Journal of Applied Analysis and Computation
Volume 9, Number 1, February 2019, 105–117
Website:http://jaac-online.com/
DOI:10.11948/2019.105

ON THE ASYMPTOTIC SOLUTION TO A TYPE OF PIECEWISE-CONTINUOUS SECOND-ORDER DIRICHLET PROBLEMS OF TIKHONOV SYSTEM*

Xutian Qi[1] and Mingkang Ni[1,†]

Abstract A Type of second-order Dirichlet problems of Tikhonov system with piecewise-continuous right hand side is studied. By using the multiscale theory and the theory of contrast structures, a first-order continuous, uniform and effective asymptotic solution of the problem is constructed. Existence of the solution is proved and the remainder is estimated. An illustrative example for explaining this method is also given.

Keywords Tikhonov System, theory of contrast structures, piecewise continuous.

MSC(2010) 93C70.

1. Statement of the Problem

In recent years, close attention has been paid to the study of singularly perturbed dynamical system with discontinuous term, which has widely appeared in the fields of physical and biological researchers. Valuable contributions and achievements were obtained by many researchers in this field [2,3,5,8,9]. At present, three main methods, i.e., the asymptotic theory, the theory of differential inequality and the geometric singular perturbation theory, are used for solving problems of discontinuous singularly perturbed dynamical system. In this paper, by the asymptotic theory, Tikhonov system was studied on the base of Nefedov [6]. For such problem, an asymptotic solution was constructed by using the method of boundary layer function [10,11]. The existence of smooth solution was proved by the “connection method” given in the theory of contrast structures [1,12,13]. Finally, an estimation of remainder of the solution was given.

The following singularly perturbed problem is considered£º

$$\begin{cases}\mu^2 y'' = F(y,z,t,\mu), \quad z'' = G(y,z,t,\mu), \quad 0\le t\le 1,\\ y(0)=y^0, \quad y(1)=y^1, \quad z(0)=z^0, \quad z(1)=z^1,\end{cases} \tag{1.1}$$

† the corresponding author. Email address:xiaovikdo@163.com(M. Ni)
1 School of Mathematical Sciences, East China Normal University, No.500 Dongchuan Rd, 200241 Shanghai, China
* The authors were supported by National Natural Science Foundation of China (11471118, 11871217) and were supported in part by Science and Technology Commission of Shanghai Municipality.

where

$$F(y,z,t,\mu)=\begin{cases}F^{(-)}(y,z,t,\mu), & 0\le t<t_0,\\ F^{(+)}(y,z,t,\mu), & t_0<t\le 1,\end{cases} \tag{1.2}$$

and $0<\mu\ll 1$ is a small parameter, t_0 is a given value, satisfying $0<t_0<1$.

Condition 1: Let $F^{(-)}(y,z,t,\mu)$ be sufficiently smooth on the set $D_1=\{(y,z,t,\mu)|\,|y|\le l,\,|z|\le l,\,t\in[0,t_0]\}$, $F^{(+)}(y,z,t,\mu)$ is sufficiently smooth on the set $D_2=\{(y,z,t,\mu)|\,|y|\le l,\,|z|\le l,\,t\in[t_0,1]\}$, and $G(y,z,t,\mu)$ is sufficiently smooth on the set $D=D_1\cup D_2$. Moreover, $F^{(-)}(y,z,t_0,\mu)\ne F^{(+)}(y,z,t_0,\mu)$ for arbitrary (y,z,t_0,μ) in D, where l is a positive constant.

Condition 2: Let the degenerate function $F(y,z,t,0)=0$ have isolated solutions in D,

$$y=\begin{cases}\varphi^{(-)}(z,t), & 0\le t<t_0,\\ \varphi^{(+)}(z,t), & t_0<t\le 1,\end{cases} \tag{1.3}$$

(which means that $F_y^{(-)}(\varphi^{(-)}(z,t),z,t,0)\ne 0$ in D_1, and $F_y^{(+)}(\varphi^{(+)}(z,t),z,t,0)\ne 0$ in D_2), and for any q_0 the following two problems:

$$\begin{cases}\bar z_0^{(\mp)}(t)''=G(\varphi^{(\mp)}(\bar z_0^{(\mp)},t),\bar z_0^{(\mp)},t,0),\\ \bar z_0^{(-)}(0)=z^0, \quad \bar z_0^{(-)}(t_0)=q_0,\\ (\bar z_0^{(+)}(t_0)=q_0, \quad \bar z_0^{(+)}(1)=z^1),\end{cases} \tag{1.4}$$

have solutions $\bar z_0^{(\mp)}(t)$ such that

$$\varphi^{(-)}(\bar z_0^{(-)}(t_0),t_0)=\varphi^{(-)}(q_0,t_0)\ne\varphi^{(+)}(q_0,t_0)=\varphi^{(+)}(\bar z_0^{(+)}(t_0),t_0).$$

Condition 3: Let the following problems have only trivial solutions:

$$\begin{cases}z^{(\mp)}(t)''=[\bar G_z(t)-\bar G_y(t)\dfrac{\bar F_z^{(\mp)}(t)}{\bar F_y^{(\mp)}(t)}]z^{(\mp)}(t),\\ z^{(-)}(0)(z^{(+)}(t_0))=0, \quad z^{(-)}(t_0)(z^{(+)}(1))=0,\end{cases} \tag{1.5}$$

where $\bar F_{(\cdot)}^{(-)}(t)=F_{(\cdot)}^{(-)}(\bar y_0^{(-)},\bar z_0^{(-)},t,0)$ and $\bar G_{(\cdot)}(t)=G_{(\cdot)}(\bar y_0^{(-)},\bar z_0^{(-)},t,0)$.

Condition 4: Let $F_y^{(\pm)}(\varphi(t),\bar z_0(t),t,0)>0,\ 0\le t\le 1$.

In general, based on the above conditions, sufficiently smooth solutions for problem (1.1) and (1.2) are expected. But considering the fact that the second derivatives are not continuous at the point $t=t_0$, and the "connection method" will be used to sew up at the discontinuous point, the solution found should just be smooth in first derivative, hence, the following definition is given:

Definition 1.1. The function pair $\{y(t,\mu),\,z(t,\mu)\}\in C^1[0,1]\cap(C^2(0,t_0)\cup C^2(t_0,1))$ satisfying problems (1.1)-(1.2) is the solution for the original problem.

2. Algorithm for Construction of Asymptotic Expansion

The original problem is classified into two auxiliary problems, namely the left and the right problems, by taking $t=t_0$ as boundary. Their solutions are found and smoothed by connection. Superscripts of "-" and "+" are added to the functions $y(t)$ and $z(t)$, by taking $y^{(-)}(t)$ and $z^{(-)}(t)$ for the left problem, and $y^{(+)}(t)$ and $z^{(+)}(t)$ for the right problem. The value of the solution at the point $t = t_0$ is assumed as:

$$\begin{cases} y(t_0, \mu) = p(\mu) = p_0 + \mu p_1 + \mu^2 p_2 + \dots, \\ z(t_0, \mu) = q(\mu) = q_0 + \mu q_1 + \mu^2 q_2 + \dots, \end{cases} \tag{2.1}$$

where the coefficients p_i, q_i, $i \in N$ are remained to be determined.

The left problem may be written as:

$$\begin{cases} \mu^2 y^{(-)}(t)'' = F^{(-)}(y^{(-)}, z^{(-)}, t, \mu), \quad z^{(-)}(t)'' = G(y^{(-)}, z^{(-)}, t, \mu), \quad 0 \le t \le t_0, \\ y^{(-)}(0, \mu) = y^0, \quad y^{(-)}(t_0, \mu) = p(\mu), \quad z^{(-)}(0, \mu) = z^0, \quad z^{(-)}(t_0, \mu) = q(\mu), \end{cases} \tag{2.2}$$

and the right problem may be written as:

$$\begin{cases} \mu^2 y^{(+)}(t)'' = F^{(+)}(y^{(+)}, z^{(+)}, t, \mu), \quad z^{(+)}(t)'' = G(y^{(+)}, z^{(+)}, t, \mu), \quad t_0 \le t \le 1, \\ y^{(+)}(t_0, \mu) = p(\mu), \quad y^{(+)}(1, \mu) = y^1, \quad z^{(+)}(t_0, \mu) = q(\mu), \quad z^{(+)}(1, \mu) = z^1. \end{cases} \tag{2.3}$$

The formal asymptotic expansion solutions of the respective constructed left (right) problems with the regular part, the left (right) boundary layer part and the internal layer part are as follows:

$$\begin{cases} y^{(-)}(t, \mu) = \bar{y}^{(-)}(t, \mu) + Ly(\tau_0, \mu) + Q^{(-)}y(\tau, \mu), \\ z^{(-)}(t, \mu) = \bar{z}^{(-)}(t, \mu) + Lz(\tau_0, \mu) + Q^{(-)}z(\tau, \mu), \end{cases} \tag{2.4}$$

$$\begin{cases} y^{(+)}(t, \mu) = \bar{y}^{(+)}(t, \mu) + Ry(\tau_1, \mu) + Q^{(+)}y(\tau, \mu), \\ z^{(+)}(t, \mu) = \bar{z}^{(+)}(t, \mu) + Rz(\tau_1, \mu) + Q^{(+)}z(\tau, \mu), \end{cases} \tag{2.5}$$

where

$$\tau_0 = \frac{t}{\mu}, \quad \tau = \frac{t - t_0}{\mu}, \quad \tau_1 = \frac{t-1}{\mu},$$

$$\bar{y}^{(\pm)}(t, \mu) = \sum_{n=0}^{+\infty} \mu^n \bar{y}_n^{(\pm)}(t), \quad \bar{z}^{(\pm)}(t, \mu) = \sum_{n=0}^{+\infty} \mu^n \bar{z}_n^{(\pm)}(t),$$

$$Ly(\tau_0, \mu) = \sum_{n=0}^{+\infty} \mu^n L_n y(\tau_0), \quad Lz(\tau_0, \mu) = \sum_{n=0}^{+\infty} \mu^n L_n z(\tau_0),$$

$$Ry(\tau_1, \mu) = \sum_{n=0}^{+\infty} \mu^n R_n y(\tau_1), \quad Rz(\tau_1, \mu) = \sum_{n=0}^{+\infty} \mu^n R_n z(\tau_1),$$

$$Q^{(\pm)}y(\tau, \mu) = \sum_{n=0}^{+\infty} \mu^n Q_n^{(\pm)} y(\tau), \quad Q^{(\pm)}z(\tau, \mu) = \sum_{n=0}^{+\infty} \mu^n Q_n^{(\pm)} z(\tau),$$

by substitution of the formal asymptotic solution of (2.4) and (2.5) into the problems of (2.2) and (2.3), and by multiscale separation of variables, the regular part, the boundary layer part and the internal layer part are obtained respectively.

Among which, the regular part is:

$$\mu^2 \bar{y}^{(\mp)}(t)'' = F^{(\mp)}(\bar{y}^{(\mp)}, \bar{z}^{(\mp)}, t, \mu), \quad \bar{z}^{(\mp)}(t)'' = G(\bar{y}^{(\mp)}, \bar{z}^{(\mp)}, t, \mu), \tag{2.6}$$

and the internal layer part is:

$$\begin{aligned}\frac{\mathrm{d}^2 Q^{(\pm)}y}{\mathrm{d}\tau^2} &= F^{(\pm)}(\bar{y}^{(\pm)}(\mu\tau)+Q^{(\pm)}y, \bar{z}^{(\pm)}(\mu\tau)+Q^{(\pm)}z, \mu\tau, \mu)\\ &\quad - F^{(\pm)}(\bar{y}^{(\pm)}(\mu\tau), \bar{z}^{(\pm)}(\mu\tau), \mu\tau, \mu),\\ Q^{(\pm)}y(0,\mu) &= p(\mu), \quad Q^{(\pm)}y(\pm\infty)=0,\\ Q^{(\pm)}z(0,\mu) &= q(\mu), \quad Q^{(\pm)}z(\pm\infty)=0.\end{aligned} \tag{2.7}$$

Since the left (right) boundary layer part is quite similar to the internal layer part, some details of the boundary layer part are omitted in the following. Problems of (2.6) and (2.7) are expanded with respect to μ, and exponential coefficients of same order of μ are compared. Then a series of second-order ordinary differential boundary values of various terms of coefficients for defining the formal asymptotic solutions are obtained. After that, uniform and valid expansion of the formal asymptotic solution is achieved by solving these problems.

Furthermore, coefficient of every term of the formal asymptotic solution is discussed. By expanding the regular term of the left problem with respect to μ, problems used respectively to define $\bar{y}_k^{(-)}(t)$, $\bar{z}_k^{(-)}(t)$, $k \ge 0$, are obtained. Assuming $\bar{z}^{(-)}(0) = z^0$, $\bar{z}_0^{(-)}(t_0) = q_0$, equations and conditions of solutions satisfied with $\bar{y}_0^{(-)}, \bar{z}_0^{(-)}$ are as follows:

$$\begin{cases} 0 = F^{(-)}(\bar{y}_0^{(-)}, \bar{z}_0^{(-)}, t, 0), \quad 0 \le t \le t_0,\\ \bar{z}_0^{(-)}(t)'' = G(\bar{y}_0^{(-)}, \bar{z}_0^{(-)}, t, 0),\\ \bar{z}_0^{(-)}(0) = z^0, \quad \bar{z}_0^{(-)}(t_0) = q_0, \end{cases} \tag{2.8}$$

it is known from condition 2 that $y = \varphi^{(-)}(z,t)$. Putting it into the second equation of (2.8), the following equation is obtained: $\bar{z}_0^{(-)}(t)'' = G(\varphi^{(-)}(\bar{z}_0^{(-)}, t), \bar{z}_0^{(-)}, t, 0)$. By combining this equation with the condition of boundary value, $\bar{z}_0^{(-)}(t)$ can be solved, and $\bar{y}_0^{(-)}(t) = \varphi^{(-)}(\bar{z}_0^{(-)}, t)$ is obtained further.

The equations and conditions of solutions for $\bar{y}_k^{(-)}$, $\bar{z}_k^{(-)}$, $k \ge 1$ are set as($\bar{y}_{-1}^{(-)} = 0$):

$$\begin{cases} \bar{y}_{k-2}^{(-)}(t)'' = \bar{F}_y^{(-)}(t)\bar{y}_k^{(-)} + \bar{F}_z^{(-)}(t)\bar{z}_k^{(-)} + \bar{F}_k^{(-)}(t),\\ \bar{z}_k^{(-)}(t)'' = \bar{G}_y(t)\bar{y}_k^{(-)} + \bar{G}_z(t)\bar{z}_k^{(-)} + \bar{G}_k(t),\\ \bar{z}_k^{(-)}(0) = -L_k z(0), \quad \bar{z}_k^{(-)}(t_0) = q_k - Q_k z^{(-)}(0), \end{cases} \tag{2.9}$$

where $\bar{F}_{(\cdot)}^{(-)}(t) = F_{(\cdot)}^{(-)}(\bar{y}_0^{(-)}, \bar{z}_0^{(-)}, t, 0)$, $\bar{G}_{(\cdot)}(t) = G_{(\cdot)}(\bar{y}_0^{(-)}, \bar{z}_0^{(-)}, t, 0)$.

Since the term $\bar{y}_{k-2}^{(-)}(t)$ is known, the expression of $\bar{y}_k^{(-)}$ as related to $\bar{z}_k^{(-)}$ can be obtained from the first equation of (2.9) as follows:

$$\bar{y}_k^{(-)}(t) = (\bar{y}_{k-2}^{(-)}(t)'' - \bar{F}_z^{(-)}(t)\bar{z}_k^{(-)} - \bar{F}_k^{(-)}(t))/\bar{F}_y^{(-)}(t),$$

by putting it into (2.9) and after some simplification, the following form is obtained:

$$\begin{cases} \bar{z}_k^{(-)}(t)'' = [\bar{G}_z(t) - \bar{G}_y(t)\dfrac{\bar{F}_z^{(-)}(t)}{\bar{F}_y^{(-)}(t)}]\bar{z}_k^{(-)} + [\dfrac{(\bar{y}_{k-2}^{(-)}(t)'' - \bar{F}_k^{(-)}(t))\bar{G}_y(t)}{\bar{F}_y^{(-)}(t)} + \bar{G}_k(t)],\\ \bar{z}_k^{(-)}(0) = -L_k z(0), \quad \bar{z}_k^{(-)}(t_0) = q_k - Q_k^{(-)} z(0), \end{cases} \tag{2.10}$$

as from condition 3, the existence of trivial solution for the linear part of the problem (2.10) is shown, hence solution of problem (2.10) exists Lin etc [4, p65-69]. Meanwhile, it is obvious that the problem (2.10) depends on $L_k z(0)$, $Q_k^{(-)} z(0)$, and the solution of the problem can be found by the Green function method after solving $L_k z(\tau_0)$, $Q_k^{(-)} z(\tau)$. Thus, the terms of $\bar{y}_k^{(-)}(t)$, $\bar{z}_k^{(-)}(t)$, $k \geq 0$ can be defined. For the right problems, similar discussion can be made, and the details are omitted.

The internal layer part is now to be considered. It should be pre-stated that the consensus of " $\pm$ " is strictly adhered in the following discussion, except in special cases. Problem of (2.7) is expanded in respect to the order of μ, and problems to define respectively the terms of $Q_k^{(\pm)} y(\tau)$, $Q_k^{(\pm)} z(\tau)$, $k \geq 0$ are obtained. In which, equations and conditions of solution satisfied with $Q_0^{(\pm)} y(\tau)$, $Q_0^{(\pm)} z(\tau)$ are as follows:

$$\begin{cases} \dfrac{\mathtt{d}^2 Q_0^{(\pm)} y}{\mathtt{d}\tau^2} = F^{(\pm)}(\bar{y}_0^{(\pm)}(t_0) + Q_0^{(\pm)} y, \bar{z}_0^{(\pm)}(t_0) + Q_0^{(\pm)} z, t_0, 0), \quad \dfrac{\mathtt{d}^2 Q_0^{(\pm)} z}{\mathtt{d}\tau^2} = 0, \\ Q_0^{(\pm)} y(0) = p^0 - \bar{y}_0^{(\pm)}(t_0), \quad Q_0^{(\pm)} y(\pm\infty) = 0, \\ Q_0^{(\pm)} z(0) = 0, \quad Q_0^{(\pm)} z(\pm\infty) = 0, \end{cases} \tag{2.11}$$

as known from the second equation of problem (2.11) and from the conditions of boundary value, it is evident that $Q_0^{(\pm)} z(\tau) \equiv 0$, and problem (2.11) may be rewritten as:

$$\begin{cases} \dfrac{\mathtt{d}^2 Q_0^{(\pm)} y}{\mathtt{d}\tau^2} = F^{(\pm)}(\bar{y}_0^{(\pm)}(t_0) + Q_0^{(\pm)} y, q_0, t_0, 0), \\ Q_0^{(\pm)} y(0) = p^0 - \bar{y}_0^{(\pm)}(t_0), \quad Q_0^{(\pm)} y(\pm\infty) = 0, \end{cases} \tag{2.12}$$

by setting $\tilde{y}^{(\pm)}(\tau) = \bar{y}_0^{(\pm)}(t_0) + Q_0^{(\pm)} y(\tau)$, $\tilde{u}^{(\pm)} = \dfrac{\mathtt{d}\tilde{y}^{(\pm)}}{\mathtt{d}\tau}$, problem (2.12) may be written as:

$$\begin{cases} \dfrac{\mathtt{d}\tilde{u}^{(\pm)}}{\mathtt{d}\tau} = F^{(\pm)}(\tilde{y}^{(\pm)}, q_0, t_0, 0), \quad \dfrac{\mathtt{d}\tilde{y}^{(\pm)}}{\mathtt{d}\tau} = \tilde{u}^{(\pm)}, \\ \tilde{y}^{(\pm)}(0) = p_0, \quad \tilde{y}^{(\pm)}(\pm\infty) = \bar{y}_0^{(\pm)}(t_0), \end{cases} \tag{2.13}$$

the eigenvalue of problem (2.13) at the point $(\bar{y}_0^{(\pm)}(t_0), 0)$ is $\lambda = \pm\sqrt{F_y^{(\pm)}(\tilde{y}^{(\pm)}, q_0, t_0, 0)}$, hence, as known from condition 4, the point $(\bar{y}_0^{(\pm)}(t_0), 0)$ are all saddle points.

By dividing the left(right) part of the first equation by the left(right) part of the second equation in (2.13), the following equation is obtained:

$$\frac{\mathtt{d}\tilde{u}^{(\pm)}}{\mathtt{d}\tilde{y}^{(\pm)}} = \frac{F^{(\pm)}(\tilde{y}^{(\pm)}, q_0, t_0, 0)}{\tilde{u}^{(\pm)}},$$

integrating this equation, explicit formulas for the unstable manifold $\Omega^{(-)}$ of the fixed point $\tilde{y}^{(-)} = \bar{y}_0^{(-)}(t_0)$ and for the stable manifold $\Omega^{(+)}$ of the fixed point $\tilde{y}^{(+)} = \bar{y}_0^{(+)}(t_0)$ can be obtained:

$$\Omega^{(-)} : \tilde{u}^{(-)} = \sqrt{2} \left[\int_{\bar{y}_0^{(-)}(t_0)}^{\tilde{y}^{(-)}(\tau)} F^{(-)}(y, q_0, t_0, 0) \, \mathtt{d}y \right]^{\frac{1}{2}},$$

$$\Omega^{(+)}: \tilde{u}^{(+)} = \sqrt{2}\left[\int_{\bar{y}_0^{(+)}(t_0)}^{\tilde{y}^{(+)}(\tau)} F^{(+)}(y, q_0, t_0, 0)\, \mathtt{d}y\right]^{\frac{1}{2}}.$$

The solvability condition for problem (2.12) can be formulated as the condition that the boundary values belong to the domain of attraction of the corresponding roots of the reduced equation. It can be written as follows:

Condition 5: Assume that $\{\tilde{y}^{(\pm)} = p_0\} \bigcap \Omega^{(-)} \neq \emptyset$, and $\{\tilde{y}^{(+)} = p_0\} \bigcap \Omega^{(+)} \neq \emptyset$, which means that

$$\int_{\bar{y}_0^{(-)}(t_0)}^{\tilde{y}^{(-)}} F^{(-)}(y, q_0, t_0, 0)\, \mathtt{d}y > 0 \text{ for all } \tilde{y}^{(-)} \in (\bar{y}_0^{(-)}(t_0), p_0],$$

$$\text{and } \int_{\bar{y}_0^{(+)}(t_0)}^{\tilde{y}^{(+)}} F^{(+)}(y, q_0, t_0, 0)\, \mathtt{d}y > 0 \text{ for all } \tilde{y}^{(+)} \in [p_0, \bar{y}_0^{(+)}(t_0)).$$

Based on the work of Ni etc [7, p24-29], if condition 5 is satisfied, problem (2.11) has the solution $Q_0^{(\pm)}y(\tau)$, and

$$|Q_0^{(-)}y(\tau)| \leq ce^{\kappa\tau}, \quad |Q_0^{(+)}y(\tau)| \leq ce^{-\kappa\tau},$$

where c and κ are given real numbers.

Although the details of the boundary layer part are not expounded, in fact, conditions, similar to condition 5, are also necessary to be fulfilled and consensus of the proposed conditions is same with the only difference of expressing the variables.

After the discussion on $Q_0^{(\pm)}(\tau)$, each term of $Q_k^{(\pm)}(\tau)$, $k \geq 1$ will be studied one by one. As mentioned previously in the expanding of the problem (2.7) with respect to μ, equations and conditions for solution used to define $Q_k^{(\pm)}y(\tau)$, $Q_k^{(\pm)}z(\tau)$, $k \geq 1$ are obtained as follows($Q_{-1}^{(\pm)}y = 0$, $Q_{-1}^{(\pm)}z = 0$):

$$\begin{cases} \dfrac{\mathtt{d}^2 Q_k^{(\pm)}y}{\mathtt{d}\tau^2} = \tilde{F}_y^{(\pm)}(\tau)Q_k^{(\pm)}y + \tilde{F}_z^{(\pm)}(\tau)Q_k^{(\pm)}z + Q_k^{(\pm)}F(\tau), \\ \dfrac{\mathtt{d}^2 Q_k^{(\pm)}z}{\mathtt{d}\tau^2} = \tilde{G}_y(\tau)Q_{k-2}^{(\pm)}y + \tilde{G}_z(\tau)Q_{k-2}^{(\pm)}z + Q_k^{(\pm)}G(\tau), \\ Q_k^{(\pm)}y(0) = p_k - \bar{y}_k^{(\pm)}(t_0), \quad Q_k^{(\pm)}y(\pm\infty) = 0, \\ Q_k^{(\pm)}z(0) = q_k - \bar{z}_k^{(\pm)}(t_0), \quad Q_k^{(\pm)}z(\pm\infty) = 0, \end{cases} \tag{2.14}$$

where $\tilde{F}_{(\cdot)}^{(\pm)}(\tau) = F_{(\cdot)}^{(\pm)}(\bar{y}_0^{(\pm)}(t_0) + Q_0^{(\pm)}y(\tau), q_0, t_0, 0)$, $\tilde{G}_{(\cdot)}(\tau) = G_{(\cdot)}(\bar{y}_0^{(\pm)}(t_0) + Q_0^{(\pm)}y(\tau), q_0, t_0, 0)$, while $Q_k^{(\pm)}F(\tau)$, $Q_k^{(\pm)}G(\tau)$ are known functions depending on $Q_j^{(\pm)}y(\tau)$, $Q_j^{(\pm)}z(\tau)$, $\bar{y}_j^{(\pm)}(t_0)$, $\bar{z}_j^{(\pm)}(t_0)$, $0 \leq j \leq k-1$.

Functions $Q_j^{(\pm)}y(\tau)$ and $Q_j^{(\pm)}z(\tau)$ are exponentially decaying and $\bar{y}_j^{(\pm)}(t_0)$ and $\bar{z}_j^{(\pm)}(t_0)$ are bounded, therefore there exist positive numbers c, κ such that

$$|Q_k^{(-)}F(\tau)| \leq ce^{\kappa\tau}, \ |Q_k^{(+)}F(\tau)| \leq ce^{-\kappa\tau}, \ |Q_k^{(-)}G(\tau)| \leq ce^{\kappa\tau}, \ |Q_k^{(+)}G(\tau)| \leq ce^{-\kappa\tau}.$$

Noticing that the expression of $Q_k^{(\pm)}z(\tau)$ can be found by integrating twice, and $\tilde{u}^{(\pm)}(\tau)$ is a solution of corresponding homogeneous problem of (2.14), by Liouville formula and variation of constants formula Vasil'eva etc [15, p44], the solution of the above problem have the following forms:

$$Q_k^{(\pm)}z(\tau) = \int_{\pm\infty}^{\tau}\int_{\pm\infty}^{\eta}[\tilde{G}_y(s)Q_{k-2}^{(\pm)}y + \tilde{G}_z(s)Q_{k-2}^{(\pm)}z + Q_k^{(\pm)}G(s)]\,\mathtt{d}s\,\mathtt{d}\eta,$$

$$Q_k^{(\pm)}y(\tau) = (p_k - \bar{y}_k^{(\pm)}(t_0))\frac{\tilde{u}^{(\pm)}(\tau)}{\tilde{u}^{(\pm)}(0)} + H_k^{(\pm)}(\tau), \quad \tau \le 0, \tag{2.15}$$

$$H_k^{(\pm)}(\tau) = \tilde{u}^{(\pm)}(\tau)\int_0^{\tau}(\tilde{u}^{(\pm)})^{-2}\int_{\pm\infty}^{\eta}\tilde{u}^{(\pm)}(s)(\tilde{F}_z^{(\pm)}(s)Q_k^{(\pm)}z(s) + Q_k^{(\pm)}F(s))\,\mathtt{d}s\,\mathtt{d}\eta,$$

and estimations for $Q_k^{(\pm)}y(\tau)$, $Q_k^{(\pm)}z(\tau)$ are obtained, and real numbers of c, κ exist, thus giving:

$$|Q_k^{(-)}y(\tau)| \le ce^{\kappa\tau}, \quad |Q_k^{(-)}z(\tau)| \le ce^{\kappa\tau}, \tag{2.16}$$

$$|Q_k^{(+)}y(\tau)| \le ce^{-\kappa\tau}, \quad |Q_k^{(+)}z(\tau)| \le ce^{-\kappa\tau}. \tag{2.17}$$

Although every term of the formal asymptotic solution was discussed, some parameters, such as p_i and q_i, are still unknown. Ways to define these parameters will be carried on in the following. As mentioned before, the solution should be first-order smooth, which means that derivative of the solution at the point $t = t_0$ should be continuous, and the "connection method" can be used to determine these parameters. The parameters p_0 and q_0 are to be defined first. Every term in the following equations

$$\frac{\mathtt{d}}{\mathtt{d}\tau}Q_0^{(-)}y(0) = \frac{\mathtt{d}}{\mathtt{d}\tau}Q_0^{(+)}y(0), \quad z_0^{(-)}(t_0)' = z_0^{(+)}(t_0)', \tag{2.18}$$

is related to p_0. In order to have the expression in terms of p_0, the following form is set:

$$H_1(p_0, q_0) = \int_{\bar{y}_0^{(-)}(t_0)}^{p_0} F^{(-)}(y, q_0, t_0, 0)\,\mathtt{d}y - \int_{\bar{y}_0^{(+)}(t_0)}^{p_0} F^{(+)}(y, q_0, t_0, 0)\,\mathtt{d}y, \tag{2.19}$$

according to the definition of $H_1(p_0, q_0)$, its continuity is ensured by the continuity of $F^{(\pm)}(y, z, t, \mu)$. Owing to the fact that $H_1(\bar{y}_0^{(-)}(t_0), q_0) \le 0$, $H_1(\bar{y}_0^{(+)}(t_0), q_0) \ge 0$, and according to the intermediate value theorem, it is known that $H_1(p_0, q_0) = 0$ has a solution of $p_0 = p_0(q_0)$, satisfying $\bar{y}_0^{(-)}(t_0) < p_0(q_0) < \bar{y}_0^{(+)}(t_0)$, and $\frac{\mathtt{d}}{\mathtt{d}p_0}H_1(p_0, q_0) \ne 0$ because:

$$\frac{\mathtt{d}}{\mathtt{d}p_0}H_1(p_0, q_0) = \frac{F^{(-)}(p_0, q_0, t_0, 0) - F^{(+)}(p_0, q_0, t_0, 0)}{\sqrt{\int_{\bar{y}_0^{(-)}(t_0)}^{p_0} F^{(-)}(p_0, q_0, t_0, 0)\,\mathtt{d}y}} \ne 0, \tag{2.20}$$

which implies that p_0 is locally unique.

Setting:

$$H_2(q_0) = (\bar{z}_0^{(-)})'(t_0, q_0) - (\bar{z}_0^{(+)})'(t_0, q_0), \tag{2.21}$$

then q_0 can be determinated by $H_2(q_0) = 0$.

Condition 6: It is assumed that $H_2(q_0) = 0$ has a solution q_0, and:

$$\frac{\mathtt{d}}{\mathtt{d}q_0}H_2(q_0) = \frac{\mathtt{d}}{\mathtt{d}q_0}[(\bar{z}_0^{(-)})'(t_0, q_0) - (\bar{z}_0^{(+)})'(t_0, q_0)] \ne 0. \tag{2.22}$$

By virtue of condition 6, (p_0, q_0) exists and is locally unique.

After determination of p_0, q_0, p_k and q_k will be explored. Conditions to be satisfied when $k \geq 1$ are considered:

$$\bar{y}_{k-1}^{(-)}(t_0)' + \frac{\mathtt{d}}{\mathtt{d}\tau} Q_k^{(-)} y(0) = \bar{y}_{k-1}^{(+)}(t_0)' + \frac{\mathtt{d}}{\mathtt{d}\tau} Q_k^{(+)} y(0), \tag{2.23}$$

$$\bar{z}_k^{(-)}(t_0)' + \frac{\mathtt{d}}{\mathtt{d}\tau} Q_{k+1}^{(-)} z(0) = \bar{z}_k^{(+)}(t_0)' + \frac{\mathtt{d}}{\mathtt{d}\tau} Q_{k+1}^{(+)} z(0), \tag{2.24}$$

introduce (2.13) (2.15) into (2.23), and the following form is obtained after simplification:

$$\begin{aligned} p_k =& [F^{(-)}(p_0, q_0, t_0, 0) - F^{(+)}(p_0, q_0, t_0, 0)]^{-1} [F^{(-)}(p_0, q_0, t_0, 0) \bar{y}_k^{(-)}(t_0) \\ &- F^{(+)}(p_0, q_0, t_0, 0) \bar{y}_k^{(+)}(t_0) - \int_{-\infty}^{0} \tilde{u}^{(-)}(s) f^{(-)}(s)\, \mathtt{d}s + \int_{+\infty}^{0} \tilde{u}^{(+)}(s) f^{(+)}(s)\, \mathtt{d}s \\ &- \tilde{u}^{(-)}(0) \bar{y}_{k-1}^{(-)}(t_0)' + \tilde{u}^{(+)}(0) \bar{y}_{k-1}^{(+)}(t_0)'], \end{aligned}$$

where $f^{(-)}(s) = \tilde{F}_z^{(-)}(s) Q_k^{(-)} z(s) + Q_k^{(-)} F(s)$, $f^{(+)}(s) = \tilde{F}_z^{(+)}(s) Q_k^{(+)} z(s) + Q_k^{(+)} F(s)$.

Consider (2.24), since

$$\frac{\mathtt{d} Q_{k+1}^{(\pm)} z(0)}{\mathtt{d}\tau} = \int_{\pm\infty}^{0} \tilde{G}_y(s) Q_{k-1}^{(\pm)} y(s) + \tilde{G}_z(s) Q_{k-1}^{(\pm)} z(s) + Q_{k+1}^{(\pm)} G\, \mathtt{d}s, \tag{2.25}$$

is not related to q_k, denote $A = \dfrac{\mathtt{d} Q_{k+1}^{(+)} z(0)}{\mathtt{d}\tau} - \dfrac{\mathtt{d} Q_{k+1}^{(-)} z(0)}{\mathtt{d}\tau}$, it is obvious that A is a constant, hence (2.24) can be rewritten as:

$$\bar{z}_k^{(-)}(t_0)' - \bar{z}_k^{(+)}(t_0)' = A. \tag{2.26}$$

Lemma 2.1. $\dfrac{d}{dq_k} [\bar{z}_k^{(-)}(t_0)' - \bar{z}_k^{(+)}(t_0)'] = \dfrac{d}{dq_0} [\bar{z}_0^{(-)}(t_0)' - \bar{z}_0^{(+)}(t_0)']$ *is true for arbitrary* $k \geq 0$.

Proof. As known from (2.10), the equation used to define $\bar{z}_k^{(-)}(t)$ may be written in the following form:

$$\bar{z}_k^{(-)}(t)'' = A(t) \bar{z}_k^{(-)}(t) + B(t), \tag{2.27}$$

where $A(t) = \bar{G}_z(t) - \bar{G}_y(t) \dfrac{\bar{F}_z^{(-)}(t)}{\bar{F}_y^{(-)}(t)}$, $B(t) = \dfrac{(\bar{y}_{k-2}^{(-)}(t)'' - \bar{F}_k^{(-)}(t)) \bar{G}_y(t)}{\bar{F}_y^{(-)}(t)} + \bar{G}_k(t)$.

Set $\xi = \dfrac{\mathtt{d}}{\mathtt{d}q_k} \bar{z}_k^{(-)}(t)$, and differentiate both sides of equation (2.27) with respect to q_k, problem used for defining ξ is obtained:

$$\xi'' = A(t)\xi, \quad \xi(0) = 0, \quad \xi(t_0) = 1. \tag{2.28}$$

Meanwhile, set $\eta = \dfrac{\mathtt{d}}{\mathtt{d}q_0} \bar{z}_0^{(-)}(t)$, and differentiate both sides of equation (2.8) with respect to q_0, problem used for defining η is obtained:

$$\eta'' = A(t)\eta, \quad \eta(0) = 0, \quad \eta(t_0) = 1. \tag{2.29}$$

By virtue of condition 2, both problem (2.28) and (2.29) have unique solution:

$$\frac{\mathtt{d}}{\mathtt{d}q_k}\bar{z}_k^{(-)}(t_0) = \frac{\mathtt{d}}{\mathtt{d}q_0}\bar{z}_0^{(-)}(t_0).$$

Similarly, it can be proved:

$$\frac{\mathtt{d}}{\mathtt{d}q_k}\bar{z}_k^{(+)}(t_0) = \frac{\mathtt{d}}{\mathtt{d}q_0}\bar{z}_0^{(+)}(t_0),$$

□

Since both the equation and boundary value conditions of problem (2.10) for defining $\bar{z}_k^{(-)}$ are linear, the following form is obtained:

$$\bar{z}_k^{(-)}(t_0)' - \bar{z}_k^{(+)}(t_0)' = \frac{\mathtt{d}}{\mathtt{d}q_k}[\bar{z}_k^{(-)}(t_0)' - \bar{z}_k^{(+)}(t_0)']q_k. \tag{2.30}$$

As known from Lemma, equation (2.26) may be transformed into:

$$\frac{\mathtt{d}}{\mathtt{d}q_0}[\bar{z}_0^{(-)}(t_0)' - \bar{z}_0^{(+)}(t_0)']q_k = A,$$

hence,

$$q_k = [\frac{\mathtt{d}}{\mathtt{d}q_0}\bar{z}_0^{(-)}(t_0)' - \frac{\mathtt{d}}{\mathtt{d}q_0}\bar{z}_0^{(+)}(t_0)']^{-1}A, \tag{2.31}$$

then, the parameters p_k and q_k are defined.

3. Existence of Solutions and Estimation of Remainder

The left and right problems above are all ordinary problem of boundary value of Tikhonov system. Both the existence of solutions and the estimation of remainder are obvious (see Vasil'eva etc [10, p97] and Vasil'eva etc [14, p36–44]), what needs prompt consideration are the existence and uniqueness and the asymptotic estimation of internal layer. Thus, the assumption (2.1) is rewritten as:

$$\begin{cases} y(t_0,\mu) = p(\mu) = p_0 + \mu p_1 + \mu^2 p_2 + \ldots + \mu^{n+1}(p_{n+1} + \delta_1), \\ z(t_0,\mu) = q(\mu) = q_0 + \mu q_1 + \mu^2 q_2 + \ldots + \mu^{n+1}(q_{n+1} + \delta_2), \end{cases} \tag{3.1}$$

where $p_j, q_j,\ j \le n+1$ are known quantities. According to existence of solutions and estimation of remainder of left and right problems, the expression of asymptotic solution is given as:

$$\begin{cases} y^{(-)}(t,\mu) = \sum\limits_{k=0}^{n+1}\mu^k(\bar{y}_k^{(-)}(t) + L_k y(\tau_0) + Q_k^{(-)}y(\tau) + O(\mu^{n+1}), \quad 0 \le t \le t_0, \\ y^{(+)}(t,\mu) = \sum\limits_{k=0}^{n+1}\mu^k(\bar{y}_k^{(+)}(t) + R_k y(\tau_1) + Q_k^{(+)}y(\tau) + O(\mu^{n+1}), \quad t_0 \le t \le 1, \\ z^{(-)}(t,\mu) = \sum\limits_{k=0}^{n+1}\mu^k(\bar{z}_k^{(-)}(t) + L_k z(\tau_0) + Q_k^{(-)}z(\tau) + O(\mu^{n+1}), \quad 0 \le t \le t_0, \\ z^{(+)}(t,\mu) = \sum\limits_{k=0}^{n+1}\mu^k(\bar{z}_k^{(+)}(t) + R_k z(\tau_1) + Q_k^{(+)}z(\tau) + O(\mu^{n+1}), \quad t_0 \le t \le 1. \end{cases} \tag{3.2}$$

Since $y(t,\mu), z(t,\mu)$ are first-order smooth at the point $t=t_0$, above solutions should satisfy:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y^{(-)}(t_0,\mu) = \frac{\mathrm{d}}{\mathrm{d}t} y^{(+)}(t_0,\mu), \\ \frac{\mathrm{d}}{\mathrm{d}t} z^{(-)}(t_0,\mu) = \frac{\mathrm{d}}{\mathrm{d}t} z^{(+)}(t_0,\mu). \end{cases} \tag{3.3}$$

Let $M(p,\mu) = \frac{\mathrm{d}}{\mathrm{d}t} y^{(-)}(t_0,\mu) - \frac{\mathrm{d}}{\mathrm{d}t} y^{(+)}(t_0,\mu)$, $N(q,\mu) = \frac{\mathrm{d}}{\mathrm{d}t} z^{(-)}(t_0,\mu) - \frac{\mathrm{d}}{\mathrm{d}t} z^{(+)}(t_0,\mu)$, and after some simplification, the following form is obtained:

$$\begin{aligned} M(p,\mu) &= \mu^{n+1}[\bar{y}_n^{(-)}(t_0)' - \bar{y}_n^{(+)}(t_0)' + \frac{\mathrm{d}}{\mathrm{d}\tau} Q_{n+1}^{(-)} y(0) - \frac{\mathrm{d}}{\mathrm{d}\tau} Q_{n+1}^{(+)} y(0)] + O(\mu^{n+1}) \\ &= \mu^{n+1}[\frac{F^{(-)}(p_0,q_0,t_0,0) - F^{(+)}(p_0,q_0,t_0,0)}{\tilde{u}(0)} \delta_1] + O(\mu^{n+1}), \\ N(q,\mu) &= \mu^{n+1}[\bar{z}_n^{(-)}(t_0)' - \bar{z}_n^{(+)}(t_0)' + \frac{\mathrm{d}}{\mathrm{d}\tau} Q_{n+1}^{(-)} z(0) - \frac{\mathrm{d}}{\mathrm{d}\tau} Q_{n+1}^{(+)} z(0)] + O(\mu^{n+1}) \\ &= \mu^{n+1}[\frac{\mathrm{d}}{\mathrm{d}q_0}[\bar{z}_0^{(-)}(t_0)' - \bar{z}_0^{(+)}(t_0)']\delta_2] + O(\mu^{n+1}), \end{aligned}$$

based on condition 1, $F^{(-)}(p_0,q_0,t_0,0) \neq F^{(+)}(p_0,q_0,t_0,0)$, which means that the coefficient of δ_1 is not equal to zero, thus the unique solution $\delta_1 = \delta_1^* = O(1)$ can be found. Similarly, based on condition 6, $\frac{\mathrm{d}}{\mathrm{d}q_0}[\bar{z}_0^{(-)}(t_0)' - \bar{z}_0^{(+)}(t_0)'] \neq 0$, which means that the unique solution $\delta_2 = \delta_2^* = O(1)$ can be found. Hence, the smoothness of the solution at the point $t = t_0$ is guaranteed by these facts.

Theorem 3.1. *Under conditions 1-6, asymptotic solutions of the problems expounded in this paper exists on [0,1], and can be expressed in the following form:*

$$y(t,\mu) = \begin{cases} y^{(-)}(t,\mu) = \sum_{k=0}^{n+1} \mu^k (\bar{y}_k^{(-)}(t) + L_k y(\tau_0) + Q_k^{(-)} y(\tau) + O(\mu^{n+1}), \ 0 \le t \le t_0, \\ y^{(+)}(t,\mu) = \sum_{k=0}^{n+1} \mu^k (\bar{y}_k^{(+)}(t) + R_k y(\tau_1) + Q_k^{(+)} y(\tau) + O(\mu^{n+1}), \ t_0 \le t \le 1, \end{cases}$$

$$z(t,\mu) = \begin{cases} z^{(-)}(t,\mu) = \sum_{k=0}^{n+1} \mu^k (\bar{z}_k^{(-)}(t) + L_k z(\tau_0) + Q_k^{(-)} z(\tau) + O(\mu^{n+1}), \ 0 \le t \le t_0, \\ z^{(+)}(t,\mu) = \sum_{k=0}^{n+1} \mu^k (\bar{z}_k^{(+)}(t) + R_k z(\tau_1) + Q_k^{(+)} z(\tau) + O(\mu^{n+1}), \ t_0 \le t \le 1. \end{cases}$$

4. Example

The Dirichlet problem with boundary value conditions is considered:

$$\begin{cases} \mu^2 y'' = F(y,z,t,\mu), \quad z'' = y + z, \quad 0 \le t \le 1, \\ y(0) = -1, \quad y(1) = 2, \quad z(0) = 0, \quad z(1) = 1, \end{cases} \tag{4.1}$$

where

$$F(y,z,t,\mu) = \begin{cases} y + z + 2, \quad 0 \le t \le \frac{1}{2}, \\ y + z - 2, \quad \frac{1}{2} \le t \le 1. \end{cases} \tag{4.2}$$

Step 1: When $\mu = 0$, $\varphi^{(-)}(z) = -z - 2$, $\varphi^{(+)}(z) = 2 - z$ can be found.

Step 2: Introduce $\varphi^{(\pm)}$ into problem (4.1), two problems used to define $\bar{z}_0^{(\pm)}$ are obtained:

$$\bar{z}_0^{(-)}(t)'' = -2, \quad \bar{z}_0^{(-)}(0) = 0, \quad \bar{z}_0^{(-)}(\frac{1}{2}) = q_0, \tag{4.3}$$

$$\bar{z}_0^{(+)}(t)'' = 2, \quad \bar{z}_0^{(+)}(\frac{1}{2}) = q_0, \quad \bar{z}_0^{(+)}(1) = 1, \tag{4.4}$$

the following form is obtained by solving (4.3) and (4.4):

$$\begin{cases} \bar{z}_0^{(-)}(t) = -t^2 + (2q_0 + \frac{1}{2})t, \\ \bar{z}_0^{(+)}(t) = t^2 + (\frac{1}{2} - 2q_0)t + 2q_0 - \frac{1}{2}, \end{cases} \tag{4.5}$$

from the formula (2.21), $q_0 = \dfrac{1}{2}$ is found, hence:

$$\begin{cases} \bar{z}_0^{(-)}(t) = -t^2 + \frac{3}{2}t, \\ \bar{z}_0^{(+)}(t) = t^2 - \frac{1}{2}t + \frac{1}{2}, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \bar{y}_0^{(-)}(t) = t^2 - \frac{3}{2}t - 2, \\ \bar{y}_0^{(+)}(t) = -t^2 + \frac{1}{2}t + \frac{3}{2}. \end{cases} \tag{4.7}$$

Step 3: By substitution of (4.7) into left and right boundary value problem, two problems used to define $L_0y(\tau_0)$, $R_0y(\tau_1)$ are obtained:

$$\frac{\mathtt{d}^2 L_0 y}{\mathtt{d}\tau_0^2} = L_0 y, \quad L_0 y(0) = 1, \quad L_0 y(+\infty) = 0, \tag{4.8}$$

$$\frac{\mathtt{d}^2 R_0 y}{\mathtt{d}\tau_1^2} = R_0 y, \quad R_0 y(0) = 1, \quad R_0 y(-\infty) = 0, \tag{4.9}$$

and by solving problems (4.8) and (4.9), $L_0y(\tau_0) = e^{-\tau_0}$, $R_0y(\tau_1) = e^{\tau_1}$ are obtained.

Step 4: By substitution of (4.7) into problem (2.11), two problems used to define $Q_0^{(-)}y(\tau)$, $Q_0^{(+)}y(\tau)$ are obtained:

$$\frac{\mathtt{d}^2 Q_0^{(-)} y}{\mathtt{d}\tau^2} = Q_0^{(-)} y, \quad Q_0^{(-)} y(0) = p_0 + \frac{5}{2}, \quad Q_0^{(-)} y(-\infty) = 0, \tag{4.10}$$

$$\frac{\mathtt{d}^2 Q_0^{(+)} y}{\mathtt{d}\tau^2} = Q_0^{(+)} y, \quad Q_0^{(+)} y(0) = p_0 - \frac{3}{2}, \quad Q_0^{(+)} y(+\infty) = 0, \tag{4.11}$$

by solving problems (4.10) and (4.11), $Q_0^{(-)}y(\tau) = (p_0 + \dfrac{5}{2})e^{\tau}$, $Q_0^{(+)}y(\tau) = (p_0 - \dfrac{3}{2})e^{-\tau}$ are found.

According to formula (2.18), $p_0 = -\dfrac{1}{2}$. Hence, $Q_0^{(-)}y(\tau) = 2e^{\tau}$, $Q_0^{(+)}y(\tau) = -2e^{-\tau}$.

Step 5:
The solutions of problem (4.1)-(4.2) can be written in the following form:

$$y(t,\mu)=\begin{cases} t^2-\dfrac{3}{2}t-2+e^{-\frac{t}{\mu}}+2e^{\frac{t-\frac{1}{2}}{\mu}}+O(\mu), & 0\le t\le \dfrac{1}{2},\\ -t^2+\dfrac{1}{2}t+\dfrac{3}{2}+e^{\frac{t-1}{\mu}}-2e^{\frac{\frac{1}{2}-t}{\mu}}+O(\mu), & \dfrac{1}{2}\le t\le 1, \end{cases} \tag{4.12}$$

$$z(t,\mu)=\begin{cases} -t^2+\dfrac{3}{2}t+O(\mu), & 0\le t\le \dfrac{1}{2},\\ t^2-\dfrac{1}{2}t+\dfrac{1}{2}+O(\mu), & \dfrac{1}{2}\le t\le 1. \end{cases} \tag{4.13}$$

Acknowledgements. The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

References

[1] V. F. Butuzov, A. B. Vasil'eva and N. N. Nefedov, *Asymptotic theory of contrasting structures*, Automation & Remote Control, 1997, 58(7), 1068–1091.

[2] C. A. Buzzi, P. R. da Silva and M. A. Teixeira, *A Singular approach to discontinuous vector fields on the plane*, Journal of Differential Equations, 2006, 231(2), 633–655.

[3] Y. Cheng and Q. Zhang, *Local analysis of the local discontinuous Galerkin method with generalized alternating numerical flux for one-dimensional singularly perturbed problem*, 2017, 72(2), 792–819.

[4] W. Lin, Z. Wang and J. Zhang, *Ordinary Differential Equations*, Science Press, Beijing, 2003.

[5] N. T. Levashova, N. N. Nefedov and A. O. Orlov, *Time-independent reaction-diffusion equation with a discontinuous reactive term*, Computational Mathematics and Mathematical Physics, 2017, 57(5), 854–866.

[6] N. N. Nefedov and M. Ni, *Internal layers in the one-dimensional reaction-diffusion equation with a discontinuous reactive term*, Computational Mathematics & Mathematical Physics, 2015, 55(12), 2001–2007.

[7] M. Ni and W. Ni, *The Asymptotic theory of singular perturbation problems*, Higher Education Press, Beijing, 2009.

[8] O. E. Omel'chenko, L. Recke and V. F. Butuzov, *Time-periodic boundary layer solutions to singularly perturbed parabolic problems*, Journal of Differential Equations, 2017, 262(9), 4823–4862.

[9] T. Prabha, M. Chandru and V. Shanthi, *Hybird difference scheme for singularly perturbed reaction-convection-diffusion problem with boundary and interior layers*, Applied Mathematics and Computation, 2017, 314, 237–256.

[10] A. B. Vasil'eva and V. F. Butuzov, *Asymptotic expansions of solutions of singularly perturbed equations(in Russian)*, Nauka, Moscow, 1973.

[11] A. B. Vasil'eva, V. F. Butuzov and N. N. Nefedov, *Singularly perturbed problems with boundary and internal layers*, Proceedings of the Steklov Institute of Mathematics, 2010, 268(1), 258–273.

[12] A. B. Vasil'eva, V. F. Butuzov and N. N. Nefedov, *Contrasting structures in singularly perturbed problems*, Fundam. Prikl. Mat, 1998, 4(3), 799–851.

[13] A. B. Vasil'eva, V. F. Butuzov and N. N. Nefedov, *Singularly perturbed problems with boundary and internal layers*, Proceedings of the Steklov Institute of Mathematics, 2010, 268(1), 258–273.

[14] A. B. Vasil'eva and V. F. Butuzov, *Asymptotic methods of singular perturbation theory(in Russian)*, Vysshaya Shkola, Moscow, 1990.

[15] A. B. Vasil'eva and V. F. Butuzov, *Asymptotic methods in singularly perturbed problems(in Russian)*, Yaroslavl, 2014.