_{-1}^0 and δ_{N+1}^0 are calculated from (3.10) using the boundary conditions $U_N''(a,0) = 0$ and $U_N''(b,0) = 0$

$$\begin{aligned}
&\delta_{-1}^0 - 2\delta_0^0 + \delta_1^0 = 0, \\
&\delta_{N-1}^0 - 2\delta_N^0 + \delta_{N+1}^0 = 0.
\end{aligned}$$

Now, a new $(N+1) \times (N+1)$ dimensional solvable matrix is obtained for δ_m^0 parameters.

$$\begin{bmatrix} 6 & 0 & 0 & & \\ 1 & 4 & 1 & & \\ & & \ddots & & \\ & & 1 & 4 & 1 \\ & & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \delta_0^0 \\ \delta_1^0 \\ \vdots \\ \delta_{N-1}^0 \\ \delta_N^0 \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{N-1} \\ U_N \end{bmatrix}.$$

3.2. Von Neumann Stability Analysis

Firstly to investigate the stability of the scheme given in Eq.(3.7) the expression $\delta_m^n = e^{i\beta mh}\xi^n$ is written in Eq.(3.7) for halved time step. If the necessary operations are carried out, one obtains the following equations

$$\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right) = \frac{X - Yi}{X + Yi},$$

$$a_1 = 1 - \frac{6\mu}{h^2} - \frac{3\Delta t}{2h},\ b_1 = 4 + \frac{12\mu}{h^2},\ c_1 = 1 - \frac{6\mu}{h^2} + \frac{3\Delta t}{2h},$$

where $X = (a_1 + c_1)\cos\beta h$ and $Y = (c_1 - a_1)\sin\beta h$. Since the stability condition is provided by inequality $\left|\rho_A\left(\frac{\xi^{n+1}}{\xi^n}\right)\right| \leq 1$, the scheme given in Eq.(3.7) is unconditionally stable. Secondly, for the stability of the scheme given in Eq.(3.8) once the U in the term UU_x has been linearized, z_m will then behave as a local constant and the von Neumann method will become feasible to investigate the stability of the difference scheme given in Eq.(3.8). If the expression $\delta_m^n = e^{i\beta mh}\xi^n$ is written in Eq.(3.8) and the necessary operations are performed, the following equation

$$\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right) = \frac{p - iq}{p + iq},$$

$$a_2 = 1 - \frac{6\mu}{h^2} - \frac{3z_m\Delta t\epsilon}{2h},\ b_2 = 4 + \frac{12\mu}{h^2},\ c_2 = 1 - \frac{6\mu}{h^2} + \frac{3z_m\Delta t\epsilon}{2h}$$

is obtained, where $p = b_2 + (a_2 + c_2)\cos\beta h$ and $q = (a_2 - c_2)\sin\beta h$. Since von Neumann provides the stability condition as $|\xi| \leq 1$, the scheme is unconditionally stable. Thus, the Strang splitting method is unconditionally stable since it provides the condition

$$|\rho(\xi)| \leq \left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)\right|\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)\right| \leq 1.$$

The readers may be suspicious about constructing the solutions from both linear and nonlinear solutions because they produce smooth and discontinuous shock solution within a certain time interval depending on the initial condition. Thus, we refer the readers to two important articles on this topic. Holden *et al.* [9] have stated that if the initial data are sufficiently regular, the Strang splitting method converges to the smooth solution of the full equation if the splitting step size is under control. Another study by Holden *et al.* [10] proved that splitting solution converges to the weak solution of the full equation assuming that the splitting procedure is convergent. As pointed out, lower order splitting methods converge under these conditions and similar results for higher order methods can be expected using stronger assumptions on the smoothness. [26]

4. Numerical Examples and Their Results

We have considered three test problems to observe the effectiveness of the present method. The solution of each problem with cubic B-spline collocation method gives $(N+1) \times (N+1)$ tridiagonal band matrix systems which can be easily and effectively solved by Thomas algorithm. In order to see the difference between numerical solution and analytic solution, we have used the error norms defined as follows

$$L_2 = \sqrt{h\sum_{j=1}^{N}\left[U_j^{exact} - U_j\right]^2},$$
$$L_\infty = \max_j \left|U_j^{exact} - U_j\right|.$$

The RLW equation given in (1.1) satisfies three invariants known as mass, monentum and energy given as follows

$$I_1 = \int_{-\infty}^{+\infty} U dx \simeq h\sum_{j=1}^{N} U_j^n,$$
$$I_2 = \int_{-\infty}^{+\infty} \left[U^2 + \mu(U_x)^2\right] dx \simeq h\sum_{j=1}^{N}\left[(U_j^n)^2 + \mu((U_x)_j^n)^2\right],$$
$$I_3 = \int_{-\infty}^{+\infty} \left[U^3 + 3(U)^2\right] dx \simeq h\sum_{j=1}^{N}\left[(U_j^n)^3 + 3(U_j^n)^2\right].$$

4.1. Single Solitary Movement

Analytical solution for single soliton wave solution of RLW equation is

$$U(x,t) = 3c\sec h^2\left[k(x - x_0 - vt)\right],$$

where $k = \frac{1}{2}(\frac{\varepsilon c}{\mu(1+\varepsilon c)})^{1/2}$, $v = 1 + \varepsilon c$ is wave velocity and $3c$ is wave amplitude. The following initial

$$U(x, 0) = 3c \sec h^2 [k(x - x_0)]$$

and the boundary conditions

$$U(a, t) = U(b, t) = 0$$

are used at the boundaries. Analytic values of the invariants for this problem are

$$I_1 = \frac{6c}{k},\ I_2 = \frac{12c^2}{k} + \frac{48kc^2\mu}{5},\ I_3 = \frac{36c^2}{k}(1 + \frac{4c}{5})$$

given by Zaki [27]. In order to be able to make a comparison with the previous studies, all calculations are made with the values of $\varepsilon = 1, \mu = 1, x_0 = 0$ and $\Delta t = 0.1$. In Table 1, the values of the error norms L_2 and L_∞ and the invariant values at different times in the region $-40 \le x \le 60$ and their comparison with different studies in the literature are given.

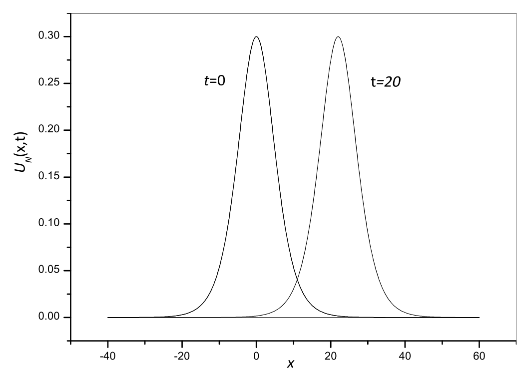

Figure 1. The profile of the single solitary wave at $t = 0$ and $t = 20$.

In Tables 1 and 2, a comparison of the results of the present study with those using and without using splitting techniques has been presented. Table 1 shows a comparison of the calculated invariants and the error norms L_2 and L_∞ for $c = 0.1$ in the region $-40 \le x \le 60$ at various times with some of those available in the literature. As it is clearly seen from Table 1, the invariants remain almost the same as time progresses. Moreover, although the calculated error norm L_2 is greater than that given in Ref. [5]; our error norm L_2 is less than some of those used in our comparison using higher degree spline functions. But, our calculated error norm L_∞ is better than all of those used in our comparison. Again from the table it can be seen that our error norms are better than those calculated using splitting technique (w-Sp) and given in [3]. In Table 2, several comparisons have been made with some of those in the literature at time $t = 20$ for values of $c = 0.1$, $c = 0.03$ and $h = 0.125$ and $h = 0.1$. It is obviously seen from the table that our error norms are relatively low and our results are better than theirs for $c = 0.1$ and in good agreement for $c = 0.03$ even though some of them use higher degrees of spline functions. The error norms L_2 and L_∞ for $c = 0.03$ are generally less than those in other studies, however, greater than those in Ref. [27]. When compared with those in Ref. [20], while our error norm L_2 is worse, our error norm L_∞ is better.

Table 1. A comparison of the invariants and error norms calculated at various times for values of $h = 0.125$, $\Delta t = 0.1$, $c = 0.1$, $\varepsilon = \mu = 1$ in the region $-40 \leq x \leq 60$ for single solitary wave. (w-Sp stands for with splitting)

Time	Method	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
$t = 0$	Present	3.979927	0.810462	2.579007	0.000	0.000
	[4]	3.97993	0.810461	2.57901	0.002	0.007
	[3]w-Sp	3.9799271	0.8104625	2.5790075	0.000	0.000
	[14]	3.9799	0.8104	2.5790	0.000	0.000
	[21]	3.9799271	0.8104625	2.5790075	0.000	0.000
	[5]	3.979927	0.810463	2.579007	0.000	0.000
$t = 4$	Present	3.979954	0.810462	2.579007	0.016917	0.007210
	[4]	3.98039	0.810610	2.57950	0.116	0.054
	[3]w-Sp	3.9799286	0.8104623	2.5790069	0.09380	0.03954
	[14]	3.9800	0.8104	2.5790	0.36556	0.14725
	[21]	3.9799299	0.8104624	2.5790073	0.04080	0.01563
	[5]	3.977092	0.809641	2.576296	0.0006	0.1458
$t = 8$	Present	3.979971	0.810462	2.579007	0.032897	0.014087
	[4]	3.98083	0.810752	2.57996	0.224	0.100
	[3]w-Sp	3.9799259	0.8104621	2.5790062	0.17858	0.07224
	[14]	3.9800	0.8104	2.5791	0.73694	0.30090
	[21]	3.9799282	0.8104624	2.5790070	0.08049	0.03152
	[5]	3.973316	0.808320	2.571938	0.0026	0.5786
$t = 12$	Present	3.979984	0.810462	2.579007	0.047378	0.019864
	[4]	3.98125	0.810884	2.58041	0.325	0.139
	[3]w-Sp	3.9799226	0.8104619	2.5790056	0.24991	0.09654
	[14]	3.9800	0.8104	2.5791	1.0940	0.44214
	[21]	3.9799259	0.8104623	2.5790068	0.11909	0.04663
	[5]	3.979106	0.806774	2.566836	0.0064	0.9223
$t = 16$	Present	3.979987	0.810462	2.579007	0.060388	0.024692
	[4]	3.98165	0.811014	2.58083	0.417	0.171
	[3]w-Sp	3.9799142	0.8104617	2.5790049	0.30791	0.11459
	[14]	3.9800	0.8104	2.5791	1.4340	0.56944
	[21]	3.9799182	0.8104622	2.5790065	0.15629	0.06060
	[5]	3.965344	0.805461	2.562505	0.0115	1.2148
$t = 20$	Present	3.979962	0.810462	2.579007	0.072292	0.028834
	[4]	3.98206	0.811164	2.58133	0.511	0.198
	[3]w-Sp	3.9798830	0.8104616	2.5790043	0.35489	0.12848
	[14]	3.9800	0.8104	2.5791	1.4340	0.56944
	[21]	3.9798879	0.8104622	2.5790063	0.19199	0.07344
	[5]	3.961597	0.804185	2.558292	0.0184	1.5664

Table 2. The error norms L_2 and L_∞ and invariants at time $t = 20$ for values of $\Delta t = 0.1$, $h = 0.125$, 0.1, $c = 0.1$, 0.03 for single solitary wave. (w-Sp stands for with splitting)

Method	$L_2 \times 10^3$	$L_\infty \times 10^3$	I_1	I_2	I_3
$c = 0.1$					
Present	0.072292	0.028834	3.979962	0.810462	2.579007
MQ [11]	0.206910	0.078027	3.9798831	0.81046248	2.5790074
IMQ [11]	0.206912	0.078027	3.9798725	0.81046248	2.5790074
IQ [11]	0.206913	0.078027	3.9798824	0.81046248	2.5790074
GA [11]	0.206911	0.078027	3.9798831	0.81046248	2.5790074
TPS [11]	0.207147	0.078152	3.9798826	0.81046247	2.5790073
[2]	0.30	0.116	3.979883	0.81027618	2.57839258
[5]	0.0184	1.5664	3.961597	0.804185	2.558292
[22]	0.192	0.073	3.97989	0.81046	2.57901
[15]	0.1797446	0.06.799314	3.979913	0.810462	2.579007
[23](QBCM1)	0.215	0.083	3.97995	0.81046	2.57901
[23](QBCM2)w-Sp	0.357	0.129	3.97995	0.81046	2.57901
[6]	0.219	0.086	3.97988	0.810465	2.57901
[27]w-Sp	0.71913	0.25398	3.97989	0.80925	2.57501
[20](SBCM2)w-Sp	0.357	0.130	3.97995	0.81046	2.57901
[20](SEBCM2)w-Sp	0.357	0.130	3.97996	0.81046	2.57901
$c = 0.03$					
Present	0.525073	0.198401	2.109032	0.127302	0.388806
[2]	0.57	0.432	2.104584	0.12729366	0.3887776
[4]	0.535	0.198	2.10906	0.127305	0.388815
[3](QBGM1)	0.558	0.205	2.10460	0.12730	0.38880
[3](QBGM2)w-Sp	0.566	0.207	2.10457	0.12730	0.38880
[27]w-Sp	0.24185	0.12464	2.10741	0.127230	0.38856
[16]	0.550	0.234	2.10461	0.12730	0.38880
[23](QBCM2)W-SP	0.356	0.295	2.10831	0.12913	0.38881
[20](SBCM2)W-SP	0.444	0.419	2.10849	0.12730	0.38881
[20](SEBCM2)W-SP	0.552	0.402	2.10899	0.12730	0.38881
$h = 0.1$(Present)	0.636369	0.233236	2.109509	0.127303	0.388807
[3](QBGM1)	0.560	0.205	2.10459	0.12730	0.38880
[3](QBGM2)W-SP	0.567	0.208	2.10456	0.12730	0.38880
[21]	0.539	0.198	2.10770	0.12730	0.38880
[22]Space Splitting	0.541	0.199	2.10707	0.12730	0.38880
[19]	0.57247	0.36498	2.103622	0.127184	0.3884398
[20](SBCM2)w-Sp	0.556	0.419	2.10904	0.12730	0.38881
[20](SEBCM2)w-Sp	0.647	0.428	2.10943	0.12730	0.38881

4.2. Interaction of Two Solitary Waves

To observe the interaction of two positive solitary waves, RLW equation is taken with boundary conditions $U \to 0$ as $x \to \pm\infty$ and with the initial condition

$$U(x,0) = 3c_1 \sec h^2\left[k_1(x-x_1)\right] + 3c_2 \sec h^2\left[k_2(x-x_2)\right], \qquad (4.1)$$
$$k_1 = \frac{1}{2}\sqrt{\varepsilon c_1/(1+\epsilon c_1)}, k_2 = \frac{1}{2}\sqrt{\varepsilon c_2/(1+\varepsilon c_2)}.$$

Eq. (4.1) denotes the first wave placed at $x = x_1$ with amplitude $3c_1$ and the second wave placed at $x = x_2$ with a amplitude $3c_2$. As it is known, the speed of a wave with larger amplitude is greater than that of another wave whose amplitude is smaller. As a result, choosing $x_1 < x_2$ and $c_2 < c_1$ will ensure the interaction of the two waves as time progresses. In order to observe this phenomenon, parameters in the range of $-200 \leq x \leq 400$ are taken as $x_1 = -177$, $x_2 = -147$, $c_1 = 0.2$, $c_2 = 0.1$, $\varepsilon = 1$, $\mu = 1$, $h = 0.12$ and $\Delta t = 0.1$. From $t = 0$ to $t = 400$, the two simulations have been conducted. As it can be seen in the following figures, the wave having a large amplitude at the time $t = 0$ is located to the left of the wave having small amplitude. As the time progresses, it has been observed that the naturally wave having a large amplitude has interacted into the small wave and then left it behind. At the start, the amplitude of the large wave is 0.6 and the amplitude of the small wave is 0.3. After the interaction, it is seen that the amplitude of the large wave is 0.599797 at $x = 311.44$ and the amplitude of the small wave is 0.299904 at $x = 281.560$ for time $t = 400$.

Table 3. The invariants calculated in the end of the interaction of two solitary waves and a comparison with those in Ref. [6]

t	I_1	I_2	I_3	I_1 [6]	I_2 [6]	I_3 [6]
0	9.858245	3.244789	10.778329	9.85825	3.24481	10.77833
40	9.861351	3.244790	10.778328	9.85833	3.24482	10.77836
80	9.861624	3.244790	10.778323	9.85832	3.24482	10.77834
120	9.861648	3.244791	10.778314	9.85833	3.24486	10.77843
160	9.861650	3.244794	10.778300	9.85833	3.24491	10.77852
200	9.861650	3.244796	10.778290	9.85830	3.24492	10.77851
240	9.861650	3.244794	10.778299	9.85830	3.24489	10.77846
280	9.861650	3.244790	10.778312	9.85829	3.24484	10.77834
320	9.861649	3.244788	10.778319	9.85832	3.24482	10.77833
360	9.861649	3.244798	10.778320	9.85829	3.24479	10.77823
400	9.861648	3.244787	10.778320	9.85830	3.24478	10.77819

Another interaction problem is that the boundary condition is taken as $U \to 0$ when $x \to \pm\infty$ and the initial condition is taken as follows

$$U(x,0) = \sum_{j=1}^{2} 3A_j \sec h^2\left[k_j(x-x_j)\right], \quad A_j = \frac{4k_j^2}{(1-4k_j^2)}, j = 1, 2.$$

In order to observe this interaction problem, we have worked over the region $0 \leq x \leq 120$ and with the parameters $x_1 = 15$, $x_2 = 35$, $k_1 = 0.4$, $k_2 = 0.3$, $\varepsilon = 1$, $\mu = 1$, $h = 0.3$ and $\Delta t = 0.1$. As in the previous interaction problem, the wave with the large amplitude is at the left of the small wave at time $t = 0$, but as

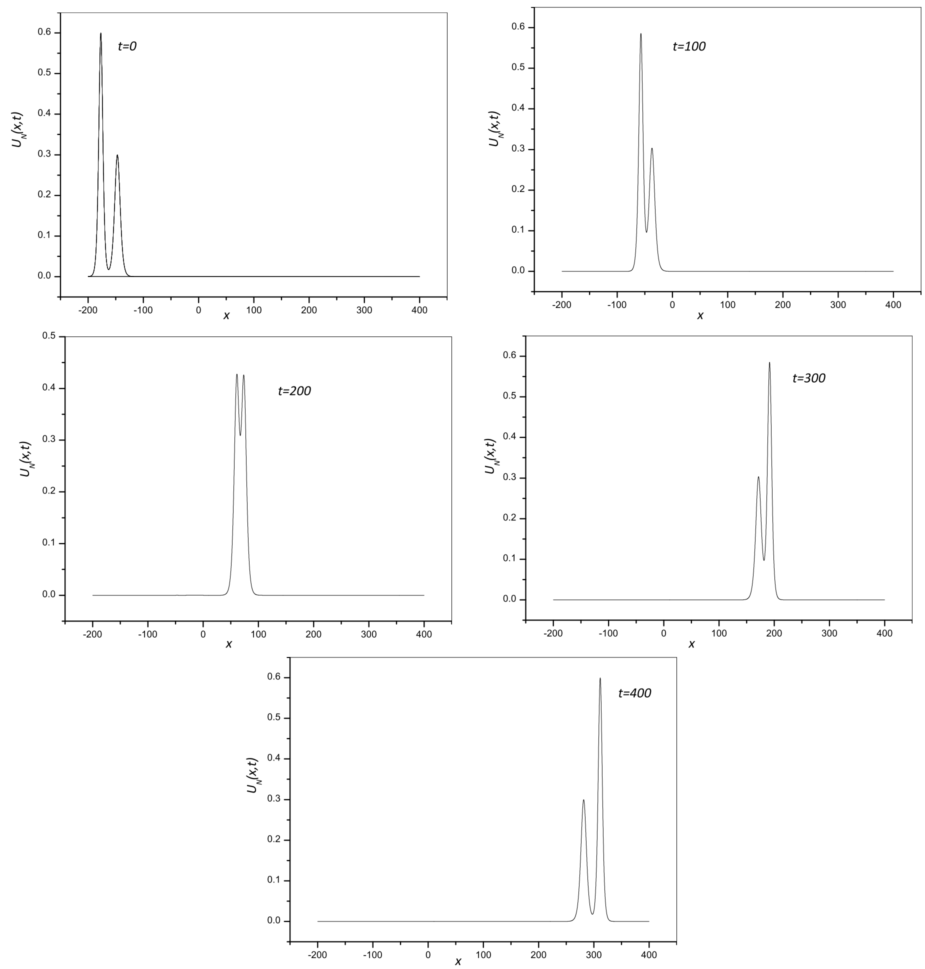

Figure 2. The interaction of two solitary waves in the time interval $0 \leq t \leq 400$.

Table 4. The invariants of the second interaction problem and a comparison with those in Ref. [13]

t	I_1	I_2	I_3	I_1 [13]	I_2 [13]	I_3 [13]
0	37.916522	120.522769	744.081209	37.91648	120.35150	744.08140
2	37.910586	120.486493	744.079530	37.91682	120.35710	744.03870
4	37.904506	120.449938	744.079734	37.91697	120.35840	744.01100
6	37.898508	120.413763	744.078250	37.91709	120.35830	743.97960
8	37.892728	120.379162	744.067803	37.91719	120.35700	743.86790
10	37.887731	120.353147	744.019816	37.91727	120.36380	743.42020
12	37.884625	120.350565	743.886930	37.91733	120.39150	742.33870
14	37.883441	120.357086	743.792636	37.91736	120.41560	741.57810
16	37.882139	120.333975	743.919530	37.91740	120.38860	742.48890
18	37.878825	120.300550	744.034627	37.91741	120.36530	743.47520
20	37.873716	120.266184	744.074575	37.91744	120.35990	743.86380
22	37.867862	120.230706	744.086166	37.91745	120.35940	743.97500
24	37.861791	120.194639	744.090269	37.91746	120.35950	744.00370
25	37.858731	120.176525	744.091498	37.91745	120.35950	744.00850

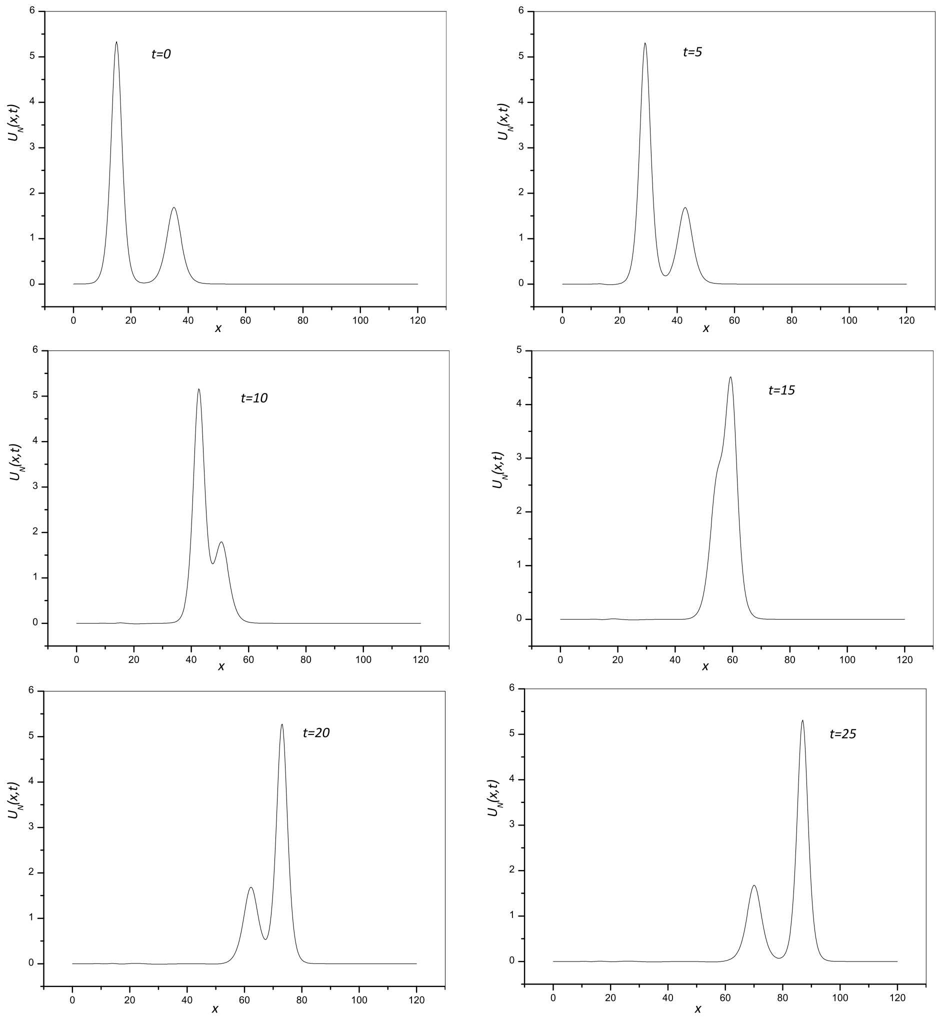

Figure 3. The interaction of two soliatry waves in the time interval $0 \leq t \leq 25$.

the time progresses, the wave with larger amplitude catches the small wave after a certain time, before interaction the amplitude of the larger solitary wave is 5.33338 while the amplitude of the smaller one is 1.68598 at time $t = 0$. The amplitudes measured at time $t = 25$ after the interaction are 5.301403 at $x = 87.0$ and 1.676635 at $x = 70.2$, respectively.

In Tables 3 and 4, we have compared the invariants of two interaction problems available in the literature with those given in Refs. [6] and [13]. As it is seen from these tables, our results are in good agreement with those in these studies.

4.3. The Development of Undular Bore

As a final test problem, the physical boundary conditions of the RLW equation is taken as $U \to U_0$ with $x \to -\infty$ and $U \to 0$ with $x \to \infty$ and with the initial condition

$$U(x,0)=\frac{U_0}{2}\left[1-\tanh(\frac{x-x_0}{d})\right].$$

where $U(x,0)$ measures the height of the water surface at time $t=0$, U_0 is the magnitude of the change in water level centered at $x=x_0$, and d measures the change in vertical dimension. Under the above physical conditions, the invariants I_1, I_2, I_3 are not stable, but they increase linearly in the following proportions during simulations as given in [27].

$$M_1=\frac{d}{dt}I_1=\frac{d}{dt}\int_{-\infty}^{+\infty}U dx=U_0+\frac{\varepsilon}{2}U_0^2,$$

$$M_2=\frac{d}{dt}I_2=\frac{d}{dt}\int_{-\infty}^{+\infty}\left\{U+\mu\left(U_x\right)^2\right\}dx=U_0^2+\frac{2\varepsilon}{3}U_0^3,$$

$$M_3=\frac{d}{dt}I_3=\frac{d}{dt}\int_{-\infty}^{+\infty}\left(U^3+3U^2\right)dx=3U_0^2+(1+2\varepsilon)U_0^3+\frac{3\varepsilon}{4}U_0^4.$$

In order to investigate this problem, all the calculations have to be carried out with $\varepsilon=1.5$, $\mu=\frac{1}{6}$, $U_0=0.1$, $x_0=0$, $h=0.24$, $\Delta t=0.1$, $d=2,5$ and $-36\leq x\leq 300$. The problem is investigated until $t=250$ and the values of I_1, I_2, I_3 of leading undulation for $d=2,5$ wave position and amplitudes are given in Table 5. Numerical variation in I_1, I_2, I_3 values are obtained as $d=2$ for $M_1=0.1075$, $M_2=0.011$, $M_3=0.034095$ and $d=5$ for $M_1=0.1075$, $M_2=0.011$, $M_3=0.0341$.These values are in agreement with the theoretical values $M_1=0.1075$, $M_2=0.011$, $M_3=0.034113$. The values I_1, I_2, I_3 are observed to increase linearly with respect to M_1, M_2, M_3 values, respectively.

5. Conclusions

In this study, we have used finite element cubic B-spline collocation method by combining with the Strang splitting technique which is one of the time splitting techniques in order to solve the RLW equation. We have converted the main problem into two sub problems, and applied our numerical method for obtaining their schemes and von Neumann stability analysis to both of the sub problems. Then we reached to the solution by applying Strang splitting technique to these schemes. We have investigated single solitary wave movement and seen that the wave moves toward right almost without chancing its shape in time. We have successfully achieved the interaction problem of two solitary waves for various parameters in different regions. In wave undulation problem, we have observed that the waves steepen in view of initial condition, the amplitude of the waves increase as time progress, and the waves disappear suddenly after this period of time. Finally, we have calculated the error norms L_2 and L_∞ and seen that they are very low. As a result, the present method is an effective and efficient one in obtaining numerical solutions for a wide range of nonlinear problems.

Table 5. The invariants, wave positions and amplitudes for undular bore problem

t	I_1	I_2	I_3	x	amplitute
$d = 2$					
0	3.612000	0.351478	1.088220	–	–
50	8.987000	0.901476	2.793248	48.9600	0.139631
100	14.362000	1.451475	4.498036	102.7200	0.159078
150	19.737000	2.001473	6.202710	156.9600	0.170753
200	25.112000	2.551471	7.907342	211.2000	0.177385
250	30.486999	3.101469	9.611957	265.9200	0.181884
$d = 5$					
0	3.612000	0.336311	1.040970	–	–
50	8.987000	0.886311	2.746457	48.4800	0.110283
100	14.362000	1.436310	4.451662	102.2400	0.136901
150	19.737000	1.986309	6.156542	156.2400	0.157464
200	25.112000	2.536307	7.861249	210.7200	0.170243
250	30.487000	3.086305	9.565893	264.9600	0.177727
$d = 2$ [6]					
0	3.6120001	0.35148	1.08822	–	–
50	8.9869997	0.90144	2.79314	48.96000	0.13960
100	14.3619987	1.45140	4.49778	102.72000	0.15900
150	19.7369989	2.00134	6.20229	156.96000	0.17065
200	25.1119971	2.55128	7.90675	211.20000	0.17735
250	30.4869971	3.10123	9.61118	265.92000	0.18177
$d = 5$ [6]					
0	3.6120002	0.33631	1.04097	–	–
50	8.9870004	0.88630	2.74643	48.48000	0.11028
100	14.3619996	1.43628	4.45156	102.24000	0.13686
150	19.7369994	1.98624	6.15631	156.24000	0.15741
200	25.1119996	2.53618	7.86086	210.72000	0.17012
250	30.4869998	3.08613	9.56533	264.96000	0.17767

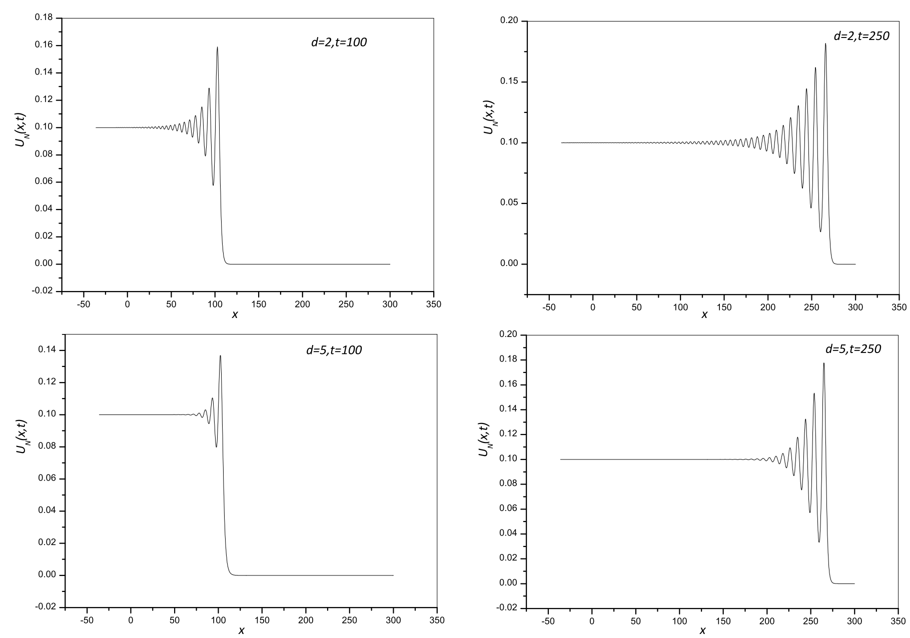

Figure 4. Simulation of the undular bore at times $t = 100$ and $t = 250$ for values of $d = 2$ and $d = 5$.

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